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Black hole squeezers

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We show that the gravitational quasinormal modes (QNMs) of a Schwarzschild black hole play the role of a multimode squeezer that can generate particles. For a minimally coupled scalar field, the QNMs "squeeze" the initial state of the scalar field (even for the vacuum) and produce scalar particles. The maximal squeezing amplitude is inversely proportional to the cube of the imaginary part of the QNM frequency, implying that the particle generation efficiency is higher for lower decaying QNMs. Our results show that the gravitational perturbations can amplify Hawking radiation.

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I. INTRODUCTION

Particle creation is a fundamental phenomenon characteristic of quantum field theory in curved spacetime [1–3]. It typically occurs in highly dynamical spacetimes: in the very early Universe, initial quantum fluctuations of curvature can be amplified by the exponentially expanding Universe to form tiny perturbations on the background spacetime [4,5]. These play a crucial role in explaining the anisotropies of the cosmic microwave background (CMB) and the formation of the large scale structure of the observed Universe. Primordial gravitational waves are also expected to be generated in the early Universe and their detection is one of the main targets of modern astrophysics. In addition to dynamical spacetimes, stationary or static spacetimes can also create quantum particles. Well-known examples include the superradiance from a rotating black hole [6–8], the Unruh-Davies [9,10] radiation observed by a uniformly accelerated observer, and the Hawking radiation from a Schwarzschild black hole [11]. Although Hawking particles were observed in an analogue system recently [12], detection of Hawking radiation from a real black hole remains elusive because the temperature of a solar mass black hole (10^{-8} °K) is much lower than the temperature of the CMB (∼2.7 °K).

One question of particular interest is whether gravitational waves (ripples of spacetime) can create quantum particles. Although a dynamical spacetime characteristically generates particles, it has been shown that particle creation by plane gravitational waves is forbidden [13–16]. A similar statement applies to electromagnetic waves: electron-positron pairs cannot be produced by plane electromagnetic waves, no matter how strong we make the electromagnetic field [17]; otherwise momentum conservation would be violated. However, pair production of electrons and positrons is possible if a nucleus is introduced to balance the momentum [18].

In the gravitational wave case, one might expect that an analog to the nucleus, e.g., a black hole, has to be introduced to allow particle creation. From a theoretical perspective, a detailed study of this issue is important and necessary to determine whether and to what degree gravitational perturbations in a black hole spacetime can create particles. There are now three gravitational wave events that have originated from the coalescence of two black holes that have been directly detected by the Laser Interferometer Gravitational-Wave Observatory (LIGO) [19–21]. The observed gravitational wave signals reveal that the spacetime changes dramatically when two black holes merge into one, with a large amount of energy of order a few solar rest masses carried away by the emitted gravitational waves. If particles, e.g., photons, can be produced by gravitational perturbations, they will travel along with the gravitational waves and could be detected if the particle creation efficiency is high enough. From an observational perspective, it is therefore also very important to have a thorough study of this problem.

In this paper we address this question. Instead of studying the whole process of the coalescence of two black holes, a very complicated situation requiring numerical relativity [22], we study the final stage of merging: the ring-down stage. At the ring-down stage, the quasinormal oscillations of the black hole are dominant, which are known as the gravitational quasinormal modes (QNMs). We are interested in the effects of the gravitational quasinormal modes (QNMs) of a black hole, which have been extensively studied for decades [23–28] and to which analytic techniques can be applied. We consider a massless Hermitian scalar field that propagates in the Schwarzschild background spacetime with quasinormal perturbations. The

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scalar field is assumed to be minimally coupled with the spacetime. The coupling can be divided into two parts: with the Schwarzschild background spacetime and with the QNMs. Though the former is well studied, the latter is somewhat less understood and is the main content of this paper.

We derive the interaction Hamiltonian for the scalar field, which implies that the QNMs play the role of a multimode squeezer. We show that the QNMs "squeeze" the initial state (vacuum or thermal state) of the scalar field and produce particles. In this sense black holes themselves can be quantum squeezers.

Our paper is organized as follows. In Sec. II, we briefly review the quantization of a massless scalar field in the Schwarzschild background spacetime. In Sec. III, we review the gravitational QNMs for Schwarzschild black holes and list some important results for our calculations. In Sec. IV, we study the coupling between the scalar field and the gravitational QNMs, and derive the interaction Hamiltonian for the scalar field, based on which we show that the gravitational perturbations around a Schwarzschild black hole create particles. In Sec. V, we provide a crude estimate of the total particle number. We summarize with some concluding remarks in Sec. VI. Throughout this paper we work with units $\hbar = G = c = 1$.

II. SCALAR FIELD IN CURVED SPACETIME

We consider a Hermitian massless scalar field Φ that minimally couples to the curved spacetime with metric $g_{\mu\nu}$. The Lagrangian density for the scalar field is [1]

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi), \tag{1}$$

where g is the determinant of $g_{\mu\nu}$. We assume that the metric $g_{\mu\nu}$ can be decomposed into a background part $g_{B\mu\nu}$ and a perturbation $h_{\mu\nu}$, namely, $g_{\mu\nu} = g_{B\mu\nu} + h_{\mu\nu}$. The background metric usually possesses some symmetries (time-translation invariance, rotational invariance, etc.), and the dynamics of the scalar field in the background spacetime is well established. The perturbation $h_{\mu\nu}$ is assumed to be small so that perturbation theory is applicable. Expanding the Lagrangian density Eq. (1) with respect to $h_{\mu\nu}$ and keeping terms to first order, we find

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \tag{2}$$

where the background part $\mathcal{L}_0$ and perturbed part $\mathcal{L}_1$ are

$$\mathcal{L}_0 = \frac{1}{2}\sqrt{-g_B}g_B^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi),$$
$$\mathcal{L}_1 = \frac{1}{4}\sqrt{-g_B}(h^\alpha{}_\alpha g_B^{\mu\nu} - 2h^{\mu\nu})(\partial_\mu\Phi)(\partial_\nu\Phi),$$

with g_B the determinant of the background metric and $h^\alpha{}_\alpha \equiv g_{B\alpha\beta}h^{\alpha\beta}$ the trace of the metric perturbation. Note that we use the convention $h^{\mu\nu} \equiv g_B^{\mu\alpha}g_B^{\nu\beta}h_{\alpha\beta}$. In this paper, we are concerned with the Schwarzschild background spacetime, for which the line element in the Schwarzschild coordinates (t, r, θ, ϕ) is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{3}$$

where $f(r) = 1\text{–}2M/r$ and M is the mass of the Schwarzschild black hole.

The canonically conjugate field of Φ is also decomposed into a background and perturbed part,

$$\Pi = \Pi_0 + \Pi_1, \tag{4}$$

where

$$\Pi_0 = \frac{\partial\mathcal{L}_0}{\partial(\partial_t\Phi)} = \sqrt{-g_B}g_B^{tt}(\partial_t\Phi),$$
$$\Pi_1 = \frac{\partial\mathcal{L}_1}{\partial(\partial_t\Phi)} = -\frac{1}{2}\sqrt{-g_B}[2h^{t\nu}(\partial_\nu\Phi) - h^\alpha{}_\alpha g_B^{tt}(\partial_t\Phi)].$$

The Hamiltonian density is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1, \tag{5}$$

where

$$\mathcal{H}_0 = \Pi_0(\partial_t\Phi) - \mathcal{L}_0 = \frac{1}{2}\sqrt{-g_B}[g_B^{tt}(\partial_t\Phi)^2 - g_B^{ij}(\partial_i\Phi)(\partial_j\Phi)] \tag{6}$$

is the unperturbed Hamiltonian density and

$$\mathcal{H}_1 = \Pi_1(\partial_t\Phi) - \mathcal{L}_1 = -\frac{1}{2}\sqrt{-g_B}[h^{tt}(\partial_t\Phi)^2 - h^{ij}(\partial_i\Phi)(\partial_j\Phi)] + \frac{1}{2}h^\alpha{}_\alpha\mathcal{H}_0 \tag{7}$$

is the perturbed Hamiltonian density. For the Schwarzschild background spacetime, $\sqrt{-g_B} = r^2\sin\theta$, so the perturbed Hamiltonian is

$$H_1 = \int \mathrm{d}^3x\mathcal{H}_1 = \frac{1}{2}\int_{2M}^{\infty}\mathrm{d}r\int_{4\pi}\mathrm{d}\Omega r^2\Big\{-h^{tt}(\partial_t\Phi)^2 + h^{ij}(\partial_i\Phi)(\partial_j\Phi) + \frac{1}{2}h^\alpha{}_\alpha[g_B^{tt}(\partial_t\Phi)^2 - g_B^{ij}(\partial_i\Phi)(\partial_j\Phi)]\Big\} \tag{8}$$

where $\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$.

The dynamics of the scalar field on the background spacetime is determined by the unperturbed Lagrangian

density $\mathcal{L}_0$, from which one can derive the Klein-Gordon equation [1],

$$\frac{1}{\sqrt{-g_B}}\partial_\mu(\sqrt{-g_B}g_B^{\mu\nu}\partial_\nu\Phi)=0. \tag{9}$$

The normal-mode solutions to Eq. (9) can be decomposed as

$$\phi_{\omega lm}(t,r,\theta,\phi)=\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega t}Y_{lm}(\theta,\phi)R_{\omega l}(r)/r \tag{10}$$

where $\omega>0$ is the frequency of the mode and $Y_{lm}(\theta,\phi)$ is the spherical harmonic. The radial function $R_{\omega l}(r)$ satisfies

$$-\frac{d^2R_{\omega l}}{dr_*^2}+V_l^{(s)}(r)R_{\omega l}=\omega^2R_{\omega l}, \tag{11}$$

where $V_l^{(s)}(r)$ is the effective potential,

$$V_l^{(s)}(r)=f(r)\left(\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right). \tag{12}$$

Here r_* is the tortoise coordinate

$$dr_*=dr/f(r),\qquad r_*=r+2M\ln(r/2M-1). \tag{13}$$

Note that $r_*\to-\infty$ corresponds to the event horizon of the Schwarzschild black hole. An implicit relation $r=r(r_*)$ can be derived and substituted into the effective potential $V_l^{(s)}$, Eq. (12), so that the effective potential can be considered as a function of r_*.

We only consider the field outside the event horizon, where there exist two sets of orthonormal modes that can completely represent the field. They are the upcoming and ingoing modes, denoted as $\phi^{\rm up}_{\omega lm}$ and $\phi^{\rm in}_{\omega lm}$, respectively. The asymptotic behavior for the radial part of the upcoming mode $R^{\rm up}_{\omega l}$ is

$$R^{\rm up}_{\omega l}\sim\begin{cases}B^{\rm up}_{\omega l}e^{i\omega r_*}, & r_*\to+\infty;\\ e^{i\omega r_*}+A^{\rm up}_{\omega l}e^{-i\omega r_*}, & r_*\to-\infty,\end{cases} \tag{14}$$

and for the radial part of the ingoing mode $R^{\rm in}_{\omega l}$ is

$$R^{\rm in}_{\omega l}\sim\begin{cases}e^{-i\omega r_*}+A^{\rm in}_{\omega l}e^{i\omega r_*}, & r_*\to+\infty;\\ B^{\rm in}_{\omega l}e^{-i\omega r_*}, & r_*\to-\infty.\end{cases} \tag{15}$$

Here $A^{\rm up}_{\omega l}$ ($A^{\rm in}_{\omega l}$) and $B^{\rm up}_{\omega l}$ ($B^{\rm in}_{\omega l}$) are the reflection and transmission amplitudes of the upcoming (ingoing) modes, respectively. They satisfy the following Wronskian relations [29],

$$\begin{aligned}|A^{\rm up}_{\omega l}|^2&=1-|B^{\rm up}_{\omega l}|^2,\\ |A^{\rm in}_{\omega l}|^2&=1-|B^{\rm in}_{\omega l}|^2,\\ |A^{\rm up}_{\omega l}|&=|A^{\rm in}_{\omega l}|,\qquad B^{\rm up}_{\omega l}=B^{\rm in}_{\omega l}.\end{aligned} \tag{16}$$

The upcoming modes $\phi^{\rm up}_{\omega lm}$ and ingoing modes $\phi^{\rm in}_{\omega lm}$ are chosen to satisfy the orthonormality relations,

$$\begin{aligned}\langle\phi^{\rm up}_{\omega lm},\phi^{\rm up}_{\omega' l'm'}\rangle&=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'},\\ \langle\phi^{\rm up*}_{\omega lm},\phi^{\rm up*}_{\omega' l'm'}\rangle&=-\delta(\omega-\omega')\delta_{ll'}\delta_{mm'},\\ \langle\phi^{\rm in}_{\omega lm},\phi^{\rm in}_{\omega' l'm'}\rangle&=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'},\\ \langle\phi^{\rm in*}_{\omega lm},\phi^{\rm in*}_{\omega' l'm'}\rangle&=-\delta(\omega-\omega')\delta_{ll'}\delta_{mm'},\\ \langle\phi^{\rm up}_{\omega lm},\phi^{\rm in}_{\omega' l'm'}\rangle&=0,\\ \langle\phi^{\rm up*}_{\omega lm},\phi^{\rm in*}_{\omega' l'm'}\rangle&=0.\end{aligned} \tag{17}$$

Here $\langle,\rangle$ represents the Klein-Gordon inner product [1], which is defined on a spacelike hypersurface $t=$ const as

$$\langle\varphi,\chi\rangle=i\int_{2M}^{\infty}dr\frac{r^2}{f(r)}\int_{4\pi}d\Omega(\varphi^*\partial_t\chi-\chi\partial_t\varphi^*) \tag{18}$$

for any two solutions φ and χ of the Klein-Gordon equation (9).

