Noncommutative AdS_2/CFT_1 duality: The case of massless scalar fields

A. Pinzul[1,*] and A. Stern[2,†]

[1]*Universidade de Brasília, Instituto de Física 70910-900 Brasília, DF, Brazil and International Center of Physics, C.P. 04667 Brasília, DF, Brazil*
[2]*Department of Physics, University of Alabama, Tuscaloosa, Alabama 35487, USA*
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We show how to construct correlators for the CFT_1 which is dual to noncommutative AdS_2 ($nc\mathrm{AdS}_2$). We do it explicitly for the example of the massless scalar field on Euclidean $nc\mathrm{AdS}_2$. $nc\mathrm{AdS}_2$ is the quantization of AdS_2 that preserves all the isometries. It is described in terms of the unitary irreducible representations, more specifically discrete series representations, of $so(2,1)$. We write down symmetric differential representations for the discrete series and then map them to functions on the Moyal-Weyl plane. The Moyal-Weyl plane has a large distance limit which can be identified with the boundary of $nc\mathrm{AdS}_2$. Killing vectors can be constructed on $nc\mathrm{AdS}_2$ which reduce to the AdS_2 Killing vectors near the boundary. We, therefore, conclude that $nc\mathrm{AdS}_2$ is asymptotically AdS_2, and so the AdS/CFT correspondence should apply. For the example of the massless scalar field on Euclidean $nc\mathrm{AdS}_2$, the on-shell action, and resulting two-point function for the boundary theory, are computed to leading order in the noncommutativity parameter. The computation is nontrivial because nonlocal interactions appear in the Moyal-Weyl description. Nevertheless, the result is remarkably simple and agrees with that of the commutative scalar field theory, up to a rescaling.

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I. INTRODUCTION

The AdS/CFT correspondence [1] has been one of the main themes in theoretical physics for the last 20 years (see, e.g. [2] for some recent review). This conjectured correspondence is the explicit realization of the holographic principle [3,4]. In the case of the AdS/CFT correspondence this principle is realized in the form of the weak/strong duality between the quantum gravity in the bulk of an asymptotically AdS space and a conformal field theory (CFT) on the conformal boundary of this space. The original proposal was made for the case of $AdS_5 \times S^5$ geometry, in addition to a variety of asymptotically AdS spaces of different dimensions. A well studied case is that of AdS_3/CFT_2 correspondence. This is due to the fact that the conformal symmetry in two dimensions is infinite dimensional and, as a consequence, the corresponding CFTs are very well studied. It would seem that going one dimension down should simplify things even more. Unfortunately, this is not the case. AdS_2/CFT_1 correspondence [5] appears far from being settled. There are several reasons why this seemingly simple case is more complicated on both sides of the duality. For example, the geometry of AdS_2 is distinct from AdS_n, $n > 2$ because it has two disconnected timelike boundaries. On the CFT side, there is a realization of CFT_1 (which is actually conformal quantum mechanics rather then field theory), the de Alfaro-Fubini-Furlan (dAFF) model, which has been known for some time [6]. Although it lacks an $SO(2,1)$-invariant ground state, it was argued in [7] that despite this fact one still can have correlators consistent with the correspondence. Another realization of CFT_1 is matrix quantum mechanics, which is obtained from the dimensional reduction of ten-dimensional super-Yang-Mills theory [8–12]. Recently a completely different realization of AdS_2/CFT_1 was suggested in [13,14]. There it was conjectured that gravity on (nearly) AdS_2 is dual to the so-called Sachdev-Ye-Kitaev models (see references in [13,14]). Though this proposal has attracted considerable attention, in general, the case of AdS_2/CFT_1 correspondence is still begging for better understanding. In this situation any effort in this direction should be welcome.

In this paper, we want to study aspects of the AdS_2/CFT_1 correspondence in a noncommutative setting, namely when the geometry on the gravity side of the correspondence is replaced by the noncommutative version of (Euclidean) AdS_2. In this regard, two questions naturally arise:

(1) Why would one like to make the geometry noncommutative?
(2) How can we study the noncommutative generalization of the correspondence when, as we mentioned above, even the commutative case is not yet settled?

Concerning the first question, we can argue as follows. There is a general belief (supported by multiple arguments) [15] that the quasiclassical regime of quantum gravity should appear as a quantum field theory on some noncommutative background. In this regard, making the AdS_2 space noncommutative should correspond to the inclusion of some quantum gravitational corrections. Since it is

*apinzul@unb.br
†astern@ua.edu

conjectured that the AdS/CFT correspondence is exact even at the quantum level, it is worthwhile to take noncommutative effects into account in order to see explicitly how (or if) the correspondence applies. AdS_2 is an ideal candidate for examining noncommutative effects. This is because it is possible to construct a noncommutative version, $nc\mathrm{AdS}_2$, of AdS_2 in such a manner that preserves the isometry group $SO(2,1)$ [16–19]. (This does not mean that the Killing vectors retain their commutative form under the deformation.) A similar example of a noncommutative space is the fuzzy sphere S_F^2 [20–27]. (See [23] for an explicit efficient construction for recovering the commutative limit.) Like $nc\mathrm{AdS}_2$, it has the feature of an undeformed isometry, which proved to be both physically and mathematically useful. Unlike S_F^2, the notion of a boundary can be defined for $nc\mathrm{AdS}_2$, and this is done purely in terms of states of the unitary irreducible representations (UIRR's) of $SO(2,1)$, or more generally its universal cover. The noncommutative version of Killing vectors for AdS_2 reduces to the commutative form at the boundary. In this sense, $nc\mathrm{AdS}_2$ can be said to be asymptotically AdS_2, and the AdS/CFT correspondence principle should then be applicable.

We have only a partial answer to the second question. Of course, we will not be able to construct the full correspondence. Instead, our goal is more modest: We want to study the perturbative corrections to the correlator functions of operators on the boundary induced by the bulk-to-boundary and bulk-to-bulk propagators, and see if they preserve the form which is compatible with conformal symmetry. It is possible that the conformal symmetry gets deformed, and this was recently shown in [28] where a model of conformal quantum mechanics in κ-spacetime was considered. This led to noncommutative corrections to the scaling dimensions. We will see, on the other hand, that such a result does not follow from our construction of $nc\mathrm{AdS}_2$, which is essentially unique when one insists on preserving the isometry group when passing to the noncommutative theory.

We shall assume the usual prescription for the AdS/CFT correspondence, namely, that the connected correlation functions for operators $\mathcal{O}$ spanning the CFT are generated by the on-shell field theory action on the corresponding asymptotically AdS space, and that the boundary values ϕ_0 of the fields are sources associated with $\mathcal{O}$. In this article, we specialize to the case of a single massless scalar field. This provides a particularly simple example, in part because of the fact that solutions to the field equation on AdS_2 are regular at the boundary, i.e., $|\phi_0| < \infty$. Moreover, we find that this property is preserved when passing to the noncommutative theory.

Before going to the noncommutative theory, we first briefly recall how the correspondence works for a massless scalar field $\Phi^{(0)}$ on Euclidean AdS_2. One starts with the action

$$S[\Phi^{(0)}] = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}_+} dtdz\{(\partial_z\Phi^{(0)})^2 + (\partial_t\Phi^{(0)})^2\}, \quad (1.1)$$

where it is convenient to use Fefferman-Graham coordinates, (z,t), $z\geq 0, -\infty < t < \infty$, which we review in Sec. II. The AdS_2 boundary occurs at $z=0$. Variations $\delta\Phi^{(0)}$ of $\Phi^{(0)}$ in (1.1) give

$$\delta S[\Phi^{(0)}] = -\int_{\mathbb{R}\times\mathbb{R}_+} dtdz\delta\Phi^{(0)}(\partial_z^2 + \partial_t^2)\Phi^{(0)} - \int_{\mathbb{R}} dt(\partial_z\Phi^{(0)}\delta\Phi^{(0)})|_{z=0}. \quad (1.2)$$

Extremizing the action with Dirichlet boundary conditions yields the field equation

$$\Box\Phi^{(0)} = (\partial_z^2 + \partial_t^2)\Phi^{(0)} = 0. \quad (1.3)$$

Since the equation is second order, we should impose two boundary conditions to obtain a unique solution. Solutions which are everywhere (and in particular at $z\to\infty$) regular can be expressed in terms of the boundary value of the field, $\phi_0(t) = \Phi^{(0)}(0,t)$, using the boundary-to-bulk propagator [29][1]

$$\Phi^{(0)}(z,t) = \int_{\mathbb{R}} dt' K(z,t;t')\phi_0(t'),$$
$$K(z,t;t') = \frac{z/\pi}{z^2 + (t-t')^2}. \quad (1.4)$$

Denote such solutions by $\Phi_{\mathrm{sol}}[\phi_0]$. They are then substituted back into the action (1.1), which can also be written as

$$S[\Phi^{(0)}] = -\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}_+} dtdz\Phi^{(0)}\Box\Phi^{(0)} - \frac{1}{2}\int_{\mathbb{R}} dt(\Phi^{(0)}\partial_z\Phi^{(0)})|_{z=0} \quad (1.5)$$

to obtain the on-shell action. This leaves only the boundary term

[1]This result is simple to show in two dimensions: (1.4) is a solution to the field equation (1.3) since it can be written as $\Phi^{(0)}(z,t) = f(t+iz) + g(t-iz)$, where $f(t+iz) = \frac{i}{2\pi}\int_{\mathbb{R}} dt' \frac{\phi_0(t')}{t+iz-t'}$ and $g(t-iz) = f(t+iz)^*$. In the limit $z\to 0$, the Sokhotski formula gives $\frac{1}{t+iz-t'} \to -i\pi\delta(t-t') + \mathcal{P}(\frac{1}{t-t'})$, where $\mathcal{P}$ denotes the principal value, and so $\lim_{z\to 0}\Phi^{(0)}(z,t) = \phi_0(t)$. To see that (1.4) is regular at $z\to\infty$, we can write it as

$$\Phi^{(0)}(z,t) = \frac{\pi}{z}\int_{\mathbb{R}} dt' \frac{\phi_0(t')}{1 + \frac{(t-t')^2}{z^2}},$$

which for suitable ϕ_0 tends to zero as $z\to\infty$.

