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Aharonov-Bohm effect on entanglement entropy in conformal field theory

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We consider the Aharonov-Bohm effect on entanglement entropy for one interval in $(1+1)$ dimensional conformal field theory on a one dimensional ring. The magnetic field is confined inside the ring, i.e., there is a Wilson loop on the ring. The Aharonov-Bohm phase factor which is proportional to the Wilson loop is represented as insertion of twist operators. We compute exactly the Rényi entropy from a four point function of twist operators in a free charged scalar field.

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I. INTRODUCTION

The entanglement entropy in the quantum field theory plays important roles in many fields of physics including the string theory [1–12], condensed matter physics [13–15], lattice gauge theories [16,17], and the physics of the black hole [18–23]. The entanglement entropy is a useful quantity which characterize quantum properties of given states.

For a given density matrix ρ of the total system, the entanglement entropy of the subsystem Ω is defined as

$$S_\Omega = -\mathrm{Tr}\rho_\Omega \ln \rho_\Omega, \tag{1}$$

where $\rho_\Omega = \mathrm{Tr}_{\Omega^c}\rho$ is the reduced density matrix of the subsystem Ω and Ω^c is the complement of Ω. The Rényi entropy $S_\Omega^{(n)}$ is defined as

$$S_\Omega^{(n)} = \frac{1}{1-n}\ln \mathrm{Tr}\rho_\Omega^n. \tag{2}$$

The limit $n \to 1$ coincides with the entanglement entropy $\lim_{n=1} S_\Omega^{(n)} = S_\Omega$.

On the other hand, the Aharonov-Bohm (AB) effect is a fundamental quantum phenomenon in which an electrically charged particle is affected by an electromagnetic potential A_μ, despite being confined to a region in which both the magnetic and electric field are zero.

In this paper, we consider the dependence of entanglement entropy with the AB phase. In particular, we consider $(1+1)$ dimensional conformal field theory on a one dimensional ring and study how the entanglement entropy for one interval on the ring is affected by a magnetic field enclosed by it (see Fig. 1). The Aharonov-Bohm phase factor can be represented as a twisted boundary condition by a gauge transformation. Thus, the twisted boundary condition is represented as insertion of twist operators. We compute exactly the Rényi entropy from a four point function of twist operators in a free charged scalar field.

The Aharonov-Bohm effect on entanglement entropy was studied in [24]. In [24], entanglement entropy for free charged scalar and Dirac fields in an annular strip on two dimensional cylinder was studied. Entanglement entropy in quantum field theories with twisted boundary conditions was studied in [25–28].

II. THE AHARONOV-BOHM EFFECT ON ENTANGLEMENT ENTROPY IN 2D CFT

We consider $(1+1)$ dimensional conformal field theory on a one dimensional ring whose circumference is L. The space coordinate x has the periodicity $x \sim x + L$. We analyze a complex scalar field, ϕ, charged with respect to an external gauge field, A_μ, which is pure gauge on the ring. We assume that $\phi(x)$ has the periodicity $\phi(x) = \phi(x+L)$. We choose a constant gauge field, A_x, in the x direction. We can eliminate it by a gauge transformation

$$\phi'(x) = e^{-iq\int^x dx' A_x}\phi(x), \tag{3}$$

where q is a charge of ϕ. The scalar field has now the following boundary condition

$$\phi'(x+L) = e^{-iqA_x L}\phi'(x) \equiv e^{-i2\pi\nu}\phi'(x), \tag{4}$$

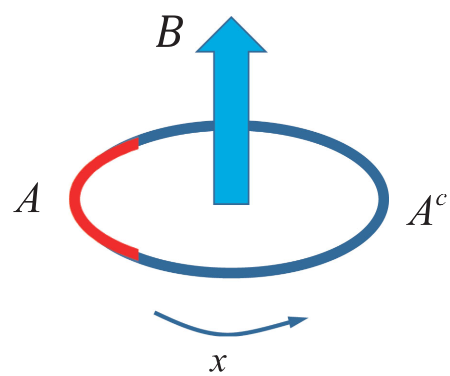

FIG. 1. One dimensional ring studied in this paper. The circumference of the ring is L and the subsystem A is one interval whose length is l. The space coordinate x has the periodicity $x \sim x + L$. The magnetic field is confined inside the ring and induces the Aharonov-Bohm phase on the ring.