In the canonical quantization procedure, the scalar field Φ is regarded as an operator, satisfying certain canonical commutation relations, and is expanded as

$$\Phi=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty}d\omega(\hat{a}_{\omega lm}\phi^{\rm up}_{\omega lm}+\hat{b}_{\omega lm}\phi^{\rm in}_{\omega lm}+\text{H.c.}), \tag{19}$$

where H.c. represents the Hermitian conjugate. The operators $\hat{a}_{\omega lm}$ and $\hat{b}_{\omega lm}$ represent upcoming and ingoing modes, respectively. They satisfy the boson commutation relations

$$\begin{aligned}[\hat{a}_{\omega lm},\hat{a}^\dagger_{\omega' l'm'}]&=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'},\\ [\hat{b}_{\omega lm},\hat{b}^\dagger_{\omega' l'm'}]&=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'},\\ [\hat{a}_{\omega lm},\hat{b}_{\omega' l'm'}]&=[\hat{a}_{\omega lm},\hat{b}^\dagger_{\omega' l'm'}]=0.\end{aligned} \tag{20}$$

III. GRAVITATIONAL QUASINORMAL MODES

A Schwarzschild black hole is a static and spherically symmetric spacetime that is described by the Schwarzschild metric Eq. (3). Taking this metric to be the background metric $g_{B\mu\nu}$, gravitational perturbations $h_{\mu\nu}=g_{\mu\nu}-g_{B\mu\nu}$ can arise through various physical processes, such as a star falling into the black hole. The equations governing the evolution of the perturbations

were first derived by Regge and Wheeler [30], and Zerilli [31], in what is known as the Regge-Wheeler-Zerilli (RWZ) gauge. Due to the time-translation and rotational invariance of the Schwarzschild metric, the perturbations can be decomposed into eigenmodes with definite frequency and angular momentum. Furthermore, they can be classified as two distinct types: odd-parity (or magnetic-parity) and even-parity (or electric-parity) perturbations.

In the RWZ gauge, the odd-parity perturbations are characterized by two functions $\tilde{h}_0(r)$ and $\tilde{h}_1(r)$. The nonzero components of $h_{\mu\nu}$ are

$$h^{(o)}_{tA} = \tilde{h}_0(r)e^{-i\omega t}X^{lm}_A(\theta,\phi),$$
$$h^{(o)}_{rA} = \tilde{h}_1(r)e^{-i\omega t}X^{lm}_A(\theta,\phi), \tag{21}$$

where $A = \{\theta,\phi\}$. Here ω is the frequency of the perturbations, and X^{lm}_A is the odd-parity vector spherical harmonic on the unit two-sphere [32],

$$X^{lm}_\theta = -\csc\theta Y^{lm}{}_{,\phi}, \qquad X^{lm}_\phi = \sin\theta Y^{lm}{}_{,\theta}, \tag{22}$$

where $Y^{lm}(\theta,\phi)$ is the scalar spherical harmonic. The two functions $\tilde{h}_0(r)$ and $\tilde{h}_1(r)$ are not independent and can be expressed in terms of a single scalar function $Q(r)$ as [30]

$$\tilde{h}_0 = -\frac{f}{i\omega}\frac{d}{dr}(rQ), \qquad \tilde{h}_1 = \frac{rQ}{f}. \tag{23}$$

The scalar function $Q(r)$ satisfies the equation

$$-\frac{d^2Q}{dr_*^2} + V^{(o)}_l Q = \omega^2 Q, \tag{24}$$

where

$$V^{(o)}_l(r) = f(r)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right) \tag{25}$$

is the odd-parity effective potential.

In the RWZ gauge, the even-parity perturbations are characterized by three functions: $\tilde{H}_0(r), \tilde{H}_1(r)$, and $\tilde{K}(r)$. The nonzero components of $h_{\mu\nu}$ are

$$h^{(e)}_{tt} = f(r)\tilde{H}_0(r)e^{-i\omega t}Y^{lm}, \qquad h^{(e)}_{rr} = \frac{\tilde{H}_0(r)}{f(r)}e^{-i\omega t}Y^{lm},$$
$$h^{(e)}_{tr} = \tilde{H}_1(r)e^{-i\omega t}Y^{lm}, \qquad h^{(e)}_{AB} = r^2\Omega_{AB}\tilde{K}(r)e^{-i\omega t}Y^{lm}, \tag{26}$$

where $\Omega_{AB} = \mathrm{diag}\{1,\sin^2\theta\}$ is the metric on the unit two-sphere. $\tilde{H}_0(r), \tilde{H}_1(r)$, and $\tilde{K}(r)$ can be expressed in terms of the Zerilli function $Z(r)$ as [33]

$$\tilde{K} = \left[\frac{\lambda(\lambda+1)r^2 + 3\lambda Mr + 6M^2}{r^2(\lambda r + 3M)}\right]Z + \sqrt{f}Z_{,r},$$
$$\tilde{H}_1 = -i\omega\left[\frac{\lambda r^2 - 3\lambda Mr - 3M^2}{(r-2M)(\lambda r+3M)}\right]Z - i\omega rZ_{,r},$$
$$\tilde{H}_0 = \left[\frac{\lambda r(r-2M) - \omega^2 r^4 + M(r-3M)}{(r-2M)(\lambda r+3M)}\right]\tilde{K} + \left[\frac{(\lambda+1)M - \omega^2 r^3}{i\omega r(\lambda r+3M)}\right]\tilde{H}_1, \tag{27}$$

where

$$\lambda = \frac{1}{2}(l-1)(l+2). \tag{28}$$

The Zerilli function satisfies the equation

$$-\frac{d^2Z}{dr_*^2} + V^{(e)}_l Z = \omega^2 Z \tag{29}$$

with the even-parity effective potential

$$V^{(e)}_l(r) = f(r)\left[\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2Mr^2 + 18\lambda M^2 r + 18M^3}{r^3(\lambda r+3M)^2}\right]. \tag{30}$$

The boundary conditions for the QNMs are that on the event horizon, there is only an ingoing wave

$$Q(Z) \sim e^{-i\omega r_*}, \qquad r_* \to -\infty (r \to 2M) \tag{31}$$

and at spatial infinity there is only an outgoing wave

$$Q(Z) \sim e^{i\omega r_*}, \qquad r_* \to +\infty (r \to +\infty). \tag{32}$$

The above boundary conditions imply that the perturbations are dissipative: waves can escape either to infinity or into the black hole. The frequencies of the QNMs are complex,

$$\omega = \omega_R - i\omega_I, \tag{33}$$

where ω_I is positive and characterizes the decay of the QNMs. For a Schwarzschild black hole, there is a discrete infinity of QNMs. The QNM frequencies depend on l and an integer n called the overtone number [34,35].

Based on the Newman-Penrose (NP) null-tetrad formalism [36], another approach has been developed to study the gravitational perturbations in a Schwarzschild background [23,37]. This more general method has been

generalized to study the neutrino, electromagnetic, and gravitational perturbations in a Kerr background spacetime [38–41]. In this framework, gravitational perturbations are represented by two field quantities ψ_s with $s = \pm 2$, which are related to the Weyl scalars [36] and satisfy the Teukolsky master equation [38]. For the Schwarzschild case, one can take the limit $a \to 0$ (a is the angular momentum per unit mass of the Kerr black hole) in the master equation to obtain the corresponding field equation for ψ_s. This quantity can be decomposed as

$$\psi_s = e^{-i\omega t}{}_sY_{lm}(\theta,\phi)\,{}_sR_{\omega l}(r), \tag{34}$$

where ${}_sY_{lm}$ is the spin-weighted spherical harmonic [42], and ${}_sR_{\omega l}(r)$ is the radial function satisfying the equation

$$\Delta_r^{-s}\frac{d}{dr}\left(\Delta_r^{s+1}\frac{d}{dr}\right){}_sR_{\omega l}(r) + \left[\frac{r^4\omega^2 - 2isr^2(r-M)\omega}{\Delta_r} + 4is\omega r - (l-s)(l+s+1)\right]{}_sR_{\omega l}(r) = 0, \tag{35}$$

where $\Delta_r = r(r-2M)$. At a large distance from the black hole, the asymptotic solutions of ${}_sR_{\omega l}$ are

$${}_sR_{\omega l} \sim \frac{e^{-i\omega r_*}}{r} \quad \text{and} \quad {}_sR_{\omega l} \sim \frac{e^{i\omega r_*}}{r^{2s+1}}, \tag{36}$$

whereas very close to the event horizon,

$${}_sR_{\omega l} \sim \Delta_r^{-s}e^{-i\omega r_*} \quad \text{and} \quad {}_sR_{\omega l} \sim e^{i\omega r_*}. \tag{37}$$

The QNM boundary conditions (only outgoing waves at spatial infinity, only ingoing waves at the future horizon) imply

$${}_sR_{\omega l} \sim \begin{cases} C_{\omega l}\Delta_r^{-s}e^{-i\omega r_*}, & r_* \to -\infty; \\ D_{\omega l}e^{i\omega r_*}/r^{2s+1}, & r_* \to +\infty, \end{cases} \tag{38}$$

where $C_{\omega l}$ and $D_{\omega l}$ are the amplitudes of the QNM at the event horizon and spatial infinity, respectively.

The explicit expressions for the components of the metric perturbation $h_{\mu\nu}$ are very important when considering the coupling between the gravitational perturbations and the quantum fields. Chrzanowski, Cohen, and Kegeles (CCK) developed a procedure for reconstructing the metric perturbation $h_{\mu\nu}$ in the ingoing and outgoing radiation gauges from the field quantity ψ_s [43–45]. Roughly speaking, the CCK procedure consists of two steps: the first step is to relate the field quantity ψ_s to the so-called Hertz potential Ψ, which also satisfies the master equation with spin weight $s = -2$; the second step is to find the relation between $h_{\mu\nu}$ and the Hertz potential Ψ [43]. The first explicit calculation of the relation between the Hertz potential Ψ and ψ_s for the Schwarzschild black hole was done by Lousto and Whiting [46]. Generalization to the Kerr black hole was performed by Ori [47], Yunes, and Gonzalez [48].

In the ingoing and outgoing radiation gauges, the trace of the metric perturbation $h^\alpha{}_\alpha$ vanishes in the whole spacetime [43]. The perturbation $h_{\mu\nu}$ is transverse at past null infinity and at the future horizon in the ingoing radiation gauge. Therefore it is a suitable gauge to study the gravitational effects near the event horizon, while in the outgoing radiation gauge, $h_{\mu\nu}$ is transverse at future null infinity and at the past horizon. It is therefore a suitable gauge for studying gravitational effects at spatial infinity, e.g., gravitational waves emitted by a black hole. Since it is reasonable to expect that the interaction between the metric perturbation and the quantum fields is strong near the event horizon, we therefore work in the ingoing radiation gauge throughout this paper.

Following the CCK procedure, Nichols *et al.* [33] derived explicit expressions for the metric perturbation $h_{\mu\nu}$ in the Schwarzschild background spacetime in the ingoing radiation gauge. For the odd (magnetic)-parity perturbations, the nonzero components are

$$\begin{aligned} h^{(o)}_{tA} &= -fh^{(o)}_{rA} = \frac{\sqrt{D}}{2f\sqrt{2l(l+1)}}\Re\left\{\left[\frac{d}{dr_*}{}_{-2}R_{\omega l} - \left(i\omega + \frac{2f}{r}\right){}_{-2}R_{\omega l}\right]({}_{-1}Y_{lm}m_A + {}_1Y_{lm}m_A^*)e^{-i\omega t}\right\}, \\ h^{(o)}_{AB} &= \frac{1}{f^2}\Re\left\{\left[(i\omega r^2 - M)\frac{d}{dr_*}{}_{-2}R_{\omega l} - \left(\frac{1}{2}\mu^2 f - i\omega(-3r+7M) - \omega^2 r^2\right){}_{-2}R_{\omega l}\right]({}_{-2}Y_{lm}m_Am_B - {}_2Y_{lm}m_A^*m_B^*)e^{-i\omega t}\right\}, \end{aligned} \tag{39}$$

where $D = (l+2)!/(l-2)!$, $\mu^2 = (l-1)(l+2)$, $\Re$ represents the real part of a function, and $m_A = \frac{1}{\sqrt{2}}(1, i\sin\theta)$ is a vector on the unit sphere with its index raised by the metric Ω^{AB}. For the even (electric)-parity perturbations, the nonzero components are

$$h^{(e)}_{tt} = -f h^{(e)}_{tr} = f^2 h^{(e)}_{rr} = -\frac{2\sqrt{D}}{r^2}\Re\{{}_{-2}R_{\omega l}Y_{lm}e^{-i\omega t}\},$$
$$h^{(e)}_{tA} = -f h^{(e)}_{rA} = \frac{\sqrt{D}}{2f\sqrt{2l(l+1)}}\Re\left\{\left[\frac{d}{dr_*}{}_{-2}R_{\omega l} - \left(i\omega + \frac{2f}{r}\right){}_{-2}R_{\omega l}\right]({}_{-1}Y_{lm}m_A - {}_{1}Y_{lm}m^*_A)e^{-i\omega t}\right\},$$
$$h^{(e)}_{AB} = \frac{1}{f^2}\Re\left\{\left[(i\omega r^2 - M)\frac{d}{dr_*}{}_{-2}R_{\omega l} - \left(\frac{1}{2}\mu^2 f - i\omega(-3r+7M) - \omega^2 r^2\right){}_{-2}R_{\omega l}\right]({}_{-2}Y_{lm}m_A m_B + {}_{2}Y_{lm}m^*_A m^*_B)e^{-i\omega t}\right\}. \tag{40}$$

Note that the metric perturbation in the ingoing radiation gauge is related to that in the RWZ gauge, Eqs. (21) and (26), by a gauge transformation.