$$S[\Phi_{\rm sol}[\phi_0]] = -\frac{1}{2}\int_{\mathbb{R}} dt \Phi_{\rm sol}[\phi_0]\partial_z\Phi_{\rm sol}[\phi_0]|_{z=0} \quad (1.6)$$

$$= -\frac{1}{2\pi}\int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \frac{\phi_0(t)\phi_0(t')}{(t-t')^2}. \quad (1.7)$$

In the AdS/CFT correspondence, one identifies $S[\Phi_{\rm sol}[\phi_0]]$ with the generating functional of the n-point connected correlation functions for the operators $\mathcal{O}$ associated with ϕ_0. Here, both $\mathcal{O}$ and ϕ_0 are functions of only t,

$$\langle \mathcal{O}(t_1)\cdots\mathcal{O}(t_n)\rangle = \frac{\delta^n S[\Phi_{\rm sol}[\phi_0]]}{\delta\phi_0(t_1)\cdots\delta\phi_0(t_n)}\bigg|_{\phi_0=0}. \quad (1.8)$$

So the two-point function in this example is

$$\langle \mathcal{O}(t)\mathcal{O}(t')\rangle = -\frac{1}{2\pi}\frac{1}{(t-t')^2}. \quad (1.9)$$

The goal of this article is to repeat the above procedure for scalar fields on Euclidean $nc\mathrm{AdS}_2$. The action (1.1) is replaced by an operator trace. No additional terms analogous to the Gibbons-Hawking-York boundary term[30] need to be added to the action for the variational problem to be well defined. The field equation (1.3) gets replaced by an equation involving infinitely many derivatives in t, but still only two derivatives in z. Then again only two boundary conditions on a t-slice are required to obtain unique solutions. Regular solutions can be found order by order in the noncommutativity parameter, which can once again be expressed in terms of its boundary values ϕ_0. Following an analogous procedure to the above, we obtain the leading-order correction to the two-point function (1.9).

The outline of the article is as follows: In Sec. II, we review Euclidean AdS_2 for which we consider two different parametrizations, one are what we call canonical coordinates and the other are Fefferman-Graham coordinates. A Poisson bracket is attached to AdS_2 in a manner consistent with the isometries. The Poisson brackets imply that the time is canonically conjugate to the radial coordinate, which is conventionally interpreted as the energy scale for the boundary CFT. In Sec. III, we "quantize" the Poisson manifold, and as we indicated previously, we do it in a manner that preserves the AdS_2 isometries. The result is $nc\mathrm{AdS}_2$, which is described by the UIRR's of the universal cover of $SU(1,1)$. Of the different nontrivial UIRR's, i.e., the principal, supplemental, and discrete series, only the discrete series has a limit back to *Euclidean* AdS_2, and it is the subject of Sec. IV.[2] Following [31], we utilize properties of the generalized Laguerre polynomials to write down a symmetric differential representations for the discrete series. The differential operators can then be mapped to functions on the Moyal-Weyl plane, and so one arrives at a convenient Moyal-Weyl description of $nc\mathrm{AdS}_2$. Furthermore, a boundary can be defined on the Moyal-Weyl plane which coincides with the boundary of AdS_2 in the commutative limit. The Killing vectors for AdS_2 have a straightforward analogue in the noncommutative theory and are constructed in Sec. V. They are realized by infinite-order derivative operators on the Moyal-Weyl plane, and as stated above, they preserve the isometry algebra and reduce to the commutative form near the boundary. We explore massless scalar field theory on $nc\mathrm{AdS}_2$ in Sec. VI. An explicit expression for the dynamics of the massless scalar field on $nc\mathrm{AdS}_2$ is given. Although it describes a free scalar field on $nc\mathrm{AdS}_2$, after being mapped to the Moyal-Weyl plane the scalar field picks up nontrivial nonlocal interactions with the background. Just as with the case of the Killing vectors, the field equation essentially reduces to the commutative equation near the boundary. The field equation can be consistently obtained from the action principle upon imposing Dirichlet boundary conditions, and this is because we find no noncommutative corrections to the boundary term from variations of the action. The on-shell action, and resulting two-point function for the boundary theory, are computed in Sec. VII to leading order in the noncommutativity parameter. We find that the results agree with those of the commutative scalar field theory, up to a rescaling. In the Appendix, we collect some useful results about the Moyal-Weyl star product used in the calculations presented in the Secs. V, VI, and VII.

[2] The principal series has a limit to Lorentzian AdS_2, [17] while the supplemental series has no continuum limit.

II. EUCLIDEAN AdS_2: CANONICAL COORDINATES VERSUS FEFFERMAN-GRAHAM COORDINATES

AdS_2 can be defined in terms of embedding coordinates X^μ, $\mu = 0, 1, 2$, along with a scale parameter ℓ_0. In the case of the Euclidean version of AdS_2, X^μ span three-dimensional Minkowski space with invariant interval $ds^2 = dX^\mu dX_\mu$, where indices raised and lowered using the ambient metric tensor $\eta = \mathrm{diag}(1,1,-1)$. AdS_2 is defined by the constraint

$$X^\mu X_\mu = -\ell_0^2, \quad (2.1)$$

and $\ell_0^2 > 0$. The constraint describes a double-sheeted hyperboloid embedded in 3d Minkowski space-time. AdS_2 has three Killing vectors K^μ which generate the $SO(2,1)$ isometry group and get identified with the generators of the global conformal symmetry on the boundary. Thus

$$[K^\mu, K^\nu] = \epsilon^{\mu\nu\rho}K_\rho. \quad (2.2)$$

Our convention for the Levi-Civita symbol is $\epsilon^{012} = 1$. The action of the Killing vectors on the embedding coordinates is

$$(K^\mu X^\nu) = \epsilon^{\mu\nu\rho} X_\rho. \tag{2.3}$$

For the purpose of quantization we attach a Poisson bracket to the AdS manifold. In two dimensions, one can introduce a Poisson bracket which respects the isometry group and, therefore, the global conformal symmetry at the boundary. Expressed in terms of the embedding coordinates it is

$$\{X^\mu, X^\nu\} = \epsilon^{\mu\nu\rho} X_\rho. \tag{2.4}$$

In comparing with (2.3), the action of Killing vectors on functions on AdS_2 can the be written as $K^\mu = \{X^\mu, \cdot\}$.

Two choices of coordinates on the surface are useful for us. One choice is $\{(x, y), -\infty < x, y < \infty\}$, defined by

$$X^0 = -y, \qquad X^1 = -\frac{1}{2\ell_0} e^{-x} y^2 + \ell_0 \sinh x,$$
$$X^2 = -\frac{1}{2\ell_0} e^{-x} y^2 - \ell_0 \cosh x. \tag{2.5}$$

It covers a single hyperboloid ($X^2 < 0$). In terms of these coordinates, the metric tensor induced on the surface from the 3d Minkowski metric is given by

$$ds^2 = \ell_0^2 dx^2 + (dy - y dx)^2, \tag{2.6}$$

while the three Killing vectors are

$$K^0 = \partial_x, \qquad K^1 = \frac{1}{\ell_0} e^{-x} y \partial_x - X^2 \partial_y,$$
$$K^2 = \frac{1}{\ell_0} e^{-x} y \partial_x - X^1 \partial_y. \tag{2.7}$$

The coordinates (x, y) have the feature that they are canonically conjugate. That is, upon assuming that

$$\{x, y\} = 1, \tag{2.8}$$

and using (2.5), we recover the invariant Poisson brackets (2.4). For this reason refer to (x, y) as canonical coordinates.

A more familiar parametrization of the hyperboloid is given by the Fefferman-Graham coordinates (z, t)

$$z = e^{-x}, \qquad t = \frac{1}{\ell_0} e^{-x} y. \tag{2.9}$$

Whereas the canonical coordinates span $\mathbb{R}^2$, (z, t) cover the half-plane, $z \geq 0, -\infty < t < \infty$. $r = z^{-1}$ can be regarded as a radial variable. It can be expressed linearly in terms of the embedding coordinates,

$$r = z^{-1} = \frac{1}{\ell_0}(X^1 - X^2). \tag{2.10}$$

The AdS_2 boundary is the open curve at $z = 0$ or $r \to \infty$. In terms of the canonical coordinates, the boundary corresponds to both x and y going to infinity, with $e^{-x} y$ finite. The metric tensor when expressed in Fefferman-Graham coordinates is given by

$$ds^2 = \frac{\ell_0^2}{z^2}(dz^2 + dt^2), \tag{2.11}$$

and the Killing vectors take the form

$$K^- = -\partial_t, \qquad K^0 = -t\partial_t - z\partial_z,$$
$$K^+ = (z^2 - t^2)\partial_t - 2zt\partial_z, \tag{2.12}$$

where $K^\pm = K^2 \pm K^1$. We see that in the limit $z \to 0$, one recovers the standard form for the global conformal symmetry generators on the boundary

$$K^- \to -\partial_t, \qquad K^0 \to -t\partial_t, \qquad K^+ \to -t^2\partial_t. \tag{2.13}$$

They generate, respectively, translations, dilatations and special conformal transformations on the boundary. In terms of the Fefferman-Graham coordinates the Poisson bracket which yields the $so(2,1)$ Lie algebra (2.4) is

$$\{t, z\} = \frac{1}{\ell_0} z^2. \tag{2.14}$$

From (2.10), it also follows that

$$\{r, t\} = \frac{1}{\ell_0}. \tag{2.15}$$

In the AdS/CFT correspondence the radial variable is often regarded as the energy scale for the boundary CFT, and so it is reasonable to find that it is canonically conjugate to the time t. Note that in passing to the quantum theory we cannot simply replace the variables r and t with self-adjoint operators since r is only defined on the half-line. An alternative way to proceed to the quantum theory will be given in the following section.