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where we defined $\nu \equiv \frac{q}{2\pi}\Phi = \frac{q}{2\pi}LA_x$ and $\Phi \equiv \oint dx' A_x = LA_x$. The integral Φ is the magnetic flux inside the ring and ν is the Aharonov-Bohm (AB) phase. Now we consider the Rényi entropy, $S_A^{(n)} = \frac{1}{1-n}\ln \mathrm{Tr}\rho_A^n$, for one interval whose length is l. We compute the Rényi entropy by using the replica method and the Euclidean path integral [15]. The Euclidean coordinate is $w = x + i\tau$, where x is the space coordinate and has periodicity $x \sim x + L$, and τ is the Euclidean time ($-\infty < \tau < \infty$). We define the subsystem A to be the interval $x_1 \leq x \leq x_2$, $\tau_1 = \tau_2 = 0$, $x_2 - x_1 = l$, where $w_1 = x_1$ and $w_2 = x_2$ are endpoints of the interval. The Rényi entropy is expressed as the expectation value of twist operators,

$$\mathrm{Tr}[\rho_A^n] = \langle \mathcal{T}_n(w_1, \bar{w}_1)\tilde{\mathcal{T}}_n(w_2, \bar{w}_2)\rangle_\nu, \tag{5}$$

where $\langle \ldots \rangle_\nu$ is the expectation value under the boundary condition (4), and $\mathcal{T}_n$ and $\tilde{\mathcal{T}}_n$ are the twist operators whose action is

$$\mathcal{T}_n : \phi'_i \to \phi'_{i+1} \pmod{n}, \quad \tilde{\mathcal{T}}_n : \phi'_{i+1} \to \phi'_i \pmod{n}, \tag{6}$$

here ϕ_i denotes the ith replica field. To compute (5), we use the conformal map $z = e^{-i\frac{2\pi}{L}w}$. From (4), the scalar field in the z plane has the following boundary condition,

$$\phi'(e^{i2\pi}z, e^{-i2\pi}\bar{z}) = e^{i2\pi\nu}\phi'(z, \bar{z}). \tag{7}$$

The boundary condition (7) can be expressed by inserting twist operators σ_ν and $\sigma_{1-\nu}$ at $z = 0$ and $z = \infty$ (See Fig. 2). The action of σ_α is

$$\sigma_\alpha : \phi'_i \to e^{i2\pi\alpha}\phi'_i. \tag{8}$$

Thus, we rewrite $\mathrm{Tr}[\rho_A^n]$ in (5) as

$$\mathrm{Tr}[\rho_A^n] = \left|\frac{dw_1}{dz_1}\right|^{-2h_n}\left|\frac{dw_2}{dz_2}\right|^{-2h_n} \times \frac{\langle \mathcal{T}_n(z_1)\tilde{\mathcal{T}}_n(z_2)\sigma_\nu(0)\sigma_{1-\nu}(\infty)\rangle}{\langle \sigma_\nu(0)\sigma_{1-\nu}(\infty)\rangle}, \tag{9}$$

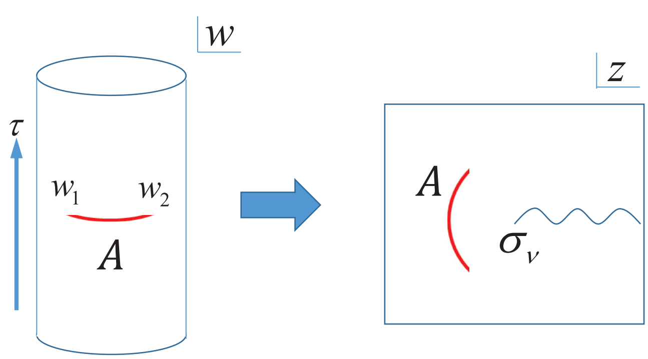

FIG. 2. The Euclidean path integral for $\mathrm{Tr}[\rho_A^n]$ in w and z coordinates. In z-plane, the boundary condition (7) can be expressed by inserting twist operators σ_ν and $\sigma_{1-\nu}$ at $z = 0$ and $z = \infty$.

where $z_{1,2} = e^{-i\frac{2\pi}{L}w_{1,2}}$ and $h_n = \frac{c}{24}(n - 1/n)$ is the conformal weight of $\mathcal{T}_n$ and $\tilde{\mathcal{T}}_n$, here c is the central charge.