IV. COUPLING BETWEEN QNMS AND SCALAR FIELD

In the absence of gravitational perturbations, the scalar field Φ evolves freely on the Schwarzschild background spacetime. Its dynamics is dominated by the unperturbed Lagrangian density $\mathcal{L}_0$. If the Schwarzschild background spacetime is perturbed, the scalar field will couple to the gravitational perturbations. Its dynamics is governed by the interaction Hamiltonian H_1, Eq. (8). Our particular interest is in whether or not gravitational perturbations in a Schwarzschild background can produce particles. We have noted earlier that plane gravitational waves do not produce particles [13–16]; were it otherwise, momentum conservation would be violated. As we shall demonstrate, the situation is different for spherical perturbations. We will show that gravitational perturbations in a Schwarzschild background do generate scalar particles and that angular momentum is conserved in this process.

In order to know the evolution of the state of the scalar field, one needs to find the explicit expression for the interaction Hamiltonian H_1 which contains only first-order terms of the components of the metric perturbation $h_{\mu\nu}$. An appropriate gauge can be chosen so that the interaction Hamiltonian H_1 takes a relatively simple form. Throughout this paper we will work in the ingoing radiation gauge. There are several advantages of choosing this gauge. First, it is straightforward to generalize the calculations to the Kerr background case. Second, it is expected that the coupling between the gravitational perturbations and the scalar field is strong around the event horizon so it is more convenient to use the ingoing radiation gauge. Third, the trace of the metric perturbation vanishes in this gauge, $h^\alpha{}_\alpha = 0$. Consequently, Eq. (8) is simplified:

$$H_1 = \frac{1}{2}\int_{2M}^{\infty} r^2 dr \int_{4\pi} d\Omega[-h^{tt}(\partial_t\Phi)^2 + h^{rr}(\partial_r\Phi)^2 + 2h^{rA}(\partial_r\Phi)(\partial_A\Phi) + h^{AB}(\partial_A\Phi)(\partial_B\Phi)]. \tag{41}$$

In what follows we will consider the effects of both the odd-parity and even-parity QNMs with frequency $\omega_0 = \omega_R - i\omega_I$ and angular momentum l_0, m_0.

A. Effects of odd-parity QNMs

1. Odd-parity interaction Hamiltonian

For simplicity, we only consider the coupling between upcoming and upcoming modes, and omit the superscript "up" without introducing any confusion. Couplings between upcoming and ingoing modes, and ingoing and ingoing modes are also possible, which we leave for future work. Since for odd-parity perturbations, $h^{(o)}_{tt} = h^{(o)}_{rr} = 0$, the relevant terms in Eq. (41) are $(\partial_r\Phi)(\partial_A\Phi)$ and $(\partial_A\Phi)(\partial_B\Phi)$.

$$(\partial_r\Phi)(\partial_A\Phi) = \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\int d\omega\int d\omega'\frac{1}{4\pi\sqrt{\omega\omega'}}\left[\hat{a}_{\omega lm}\hat{a}_{\omega' l'm'}e^{-i(\omega+\omega')t}\partial_r\left(\frac{R_{\omega l}}{r}\right)\left(\frac{R_{\omega' l'}}{r}\right)Y_{lm}(\partial_A Y_{l'm'}) + \hat{a}_{\omega lm}\hat{a}^\dagger_{\omega' l'm'}e^{-i(\omega-\omega')t}\partial_r\left(\frac{R_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right)Y_{lm}(\partial_A Y^*_{l'm'}) + \text{H.c.}\right]. \tag{42}$$

$$(\partial_A\Phi)(\partial_B\Phi) = \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\int d\omega\int d\omega'\frac{1}{4\pi\sqrt{\omega\omega'}}\left[\hat{a}_{\omega lm}\hat{a}_{\omega' l'm'}e^{-i(\omega+\omega')t}\left(\frac{R_{\omega l}}{r}\right)\left(\frac{R_{\omega' l'}}{r}\right)(\partial_A Y_{lm})(\partial_B Y_{l'm'}) + \hat{a}_{\omega lm}\hat{a}^\dagger_{\omega' l'm'}e^{-i(\omega-\omega')t}\left(\frac{R_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right)(\partial_A Y_{lm})(\partial_B Y^*_{l'm'}) + \text{H.c.}\right]. \tag{43}$$

From Eq. (39) we see that $h^{(o)}_{rA}$ and $h^{(o)}_{AB}$ contain terms that are proportional to $e^{-i\omega_0 t} = e^{-\omega_I t}e^{-i\omega_R t}$ and $(e^{-i\omega_0 t})^* = e^{-\omega_I t}e^{i\omega_R t}$. When multiplying with $(\partial_r\Phi)(\partial_A\Phi)$ and $(\partial_A\Phi)(\partial_B\Phi)$ we get terms containing factors

$$e^{\pm i(\omega_R-\omega-\omega')t}, \qquad e^{\pm i(\omega_R+\omega+\omega')t}, \qquad e^{\pm i(\omega_R+\omega-\omega')t}, \qquad e^{\pm i(\omega_R-\omega+\omega')t}.$$

In the rotating-wave approximation, terms with the lowest frequency oscillations $e^{\pm i(\omega_R-\omega-\omega')t}$ dominate over more highly oscillatory terms. This approximation ensures that the energy is approximately conserved, $\omega_R \approx \omega + \omega'$. Substituting Eqs. (39), (42), and (43) into Eq. (41) we have in this approximation

$$H^{(o)}_1 \approx \frac{1}{16\pi}e^{-\omega_I t}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\int d\omega\int d\omega'\frac{1}{\sqrt{\omega\omega'}}[e^{-i(\omega_R-\omega-\omega')t}\hat{a}^\dagger_{\omega lm}\hat{a}^\dagger_{\omega' l' m'}(\mathcal{I}_{r1}\mathcal{I}_{a1}+\mathcal{I}_{r2}\mathcal{I}_{a2})+\text{H.c.}], \quad (44)$$

where $\mathcal{I}_{r1}$ and $\mathcal{I}_{r2}$ are the radial integrals,

$$\mathcal{I}_{r1} = -\frac{\sqrt{D_0}}{\sqrt{2l_0(l_0+1)}}\int_{2M}^{\infty} dr\frac{1}{f}\left[\frac{d}{dr_*}{}_{-2}R_{\omega_0 l_0} - \left(i\omega_0+\frac{2f}{r}\right){}_{-2}R_{\omega_0 l_0}\right]\partial_r\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right), \quad (45)$$

$$\mathcal{I}_{r2} = \int_{2M}^{\infty} dr\frac{1}{r^2 f^2}\left[(i\omega_0 r^2 - M)\frac{d}{dr_*}{}_{-2}R_{\omega_0 l_0} - \left(\frac{1}{2}\mu_0^2 f - i\omega_0(-3r+7M) - \omega_0^2 r^2\right){}_{-2}R_{\omega_0 l_0}\right]\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right), \quad (46)$$

and $\mathcal{I}_{a1}$ and $\mathcal{I}_{a2}$ are the angular integrals,

$$\mathcal{I}_{a1}(l_0,m_0;l,m;l',m') = \int_{4\pi} d\Omega({}_{-1}Y_{l_0 m_0}m^A + {}_{1}Y_{l_0 m_0}m^{A*})Y^*_{lm}(\partial_A Y^*_{l'm'}), \quad (47)$$

$$\mathcal{I}_{a2}(l_0,m_0;l,m;l',m') = \int_{4\pi} d\Omega({}_{-2}Y_{l_0 m_0}m^A m^B - {}_{2}Y_{l_0 m_0}m^{A*}m^{B*})(\partial_A Y^*_{lm})(\partial_B Y^*_{l'm'}). \quad (48)$$

Since the Hamiltonian (44) is quadratic in creation (and annihilation) operators, it is clear that it describes multimode squeezing. The quantity $\mathcal{I}_{r1}\mathcal{I}_{a1}+\mathcal{I}_{r2}\mathcal{I}_{a2}$ plays the role of a phase matching function, the nonzero value of which would imply that gravitational perturbations generate quantum particles.

2. Radial and angular integrals

Since there are no analytic solutions for the radial functions $R_{\omega l}$ and ${}_{-2}R_{\omega_0 l_0}$, it is therefore impossible to find analytic results for the radial integrals $\mathcal{I}_{r1}$ and $\mathcal{I}_{r2}$. One might expect the radial integrals can be calculated numerically. It turns out that the calculation of the radial integrals is not trivial because of the peculiar property of the radial function of the QNMs. From the boundary conditions for the QNMs, Eq. (38), the radial function of the QNMs is proportional to $e^{\omega_I r_*}$ when $r_* \to +\infty$, and it is proportional to $e^{-\omega_I r_*}$ when $r_* \to -\infty$. In both limits the radial function of the QNMs is divergent and the radial integrals are not well defined. Leaver [49] proposed a method to overcome this difficulty by exploiting the analyticity of the integrand in r_*. A new contour (see Appendix A for details) is chosen such that the integral along this contour is finite. Sun and Price [50] discussed in detail how to construct Leaver's contour by analytic continuation and restored a factor that is missing in [49]. Similar techniques were also used by Yang *et al.* [51] to define the inner product of the radial function of the QNMs. In this paper, we follow the method of Leaver (taking into account the missing factor) to regularize the radial integral to obtain a finite result.

By using Leaver's method, the radial integral can, in principle, be calculated numerically. In order to obtain an approximately analytic result, we assume that the main contribution to the integration is from the region near the event horizon, that is, $r \sim 2M$. This is because the coupling between the QNMs and scalar field near the horizon is expected to be stronger. This assumption can also be justified by looking at the asymptotic behavior of the integrand along the contour at infinity, which is exponentially suppressed (see Appendix A for details). To further simplify the result, we assume that the imaginary parts of the QNM frequencies are small. This is a rather crude approximation because the imaginary parts of the QNM frequencies of a Schwarzschild black hole are not so small. However, this approximation is adequate for the purpose of this paper.

Taking into account all the above approximations, we find

$$\mathcal{I}_{r1} \approx -\frac{\sqrt{D_0}}{\sqrt{2l_0(l_0+1)}} \frac{16\pi i M\Omega\Omega_0}{(i\Delta)^2} \frac{A^*_{\omega l} A^*_{\omega' l'} C_{\omega_0 l_0}}{1-e^{2\pi\Omega_0}} \tag{49}$$

and

$$\mathcal{I}_{r2} \approx \frac{16\pi i M\Omega_0^2}{(i\Delta)^2} \frac{A^*_{\omega l} A^*_{\omega' l'} C_{\omega_0 l_0}}{1-e^{2\pi\Omega_0}}, \tag{50}$$

where $\Omega = 2M\omega$, $\Omega_0 = 2M\omega_0$, $\Omega' = 2M\omega'$, and $\Delta = \Omega_0 - \Omega - \Omega'$. Equations (49) and (50) are valid when $\Delta \sim 0$.

Fortunately, the angular integrals $\mathcal{I}_{a1}$ and $\mathcal{I}_{a2}$ can be calculated analytically. In Appendix B, we derive the explicit expressions for the spin-weighted spherical harmonics with spin weights ± 1 and ± 2 in terms of the Legendre functions, as shown by Eqs. (B8) and (B9). In Appendix C, we substitute Eqs. (B8) and (B9) into Eqs. (47) and (48) to obtain the angular integrals $\mathcal{I}_{a1}$ and $\mathcal{I}_{a2}$.

For simplicity we consider a special case where the angular momentum of the QNMs along the z direction is zero, that is, $m_0 = 0$. This simplifies the calculation a lot and is sufficient to demonstrate quantum particle generation by the QNMs. The resulting angular integrals $\mathcal{I}_{a1}$ and $\mathcal{I}_{a2}$ are given by Eqs. (C7) and (C8). Note that the integration over ϕ gives rise to a δ function $\delta_{m',-m}$, which implies that the produced pair of particles have opposite angular momentum along the z direction. This is not surprising given that $m_0 = 0$ and is an indication of angular momentum conservation in the particle production process.