We note that if do yet another change of coordinates from (z, t) to complex coordinates $\zeta = t + iz$ and $\bar{\zeta} = t - iz$, the Killing vectors become $K^\mu = L^\mu + \bar{L}^\mu$, where L^μ, along with their complex conjugates $\bar{L}^\mu$, are the standard global conformal symmetry generators on the complex plane

$$L^- = -\partial_\zeta, \qquad L^0 = -\zeta\partial_\zeta, \qquad L^+ = -\zeta^2\partial_\zeta. \tag{2.16}$$

The Poisson bracket (2.14) written in terms of ζ and $\bar{\zeta}$ will become

$$\{\zeta, \bar{\zeta}\} = \frac{i}{2\ell_0}(\zeta - \bar{\zeta})^2. \tag{2.17}$$

This bracket can be quantized using the methods of [23] to produce a star-product written directly in terms of the Fefferman-Graham coordinates (it is expected to be highly nontrivial). In this paper we will not follow this line.

III. $nc\mathrm{AdS}_2$

There is a straightforward quantization of the Poisson manifold defined in the previous section, and the result is $nc\mathrm{AdS}_2$ [16–19]. The first step is to replace the three embedding coordinates X^μ by Hermitian operators $\hat{X}^\mu$. The analogue of the constraint (2.1) in this setting is

$$\hat{X}^\mu \hat{X}_\mu = -\ell^2 \mathbb{1}, \tag{3.1}$$

where $\mathbb{1}$ is the identity and $\ell^2 > 0$ in the Euclidean version of $nc\mathrm{AdS}_2$. Furthermore, following the usual quantization procedure, the Poisson brackets (2.4) are promoted to commutation relations,

$$[\hat{X}^\mu, \hat{X}^\nu] = i\alpha\epsilon^{\mu\nu\rho}\hat{X}_\rho. \tag{3.2}$$

α and ℓ are two real parameters with units of length. (3.1) and (3.2) define $nc\mathrm{AdS}_2$, which is a solution to certain matrix models [17–19], which we describe below. The commutation relations (3.2) define the $so(2,1)$ algebra, while (3.1) fixes a value of the $so(2,1)$ Casimir operator. Analogous to (2.10), one can construct an operator analogue of the radial coordinate from the Hermitian operators $\hat{X}^\mu$

$$\hat{r} = \frac{1}{\ell}(\hat{X}^1 - \hat{X}^2). \tag{3.3}$$

We obtain the spectrum and eigenfunctions of this operator in Sec. IV.

Both (3.1) and (3.2) are preserved under the action of $SO(2,1)$, $\hat{X}^\mu \to R^\mu{}_\nu \hat{X}^\nu$, where R is a $SO(2,1)$ matrix. This is the analogue of isometry transformations on AdS_2. We shall construct the noncommutative analogues of the Killing vectors (2.7) and (2.12) which generate such transformations in Sec. V. In addition to the $SO(2,1)$ symmetry, Eqs. (3.1) and (3.2) are invariant under unitary transformations $\hat{X}^\mu \to \hat{U}\hat{X}^\mu\hat{U}^\dagger$, where $\hat{U}$ is a unitary operator.

To show that (3.1) and (3.2) can be obtained from matrix models [17–19] one can introduce three infinite-dimensional Hermitian matrices Y^μ, $\mu = 0, 1, 2$, with an action S_M consisting of two terms

$$S_M(Y) = \mathrm{Tr}\left(-\frac{1}{4}[Y_\mu, Y_\nu][Y^\mu, Y^\nu] - \frac{2}{3}i\alpha\epsilon_{\mu\nu\lambda}Y^\mu Y^\nu Y^\lambda\right), \tag{3.4}$$

where Tr denotes a matrix trace and we again assume the ambient metric $\eta_{\mu\nu} = \mathrm{diag}(1,1,-1)$. Dynamics can be defined by adapting a variational principle to this system. Extremizing S_M with respect to variations in Y_μ leads to

$$[[Y^\mu, Y^\nu], Y_\nu] - i\alpha\epsilon^{\mu\nu\lambda}[Y_\nu, Y_\lambda] = 0. \tag{3.5}$$

They are clearly solved by setting $Y^\mu = \hat{X}^\mu$. Like $nc\mathrm{AdS}_2$, the matrix equations (3.5) possess $SO(2,1)$ invariance, as well as invariance under unitary transformations (where $\hat{U}$ now denotes an infinite-dimensional unitary matrix). The matrix equations have an additional translational symmetry, $Y^\mu \to Y^\mu + v^\mu \mathbb{1}$, where $\mathbb{1}$ is the unit matrix and v^μ are real, which is broken by the $nc\mathrm{AdS}_2$ solution. Other matrix models have $nc\mathrm{AdS}_2$ solutions. For example, one can add a mass term, $\mathrm{Tr}Y^\mu Y_\mu$, to (3.4), and consequently a linear term to the equations of motion (3.5), as was done in [19]. This term explicitly breaks the translation symmetry.

To recover AdS_2 from $nc\mathrm{AdS}_2$, we need to define the commutative limit. It is $(\alpha, \ell) \to (0, \ell_0)$. In that limit, (3.1), (3.2) and (3.3) go to (2.1), (2.4) and (2.10), respectively. Here α plays the role of $\hbar$. It will also be necessary to define the notion of a boundary limit in the noncommutative theory. A natural choice for this is that the limit of the expectation value of $\hat{r}$ becomes large. This limit can be made more precise upon specifying the Hilbert space of the system, which we do below.

The states of $nc\mathrm{AdS}_2$ belong to unitary irreducible representations of $SO(2,1)$, or equivalently $SU(1,1)$,[3] which are the principal, supplemental, and discrete series representations. They are in general labeled by two parameters, which we denote by ϵ_0 and k. One can take a basis in a given representation to be eigenvectors $\{|\epsilon_0, k, m>,\ m = \mathrm{integer}\}$ of $\hat{X}^2$. The integer m is raised and lowered by $\hat{X}_+ = \hat{X}^1 + i\hat{X}^0$ and $\hat{X}_- = \hat{X}^1 - i\hat{X}^0$, respectively. Thus,

$$\hat{X}_+|\epsilon_0, k, m\rangle = -\alpha c_m|\epsilon_0, k, m+1\rangle, \tag{3.6}$$

$$\hat{X}_-|\epsilon_0, k, m\rangle = -\alpha c_{m-1}|\epsilon_0, k, m-1\rangle, \tag{3.7}$$

$$\hat{X}^2|\epsilon_0, k, m\rangle = -\alpha(\epsilon_0 + m)|\epsilon_0, k, m\rangle, \tag{3.8}$$

where the coefficient c_m is

$$c_m = \sqrt{(k + \epsilon_0 + m + 1)(\epsilon_0 - k + m)} \tag{3.9}$$

which ensures that the basis vectors are orthonormal $\langle\epsilon_0, k, m|\epsilon_0, k, m'\rangle = \delta_{m,m'}$. For any irreducible representation the Casimir operator is fixed by

$$\hat{X}^\mu\hat{X}_\mu|\epsilon_0, k, m\rangle = -\alpha^2 k(k+1)|\epsilon_0, k, m\rangle. \tag{3.10}$$

[3]More precisely, it is the universal cover of these groups, because we our only concerned with representations of the commutation algebra (3.2).

Upon comparing with (3.1), we then get

$$\left(k+\frac{1}{2}\right)^2=\frac{1}{4}+\frac{\ell^2}{\alpha^2}. \tag{3.11}$$

We note that the right-hand side of (3.11) diverges in the commutative limit. Therefore, the commutative limit corresponds to the limit of representations with $k\to\pm\infty$. From (3.3), the expectation value of the radial position vector $\hat r$ for any eigenvector $|\epsilon_0,k,m\rangle$ is $\frac{\alpha}{\ell}(\epsilon_0+m)$. Since the expectation value grows with m one can associate the boundary of $nc\mathrm{AdS}_2$ with $m\to\infty$.

As is well known there are three different types of nontrivial unitary irreducible representations of $SU(1,1)$: the principal, supplemental, and discrete series representations. These series are distinguished by their allowed values for k. The principal series representation has $k=-\frac{1}{2}-i\rho$, where ρ is real. This means that Casimir in (3.10) is positive and ℓ in (3.1) is imaginary. This corresponds to the Lorentzian version of $nc\mathrm{AdS}_2$ which we are not considering here. Moreover the limit $\rho\to\infty$, $\alpha\to 0$ yields Lorentzian AdS_2, which was pointed out in [17]. As our interest is in recovering Euclidean AdS_2, we do not examine the principal series. The supplemental series has k real, but restricted to $-\frac{1}{2}<k<0$. The Casimir in (3.10) is again positive and ℓ is imaginary. But since we cannot take the limit $k\to\infty$ in this case, the supplemental series has no commutative limit.[4] We can say that this case describes purely quantum Lorentzian $nc\mathrm{AdS}_2$. For these reasons we shall also not consider the supplemental series. We note that m ranges over all positive and negative integers for the principal and supplemental series. This means that the expectation values of $\hat r$ are not restricted to being positive. Moreover, $m\to\infty$ and $m\to-\infty$ are permissible limits of the states, which can be associated with two boundaries for the noncommutative version of Lorentzian AdS_2.