III. CHARGED FREE SCALAR FIELD

We apply (9) to a free charged scalar field. For the free scalar field, it is useful to use the following Fourier transformation,

$$\tilde{\phi}_k \equiv \sum_{j=0}^{n-1} e^{i2\pi\frac{k}{n}j}\phi'_j. \tag{10}$$

For free fields, the Fourier transformation diagonalizes the action of $\mathcal{T}_n$, $\tilde{\mathcal{T}}_n$ and σ_α simultaneously,

$$\begin{aligned} \mathcal{T}_n &: \tilde{\phi}_k \to e^{i2\pi\frac{k}{n}}\tilde{\phi}_k, \\ \tilde{\mathcal{T}}_n &: \tilde{\phi}_k \to e^{-i2\pi\frac{k}{n}}\tilde{\phi}_k, \\ \sigma_\alpha &: \tilde{\phi}_k \to e^{i2\pi\alpha}\tilde{\phi}_k. \end{aligned} \tag{11}$$

Thus, the four point function of the twist operators in (9) become

$$\begin{aligned} &\frac{\langle \mathcal{T}_n(z_1)\tilde{\mathcal{T}}_n(z_2)\sigma_\nu(0)\sigma_{1-\nu}(\infty)\rangle}{\langle \sigma_\nu(0)\sigma_{1-\nu}(\infty)\rangle} \\ &= \prod_{k=1}^{n-1}\frac{\langle \sigma_{k/n}(z_1)\sigma_{1-k/n}(z_2)\sigma_\nu(0)\sigma_{1-\nu}(\infty)\rangle}{\langle \sigma_\nu(0)\sigma_{1-\nu}(\infty)\rangle} \\ &= \prod_{k=1}^{n-1}\frac{\langle \sigma_{k/n}(0)\sigma_{1-k/n}(x)\sigma_\nu(1)\sigma_{1-\nu}(\infty)\rangle}{\langle \sigma_\nu(1)\sigma_{1-\nu}(\infty)\rangle} \end{aligned} \tag{12}$$

where we used the conformal map $f(z) = 1 - z/z_1$ and $x = f(z_2) = 1 - e^{-i\frac{2\pi l}{L}}$ is the cross ratio of the four points ($|x| = 2|\sin\frac{\pi l}{L}|$). From (9) and (12), we obtain the Rényi entropy,

$$\begin{aligned} S_A^{(n)} &= \frac{1}{1-n}\ln \mathrm{Tr}[\rho_A^n] \\ &= \frac{1}{1-n}\sum_{k=0}^{n-1}\ln\frac{\langle \sigma_{k/n}(0)\sigma_{1-k/n}(x)\sigma_\nu(1)\sigma_{1-\nu}(\infty)\rangle}{\langle \sigma_\nu(1)\sigma_{1-\nu}(\infty)\rangle} \end{aligned} \tag{13}$$

where we used $|\frac{dw_{1,2}}{dz_{1,2}}| = \frac{2\pi}{L}$ and omitted the irrelevant constant.

The four point function of twist operators also appear in the calculation of Rényi entropy of two disjoint intervals in free scalar field theory [29]. In the case of two disjoint intervals, the necessary four point function is $\langle \sigma_{k/n}(0)\sigma_{1-k/n}(x)\sigma_{k/n}(1)\sigma_{1-k/n}(\infty)\rangle$. In our case, we need the more general four point function $\langle \sigma_{k_1/n}(0)\sigma_{1-k_1/n}(x)\sigma_{k_3/n}(1)\sigma_{1-k_3/n}(\infty)\rangle$. In the following we will use the results of the four point function of the twist operators by Knizhnik [30]. We give derivation and different expression of $\langle \sigma_{k_1/n}(0)\sigma_{1-k_1/n}(x)\sigma_{k_3/n}(1) \times \sigma_{1-k_3/n}(\infty)\rangle$ by another method in the Appendix B. Note that there are a series of papers from late eighties about conformal field theories on orbifold (e.g., [31–33]) that are probably useful for more complicated cases.