As an example, we calculate the angular integrals $\mathcal{I}_{a1}$ and $\mathcal{I}_{a2}$ for a QNM with angular momentum $(l_0, m_0) = (2, 0)$ and a pair of scalar particles, the first of which has angular momentum $(l, m) = (1, 1)$ and the other $(l', -1)$. We find that the particle (1,1) only couples with those particles with $l' = 2$; namely, the only nonzero $\mathcal{I}_{a1}$ and $\mathcal{I}_{a2}$ are

$$\begin{aligned} \mathcal{I}_{a1}(2,0;1,1;2,-1) &= -\frac{1}{2}\sqrt{\frac{3}{\pi}}, \\ \mathcal{I}_{a2}(2,0;1,1;2,-1) &= \sqrt{\frac{3}{2\pi}}. \end{aligned} \tag{51}$$

3. QNM as multimode squeezer

In Sec. IV A 2, we have analytically calculated the angular integrals and derived approximately analytic expressions for the radial integrals for odd-parity QNMs. We thus can obtain the interaction Hamiltonian $H_1^{(o)}$, which dominates the evolution of the scalar field. In the following we will estimate the strength of the coupling between the QNMs and the scalar field.

The time evolution operator is

$$\hat{U}^{(o)} = \hat{T} \exp\left\{-i \int_0^\infty \mathrm{d}t H_1^{(o)}(t)\right\}, \tag{52}$$

where $\hat{T}$ is the time ordering operator. In the low squeezing regime [52], the time ordering is not important, so we can approximate the time evolution operator as

$$\hat{U}^{(o)} \approx \exp\left\{-i \int_0^\infty \mathrm{d}t H_1^{(o)}(t)\right\}, \tag{53}$$

and the integration can be directly carried out. Using

$$\int_0^\infty \mathrm{d}t e^{-\omega_I t} e^{-i(\omega_R - \omega - \omega')t} = \frac{1}{i[(\omega_R - \omega - \omega') - i\omega_I]},$$

we have

$$\begin{aligned} \hat{U}^{(o)} &= \bigotimes_{lm} \bigotimes_{l'} \hat{U}^{(o)}_{lml'}, \\ \hat{U}^{(o)}_{lml'} &\approx \exp\left\{-i \int \mathrm{d}\omega \int \mathrm{d}\omega' [\mathcal{F}^{(o)}_{lml'}(\omega,\omega') \hat{a}^\dagger_{\omega lm} \hat{a}^\dagger_{\omega' l',-m} \right. \\ &\quad \left. + \mathcal{F}^{(o)*}_{lml'}(\omega,\omega') \hat{a}_{\omega lm} \hat{a}_{\omega' l',-m}]\right\} \end{aligned} \tag{54}$$

where

$$\begin{aligned} \mathcal{F}^{(o)}_{lml'}(\omega,\omega') &= \frac{4iM^3 A^*_{\omega l} A^*_{\omega' l'} C_{\omega_0 l_0}}{\sqrt{\Omega\Omega'}(1-e^{2\pi\Omega_0})} \frac{1}{(i\Delta)^3} \\ &\quad \times \left[\Omega_0^2 \mathcal{I}_{a2} - \frac{\sqrt{D_0}}{\sqrt{2l_0(l_0+1)}} \Omega\Omega_0 \mathcal{I}_{a1}\right]. \end{aligned} \tag{55}$$

It is evident that Eq. (54) represents a multimode squeezing operator and $\mathcal{F}^{(o)}_{lml'}(\omega,\omega')$ is known as the joint frequency distribution. Equation (55) shows that the joint frequency distribution $\mathcal{F}^{(o)}_{lml'}(\omega,\omega')$ is not zero, indicating that there will be scalar particle creation. If the initial state of the scalar field is a vacuum state (the Boulware vacuum), the QNMs squeeze the vacuum and produce a squeezed vacuum state; if the initial state is a thermal state, e.g., Hawking thermal radiation, it is amplified due to the QNM squeezing. In general, the QNMs squeeze any state of the scalar field and amplify it, producing scalar particles. The particle production energy comes from the QNMs. This is our main result.

In Eq. (55), $A_{\omega l}$ is the reflection amplitude [see Eq. (14)] of the upcoming scalar field mode determined by the effective potential (12). For a given l, the effective potential peaks around $r_* = 2M$ with its maximum depending on l (higher for larger l).

Unfortunately, there is no analytic expression for the reflection amplitude. However, we can infer the qualitative

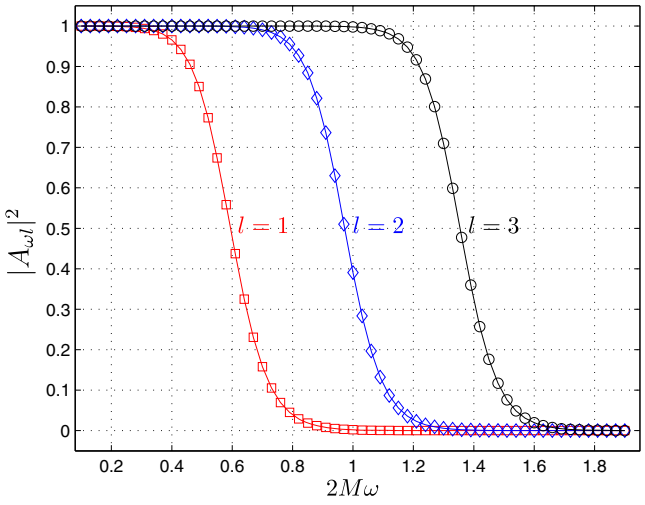

FIG. 1. Reflection coefficient for the scalar field modes.

behavior of $A_{\omega l}$. When the frequency of the field mode is lower than the maximum of the effective potential, most of the field mode is reflected, whereas the field mode almost penetrates through the potential if the frequency is higher than the potential maximum. Figure 1 shows the numerical results of the reflection coefficients $|A_{\omega l}|^2$.

After knowing the behavior of the reflection amplitude, we can study the behavior of the joint frequency distribution. The factor Δ is defined as

$$\Delta \equiv \Omega_0 - \Omega - \Omega' = (\Omega_R - \Omega - \Omega') - i\Omega_I,$$

where $\Omega_R = 2M\omega_R$, $\Omega_I = 2M\omega_I$. Therefore, we have

$$\frac{1}{|\Delta|^3} = \frac{1}{[(\Omega_R - \Omega - \Omega')^2 + \Omega_I^2]^{3/2}}. \tag{56}$$

This is a distribution with respect to Ω and Ω' that peaks along the line $\Omega + \Omega' = \Omega_R$, the maximum of which is $1/\Omega_I^3$. Here $\Omega_R - \Omega - \Omega'$ can be considered as the frequency detuning, and Ω_I can be considered as the decay rate which also characterizes the width of the distribution Eq. (56). If Ω_I is small, the distribution Eq. (56) is nonzero only for $\Omega + \Omega' \approx \Omega_R$. This is an indication of energy conservation: the sum of the frequencies of the pair of scalar particles is equal to the real part of the QNM frequency. Figure 2 shows an example of the absolute value of the joint frequency distribution $|\mathcal{F}^{(o)}_{lml'}(\omega, \omega')|$. We can see that basically $|\mathcal{F}^{(o)}_{lml'}(\omega, \omega')|$ follows the energy-conservation line $\Omega + \Omega' = \Omega_R$. The high-frequency part is suppressed by the reflection amplitude $A_{\omega l}$, while in the low-frequency regime, $|A_{\omega l}|$ is almost one and the factor $1/\sqrt{\Omega\Omega'}$ dominates. The latter is annoying because that means the joint frequency distribution is divergent at $\Omega = 0$ or $\Omega' = 0$. Here we assume that there exists a

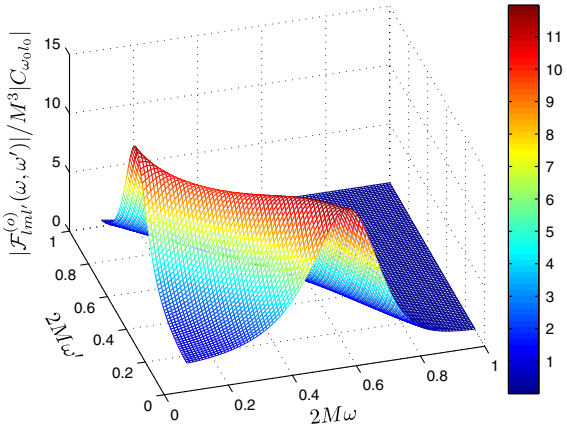

FIG. 2. Modulus of the joint frequency distribution. QNM: $\Omega_R = 0.7474$, $\Omega_I = 0.178$, $(l_0, m_0) = (2, 0)$. Scalar particle one: $(l, m) = (1, 1)$; scalar particle two: $(l', m') = (2, -1)$.

low-frequency cutoff so that the joint frequency distribution is finite.

We can compute a crude estimate of the maximum of the joint frequency distribution. Assume that in Eq. (55), $\Omega \sim \Omega' \sim \Omega_R/2$, $|A_{\omega l}| \sim |A_{\omega' l'}| \sim 1/\sqrt{2}$, and the contribution from the angular integral part is at the order of unity; we then find

$$|\mathcal{F}^{(o)}_{lml'}|_{\max} \sim 4M^3 \left(\frac{\Omega_R}{\Omega_I}\right)\left(\frac{1}{\Omega_I^2}\right)\frac{|C_{\omega_0 l_0}|}{|1 - e^{2\pi(\Omega_R - i\Omega_I)}|}. \tag{57}$$

This is an approximate relation between the squeezing amplitude (or the coupling strength) and various parameters of the black hole and the QNM. Here Ω_R is the resonance frequency of the QNM and Ω_I characterizes the decay rate. If we make an analogy with an optical cavity [53], Ω_R/Ω_I can be considered as the quality factor of the QNM. We see that the squeezing amplitude is proportional to the cube of the black hole mass, the amplitude of the QNM at the event horizon, and the quality factor of the QNM, and is inversely proportional to the square of the decay rate. This means the coupling strength is stronger for longer-lasting QNMs, larger-amplitude QNMs, and bigger black holes. However, for Schwarzschild black holes the QNMs decay very fast and the least-damped QNM is the fundamental QNM for which the overtone number is $n = 0$ [27]. For example, for the fundamental QNM of $(l_0, m_0) = (2, 0)$, $\Omega_R = 0.7474$ and $\Omega_I = 0.178$ [27]. Substituting these into Eq. (57), we find

$$|\mathcal{F}^{(o)}_{lml'}|_{\max} \sim 5M^3|C_{\omega_0 l_0}|. \tag{58}$$

B. Effects of even-parity QNMs

The procedure to evaluate the coupling between the scalar field and the even-parity QNMs is similar. For the even-parity perturbations, $h^{(e)}_{tt} \neq 0$, $h^{(e)}_{rr} \neq 0$, so we also need $(\partial_t\Phi)(\partial_t\Phi)$ and $(\partial_r\Phi)(\partial_r\Phi)$ in Eq. (41), which are

$$(\partial_t\hat{\Phi})(\partial_t\hat{\Phi}) = \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\int d\omega\int d\omega'\frac{1}{4\pi\sqrt{\omega\omega'}}\left[-\omega\omega'\hat{a}_{\omega lm}\hat{a}_{\omega' l'm'}e^{-i(\omega+\omega')t}\left(\frac{R_{\omega l}}{r}\right)\left(\frac{R_{\omega' l'}}{r}\right)Y_{lm}Y_{l'm'}\right.$$
$$\left.+\,\omega\omega'\hat{a}_{\omega lm}\hat{a}^{\dagger}_{\omega' l'm'}e^{-i(\omega-\omega')t}\left(\frac{R_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right)Y_{lm}Y^*_{l'm'}+\text{H.c.}\right]. \tag{59}$$

$$(\partial_r\hat{\Phi})(\partial_r\hat{\Phi}) = \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\int d\omega\int d\omega'\frac{1}{4\pi\sqrt{\omega\omega'}}\left[\hat{a}_{\omega lm}\hat{a}_{\omega' l'm'}e^{-i(\omega+\omega')t}\partial_r\left(\frac{R_{\omega l}}{r}\right)\partial_r\left(\frac{R_{\omega' l'}}{r}\right)Y_{lm}Y_{l'm'}\right.$$
$$\left.+\,\hat{a}_{\omega lm}\hat{a}^{\dagger}_{\omega' l'm'}e^{-i(\omega-\omega')t}\partial_r\left(\frac{R_{\omega l}}{r}\right)\partial_r\left(\frac{R^*_{\omega' l'}}{r}\right)Y_{lm}Y^*_{l'm'}+\text{H.c.}\right]. \tag{60}$$