In the case of two discrete series representations, $D^{\pm}(k)$, k can be an arbitrary negative number.[5] Therefore, the Casimir in (3.10) is negative (and hence ℓ is real) for $k<-1$, and so these representations describe Euclidean $nc\mathrm{AdS}_2$.[6] Moreover, the limit that k goes to either $+\infty$ or $-\infty$ exists, so the discrete series has a limit to Euclidean AdS_2. m takes on only positive integers (including zero) for the discrete series representation $D^+(k)$, and negative integers for $D^-(k)$, defining two distinct noncommutative analogues of AdS_2 hyperboloids.

IV. DISCRETE SERIES REPRESENTATIONS

Here following [31], we utilize properties of the generalized Laguerre polynomials to write down a symmetric differential representations of $\hat X^\mu$ for the discrete series representations $D^+(k)$ and $D^-(k)$. We do this by obtaining eigenstates of the radial coordinate operator $\hat r$ in (3.3).

We begin with $D^+(k)$. Here $\epsilon_0=-k$ is a positive number. These representations have a lowest state $|-k,k,0\rangle$, which from (3.7) is annihilated by $\hat X_-$. For brevity we denote this state by $|k,0>$, and all other states in the $\hat X^2$ eigenbasis by $|k,m\rangle$, $m=$ positive integer. Next denote the eigenvector of the radial position operator (3.3) by $|\widetilde{r,k}>_+\in D^+(k)$, and with some abuse of notation, we call the eigenvalue the Fefferman-Graham coordinate r,

$$\hat r|\widetilde{r,k}>_+=r|\widetilde{r,k}>_+, \tag{4.1}$$

$|\widetilde{r,k}>_+$ can be expanded in the $\hat X^2$ eigenbasis,

$$|\widetilde{r,k}>_+=\sum_{m=0}^{\infty}\psi^+_{k,m}(r)|k,m\rangle. \tag{4.2}$$

Recursion relations for the coefficients $\psi^+_{k,m}(r)$ follow from the definition of $\hat r$, (3.3), along with (3.6) and (3.7),

$$\begin{aligned}&\sqrt{(m+1)(m-2k)}\psi^+_{k,m+1}(r)\\&\quad+\sqrt{m(m-1-2k)}\psi^+_{k,m-1}(r)\\&\quad+2\left(k-m+\frac{\ell}{\alpha}r\right)\psi^+_{k,m}(r)=0,\end{aligned} \tag{4.3}$$

which is also valid for $m=0$ since then the second term vanishes, and so all coefficients are determined from $\psi^+_{k,0}(r)$. The recursion relations (4.3) agree with those of the generalized Laguerre polynomials $L^{(\gamma)}_m$, m being a nonnegative integer, upon setting

$$\psi^+_{k,m}(r)=\sqrt{\frac{m!}{(m-2k-1)!}}L^{(-2k-1)}_m\left(\frac{2\ell r}{\alpha}\right). \tag{4.4}$$

The domain for $L^{(\gamma)}_m$ is the half-line, and so just as in the commutative theory, $r\geq 0$. A single boundary occurs in this case, corresponding to $r\to\infty$. The dominant polynomials near the boundary have large m, which is consistent with the previous result that the expectation value of $\hat r$ grows with m.

The generalized Laguerre polynomials obey the differential equation,

[4] Also, from (3.11) it is clear that $|\ell|<\frac{\alpha}{2}$. So, again, this means that this space is an extremely quantum object without any commutative limit.

[5] If one were to specialize to UIRR's of $SU(1,1)$, rather than its universal covering group, then one can show that k is restricted to the negative half-integer numbers [32,33]. But since here we are only concerned with representations of the algebra (3.2), this restriction is not necessary.

[6] The non-negative Casimir for $k\in[-1,0)$ describes an extremely quantum space, so it does not make much sense to say that for these values of k we have a Lorenzian $nc\mathrm{AdS}_2$. In any case, we are interested in the quasiclassical regime, i.e. when k is large.

$$\zeta\frac{d^2}{d\zeta^2}L_m^{(\gamma)}(\zeta)+(\gamma+1-\zeta)\frac{d}{d\zeta}L_m^{(\gamma)}(\zeta)+mL_m^{(\gamma)}(\zeta)=0, \tag{4.5}$$

and the orthogonality conditions[7]

$$\int_0^\infty d\zeta\zeta^\gamma e^{-\zeta}L_m^{(\gamma)}(\zeta)L_n^{(\gamma)}(\zeta)=\frac{1}{n!}\Gamma(n+\gamma+1)\delta_{n,m}. \tag{4.6}$$

Upon writing $\psi^+_{k,m}(r)=(\frac{2\ell}{\alpha})^k e^{\frac{\ell}{\alpha}r}r^{k+\frac{1}{2}}u^+_{k,m}(r)$ and using (4.4), these two relations can be expressed as

$$-\frac{\alpha}{2\ell}\left(\frac{d}{dr}r\frac{d}{dr}-\frac{(k+\frac{1}{2})^2}{r}-\frac{\ell^2}{\alpha^2}r\right)u^+_{k,m}(r)=(m-k)u^+_{k,m}(r), \tag{4.7}$$

$$\int_0^\infty dru^+_{k,m}(r)u^+_{k,n}(r)=\delta_{n,m}, \tag{4.8}$$

respectively. In comparing (4.7) with the eigenvalue equation (3.8), we get a symmetric differential representation π^k of $\hat{X}^2$ on $L^2(R_+,dr)$

$$\pi^k(\hat{X}^2)=\frac{\alpha^2}{2\ell}\left(\frac{d}{dr}r\frac{d}{dr}-\frac{(k+\frac{1}{2})^2}{r}-\frac{\ell^2}{\alpha^2}r\right) \tag{4.9}$$

The corresponding differential representations for the remaining $nc\mathrm{AdS}_2$ operators $\hat{X}^0$ and $\hat{X}^1$ are obtained using $\pi^k([\hat{r},\hat{X}^2])=\frac{i\alpha}{\ell}\pi^k(\hat{X}^0)$ to get the former and then $\pi^k([\hat{X}^0,\hat{X}^2])=-i\alpha\pi^k(X^1)$ to get the latter. The results are

$$\pi^k(\hat{X}^0)=i\alpha\left(r\frac{d}{dr}+\frac{1}{2}\right), \tag{4.10}$$

$$\pi^k(\hat{X}^1)=\frac{\alpha^2}{2\ell}\left(\frac{d}{dr}r\frac{d}{dr}-\frac{(k+\frac{1}{2})^2}{r}+\frac{\ell^2}{\alpha^2}r\right). \tag{4.11}$$

As the consistency check, note that, from (3.3), (4.9), and (4.11), it follows that $\hat{r}$ is really diagonal in this representation, $\pi^k(\hat{r})=r$.

For the discrete series $D^-(k)$, $\epsilon_0=k$ and m are negative integers including zero. The radial eigenvector is

$$|\widetilde{r,k}>_-=\sum_{m=0}^{-\infty}\psi^-_{k,m}(r)\,|k,m\rangle, \tag{4.12}$$

$$\psi^-_{k,m}(r)=(-1)^m\sqrt{\frac{(-m)!}{(-m-2k-1)!}}L_{-m}^{(-2k-1)}\left(-\frac{2\ell r}{\alpha}\right), \tag{4.13}$$

which now is defined only for $r\le 0$. The boundary now is at $r\to-\infty$, where the polynomials $L_m^{(\gamma)}(\zeta)$ with large negative m dominate. The above analysis can be repeated for $D^-(k)$ to obtain expression for the symmetric differential representation of the $su(1,1)$ basis. The results are again given by (4.9) and (4.11), now acting on functions spanned by $\{u^-_{k,m}(r), r\le 0\}$, which are defined by $\psi^-_{k,m}(r)=(\frac{2\ell}{\alpha})^k e^{-\frac{\ell}{\alpha}r}(-r)^{k+\frac{1}{2}}u^-_{k,m}(r)$.