The four point function of the twist operators is given by (see Eqs. (7.22) and (7.28) in [30]),

$$\langle \sigma_{k_1/n}(0)\sigma_{k_2/n}(x)\sigma_{k_3/n}(1)\sigma_{2-(k_1+k_2+k_3)/n}(\infty)\rangle = \kappa^2 (ZZ_*)^{-1/2}, \tag{14}$$

$$Z_*(\{k_i\}|x) = |x|^{2k_1k_2/n^2}|1-x|^{2k_2k_3/n^2} \times I(-k_1/n, -k_2/n, -k_3/n, x), \tag{15}$$

$$Z(\{k_i\}|x) = Z_*(\{n-k_i\}|x), \tag{16}$$

$$I(a,b,c,x) = \int d^2z |z|^{2a}|z-x|^{2b}|z-1|^{2c}, \tag{17}$$

where κ is a constant and $\int d^2z = \int_{-\infty}^{\infty} d\mathrm{Re}z \int_{-\infty}^{\infty} d\mathrm{Im}z$. Note that $\sigma_{k/n}$ in [30] (and in (14)) is normalized as $\langle \sigma_{k/n}(0)\sigma_{1-k/n}(x)\rangle = |x|^{-2\frac{k}{n}(1-\frac{k}{n})}$. On the other hand, $\sigma_{k/n}$ in (13) is normalized as $\langle \sigma_{k/n}(0)\sigma_{1-k/n}(x)\rangle = (|x|/\epsilon)^{-2\frac{k}{n}(1-\frac{k}{n})}$, here $\epsilon \equiv a/L$ and a is the UV cutoff length. The latter normalization is usually used in calculation of Rényi entropy and gives the correct UV cutoff dependence of Rényi entropy. The integral $I(a,b,c,x)$ is calculated in the Appendix A. Note that the expression of $I(a,b,c,x)$ in the Appendix in [30] is not useful when $a+b=-1$ and we give a different expression which is useful when $a+b=-1$ in (A1). Thus, from (13)–(17), we obtain the Rényi entropy,

$$\begin{aligned} S_A^{(n)} &= \frac{1}{1-n}\sum_{k=1}^{n-1} \ln\left[\left(\frac{1}{\epsilon}\right)^{-2\frac{k}{n}(1-\frac{k}{n})} \kappa^2 (ZZ_*(k_1=k, k_2=n-k, k_3=n\nu))^{-1/2}\right] \\ &= \frac{1}{1-n}\sum_{k=1}^{n-1} \ln\left[\left(\frac{1}{\epsilon}\right)^{-2\frac{k}{n}(1-\frac{k}{n})} \kappa^2 \left(|x|^{4\frac{k}{n}(1-\frac{k}{n})}|1-x|^{4\frac{k}{n}(1-\nu)} I\left(\frac{k}{n}-1, -\frac{k}{n}, \nu-1, x\right)^2\right)^{-1/2}\right] \\ &= \frac{1}{1-n}\sum_{k=1}^{n-1} \ln\Bigg[\kappa^2 \left(\frac{|x|}{\epsilon}\right)^{-2\frac{k}{n}(1-\frac{k}{n})} |1-x|^{-2\frac{k}{n}(1-\nu)} \left(\frac{\Gamma(1-\nu)\Gamma(k/n)}{\Gamma(1+k/n-\nu)}\Big[F(1-\nu,k/n,1,x)F(1-\nu,k/n,1+k/n-\nu,1-\bar{x})\right. \\ &\quad \left. + F(1-\nu,k/n,1,\bar{x})F(1-\nu,k/n,1+k/n-\nu,1-x)\Big] + \pi\frac{\sin\pi(k/n-\nu)}{\sin(\pi k/n)\sin\pi\nu} F(1-\nu,k/n,1,x)F(1-\nu,k/n,1,\bar{x})\right)^{-1}\Bigg], \end{aligned} \tag{18}$$

where $F(\alpha,\beta,\gamma,z)$ is the Gaussian hypergeometric function,

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \times \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt, \tag{19}$$

and we used (A5) in the second equality and (A4) in the third equality in (18).

We study properties of the Rényi entropy (18). From (18), when $\nu \to 0$, $S_A^{(n)}$ diverges as

$$S_A^{(n)} \simeq \ln(1/\nu). \tag{20}$$

This divergence does not depend on the length of the subsystem, so it is the contribution of the homogeneous mode. This divergence is similar to the infrared divergence of the entanglement entropy in the massless limit in a free massive scalar field [34] and has the similar heuristic explanation. The correlation function is given by,

$$\langle \phi'(x)\phi'^*(0)\rangle = \frac{L}{2\pi} e^{-i\nu 2\pi x/L} \sum_{n=-\infty}^{\infty} \frac{e^{in2\pi x/L}}{|n-\nu|} \simeq \frac{L}{2\pi}\frac{1}{\nu}(\nu \to 0), \tag{21}$$

where we used $w = x + i\tau$ coordinates and $\tau = 0$. From (21), the typical size of the fluctuations on the homogeneous mode grows as $(1/\nu)^{1/2}$. Correspondingly, the Rényi entropy grows as the logarithm of this volume in field space [35], and becomes $S_A^{(n)} \simeq 2 \times \ln(1/\nu)^{1/2} = \ln(1/\nu)$. Note that we doubled the entropy because ϕ' is a complex field and has the real part and the imaginary part.