Substituting Eqs. (40) and (19) into (41), and taking into account the rotating wave approximation, we have

$$H_1^{(e)} \approx \frac{1}{16\pi}e^{-\omega_I t}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\int d\omega\int d\omega'\frac{1}{\sqrt{\omega\omega'}}\{e^{-i(\omega_R-\omega-\omega')t}\hat{a}^{\dagger}_{\omega lm}\hat{a}^{\dagger}_{\omega' l'm'}[(\mathcal{I}_{r3}+\mathcal{I}_{r4})\mathcal{I}_{a3}+\mathcal{I}_{r1}\mathcal{I}_{a4}+\mathcal{I}_{r2}\mathcal{I}_{a5}]$$
$$+\,\text{H.c.}\}, \tag{61}$$

where the radial integrals $\mathcal{I}_{r1}$ and $\mathcal{I}_{r2}$ are defined in Sec. IV A 1, and $\mathcal{I}_{r3}$ and $\mathcal{I}_{r4}$ are defined as

$$\mathcal{I}_{r3} = -\omega\omega'\sqrt{D_0}\int_{2M}^{\infty}dr\frac{1}{f^2}\,{}_{-2}R_{\omega_0 l_0}\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right), \tag{62}$$

$$\mathcal{I}_{r4} = -\sqrt{D_0}\int_{2M}^{\infty}dr\,{}_{-2}R_{\omega_0 l_0}\partial_r\left(\frac{R^*_{\omega l}}{r}\right)\partial_r\left(\frac{R^*_{\omega' l'}}{r}\right). \tag{63}$$

The angular integrals $\mathcal{I}_{a3}$, $\mathcal{I}_{a4}$, $\mathcal{I}_{a5}$ are

$$\mathcal{I}_{a3}(l_0,m_0;l,m;l',m') = \int_{4\pi}d\Omega Y_{l_0m_0}Y^*_{lm}Y^*_{l'm'}, \tag{64}$$

$$\mathcal{I}_{a4}(l_0,m_0;l,m;l',m') = \int_{4\pi}d\Omega({}_{-1}Y_{l_0m_0}m^A - {}_{1}Y_{l_0m_0}m^{A*})Y^*_{lm}(\partial_A Y^*_{l'm'}), \tag{65}$$

$$\mathcal{I}_{a5}(l_0,m_0;l,m;l',m') = \int_{4\pi}d\Omega({}_{-2}Y_{l_0m_0}m^Am^B + {}_{2}Y_{l_0m_0}m^{A*}m^{B*})(\partial_A Y^*_{lm})(\partial_B Y^*_{l'm'}). \tag{66}$$

The radial integrals $\mathcal{I}_{r3}$ and $\mathcal{I}_{r4}$ can be calculated similarly to that for $\mathcal{I}_{r1}$; details are in Appendix A. In the limit $i\Delta\sim 0$, we find

$$\mathcal{I}_{r3}\approx -\mathcal{I}_{r4}\approx -\frac{8\pi i\sqrt{D_0}M\Omega\Omega'}{(i\Delta)^2}\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}}{1-e^{2\pi\Omega_0}}. \tag{67}$$

This implies $\mathcal{I}_{r3}+\mathcal{I}_{r4}\approx 0$.

It turns out that $\mathcal{I}_{a3}$ can be easily obtained and expressed in terms of the 3-j symbols,

$$\mathcal{I}_{a3} = (-1)^{m_0}\sqrt{\frac{(2l_0+1)(2l+1)(2l'+1)}{4\pi}}\begin{pmatrix} l & l' & l_0\\ 0 & 0 & 0\end{pmatrix}\times\begin{pmatrix} l & l' & l_0\\ -m & -m' & m_0\end{pmatrix}, \tag{68}$$

which are zero when $m_0\neq m+m'$. Taking into account the properties of the 3-j symbols, we find that $\mathcal{I}_{a3}$ vanishes when l_0+l+l' is an odd integer.

If we consider the special case where the angular momentum of the QNMs along the z direction is zero, that is, $m_0=0$, the calculation can be significantly simplified. The resulting angular integrals $\mathcal{I}_{a4}$ and $\mathcal{I}_{a5}$ are

TABLE I. Nonzero $\mathcal{I}_{a4}$ and $\mathcal{I}_{a5}$ for a QNM $(l_0,m_0)=(2,0)$ and a pair of scalar particles: $(l,m)=(1,1)$ and $(l',-1)$.

	$l'=1$	$l'=3$
$\mathcal{I}_{a4}(2,0;1,1;l',-1)$	$\frac{1}{2}\sqrt{\frac{3}{5\pi}}$	$-4\sqrt{\frac{6}{35\pi}}$
$\mathcal{I}_{a5}(2,0;1,1;l',-1)$	$\sqrt{\frac{3}{10\pi}}$	$2\sqrt{\frac{3}{35\pi}}$

given by Eqs. (C9) and (C10). As an example, we calculate the angular integrals $\mathcal{I}_{a4}$ and $\mathcal{I}_{a5}$ for a QNM with angular momentum $(l_0, m_0) = (2, 0)$ and a pair of scalar particles, the first of which has angular momentum $(l, m) = (1, 1)$ and the other $(l', -1)$. We find that the only nonzero $\mathcal{I}_{a4}$ and $\mathcal{I}_{a5}$ are for $l' = 1$ and $l' = 3$, as shown in Table I.

The time evolution operator is

$$\hat{U}^{(\mathrm{e})} = \hat{T}\exp\left\{-i\int_0^\infty \mathrm{d}t H_1^{(\mathrm{e})}(t)\right\}, \tag{69}$$

where $\hat{T}$ is the time ordering operator; as before, in the low downconversion regime [52] time ordering is not important and we find

$$\begin{aligned}\hat{U}^{(\mathrm{e})} &= \bigotimes_{lm}\bigotimes_{l'}\hat{U}^{(\mathrm{e})}_{lml'},\\ \hat{U}^{(\mathrm{e})}_{lml'} &\approx \exp\left\{-i\int \mathrm{d}\omega\int \mathrm{d}\omega'[\mathcal{F}^{(\mathrm{e})}_{lml'}(\omega,\omega')\hat{a}^\dagger_{\omega lm}\hat{a}^\dagger_{\omega' l',-m}\right.\\ &\quad\left.+\mathcal{F}^{(\mathrm{e})*}_{lml'}(\omega,\omega')\hat{a}_{\omega lm}\hat{a}_{\omega' l',-m}]\right\},\end{aligned} \tag{70}$$

where the joint frequency distribution is

$$\begin{aligned}\mathcal{F}^{(e)}_{lml'}(\omega,\omega') &= \frac{4iM^3A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}}{\sqrt{\Omega\Omega'}(1-e^{2\pi\Omega_0})}\frac{1}{(i\Delta)^3}\\ &\quad\times\left[\Omega_0^2\mathcal{I}_{a5}-\frac{\sqrt{D_0}}{\sqrt{2l_0(l_0+1)}}\Omega\Omega_0\mathcal{I}_{a4}\right].\end{aligned} \tag{71}$$

It is evident that the joint frequency distribution $\mathcal{F}^{(\mathrm{e})}_{lml'}(\omega,\omega')$ for the coupling between the scalar field and the even-parity QNM is nonzero, showing that there is scalar particle creation. Figure 3 shows an example of the joint frequency distribution.

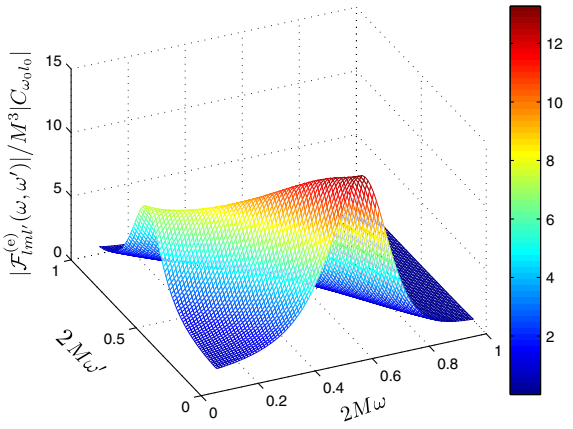

FIG. 3. Modulus of the joint frequency distribution for even-parity QNMs. QNM: $\Omega_R = 0.7474$, $\Omega_I = 0.178$, $(l_0, m_0) = (2, 0)$. Scalar particle one: $(l, m) = (1, 1)$; scalar particle two: $(l', m') = (3, -1)$.

V. PARTICLE NUMBER ESTIMATION

It is interesting to know how many particles can be produced by the gravitational perturbations around a black hole, e.g., how many particles are produced by the coalescence of two black holes in events of the type observed by LIGO. In this section, we provide a crude estimate of the total scalar particle number.

Without loss of generality, we only consider the effect of odd-parity QNMs. Using the decomposition theorem [54], the joint frequency distribution Eq. (55) can be decomposed as

$$\mathcal{F}^{(o)}_{lml'}(\omega,\omega') = \sum_k r_k e^{i\varphi_k}\psi^*_{1k}(\omega)\psi^*_{2k}(\omega'), \tag{72}$$

where $\{\psi_{1k}(\omega)\}$ and $\{\psi_{2k}(\omega')\}$ both form an orthonormal set of localized wave packet mode bases. The corresponding localized mode operators are

$$\begin{aligned}\hat{A}_{klm} &= \int \mathrm{d}\omega\psi_{1k}(\omega)\hat{a}_{\omega lm},\\ \hat{B}_{kl',-m} &= \int \mathrm{d}\omega'\psi_{2k}(\omega')\hat{a}_{\omega' l',-m}.\end{aligned} \tag{73}$$

The time evolution operator for given l, m, l' becomes

$$\begin{aligned}\hat{U}^{(o)}_{lml'} &\approx \exp\left\{\sum_k r_k e^{i\varphi_k}\hat{A}^\dagger_{klm}\hat{B}^\dagger_{kl',-m} - \mathrm{H.c.}\right\}\\ &\approx \bigotimes_k \exp\{r_k e^{i\varphi_k}\hat{A}^\dagger_{klm}\hat{B}^\dagger_{kl',-m} - \mathrm{H.c.}\}.\end{aligned} \tag{74}$$

We can see that r_k is the two-mode squeezing factor and φ_k is the squeezing angle,

$$r_k e^{i\varphi_k} = \int \mathrm{d}\omega\int \mathrm{d}\omega'\mathcal{F}^{(o)}_{lml'}(\omega,\omega')\psi_{1k}(\omega)\psi_{2k}(\omega'). \tag{75}$$

If the initial state of the scalar field is the Boulware vacuum, the particle number for $\hat{A}_{klm}$ and $\hat{B}_{kl',-m}$ modes are both $\sinh^2(r_k)$. Therefore, the total particle number is

$$N = 2\sum_{lml'}\sum_k \sinh^2(r_k). \tag{76}$$

The maximum squeezing factor $r_{\max}$ can be obtained by choosing $\psi_{1k}(\omega)$ and $\psi_{2k}(\omega')$ such that they maximally overlap with the joint frequency distribution $\mathcal{F}^{(o)}_{lml'}(\omega,\omega')$. According to the behavior of $\mathcal{F}^{(o)}_{lml'}(\omega,\omega')$, as shown by Fig. 2, we can choose $\psi_{1k}(\omega)\psi_{2k}(\omega')$ as a two-dimensional top-hat function with width ω_I and length ω_R, respectively. The maximum squeezing factor is approximately

$$r_{\max} \sim \sqrt{\omega_R \omega_I}|\mathcal{F}^{(o)}_{lml'}|_{\max}, \tag{77}$$

where $|\mathcal{F}^{(o)}_{lml'}|_{\max}$ is given by Eq. (57).

The amplitude of the ingoing gravitational waves on the horizon, $C_{\omega_0 l_0}$, remains unknown. It is related to the energy flux of the gravitational waves falling into the black hole by

$$\frac{dE_{\text{hole}}}{dt} = \frac{32(2M)^4\Omega_R^2(4\Omega_R^2+1)(\Omega_R^2+1)}{36\Omega_R^2 + l_0^2(l_0+1)^2(l_0^2+l_0-2)^2}|C_{\omega_0 l_0}|^2. \tag{78}$$

This relation is obtained by making $a \to 0$ and integrating over all angles from a similar relation [Eq. (4.44) in [41]] in the Kerr background spacetime. We assume that during time Δt, the amount of gravitational wave energy falling into the black hole is $\Delta E_{\text{hole}} \equiv \eta M_\odot$ (here $M_\odot$ is the solar mass, which is about 1.99×10^{30} kg); then

$$|C_{\omega_0 l_0}| \sim \sqrt{\frac{36\Omega_R^2 + l_0^2(l_0+1)^2(l_0^2+l_0-2)^2}{32(2M)^4\Omega_R^2(4\Omega_R^2+1)(\Omega_R^2+1)}}\sqrt{\frac{\eta M_\odot}{\Delta t}}. \tag{79}$$

As an example, we consider the fundamental QNM of a Schwarzschild black hole, for which $l_0 = 2, \Omega_R = 0.7474, \Omega_I = 0.178$. By using Eqs. (57), (77), and (79) we find

$$r_{\max} \sim 0.57 \times \sqrt{\frac{\eta M_\odot}{\Delta t}} \sim 0.4\sqrt{\eta} \times \left(\frac{10^{-5}s}{\Delta t}\right)^{1/2}. \tag{80}$$

For the first gravitational wave event, GW150914, detected by LIGO [19], the ring-down time is about 0.01 second, so

$$r_{\max} \sim \sqrt{\eta} \times 10^{-2}. \tag{81}$$

We note that even if η is at the order of one (one solar mass of gravitational wave energy falls into the black hole), the particle number created by the ring-down is very small. This is due to the fact that at the ring-down stage, the black hole oscillation dies away very quickly.