The linear operators in (4.9)–(4.11) act on $L_2(R_+,dr)$. Denote the space of square-integrable space of functions on R_+ by $\{\psi(r)\}$. It is convenient to replace r by $x=\log r$ and replace (4.9), (4.10), and (4.11) by linear operators $\tilde{\pi}^k(\hat{X}^\mu)$ that on act $L_2(R,dx)$, spanned by $\{f(x)=e^{x/2}\psi(e^x)\}$. The result can be expressed in terms of self-adjoint operators $\hat{x}$ and $\hat{y}$ on $L_2(R,dx)$, where $\hat{x}$ has a trivial action on functions, $\hat{x}f(x)=xf(x)$, and $\hat{y}$ is the self-adjoint differential operator $\hat{y}=-i\alpha\partial_x$. Then $\hat{x}$ and $\hat{y}$ satisfy the Heisenberg commutation relation

$$[\hat{x},\hat{y}]=i\alpha\mathbb{1}, \tag{4.14}$$

with $\mathbb{1}$ being the identity. For $\tilde{\pi}^k(\hat{X}^\mu)$, we get

$$\begin{aligned}\tilde{\pi}^k(\hat{X}^0)&=-\hat{y},\\ \tilde{\pi}^k(\hat{X}^1)&=-\frac{1}{2\ell}\hat{y}e^{-\hat{x}}\hat{y}-\frac{\alpha^2}{2\ell}k(k+1)e^{-\hat{x}}+\frac{\ell}{2}e^{\hat{x}},\\ \tilde{\pi}^k(\hat{X}^2)&=-\frac{1}{2\ell}\hat{y}e^{-\hat{x}}\hat{y}-\frac{\alpha^2}{2\ell}k(k+1)e^{-\hat{x}}-\frac{\ell}{2}e^{\hat{x}}.\end{aligned} \tag{4.15}$$

Since $\hat{x}$ and $\hat{y}$ satisfy (4.14), any function $\hat{\mathcal{F}}(\hat{x},\hat{y})$ can be mapped to function $\mathcal{F}(x,y)$, called a symbol, on the Moyal-Weyl plane, which we take to be spanned by commuting variables x and y. Then x and y are the symbols of $\hat{x}$ and $\hat{y}$, respectively.[8] The product $[\hat{\mathcal{F}}\hat{\mathcal{G}}](\hat{x},\hat{y})=\hat{\mathcal{F}}(\hat{x},\hat{y})\hat{\mathcal{G}}(\hat{x},\hat{y})$ of any two functions of $\hat{x}$ and $\hat{y}$ is mapped to the Moyal-Weyl star product $[\mathcal{F}\star\mathcal{G}](x,y)$, which is written down in (A1) in the Appendix. We denote the symbols of $\tilde{\pi}^k(\hat{X}^\mu)$ by $\mathcal{X}^\mu$. Then from (4.15),

$$\begin{aligned}\mathcal{X}^0&=-y.\\ \mathcal{X}^1&=-\frac{1}{2\ell}y\star e^{-x}\star y-\frac{\alpha^2}{2\ell}k(k+1)e^{-x}+\frac{\ell}{2}e^x.\\ \mathcal{X}^2&=-\frac{1}{2\ell}y\star e^{-x}\star y-\frac{\alpha^2}{2\ell}k(k+1)e^{-x}-\frac{\ell}{2}e^x.\end{aligned} \tag{4.16}$$

These are the analogues of the embedding coordinates X^μ. They do not satisfy the AdS_2 constraint (2.1) using the point-wise product. Rather using the star-product (A1), they realize the defining relations (3.1) and (3.2) for $nc\mathrm{AdS}_2$ on the Moyal plane

[7]This is defined only for $\gamma>-1$ (to avoid the logarithmic divergence at $\zeta=0$), which is satisfied in our case, $k<-1$.

[8]Here we are identifying the coordinates of the Moyal-Weyl plane with the canonical coordinates of Sec. II. This is consistent due to the fact that the two sets of coordinates coincide in the commutative limit, as we show below.

$$\mathcal{X}^\mu \star \mathcal{X}_\mu = -\ell^2, \tag{4.17}$$

$$[\mathcal{X}^\mu, \mathcal{X}^\nu]_\star = i\alpha\epsilon^{\mu\nu\rho}\mathcal{X}_\rho, \tag{4.18}$$

where $[\mathcal{F},\mathcal{G}]_\star = \mathcal{F}\star\mathcal{G} - \mathcal{G}\star\mathcal{F}$ is the star commutator of any two functions $\mathcal{F}(x,y)$ and $\mathcal{G}(x,y)$ on the Moyal-Weyl plane, and we have used (3.11). In the commutative limit $\alpha\to 0$, the star product reduces to the point-wise product, and the leading term in the star commutator is $[\mathcal{F},\mathcal{G}]_\star \to i\alpha\{\mathcal{F},\mathcal{G}\}$, where $\{,\}$ denotes the Poisson bracket defined using (2.8). Thus x and y reduce to the canonical coordinates of Sec. II. Moreover, using (3.11) one can show that $\mathcal{X}^\mu$ reduce to the AdS_2 embedding coordinates X^μ, Eq. (2.5), in the commutative limit.

V. KILLING VECTORS ON $nc\mathbf{AdS}_2$

From Sec. II, isometry transformations on AdS_2 can be obtained by taking Poisson brackets with X^μ. Given a function Φ on AdS_2 an infinitesimal variation of Φ induced by the action of the $SO(2,1)$ isometry group is

$$\delta\Phi = \epsilon_\mu(K^\mu\Phi) = \epsilon_\mu\{X^\mu,\Phi\}, \tag{5.1}$$

where K^μ are the Killing vectors on AdS_2 and ϵ_μ are infinitesimal parameters. There is a natural generalization to $SO(2,1)$ isometry transformations on $nc\mathrm{AdS}_2$, and hence to Killing vectors $\hat{K}^\mu$ on $nc\mathrm{AdS}_2$. If $\hat\Phi$ is a function on $nc\mathrm{AdS}_2$, its infinitesimal variation $\delta_{nc}\hat\Phi$ induced by the action of $SO(2,1)$ is

$$\delta_{nc}\hat\Phi = \epsilon_\mu(\hat{K}^\mu\hat\Phi) = i\epsilon_\mu[\hat{X}^\mu,\hat\Phi]. \tag{5.2}$$

Alternatively, it can be mapped to infinitesimal transformations on the Moyal-Weyl plane. If we call Φ the symbol of $\hat\Phi$ and $K^\mu_\star\Phi$ the symbol of $\hat{K}^\mu\hat\Phi$ then

$$\delta_{nc}\Phi = \epsilon_\mu(K^\mu_\star\Phi) = i\epsilon_\mu[\mathcal{X}^\mu,\Phi]_\star. \tag{5.3}$$

Using (A1) and the expressions (4.16) for $\mathcal{X}^\mu$, we get

$$\delta_{nc}\Phi = \alpha\epsilon_0\partial_x\Phi + \frac{i\epsilon_+}{2\ell}[y\star e^{-x}\star y,\Phi]_\star + \frac{i\epsilon_+\alpha^2}{2\ell}k(k+1)[e^{-x},\Phi]_\star + \frac{i\epsilon_-\ell}{2}[e^x,\Phi]_\star, \tag{5.4}$$

where $\epsilon_\pm = \epsilon_2 \pm \epsilon_1$. The variation can be explicitly computed with the help of the identities (A2) in the Appendix. One gets

$$[e^{\pm x},\Phi]_\star = \pm i\alpha e^{\pm x}\Delta_y\Phi,$$
$$[y\star e^{-x}\star y,\Phi]_\star = -i\alpha e^{-x}\left(y^2\Delta_y + 2y\partial_x S_y + \frac{\alpha^2}{4}(1-\partial_x^2)\Delta_y\right)\Phi, \tag{5.5}$$

where

$$\Delta_y\Phi(x,y) = \frac{\Phi(x,y+\frac{i\alpha}{2}) - \Phi(x,y-\frac{i\alpha}{2})}{i\alpha} = \frac{2}{\alpha}\sin\left(\frac{\alpha}{2}\partial_y\right)\Phi(x,y),$$
$$S_y\Phi(x,y) = \frac{\Phi(x,y+\frac{i\alpha}{2}) + \Phi(x,y-\frac{i\alpha}{2})}{2} = \cos\left(\frac{\alpha}{2}\partial_y\right)\Phi(x,y). \tag{5.6}$$

The noncommutative variation can then be written as $\delta_{nc}\Phi = \frac{\alpha}{2}(\epsilon_- K^-_\star + 2\epsilon_0 K^0_\star + \epsilon_+ K^+_\star)\Phi$, where the noncommutative analogues of the AdS_2 Killing vectors are

$$K^-_\star = -\ell e^x\Delta_y, \qquad K^0_\star = \partial_x,$$
$$K^+_\star = \frac{e^{-x}}{\ell}\left(2y\partial_x S_y + \left(y^2+\ell^2+\frac{\alpha^2}{4}(1-\partial_x^2)\right)\Delta_y\right). \tag{5.7}$$

By construction $K^\mu_\star$ satisfy the $so(2,1)$ Lie algebra commutation relations $[K^\mu_\star, K^\nu_\star] = \epsilon^{\mu\nu\rho}K_{\star\rho}$, where $K^2_\star = \frac{1}{2}(K^+_\star + K^-_\star)$ and $K^1_\star = \frac{1}{2}(K^+_\star - K^-_\star)$. $K^0_\star$ agrees with its commutative analogue K^0, while $K^1_\star$ and $K^2_\star$ are deformations of K^1 and K^2, (2.7), containing infinite-order polynomials in ∂_y. In the commutative limit $(\alpha,\ell)\to(0,\ell_0)$, Δ_y approaches a derivative operator ∂_y and S_y approaches the identity. It follows that we recover the AdS_2 Killing vectors in the commutative limit, $K^\mu_\star \to K^\mu$ as $\alpha\to 0$.