One important thing to note, from Eqs. (57), (77), and (79), is that $r_{\max} \propto \Omega_I^{-5/2}$, where $1/\Omega_I$ characterizes the time scale of the black hole oscillation. If this is qualitatively valid at the inspiral and merger stages, one might expect that the particle generation is quite significant because the oscillation of the black hole lasts much longer than that of the ring-down stage. For example, if the time scale is 30 times longer, $r_{\max} \sim 50$ and the total particle number $\sim 10^{42}$. However, our calculation is only valid at the ring-down stage, and further studies need to be done for the inspiral and merger stages.

VI. CONCLUSION

We studied the coupling between the gravitational QNMs of a Schwarzschild black hole and a massless scalar field, and showed that scalar particles can be produced by the gravitational perturbations. This is contrary to the plane gravitational wave case where particle creation is forbidden due to the violation of momentum conservation. In the Schwarzschild black hole case, the total angular momentum of the QNMs and the pair of particles produced is conserved. In arriving at the above conclusions, we explicitly derived the interaction Hamiltonian for the scalar field which shows that the QNMs play the role as a multimode squeezer. The QNMs squeeze the initial state of the scalar field and produce particles. If the initial state of the scalar field is a vacuum state (Boulware vacuum), then the final state is a squeezed vacuum state, indicating that the QNMs amplify the vacuum fluctuations and create particles. If initially there exists Hawking radiation (a thermal state with its temperature proportional to the surface gravity of the black hole), the QNMs squeeze the Hawking radiation and amplify it. Interestingly, the presence of gravitational perturbations results in coupling between different Hawking particles and, therefore, may build correlations between them, modifying the thermal characteristic. Modification of the Hawking radiation and Unruh radiation by fluctuating horizons was recently studied [55,56]. In realistic astrophysical situations, the CMB temperature is higher than the Hawking temperature of astrophysical black holes, so the amplification of the CMB around a black hole by the QNMs will be more significant than the amplification of Hawking radiation.

How significant the amplification is depends on the squeezing amplitude. We showed that the squeezing amplitude (57) is proportional to the amplitude of the QNMs, which is reasonable because larger gravitational perturbations would create more particles. In addition, the maximal squeezing amplitude is proportional to the cube of the black hole mass and the real part of the QNM frequency, and is inversely proportional to the cube of the imaginary part of the QNM frequency. This implies that for a given amplitude of the QNMs a larger black hole would create more particles. Furthermore, the particle creation efficiency is higher for lower decaying QNMs. For Schwarzschild black holes, the damping of the QNMs is fast. The least-damped mode has $2M\omega_I = 0.178$. For extreme Kerr black holes, there exist QNMs with very small damping rate, called zero-damping modes [57]. We expect that our result is qualitatively correct for the Kerr black holes, which implies the particle creation by gravitational perturbations around an extreme Kerr black hole is much more efficient than that around a Schwarzschild black hole.

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APPENDIX A: RADIAL INTEGRALS

It is difficult to find exactly analytic results for the radial integrals because we do not have analytic solutions for ${}_{-2}R_{\omega_0 l_0}$ and $R_{\omega l}$. However, it is possible to find approximately analytic results by using the asymptotic behavior of ${}_{-2}R_{\omega_0 l_0}$ and $R_{\omega l}$. When $r_* \to +\infty$ (spatial infinity), the integrands of the radial integrals are both proportional to $e^{i(\omega_0-\omega-\omega')r_*}$ according to Eqs. (38) and (14); when $r_* \to -\infty$ (event horizon), they are proportional to $e^{-i(\omega_0-\omega-\omega')r_*}$. The QNM frequency is a complex number $\omega_0 = \omega_R - i\omega_I$ so the integrands are proportional to $e^{\omega_I r_*}$ when $r_* \to +\infty$, and $e^{-\omega_I r_*}$ when $r_* \to -\infty$. Since $\omega_I > 0$, the integrands are divergent at spatial infinity and on the event horizon, which implies the radial integrals are not well defined. This formal divergence can be resolved by utilizing the analyticity of the integrands [49]: they are analytic in the complex r plane except at two points $r = 2M$ and $r = \infty$. A contour which encloses these two branch points $r = 2M$ and $r = \infty$ is chosen by Leaver [49] such that the integral along it is well defined and finite. Sun and Price [50] discussed in detail how to construct Leaver's contour by analytic continuation and restored a factor $1/(1 - e^{2\pi\Omega_0})$ that is missing in [49].

Noting that $r = 2M$ and $r = \infty$ are two branch points, the branch cut can be chosen as a line perpendicular to the real r axis, starting at $r = 2M$ and ending at $r = \infty$. The cut lies in the upper complex r plane if $\omega_R - \omega - \omega' > 0$, as shown in Fig. 4, and in the lower complex r plane if $\omega_R - \omega - \omega' < 0$. When $\omega_R - \omega - \omega' > 0$, the contour C begins at $r = \infty$, just to the right of the branch cut, moves downward to $r = 2M$, where it wraps and, just to the left of the branch cut, moves upward to $r = \infty$, as shown in Fig. 4. We refer to the region near $r = 2M$ as the *in* region and the region around $r = \infty$ as the *out* region, as schematically represented by the shaded region in Fig. 4. By analytically extending the integrands to the complex r plane we see that along the contour C the integrands exponentially decay in the *out* region, which thus removes the formal divergence. In addition, the exponential decay of the integrands in the out region implies that the main contributions to the integrals are from the *in* region.

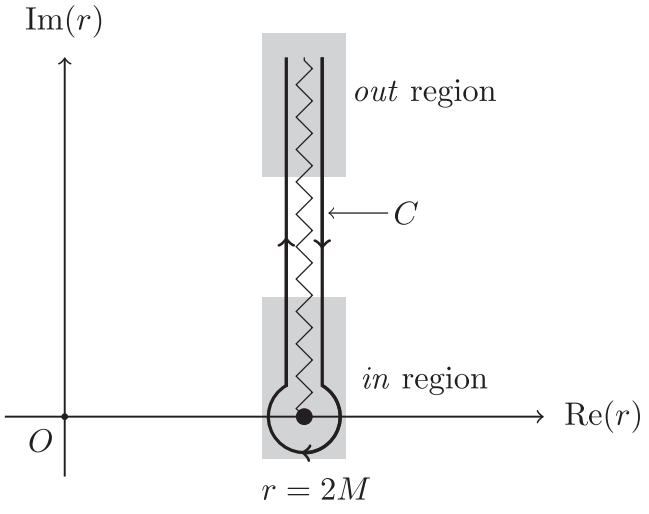

FIG. 4. The contour C and branch cut when $\omega_R - \omega - \omega' > 0$. The two shaded regions are referred to as in (close to the horizon) and out (around $r = \infty$) regions, respectively.

We describe in detail how to find the approximately analytic result for the radial integral $\mathcal{I}_{r1}$; the result for $\mathcal{I}_{r2}$ and $\mathcal{I}_{r3}$ can be obtained in a similar way. At spatial infinity ($r \to \infty$), by using Eqs. (38) and (14), the integrand of $\mathcal{I}_{r1}$ can be approximated as

$$\begin{aligned}
&\frac{1}{f}\left[\left(\frac{\mathrm{d}}{\mathrm{d}r_*} - i\omega_0 - \frac{2f}{r}\right){}_{-2}R_{\omega_0 l_0}\right]\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right)\\
&\quad\approx -B^*_{\omega l}B^*_{\omega' l'}D_{\omega_0 l_0}\left(\frac{i\omega}{f}+\frac{1}{r}\right)e^{i(\omega_0-\omega-\omega')r_*}\\
&\quad\approx -\frac{B^*_{\omega l}B^*_{\omega' l'}D_{\omega_0 l_0}}{2M}(i\Omega x^{i\Delta} + x^{-1+i\Delta})e^{i\Delta x},
\end{aligned} \tag{A1}$$

where we have defined a dimensionless radial coordinate $x = r/2M$. Near the event horizon ($r \to 2M$), according to Eqs. (38) and (14), the integrand of $\mathcal{I}_{r1}$ can be approximated as

$$\begin{aligned}
&\frac{1}{f}\left[\left(\frac{\mathrm{d}}{\mathrm{d}r_*} - i\omega_0 - \frac{2f}{r}\right){}_{-2}R_{\omega_0 l_0}\right]\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right)\\
&\quad\approx 2A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}\Big[x^{-1}(1-i\Omega_0 x)(i\Omega x-1)(x-1)^{1-i\Delta}\\
&\qquad + i\Omega(1-i\Omega_0 x)(x-1)^{-i\Delta}\Big]e^{-i\Delta x},
\end{aligned} \tag{A2}$$

where we have only kept the term proportional to $e^{-i(\omega_0-\omega-\omega')r_*}$ owing to the rotating wave approximation. As discussed before, the integration along the real r axis is formally divergent. The integrands Eqs. (A1) and (A2) are analytically extended to the whole complex r plane. Along the contour C in the *out* region, $e^{i\Delta x} \sim e^{-2M\omega_I|x|}$, which means the integrand of $\mathcal{I}_{r1}$ exponentially decays. We therefore expect that the integral $\mathcal{I}_{r1}$ is finite along the contour C. Unfortunately, we cannot find an analytic expression for the integrand on the whole contour C. Numerical techniques need to be introduced to perform the contour integration. However, it may be possible that an approximate result can be obtained by using only the asymptotic behavior of the integrand. Note that in the out region the integrand (A1) exponentially decays and contributes very little to the total integral. Introducing another exponential decaying function in the *out* region will not introduce large deviations to the integral. We therefore

replace Eq. (A1) by Eq. (A2) with the factor $e^{-i\Delta x}$ replaced by $e^{i\Delta x}$. In the *in* region, the asymptotic expression for the integrand is Eq. (A2) which dominates the contribution to the integral. In the limit of $i\Delta \sim 0$, which is the case that we are mostly interested in, $e^{-i\Delta x} \approx e^{i\Delta x}$. We thus replace $e^{-i\Delta x}$ by $e^{i\Delta x}$ in Eq. (A2) and get an approximately asymptotic expression. In summary, we approximate the original integrand by

$$\begin{aligned}&\frac{1}{f}\left[\left(\frac{\mathrm{d}}{\mathrm{d}r_*}-i\omega_0-\frac{2f}{r}\right){}_{-2}R_{\omega_0 l_0}\right]\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right)\\&\quad\approx 2A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}[x^{-1}(1-i\Omega_0 x)(i\Omega x-1)(x-1)^{1-i\Delta}\\&\qquad+i\Omega(1-i\Omega_0 x)(x-1)^{-i\Delta}]e^{i\Delta x}\end{aligned}\tag{A3}$$

along the whole contour C. Obviously, this is a very crude approximation since we have ignored the behavior of the integrand in the intermediate region. The validity of this approximation has to be verified by numerical calculation. However, we expect that this approximation provides a lower bound for the exact integral since near the horizon we replace an exponentially growing function by an exponentially decaying function. The advantage of this approximation is that we can obtain an analytic result for the radial integral $\mathcal{I}_{r1}$.

From Eq. (A3) we see that basically we need to calculate

$$\oint_C \mathrm{d}x e^{i\Delta x}(x-1)^{n_1-i\Delta}x^{n_2},\tag{A4}$$

where n_1, n_2 are two integers, and C is the contour we introduced, as shown in Fig. 4. Defining a new variable $u = i\Delta(x-1)$, we find [58]

$$\begin{aligned}&\oint_C \mathrm{d}x e^{i\Delta x}(x-1)^{n_1-i\Delta}x^{n_2}\\&\quad=(i\Delta)^{i\Delta-n_1-1}e^{i\Delta}\oint_F \mathrm{d}u e^u u^{n_1-i\Delta}\left(1+\frac{u}{i\Delta}\right)^{n_2}\\&\quad=\frac{2\pi i(-1)^{n_1+1-i\Delta}e^{i\Delta}}{\Gamma(-n_1+i\Delta)}\\&\qquad\times U(n_1+1-i\Delta,n_1+n_2+2-i\Delta,-i\Delta),\end{aligned}\tag{A5}$$

where F is the contour illustrated by Morse and Feshbach's Fig. 5.1.2 [58] (page 612), $\Gamma(z)$ is the Gamma function, and $U(a,c,z)$ is the confluent hypergeometric function [59]. [Note that the function $U_2(a,c,z)$ defined by Morse and Feshbach is related to $U(a,c,z)$ by $U_2(a,c,z) = e^{i\pi a}U(a,c,z)$.] Therefore, the radial integral $\mathcal{I}_{r1}$ can be approximated as

$$\begin{aligned}\mathcal{I}_{r1}&\approx-\frac{8\pi M i\sqrt{D_0}}{\sqrt{2l_0(l_0+1)}}\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}e^{i\Delta}}{1-e^{2\pi\Omega_0}}\left\{\frac{(-1)^{2-i\Delta}}{\Gamma(-1+i\Delta)}[\Omega\Omega_0 U(2-i\Delta,4-i\Delta,-i\Delta)+i(\Omega+\Omega_0)U(2-i\Delta,3-i\Delta,-i\Delta)\right.\\&\quad\left.-U(2-i\Delta,2-i\Delta,-i\Delta)\right]+\frac{(-1)^{1-i\Delta}}{\Gamma(i\Delta)}[\Omega\Omega_0 U(1-i\Delta,3-i\Delta,-i\Delta)+i\Omega U(1-i\Delta,2-i\Delta,-i\Delta)]\Big\}\\&\approx-\frac{8\pi M i\sqrt{D_0}}{\sqrt{2l_0(l_0+1)}}\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}e^{i\Delta}}{1-e^{2\pi\Omega_0}}\left\{\frac{(i\Delta)^{i\Delta-2}}{\Gamma(-1+i\Delta)}\left[-\frac{2\Omega\Omega_0}{i\Delta}+2\Omega\Omega_0+i(\Omega+\Omega_0)\right]\right.\\&\quad\left.-\frac{(-1)^{2-i\Delta}}{\Gamma(-1+i\Delta)}U(2-i\Delta,2-i\Delta,-i\Delta)+\frac{(i\Delta)^{i\Delta-1}}{\Gamma(i\Delta)}\left(-\frac{\Omega\Omega_0}{i\Delta}+2\Omega\Omega_0+i\Omega\right)\right\},\end{aligned}\tag{A6}$$

where we have used the fact that [60]

$$U(a,a+n+1,z)=\frac{z^{-a}}{\Gamma(a)}\sum_{k=0}^{n}\binom{n}{k}\Gamma(a+k)z^{-k},\tag{A7}$$

with $n = 0, 1, 2, \cdots$. In the limit of $i\Delta \sim 0$, from Eq. (A6), we obtain the dominant term

$$\mathcal{I}_{r1}\approx-\frac{\sqrt{D_0}}{\sqrt{2l_0(l_0+1)}}\frac{16\pi i M\Omega\Omega_0}{(i\Delta)^2}\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}}{1-e^{2\pi\Omega_0}}.\tag{A8}$$

The calculation of $\mathcal{I}_{r2}$ is very similar to that of $\mathcal{I}_{r1}$. At spatial infinity ($r \to \infty$), according to Eqs. (38) and (14), the integrand of $\mathcal{I}_{r2}$ can be approximated as

$$\frac{1}{r^2f^2}\left[(i\omega_0 r^2-M)\frac{\mathrm{d}}{\mathrm{d}r_*}{}_{-2}R_{\omega_0 l_0}-\left(\frac{1}{2}\mu_0^2 f-i\omega_0(-3r+7M)-\omega_0^2 r^2\right){}_{-2}R_{\omega_0 l_0}\right]\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right)$$
$$\approx B^*_{\omega l}B^*_{\omega' l'}D_{\omega_0 l_0}\left[\frac{3i\omega_0}{f}\left(1-\frac{1}{f}\right)+\frac{6iM\omega_0}{rf^2}-\frac{\mu_0^2}{2rf}-\frac{3M}{r^2 f}\right]e^{i(\omega_0-\omega-\omega')r_*}$$
$$\approx -\frac{1}{4M}B^*_{\omega l}B^*_{\omega' l'}D_{\omega_0 l_0}[\mu_0^2(x-1)^{-1+i\Delta}+3x^{-1}(x-1)^{-1+i\Delta}]e^{i\Delta x}. \tag{A9}$$

The dominant term is $e^{i\Delta x}/x$ when x is large. Near the event horizon ($r\to 2M$). According to Eqs. (37) and (14), the integrand of $\mathcal{I}_{r2}$ can be approximated as

$$\frac{1}{r^2f^2}\left[(i\omega_0 r^2-M)\frac{\mathrm{d}}{\mathrm{d}r_*}{}_{-2}R_{\omega_0 l_0}-\left(\frac{1}{2}\mu_0^2 f-i\omega_0(-3r+7M)-\omega_0^2 r^2\right){}_{-2}R_{\omega_0 l_0}\right]\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right)$$
$$\approx A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}\left[2\omega_0^2 r^2+i\omega_0 r+4iM\omega_0-\frac{1}{2}\mu_0^2 f-\frac{4M}{r}+\frac{4M^2}{r^2}\right]e^{-i(\omega_0-\omega-\omega')r_*}$$
$$\approx A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}\left[(2\Omega_0^2x^2+i\Omega_0 x+2i\Omega_0-2x^{-1}+x^{-2})(x-1)^{-i\Delta}-\frac{1}{2}\mu_0^2x^{-1}(x-1)^{1-i\Delta}\right]e^{-i\Delta x}. \tag{A10}$$

By analytically extending Eqs. (A9) and (A10) to the complex r plane and using the same approximation that leads to Eq. (A3), we obtain an approximate expression for the integrand of $\mathcal{I}_{r2}$ along the whole contour C,

$$\frac{1}{r^2f^2}\left[(i\omega_0 r^2-M)\frac{\mathrm{d}}{\mathrm{d}r_*}{}_{-2}R_{\omega_0 l_0}-\left(\frac{1}{2}\mu_0^2 f-i\omega_0(-3r+7M)-\omega_0^2 r^2\right){}_{-2}R_{\omega_0 l_0}\right]\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right)$$
$$\approx A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}\left[(2\Omega_0^2x^2+i\Omega_0 x+2i\Omega_0-2x^{-1}+x^{-2})(x-1)^{-i\Delta}-\frac{1}{2}\mu_0^2x^{-1}(x-1)^{1-i\Delta}\right]e^{i\Delta x}. \tag{A11}$$

We can see that in the *out* region on the contour C, the dominant term in Eq. (A11) is $x^2e^{i\Delta x}$ which is greater than that in Eq. (A9), $e^{i\Delta x}/x$. However, these two terms are both exponentially suppressed so that their contribution to the total integration is small.

We therefore expect that this approximation only introduces a small error. The main contribution to the integration comes from the *in* region where x is not large. In the limit of $i\Delta\sim 0$, which is the case that we are mostly interested in, $e^{-i\Delta x}\approx e^{i\Delta x}$. We therefore expect that Eq. (A11) is a good approximation to Eq. (A10) in the *in* region. Note that we replace an exponential growing function by an exponentially decaying function in the *in* region; the final result provides a lower bound for the exact radial integral $\mathcal{I}_{r2}$. Using Eq. (A5) we have

$$\mathcal{I}_{r2}\approx 4\pi Mi\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}e^{i\Delta}}{1-e^{2\pi\Omega_0}}\left\{\frac{(-1)^{1-i\Delta}}{\Gamma(i\Delta)}[2\Omega_0^2U(1-i\Delta,4-i\Delta,-i\Delta)+i\Omega_0U(1-i\Delta,3-i\Delta,-i\Delta)\right.$$
$$+2i\Omega_0U(1-i\Delta,2-i\Delta,-i\Delta)-2U(1-i\Delta,1-i\Delta,-i\Delta)+U(1-i\Delta,-i\Delta,-i\Delta)]$$
$$\left.-\frac{(-1)^{2-i\Delta}}{2\Gamma(-1+i\Delta)}\mu_0^2U(2-i\Delta,2-i\Delta,-i\Delta)\right\}$$
$$\approx 4\pi Mi\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}e^{i\Delta}}{1-e^{2\pi\Omega_0}}\left\{\frac{(i\Delta)^{i\Delta-1}}{\Gamma(i\Delta)}\left[\frac{4\Omega_0^2}{(i\Delta)^2}-\frac{i\Omega_0+10\Omega_0^2}{i\Delta}+8\Omega_0^2+4i\Omega_0\right]\right.$$
$$\left.-\frac{(-1)^{1-i\Delta}}{\Gamma(i\Delta)}[2U(1-i\Delta,1-i\Delta,-i\Delta)-U(1-i\Delta,-i\Delta,-i\Delta)]-\frac{(-1)^{2-i\Delta}}{2\Gamma(-1+i\Delta)}\mu_0^2U(2-i\Delta,2-i\Delta,-i\Delta)\right\}. \tag{A12}$$

The dominant term in the limit of $i\Delta\sim 0$ is

$$\mathcal{I}_{r2}\approx\frac{16\pi iM\Omega_0^2}{(i\Delta)^2}\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}}{1-e^{2\pi\Omega_0}}. \tag{A13}$$

At spatial infinity ($r \to \infty$), the integrand of $\mathcal{I}_{r3}$ can be approximated as

$$\frac{1}{f^2}\,{}_{-2}R_{\omega_0 l_0}\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right) \approx 2MB^*_{\omega l}B^*_{\omega' l'}D_{\omega_0 l_0}x^3(x-1)^{-2+i\Delta}e^{i\Delta x}. \tag{A14}$$

The dominant term is $xe^{i\Delta x}$ when x is large. Near the event horizon ($r \to 2M$), the integrand of $\mathcal{I}_{r3}$ can be approximated as

$$\frac{1}{f^2}\,{}_{-2}R_{\omega_0 l_0}\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right) \approx 4M^2A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}x^2(x-1)^{-i\Delta}e^{-i\Delta x}. \tag{A15}$$

By analytically extending Eqs. (A14) and (A15) to the complex r plane and using the same approximation as before, we obtain an approximate expression for the integrand of $\mathcal{I}_{r3}$ along the whole contour C,

$$\frac{1}{f^2}\,{}_{-2}R_{\omega_0 l_0}\left(\frac{R^*_{\omega l}}{r}\right)\left(\frac{R^*_{\omega' l'}}{r}\right) \approx 4M^2A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}x^2(x-1)^{-i\Delta}e^{i\Delta x}. \tag{A16}$$

Using Eq. (A5) we have

$$\begin{aligned}\mathcal{I}_{r3} &\approx -4\pi i\sqrt{D_0}M\Omega\Omega'\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}e^{i\Delta}}{1-e^{2\pi\Omega_0}}\frac{(-1)^{1-i\Delta}}{\Gamma(i\Delta)} \times U(1-i\Delta,4-i\Delta,-i\Delta)\\ &\approx -4\pi i\sqrt{D_0}M\Omega\Omega'\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}e^{i\Delta}}{1-e^{2\pi\Omega_0}}\frac{(i\Delta)^{i\Delta-1}}{\Gamma(i\Delta)} \times\left[4-\frac{5}{i\Delta}+\frac{2}{(i\Delta)^2}\right].\end{aligned} \tag{A17}$$

The dominant term in the limit of $i\Delta \sim 0$ is

$$\mathcal{I}_{r3} \approx -\frac{8\pi i\sqrt{D_0}M\Omega\Omega'}{(i\Delta)^2}\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}}{1-e^{2\pi\Omega_0}}. \tag{A18}$$

At spatial infinity ($r \to \infty$), the integrand of $\mathcal{I}_{r4}$ can be approximated as

$$\begin{aligned}&{}_{-2}R_{\omega_0 l_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^*_{\omega l}}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^*_{\omega' l'}}{r}\right)\\ &\approx \frac{1}{2M}B^*_{\omega l}B^*_{\omega' l'}D_{\omega_0 l_0}[-\Omega\Omega'x^3(x-1)^{-2}+x^{-1}\\ &\quad +i(\Omega+\Omega')x(x-1)^{-1}](x-1)^{i\Delta}e^{i\Delta x}.\end{aligned} \tag{A19}$$

Near the event horizon ($r \to 2M$), the integrand of $\mathcal{I}_{r4}$ can be approximated as

$$\begin{aligned}&{}_{-2}R_{\omega_0 l_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^*_{\omega l}}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^*_{\omega' l'}}{r}\right)\\ &\approx -A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}[\Omega\Omega'x^2+i(\Omega+\Omega')(x-1)\\ &\quad -x^{-2}(x-1)^2](x-1)^{-i\Delta}e^{-i\Delta x}.\end{aligned} \tag{A20}$$

By analytically extending Eqs. (A19) and (A20) to the complex r plane and using the same approximation as before, we obtain an approximate expression for the integrand of $\mathcal{I}_{r3}$ along the whole contour C,

$$\begin{aligned}&{}_{-2}R_{\omega_0 l_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^*_{\omega l}}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^*_{\omega' l'}}{r}\right)\\ &\approx -A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}[\Omega\Omega'x^2+i(\Omega+\Omega')(x-1)\\ &\quad -x^{-2}(x-1)^2](x-1)^{-i\Delta}e^{i\Delta x}.\end{aligned} \tag{A21}$$

Using Eq. (A5) we have

$$\begin{aligned}\mathcal{I}_{r4} &\approx 4\pi Mi\sqrt{D_0}\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}e^{i\Delta}}{1-e^{2\pi\Omega_0}}\left\{\frac{(-1)^{1-i\Delta}}{\Gamma(i\Delta)}\Omega\Omega'U(1-i\Delta,4-i\Delta,-i\Delta)\right.\\ &\quad +\frac{(-1)^{2-i\Delta}}{\Gamma(-1+i\Delta)}i(\Omega+\Omega')U(2-i\Delta,2-i\Delta,-i\Delta)-\left.\frac{(-1)^{3-i\Delta}}{\Gamma(-2+i\Delta)}U(3-i\Delta,2-i\Delta,-i\Delta)\right.\\ &\approx 4\pi Mi\sqrt{D_0}\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}e^{i\Delta}}{1-e^{2\pi\Omega_0}}\left\{\frac{(i\Delta)^{i\Delta-1}}{\Gamma(i\Delta)}\Omega\Omega'\left[4-\frac{5}{i\Delta}+\frac{2}{(i\Delta)^2}\right]\right.\\ &\quad +\frac{(-1)^{2-i\Delta}}{\Gamma(-1+i\Delta)}i(\Omega+\Omega')U(2-i\Delta,2-i\Delta,-i\Delta)-\left.\frac{(-1)^{3-i\Delta}}{\Gamma(-2+i\Delta)}U(3-i\Delta,2-i\Delta,-i\Delta).\right.\end{aligned} \tag{A22}$$