The noncommutative analogues of the Killing vectors can be re-expressed in Fefferman-Graham coordinates (2.9) by replacing the action of Δ_y and S_y on the fields by

$$\Delta_t\Phi(z,t) = \frac{\Phi(z,t+\frac{i\alpha z}{2\ell}) - \Phi(z,t-\frac{i\alpha z}{2\ell})}{i\alpha} = \frac{2}{\alpha}\sin\left(\frac{\alpha z}{2\ell}\partial_t\right)\Phi(z,t),$$
$$S_t\Phi(z,t) = \frac{\Phi(z,t+\frac{i\alpha z}{2\ell}) + \Phi(z,t-\frac{i\alpha z}{2\ell})}{2} = \cos\left(\frac{\alpha z}{2\ell}\partial_t\right)\Phi(z,t), \tag{5.8}$$

respectively. Then,

$$K^-_\star = -\frac{\ell}{z}\Delta_t, \qquad K^0_\star = -t\partial_t - z\partial_z,$$
$$K^+_\star = -2t(t\partial_t + z\partial_z)S_t + \frac{\ell}{z}\left(t^2 + \left(1+\frac{\alpha^2}{4\ell^2}\right)z^2\right)\Delta_t - \frac{\alpha^2 z}{4\ell}(t\partial_t + z\partial_z)^2\Delta_t. \tag{5.9}$$

We again see that $K^0_\star$ agrees with its commutative analogue K^0, while $K^+_\star$ and $K^-_\star$ are deformations of K^+ and K^-, (2.12), containing infinite-order polynomials in ∂_t. As before, the AdS_2 Killing vectors are recovered in the commutative limit, $K^\mu_\star \to K^\mu$ as $\alpha \to 0$.

The expressions (5.9) for the Killing vectors on $nc AdS_2$ can be used to examine another limit of interest, $z \to 0$, which corresponds to the boundary of $nc AdS_2$. In that limit $\Delta_t\Phi \to \frac{z}{\ell}\partial_t\Phi|_{z=0}$ and $S_t\Phi \to \Phi|_{z=0}$, and so we obtain the commutative result (2.13),

$$K^-_\star \to -\partial_t, \qquad K^0_\star \to -t\partial_t, \qquad K^+_\star \to -t^2\partial_t. \tag{5.10}$$

From $nc AdS_2$ we thus recover the standard form for the global conformal symmetry generators on the boundary. We can then say that $nc AdS_2$ is asymptotically AdS_2. Therefore, the AdS/CFT correspondence principle should be applicable. We explore this possibility in the next section with the example of massless scalar field theory.

VI. MASSLESS SCALAR FIELD THEORY ON $nc AdS_2$

Here we write down an explicit expression for the field equation for a massless scalar field on $nc AdS_2$. Although it describes a free scalar field on $nc AdS_2$, the scalar field picks up nontrivial nonlocal interactions after being mapped to the Moyal-Weyl plane. We show that these interactions disappear near the boundary. The field equation can be consistently obtained from an action principle upon imposing Dirichlet boundary conditions, and this is because we find no noncommutative corrections to the boundary term from variations of the action.

Say $\Phi^{(0)}$ is now a massless scalar field on AdS_2. The standard $SO(2,1)$ invariant action can be written in terms of Poisson brackets with the embedding coordinates

$$S[\Phi^{(0)}] = \frac{1}{2\ell_0}\int_{AdS_2} d\mu\{X^\mu, \Phi^{(0)}\}\{X_\mu, \Phi^{(0)}\}, \tag{6.1}$$

where $d\mu$ is an invariant integration measure on AdS_2. When written in terms of canonical coordinates, it becomes

$$S[\Phi^{(0)}] = \frac{1}{2\ell_0}\int_{\mathbb{R}^2} dxdy\{(y\partial_y\Phi^{(0)} + \partial_x\Phi^{(0)})^2 + \ell_0^2(\partial_y\Phi^{(0)})^2\}, \tag{6.2}$$

while it reduces to (1.1) when written in terms of Fefferman-Graham coordinates.

Upon promoting $\Phi^{(0)}$ to a field $\hat\Phi$ on $nc AdS_2$, there is an obvious generalization of (6.1) to an $SO(2,1)$ invariant action for $\hat\Phi$. It is

$$S_{nc}[\hat\Phi] = -\frac{1}{2\ell}\mathrm{Tr}[\hat X^\mu, \hat\Phi][\hat X_\mu, \hat\Phi], \tag{6.3}$$

where Tr denotes a trace operation. Here for simplicity we assume that the $nc AdS_2$ scale parameter is the same as the commutative one, $\ell = \ell_0$; i.e., ℓ has no α^2 dependence. (6.3) can be mapped to an action on the Moyal-Weyl plane,

$$S_{nc}[\Phi] = -\frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} dxdy[\mathcal{X}^\mu, \Phi]_\star \star [\mathcal{X}_\mu, \Phi]_\star, \tag{6.4}$$

where the trace has been replaced by $\frac{1}{\alpha^2}\int_{\mathbb{R}^2} dxdy$. Upon applying (4.16) and (A3) in the Appendix, one gets

$$S_{nc}[\Phi] = \frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} dxdy\{-[y,\Phi]^2_\star + [e^x, \Phi]_\star[y \star e^{-x} \star y + k(k+1)e^{-x}, \Phi]_\star\}, \tag{6.5}$$

where we are ignoring all boundary terms because for the moment we shall only be concerned with the field in the bulk. (Boundary affects are taken into account below.) Using (5.5), this becomes

$$S_{nc}[\Phi] = \frac{1}{2\ell}\int_{\mathbb{R}^2} dxdy\left\{(\partial_x\Phi)^2 + \Delta_y\Phi\left(y^2\Delta_y\Phi + 2y\partial_x S_y\Phi - \frac{\alpha^2}{4}\partial_x^2\Delta_y\Phi\right) + \alpha^2\left(k+\frac{1}{2}\right)^2(\Delta_y\Phi)^2\right\}, \tag{6.6}$$

up to boundary terms. Upon integrating by parts and using

$$\int_{\mathbb{R}^2} dxdy\left\{(\partial_x S_y\Phi)^2 - \frac{\alpha^2}{4}(\partial_x\Delta_y\Phi)^2 - (\partial_x\Phi)^2\right\} = 0,$$

it simplifies to

$$S_{nc}[\Phi] = \frac{1}{2\ell}\int_{\mathbb{R}^2} dxdy\left\{(y\Delta_y\Phi + \partial_x S_y\Phi)^2 + \left(\frac{\alpha^2}{4} + \ell^2\right)(\Delta_y\Phi)^2\right\}. \tag{6.7}$$

This is an explicit expression for the bulk action in terms of the canonical coordinates. In terms of Fefferman-Graham coordinates, the action is

$$S_{nc}[\Phi] = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}_+} dtdz\frac{1}{z^2}\left\{\left(\frac{\ell t}{z}\Delta_t\Phi - (t\partial_t + z\partial_z)S_t\Phi\right)^2 + \left(\frac{\alpha^2}{4}+\ell^2\right)(\Delta_t\Phi)^2\right\}. \tag{6.8}$$

(6.2) and (1.1) are recovered from the commutative limit, $\alpha \to 0$, of (6.7) and (6.8), respectively.

We note that as one approaches the boundary $z = 0$, the action density goes to that of a massless scalar field on commutative AdS_2, with a rescaled time parameter t. Using $\Delta_t\Phi \to \frac{z}{\ell}\partial_t\Phi|_{z=0}$ and $S_t\Phi \to \Phi|_{z=0}$ as $z \to 0$, the integrand in (6.8) goes to[9]

$$\left(1+\frac{\alpha^2}{4\ell^2}\right)(\partial_t\Phi)^2 + (\partial_z\Phi)^2, \tag{6.9}$$

as compared to the integrand in (1.1). This means that the commutative free field equation is recovered near the boundary, again with a rescaled coordinate,

$$\left(1+\frac{\alpha^2}{4\ell^2}\right)\partial_t^2\Phi + \partial_z^2\Phi \to 0, \quad \text{as } z \to 0, \tag{6.10}$$

and so Φ satisfies the equation for a massless scalar field on an asymptotically AdS_2 space.

The field equation for Φ can be written down for all z. Variations in Φ in (6.4) yield

$$\begin{aligned}\delta S_{nc}[\Phi] &= -\frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} dxdy([\mathcal{X}^\mu,\delta\Phi]_\star \star [\mathcal{X}_\mu,\Phi]_\star \\ &\quad + [\mathcal{X}^\mu,\Phi]_\star \star [\mathcal{X}_\mu,\delta\Phi]_\star) \\ &= -\frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} dxdy(2[\mathcal{X}^\mu,\delta\Phi]_\star \star [\mathcal{X}_\mu,\Phi]_\star \\ &\quad + [[\mathcal{X}^\mu,\Phi]_\star,[\mathcal{X}_\mu,\delta\Phi]_\star]_\star) \\ &= \frac{1}{\ell\alpha^2}\int_{\mathbb{R}^2} dxdy\delta\Phi \star [\mathcal{X}^\mu,[\mathcal{X}_\mu,\Phi]_\star]_\star \\ &\quad - \frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} dxdy(2[\mathcal{X}^\mu,\delta\Phi \star [\mathcal{X}_\mu,\Phi]_\star]_\star \\ &\quad + [[\mathcal{X}^\mu,\Phi]_\star,[\mathcal{X}_\mu,\delta\Phi]_\star]_\star).\end{aligned} \tag{6.11}$$

From the first term, the field equation in the bulk is

$$[\mathcal{X}^\mu,[\mathcal{X}_\mu,\Phi]_\star]_\star = 0. \tag{6.12}$$

[9] In passing from canonical coordinates to Fefferman-Graham coordinates we used the commutative formulas (2.9) (with the natural change $l_0 \to l$). On the other hand, one can reabsorb the factor in (6.9) by rescaling t (or, z) in a quantum (or noncommutative) version of (2.9). The commutative limit, of course, of this transformation must coincide with (2.9). Because this does not seem to bring any radical simplification, we will keep on using the commutative change of variables (2.9).

The remaining two terms [last two lines in (6.11)] are only defined on the boundary. This is since the Moyal star commutator of any two functions $\mathcal{F}$ and $\mathcal{G}$ on the Moyal-Weyl plane is a total divergence. Following (A4) in the Appendix, we can write the integral of $[\mathcal{F},\mathcal{G}]_\star$ over D as $\int_{\partial D}(\mathcal{V}_x dx + \mathcal{V}_y dy)$, where ∂D is the boundary of D. $\mathcal{V}_x$ and $\mathcal{V}_y$ are computed up to order α^2 in (A5). For us the boundary is located at $z = 0$, and so $\int_{\partial D}(\mathcal{V}_x dx + \mathcal{V}_y dy) = \int \mathcal{V}_t|_{z=0}dt$, where $\mathcal{V}_t = \frac{\ell}{z}\mathcal{V}_y$. To compute $\mathcal{V}_t$ for the first boundary term in (6.11) we set $\mathcal{F}$ and $\mathcal{G}$ in (A4) equal to $\mathcal{X}^\mu$ and $\delta\Phi \star [\mathcal{X}_\mu,\Phi]_\star$, respectively, and then sum over μ. At leading order in α, $\mathcal{V}_t = -\alpha^2\ell\delta\Phi\partial_z\Phi$. This is the commutative result. After some work we get that the α^2 corrections to this result go like z^n, $n \geq 1$, which then vanish after setting $z = 0$. To compute $\mathcal{V}_t$ for the second boundary term in (6.11) we set $\mathcal{F}$ and $\mathcal{G}$ in (A4) equal to $[\mathcal{X}^\mu,\Phi]_\star$ and $[\mathcal{X}_\mu,\delta\Phi]_\star$, respectively, and then sum over μ. We find that all contributions to $\mathcal{V}_t$ go like z^n, $n \geq 1$, which once again vanish after setting $z = 0$. We, thus, get that all noncommutative corrections to the boundary terms vanish. Although we have only checked this to order α^2 we expect that the result is true to all orders since they involve higher-order derivatives which will produce higher powers in z in $\mathcal{V}_t$. The boundary term in (6.11) is then just the commutative answer

$$-\int dt(\partial_z\Phi\delta\Phi)\Big|_{z=0}. \tag{6.13}$$

This means that we can fix the boundary value of the field

$$\phi_0(t) = \Phi(0,t), \tag{6.14}$$

and the variational problem is well defined for Dirichlet boundary conditions.

Alternatively, the field equation in the bulk can be found directly from the Lagrangian density (6.7) with the help of the identities

$$\begin{aligned}&\int_{\mathbb{R}^2} dxdy(\Delta_y A(x,y)B(x,y) + A(x,y)\Delta_y B(x,y)) = 0, \\ &\int_{\mathbb{R}^2} dxdy(S_y A(x,y)B(x,y) - A(x,y)S_y B(x,y)) = 0,\end{aligned} \tag{6.15}$$

which are valid up to boundary terms. Note that the first identity shows that under integration, Δ_y behaves as the usual derivative satisfying the Leibniz rule. Then the field equation following from (6.7) is

$$(\Delta_y y + \partial_x S_y)(y\Delta_y + \partial_x S_y)\Phi + \left(\frac{\alpha^2}{4}+\ell^2\right)\Delta_y^2\Phi = 0, \tag{6.16}$$

or in Fefferman-Graham coordinates,

$$\left(\ell\Delta_t\frac{t}{z}-(t\partial_t+z\partial_z)S_t\right)\left(\ell\frac{t}{z}\Delta_t-(t\partial_t+z\partial_z)S_t\right)\Phi + \left(\frac{\alpha^2}{4}+\ell^2\right)\Delta_t^2\Phi=0. \tag{6.17}$$

In both limits $\alpha\to 0$ and $z\to 0$, (6.17) reduces to a second-order differential equation. In the former, we recover the commutative answer (1.3), while in the latter, (6.17) reduces to the previously obtained result near the boundary (6.10). Although (6.17) contains infinitely many orders in derivatives with respect to t, it is only second order in derivatives in z (just as in the commutative case). Then it can be solved given sufficient data at the AdS boundary, which we do to leading order in α^2 in the next section.

VII. LEADING-ORDER SOLUTIONS AND THE CFT_1 CORRESPONDENCE

Here we compute the on-shell action and resulting two-point function for the boundary theory to leading order in the noncommutativity parameter. Expanding the field equation (6.17) up to the leading-order correction in α^2 gives

$$\Box\Phi-\frac{\alpha^2}{12\ell^2}\{t\partial_t+z^2\partial_t^2+9z\partial_z+2zt\partial_z\partial_t+3z^2\partial_z^2\}\partial_t^2\Phi + \mathcal{O}(\alpha^4)=0. \tag{7.1}$$

Using standard techniques [34], one can write down a solution to (7.1) in terms of the boundary value of the field (6.14), which we can define to be independent of α^2. We denote the solution by $\Phi_{\rm sol}[\phi_0]$. We expand $\Phi_{\rm sol}[\phi_0]$ in powers of α^2 about the commutative solution $\Phi^{(0)}$, satisfying (1.3),

$$\Phi_{\rm sol}[\phi_0]=\Phi^{(0)}+\alpha^2\Phi^{(1)}+\cdots+\alpha^{2M}\Phi^{(M)}+\cdots \tag{7.2}$$

$\Phi^{(0)}$ is solved in (1.4) using the boundary-to-bulk propagator. From (7.1), the leading-order noncommutative correction $\Phi^{(1)}$ satisfies

$$\Box\Phi^{(1)}=\frac{1}{12\ell^2}\{t\partial_t+z^2\partial_t^2+9z\partial_z+2zt\partial_z\partial_t + 3z^2\partial_z^2\}\partial_t^2\Phi^{(0)}. \tag{7.3}$$

After using (1.4) on the right-hand side, we get

$$\Box\Phi^{(1)}=\frac{z}{2\pi\ell^2}\int dt'\mathcal{F}(t,t',z)\phi_0(t'),$$
$$\mathcal{F}(t,t',z)=\frac{z^6-(t+35t')(t-t')z^4-5(t-17t')(t-t')^3z^2-3(t+3t')(t-t')^5}{((t-t')^2+z^2)^5}. \tag{7.4}$$

We now apply the bulk-to-bulk propagator [35–37],

$$G(z,t;z',t')=\frac{1}{2\pi}\tanh^{-1}\left(\frac{2zz'}{z^2+z'^2+(t-t')^2}\right), \tag{7.5}$$

satisfying $\Box G(z,t;z',t')=-\delta(z-z')\delta(t-t')$, to obtain an integral expression for $\Phi^{(1)}$

$$\Phi^{(1)}(z,t)=-\frac{1}{2\pi\ell^2}\int_0^\infty dz'z'\int dt' \times\int dt''G(z,t;,z',t')\mathcal{F}(t',t'',z')\phi_0(t''). \tag{7.6}$$

This procedure can, in principle, be repeated to get any higher-order correction $\Phi^{(M)}$ to the commutative field.

We next use (1.4) and (7.6) to compute the on-shell action. For this purpose, it is convenient to reexpress the action (6.4) as

$$S_{nc}[\Phi]=\frac{1}{2\ell\alpha^2}\int dxdy\Phi\star[\mathcal{X}^\mu,[\mathcal{X}_\mu,\Phi]_\star]_\star - \frac{1}{2\ell\alpha^2}\int dxdy[\mathcal{X}^\mu,\Phi\star[\mathcal{X}_\mu,\Phi]_\star]_\star. \tag{7.7}$$

From (6.12), the first term vanishes on-shell. The remaining term is only defined on the boundary since the Moyal star commutator is a total divergence. We can once again use (A4) in the Appendix to compute it up to order α^2 in (A5). Setting $\mathcal{F}$ and $\mathcal{G}$ in (A4) equal to $\mathcal{X}^\mu$ and $\Phi\star[\mathcal{X}_\mu,\Phi]_\star$, respectively, and summing over μ, we get $\mathcal{V}_t=\frac{\ell}{z}\mathcal{V}_y=\alpha^2\ell\Phi\partial_z\Phi$ at leading order in α. After some work we get that the α^2 corrections to this result go like z^n, $n\geq 2$, which then vanish after setting $z=0$. This means that the expression for the on-shell action receives no noncommutative corrections (at least, at order α^2)

$$S_{nc}[\Phi_{\rm sol}[\phi_0]] = -\frac{1}{2\ell\alpha^2}\int dxdy[\mathcal{X}^\mu,\Phi\star[\mathcal{X}_\mu,\Phi]_\star]_\star\Big|_{\Phi=\Phi_{\rm sol}[\phi_0]} = -\frac{1}{2}\int dt\Phi_{\rm sol}[\phi_0]\partial_z\Phi_{\rm sol}[\phi_0]\Big|_{z=0}. \tag{7.8}$$

This is identical to the commutative result (1.6).