The dominant term in the limit of $i\Delta \sim 0$ is

$$\mathcal{I}_{r4} \approx \frac{8\pi i\sqrt{D_0}M\Omega\Omega'}{(i\Delta)^2}\frac{A^*_{\omega l}A^*_{\omega' l'}C_{\omega_0 l_0}}{1-e^{2\pi\Omega_0}}. \tag{A23}$$

APPENDIX B: EXPLICIT EXPRESSIONS FOR SPIN-WEIGHTED SPHERICAL HARMONICS

In order to calculate the angular integrals, one needs to know the explicit expressions for the spin-weighted spherical harmonics ${}_{\pm1}Y_{lm}$ and ${}_{\pm2}Y_{lm}$. The spin-weighted

spherical harmonics ${}_sY_{lm}$ for integers s, l, m is defined from the spherical harmonics Y_{lm} [42],

$$ {}_sY_{lm} = \begin{cases} \sqrt{\frac{(l-s)!}{(l+s)!}}\eth^s Y_{lm}, & 0 \le s \le l; \\ \sqrt{\frac{(l+s)!}{(l-s)!}}(-1)^s \bar{\eth}^{-s} Y_{lm}, & -l \le s \le 0, \end{cases} \tag{B1} $$

where $\eth$ and $\bar{\eth}$ are the spin-raising and spin-lowering operators, respectively. Assume that η is a quantity of spin weight s; then $\eth\eta$ is a quantity of spin weight $s+1$,

$$ \eth\eta \equiv -(\sin\theta)^s \left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)[(\sin\theta)^{-s}\eta], \tag{B2} $$

and $\bar{\eth}\eta$ is a quantity of spin weight $s-1$,

$$ \bar{\eth}\eta \equiv -(\sin\theta)^{-s} \left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)[(\sin\theta)^{s}\eta]. \tag{B3} $$

According to the definition (B1), we shall derive the explicit expressions for ${}_{\pm1}Y_{lm}$ and ${}_{\pm2}Y_{lm}$ in this appendix.

If we define two differential operators $\hat{S}_\pm$ as

$$ \hat{S}_\pm \equiv -\left(\frac{\partial}{\partial\theta} \pm \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right), \tag{B4} $$

then the action of $\eth$ (spin-raising operator) and $\bar{\eth}$ (spin-lowering operator) on η, which is a quantity of spin weight s, can be written as

$$ \begin{aligned} \eth\eta &= \hat{S}_+\eta + s\cot\theta\eta, \\ \bar{\eth}\eta &= \hat{S}_+\eta - s\cot\theta\eta. \end{aligned} \tag{B5} $$

According to the definition of the spin-weighted spherical harmonics (B1), we find for $s = \pm1$

$$ {}_{\pm1}Y_{lm} = \pm\sqrt{\frac{(l-1)!}{(l+1)!}}\hat{S}_\pm Y_{lm} \tag{B6} $$

and for $s = \pm2$

$$ \begin{aligned} {}_{\pm2}Y_{lm} &= \sqrt{\frac{(l-2)!}{(l+2)!}}(\hat{S}_\pm^2 + \cot\theta\hat{S}_\pm)Y_{lm} \\ &= \sqrt{\frac{(l-2)!}{(l+2)!}}\left[\partial_\theta^2 - \cot\theta\partial_\theta \pm \frac{2i}{\sin\theta}(\partial_\theta - \cot\theta)\partial_\phi \right. \\ &\quad \left. - \frac{1}{\sin^2\theta}\partial_\phi^2\right]Y_{lm} \end{aligned} \tag{B7} $$

since Y_{lm} is of spin weight 0.

Taking into account the definition of the spherical harmonics,

$$ Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}, $$

and the recurrence relation for the associated Legendre function

$$ \partial_\theta P_l^m = \frac{1}{2}[P_l^{m+1} - (l+m)(l-m+1)P_l^{m-1}], $$

we obtain the explicit expressions for the $s = \pm1, \pm2$ spin-weighted spherical harmonics in terms of the Legendre function,

$$ {}_{\pm1}Y_{lm} = \frac{1}{2}(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}\frac{(l-m)!}{(l+m)!}}\left(\mp P_l^{m+1} \pm c_{lm}^2 P_l^{m-1} + \frac{2m}{\sin\theta}P_l^m\right)e^{im\phi}, \tag{B8} $$

$$ \begin{aligned} {}_{\pm2}Y_{lm} &= \frac{1}{4}(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-2)!}{(l+2)!}\frac{(l-m)!}{(l+m)!}}\left[P_l^{m+2} + c_{lm}^2 c_{l,m-1}^2 P_l^{m-2} - 2\left(\cot\theta \pm \frac{2m}{\sin\theta}\right)P_l^{m+1}\right. \\ &\quad \left. + 2\left(\cot\theta \pm \frac{2m}{\sin\theta}\right)c_{lm}^2 P_l^{m-1} + \left(-c_{lm}^2 - c_{l,m+1}^2 + \frac{4m^2}{\sin^2\theta} \pm \frac{8m\cot\theta}{\sin\theta}\right)P_l^m\right]e^{im\phi}, \end{aligned} \tag{B9} $$

where $c_{lm} = \sqrt{(l+m)(l-m+1)}$.

APPENDIX C: ANGULAR INTEGRALS

Equations (B8) and (B9) can be further transformed to eliminate the trigonometric functions by using the recurrence properties of the associated Legendre functions [59]. Finally, the calculation of the angular integrals is reduced to the evaluation of the integrals of the products of three associated Legendre functions,

$$ \mathcal{I}_{P3}(l_1,m_1;l_2,m_2;l_3,m_3) = \int_{-1}^{1} \mathrm{d}\mu P_{l_1}^{m_1}P_{l_2}^{m_2}P_{l_3}^{m_3}, \tag{C1} $$

where $\mu = \cos\theta$. The integral $\mathcal{I}_{P3}$ has an analytic result and is shown in Appendix D. Therefore, the angular integrals can always be calculated analytically, although the calculation is tedious in the general case.

For simplicity we consider a special case where the angular momentum of the QNMs along the z direction is zero, that is, $m_0 = 0$. This simplifies the calculation a lot

and is sufficient to demonstrate quantum particle generation by the QNMs. It is easy to show that, if we let $m = 0$ in Eqs. (B8) and (B9),

$$
\begin{aligned}
{}_{\pm 1}Y_{l0} &= \mp\sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}}P^1_l, \\
{}_{\pm 2}Y_{l0} &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-2)!}{(l+2)!}}P^2_l.
\end{aligned}
\tag{C2}
$$

From the definitions of the angular integrals, we immediately find

$$
\mathcal{I}_{a1} = \sqrt{2}\int d\Omega_{-1}Y_{l_0 0}Y^*_{lm}\left(\frac{i}{\sin\theta}\partial_\phi Y^*_{l'm'}\right), \tag{C3}
$$

$$
\begin{aligned}
\mathcal{I}_{a2} = \int d\Omega_{-2}Y_{l_0 0}\Bigg[&(\partial_\theta Y^*_{lm})\left(\frac{i}{\sin\theta}\partial_\phi Y^*_{l'm'}\right) \\
&+ \left(\frac{i}{\sin\theta}\partial_\phi Y^*_{lm}\right)(\partial_\theta Y^*_{l'm'})\Bigg].
\end{aligned}
\tag{C4}
$$

$$
\mathcal{I}_{a4} = \sqrt{2}\int d\Omega_{-1}Y_{l_0 0}Y^*_{lm}(\partial_\theta Y^*_{l'm'}), \tag{C5}
$$

$$
\begin{aligned}
\mathcal{I}_{a5} = \int d\Omega_{-2}Y_{l_0 0}\Bigg[&(\partial_\theta Y^*_{lm})(\partial_\theta Y^*_{l'm'}) \\
&+ \left(\frac{i}{\sin\theta}\partial_\phi Y^*_{lm}\right)\left(\frac{i}{\sin\theta}\partial_\phi Y^*_{l'm'}\right)\Bigg].
\end{aligned}
\tag{C6}
$$

The integration over ϕ gives rise to a δ function $\delta_{m',-m}$, which implies that the produced pair of particles have opposite angular momentum along the z direction. This is not surprising given that $m_0 = 0$, and is an indication of angular momentum conservation in the particle production process.

Using another recurrence relation

$$
\frac{m}{\sin\theta}P^m_l = -\frac{1}{2}[P^{m+1}_{l-1} + (l+m)(l+m-1)P^{m-1}_{l-1}],
$$

of the associated Legendre function, these angular integrals become

$$
\mathcal{I}_{a1} = -\sqrt{2}\pi\delta_{m',-m}K^{lml'm'}_{l_0 1}\int_{-1}^{1} d\mu P^1_{l_0}P^m_l[P^{m'+1}_{l'-1} + (l'+m')(l'+m'-1)P^{m'-1}_{l'-1}], \tag{C7}
$$

$$
\begin{aligned}
\mathcal{I}_{a2} = -\frac{\pi}{2}\delta_{m',-m}K^{lml'm'}_{l_0 2}\int_{-1}^{1} d\mu P^2_{l_0}\{&[P^{m+1}_l - (l+m)(l-m+1)P^{m-1}_l][P^{m'+1}_{l'-1} + (l'+m')(l'+m'-1)P^{m'-1}_{l'-1}] \\
&+ [P^{m+1}_{l-1} + (l+m)(l+m-1)P^{m-1}_{l-1}][P^{m'+1}_{l'} - (l'+m')(l'-m'+1)P^{m'-1}_{l'}]\},
\end{aligned}
\tag{C8}
$$

$$
\mathcal{I}_{a4} = \sqrt{2}\pi\delta_{m',-m}K^{lml'm'}_{l_0 1}\int_{-1}^{1} d\mu P^1_{l_0}P^m_l[P^{m'+1}_{l'} - (l'+m')(l'-m'+1)P^{m'-1}_{l'}], \tag{C9}
$$

$$
\begin{aligned}
\mathcal{I}_{a5} = \frac{\pi}{2}\delta_{m',-m}K^{lml'm'}_{l_0 2}\int_{-1}^{1} d\mu P^2_{l_0}\{&[P^{m+1}_l - (l+m)(l-m+1)P^{m-1}_l][P^{m'+1}_{l'} - (l'+m')(l'-m'+1)P^{m'-1}_{l'}] \\
&+ [P^{m+1}_{l-1} + (l+m)(l+m-1)P^{m-1}_{l-1}][P^{m'+1}_{l'-1} + (l'+m')(l'+m'-1)P^{m'-1}_{l'-1}]\},
\end{aligned}
\tag{C10}
$$

where the factor $\mathcal{K}^{lml'm'}_{LM}$ is defined as

$$
\mathcal{K}^{lml'm'}_{LM} = \sqrt{\frac{(2L+1)(2l+1)(2l'+1)}{(4\pi)^3}\frac{(L-M)!}{(L+M)!}\frac{(l-m)!}{(l+m)!}\frac{(l'-m')!}{(l'+m')!}}. \tag{C11}
$$

APPENDIX D: INTEGRALS OF THE PRODUCTS OF THREE LEGENDRE FUNCTIONS

We need the overlap integrals of three associated Legendre functions, Eq. (C1), in order to calculate the angular integrals. The integral $\mathcal{I}_{P3}$ was calculated by Mavromatis and Alassar [61], albeit with a phase error in their result. Dong and Lemus [62] restudied the problem and found an expression for $\mathcal{I}_{P3}$ with positive m_1, m_2, and m_3. Here we correct the phase error in [61] and give an expression that is valid for all values of m_1, m_2, and m_3.

$$\mathcal{I}_{P3}(l_1,m_1;l_2,m_2;l_3,m_3)=\frac{(|\Delta m|)!(-1)^{m_1+m_2+\min\{m_1+m_2,m_3\}}}{2^{|\Delta m|+2}\Gamma(|\Delta m|)K_{l_1m_1}K_{l_2m_2}K_{l_3m_3}}\sum_L\sum_{L'}(2L+1)(2L'+1)$$
$$\times\begin{pmatrix}l_1 & l_2 & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l_1 & l_2 & L\\ m_1 & m_2 & -m_1-m_2\end{pmatrix}\begin{pmatrix}L & l_3 & L'\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}L & l_3 & L'\\ -m_1-m_2 & m_3 & -\Delta m\end{pmatrix}$$
$$\times\frac{[1+(-1)^{L'+|\Delta m|}]\Gamma(L'/2)\Gamma((L'-|\Delta m|+1)/2)}{K_{L',|\Delta m|}\Gamma((L'+|\Delta m|+2)/2)\Gamma((L'+3)/2)}, \tag{D1}$$

where we have defined $\Delta m=m_3-m_1-m_2$, $K_{lm}=\sqrt{(l-m)!/(l+m)!}$. Note that $\min\{a,b\}$ represents the minimal value of a and b, $|l_1-l_2|\le L\le l_1+l_2$ and $|L-l_3|\le L'\le L+l_3$.

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