It remains to substitute the solution (1.4) and (7.6) into the action (7.8). This gives

$$\begin{aligned} S_{nc}[\Phi_{\rm sol}[\phi_0]] &= -\frac{1}{2}\int dt\int dt'\phi_0(t)\bigg(\partial_z K(z,t;t')\Big|_{z=0}\phi_0(t') \\ &\quad -\frac{\alpha^2}{2\pi\ell^2}\int_0^\infty dz'z'\int_{-\infty}^{\infty}dt''K(z',t;t')\mathcal{F}(t',t'',z')\phi_0(t'')+\mathcal{O}(\alpha^4)\bigg) \\ &= -\frac{1}{2\pi}\int dt\int dt'\phi_0(t)\bigg(\frac{\phi_0(t')}{(t-t')^2}-\frac{\alpha^2}{2\pi\ell^2}\int_0^\infty dz'\int_{-\infty}^{\infty}dt''\frac{z'^2\mathcal{F}(t',t'',z')\phi_0(t'')}{z'^2+(t-t')^2}+\mathcal{O}(\alpha^4)\bigg) \\ &= -\frac{1}{2\pi}\int dt\int dt'\phi_0(t)\phi_0(t')\bigg(\frac{1}{(t-t')^2}-\frac{\alpha^2}{2\pi\ell^2}\int_0^\infty dz'\int_{-\infty}^{\infty}dt''\frac{z'^2\mathcal{F}(t'',t',z')}{z'^2+(t-t'')^2}+\mathcal{O}(\alpha^4)\bigg), \end{aligned} \tag{7.9}$$

where we used the identity $\partial_z G(z,t;z',t')|_{z=0} = K(z',t;t')$. The second term in parentheses in (7.9) is the leading noncommutative correction. It can be exactly computed using the integral

$$\int_0^\infty dz\int_{-\infty}^{\infty}dt''\frac{z^2\mathcal{F}(t'',t',z)}{z^2+(t-t'')^2} = \frac{\pi/4}{(t-t')^2}. \tag{7.10}$$

This result means that the on-shell action merely undergoes an overall rescaling

$$S_{nc}[\Phi_{\rm sol}[\phi_0]] = -\frac{1}{2\pi}\int dt\int dt'\phi_0(t)\phi_0(t') \times\bigg(\bigg(1-\frac{\alpha^2}{8\ell^2}\bigg)\frac{1}{(t-t')^2}+\mathcal{O}(\alpha^4)\bigg). \tag{7.11}$$

Then from the AdS/CFT correspondence (1.8), n-point correlation functions of quantum mechanical operators $\mathcal{O}(t)$ on the one-dimensional boundary also undergo an overall rescaling at leading order in the noncommutativity parameter. For the two-point function we get

$$\langle\mathcal{O}(t)\mathcal{O}(t')\rangle = -\frac{1}{2\pi}\bigg(1-\frac{\alpha^2}{8\ell^2}\bigg)\frac{1}{(t-t')^2}+\mathcal{O}(\alpha^4). \tag{7.12}$$

Recall that at the beginning of Sec. VI, we fixed ℓ equal to the commutative length scale ℓ_0. If ℓ instead depends on α, we should replace ℓ in the leading-order correction in (7.12) by ℓ_0.

VIII. CONCLUDING REMARKS

We have shown that $nc\mathrm{AdS}_2$ has a commutative boundary, implying that $nc\mathrm{AdS}_2$ is assymptotically AdS_2. Then from general arguments the AdS/CFT correspondence should be applicable. We explicitly demonstrated this by computing the two-point function on the boundary associated with the massless scalar field on $nc\mathrm{AdS}_2$. The dynamics for the scalar field contains nontrivial nonlocal interactions, which is evident from the Moyal-Weyl plane description. These interactions vanish at the $nc\mathrm{AdS}_2$ boundary. Our leading-order results show that the introduction of noncommutativity on the AdS_2 space does not affect the boundary conformal theory, other than to generate a rescaling of the correlation functions. The conformal dimension, which is one for the commutative theory, is unaffected at leading order in α^2. Higher-order computations are feasible. If the conformal dimension remains one to all orders, the commutative and noncommutative theory are equivalent within the context of the AdS/CFT correspondence principle. Our results utilized the isometry preserving commutation relations (3.2) which defines $nc\mathrm{AdS}_2$. Different results may follow from other deformations of anti–de Sitter space. This was found recently for a κ-deformed AdS_2 spacetime [28]. There, the conformal dimension was a nontrivial function of the noncommutativity parameter.

Concerning the issue of disconnected timelike boundaries of AdS_2 [5], we find that Euclidean $nc\mathrm{AdS}_2$ selects a single boundary. This is because the boundary in this system is described in terms of states of a particular discrete series representation $D^+(k)$ (or $D^-(k)$), which has a lowest (or highest) state. As a result, the eigenvalues of the radial coordinate operator $\hat{r}$ have a lower (or upper) bound, namely zero, while the boundary corresponds to the eigenvalue going to $+\infty$ (or $-\infty$).

A number of generalizations of our work are possible. Among them is the addition of a mass term Tr $\hat{\Phi}^2$, or interaction terms Tr $\hat{\Phi}^M$ to the action (6.3) of the scalar field on $nc\mathrm{AdS}_2$. This will introduce further nonlocal interactions in the Moyal-Weyl plane description and is likely to lead to noncommutative corrections to the Breitenlohner-Freedman bound [38]. The examination of other fields on $nc\mathrm{AdS}_2$, such as spinors, gauge fields and spin-two fields is another very natural extension of our work. A Dirac operator has been proposed for $nc\mathrm{AdS}_2$, [39] which can be utilized in writing down an action for spinors. Gauge fields on AdS_2 were recently examined in [40] and it may be possible to check whether or not they have a noncommutative generalization. Within the context of the noncommutative theory, the spin-two fields should represent quantum gravity fluctuations. The massless scalar field examined in this article required no Gibbons-Hawking-York boundary term, nor holographic renormalization, as fields were asymptotically finite. Such simplifications most likely will not apply for the other field theories on $nc\mathrm{AdS}_2$.

Generalizations to $nc\mathrm{AdS}_{d+1}, d > 1$ should prove even more challenging. In this case there is no preferred choice for the Poisson brackets or the resulting quantization, both of which will necessarily break the AdS_{d+1} isometry group, and hence the conformal symmetry on the boundary. For example, it may be desirable to posit the Poisson bracket (2.15), since it states that the time is canonically conjugate to the CFT_d energy scale. However for $d > 1$ this Poisson bracket breaks the full Lorentz (or Euclidean) symmetry on the boundary. In another example, Poisson brackets on AdS_4 were given in [19] (Sec. VD2) which broke the $SO(3,2)$ isometry group to $SO(3) \otimes SO(2)$. Thus, more complicated results for the correlation functions are expected for $d > 1$.

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APPENDIX: SOME PROPERTIES OF THE MOYAL-WEYL STAR PRODUCT

Given two functions $\mathcal{F}$ and $\mathcal{G}$ on the Moyal-Weyl plane spanned by (x, y), their star product is defined by

$$[\mathcal{F} \star \mathcal{G}](x,y) = \mathcal{F}(x,y) \exp\left\{\frac{i\alpha}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \overrightarrow{\partial}_x)\right\}\mathcal{G}(x,y). \tag{A1}$$

This definition leads to the identities the following identities for the Moyal-Weyl star product

$$\mathcal{F}(x) \star = \mathcal{F}\left(x + \frac{i\alpha}{2}\overrightarrow{\partial}_y\right), \qquad \star\, \mathcal{F}(x) = \mathcal{F}\left(x - \frac{i\alpha}{2}\overleftarrow{\partial}_y\right),$$
$$\mathcal{G}(y) \star = \mathcal{G}\left(y - \frac{i\alpha}{2}\overrightarrow{\partial}_x\right), \qquad \star\, \mathcal{G}(y) = \mathcal{G}\left(y + \frac{i\alpha}{2}\overleftarrow{\partial}_x\right). \tag{A2}$$

A property of the integral of the Moyal-Weyl star product of two functions $\mathcal{F}$ and $\mathcal{G}$ on the Moyal-Weyl plane is

$$\int_{\mathbb{R}^2} dxdy\mathcal{F} \star \mathcal{G} = \int_{\mathbb{R}^2} dxdy\mathcal{F}\mathcal{G} + \text{boundary terms}. \tag{A3}$$

Correspondingly, the Moyal star commutator is a total divergence. The integral of a star commutator of any two functions $\mathcal{F}$ and $\mathcal{G}$ on the Moyal-Weyl plane can then be written as a boundary integral,

$$\int_D dxdy[\mathcal{F},\mathcal{G}]_\star(x,y) = \int_D dxdy[\partial_x\mathcal{V}_y - \partial_y\mathcal{V}_x](x,y)$$
$$= \int_{\partial D}(\mathcal{V}_x dx + \mathcal{V}_y dy), \tag{A4}$$

where D is some two-dimensional domain, with boundary ∂D. Up to order α^2, $\mathcal{V}_x$ and $\mathcal{V}_y$ are

$$\mathcal{V}_x = i\alpha\bigg(-\partial_x\mathcal{F}\mathcal{G} + \frac{\alpha^2}{24}(\partial_x^3\mathcal{F}\partial_y^2\mathcal{G} + \partial_x\partial_y^2\mathcal{F}\partial_x^2\mathcal{G}$$
$$- 2\partial_x^2\partial_y\mathcal{F}\partial_x\partial_y\mathcal{G}) + \mathcal{O}(\alpha^4)\bigg),$$
$$\mathcal{V}_y = i\alpha\bigg(-\partial_y\mathcal{F}\mathcal{G} + \frac{\alpha^2}{24}(\partial_y^3\mathcal{F}\partial_x^2\mathcal{G} + \partial_x^2\partial_y\mathcal{F}\partial_y^2\mathcal{G}$$
$$- 2\partial_x\partial_y^2\mathcal{F}\partial_x\partial_y\mathcal{G}) + \mathcal{O}(\alpha^4)\bigg). \tag{A5}$$

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