We plot the $S_A^{(n=2)}$ as a function of ν and the length of the subsystem l ($|x| = 2|\sin\frac{\pi l}{L}|$) in Figs. 3 and 4. In these figures, we have set $\kappa = 1$ and $\epsilon = 1$. The Rényi

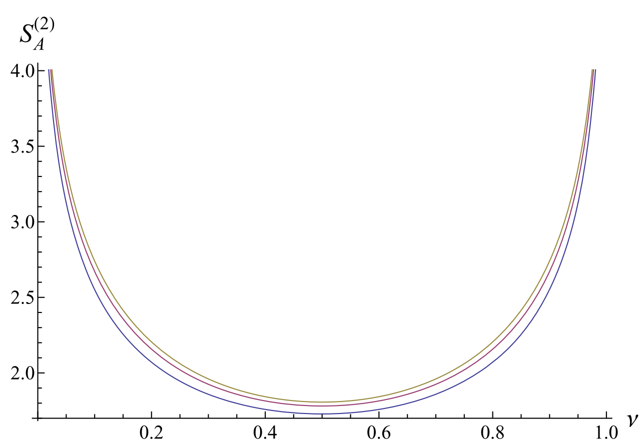

FIG. 3. The Rényi entropy $S_A^{(n=2)}$ as a function of ν. From top to bottom: $l/L = 1/3, 1/4, 1/6$. $S_A^{(n=2)}$ diverges when $\nu \to 0$ and $\nu \to 1$. $S_A^{(n=2)}$ becomes a minimum value for $\nu = 1/2$.

entropy $S_A^{(n=2)}$ diverges when $\nu \to 0$ and $\nu \to 1$ as shown in Fig. 3. $S_A^{(n=2)}$ becomes a minimum value when $\nu = 1/2$.

It is difficult to perform the analytical continuation of the Rényi entropy and to obtain the entanglement entropy because of the complexity of the expression (18). However, we can perform the analytical continuation in the limit $|x| \to 0$ and $\nu \to 0$. From (18), when $|x| \to 0$, we obtain

$$\begin{aligned} S_A^{(n)} &\simeq \frac{1}{1-n}\sum_{k=1}^{n-1} \ln\left[\left(\frac{|x|}{\epsilon}\right)^{-2\frac{k}{n}(1-\frac{k}{n})} (\ln\delta(k/n) + \ln\delta(\nu) - 2\ln|x|)^{-1}\right] \\ &= \frac{1}{3}\left(1+\frac{1}{n}\right)\ln\frac{|x|}{\epsilon} + \frac{1}{n-1}\sum_{k=1}^{n-1}\ln(\ln\delta(k/n) + \ln\delta(\nu) - 2\ln|x|), \end{aligned} \tag{22}$$

where

$$\ln\delta(y) = 2\psi(1) - \psi(y) - \psi(1-y), \tag{23}$$

here $\psi(y)$ is the digamma function, $\psi(y) = \frac{d}{dy}\ln\Gamma(y)$, and we omitted the irrelevant constant $-\ln\kappa^2$. From (22), when $|x| \to 0$ and $\nu \to 0$, we obtain

$$S_A^{(n)} \simeq \frac{1}{3}\left(1+\frac{1}{n}\right)\ln\frac{|x|}{\epsilon} + \ln\left(\frac{1}{\nu} + \ln\frac{1}{|x|^2}\right). \tag{24}$$

The first term is the same as the Rényi entropy for a free massless complex scalar field and the second term is the correction from the AB phase. Thus, when $|x| \to 0$ and $\nu \to 0$, we obtain the entanglement entropy,

$$S_A^{(n=1)} \simeq \frac{2}{3}\ln\frac{|x|}{\epsilon} + \ln\left(\frac{1}{\nu} + \ln\frac{1}{|x|^2}\right). \tag{25}$$

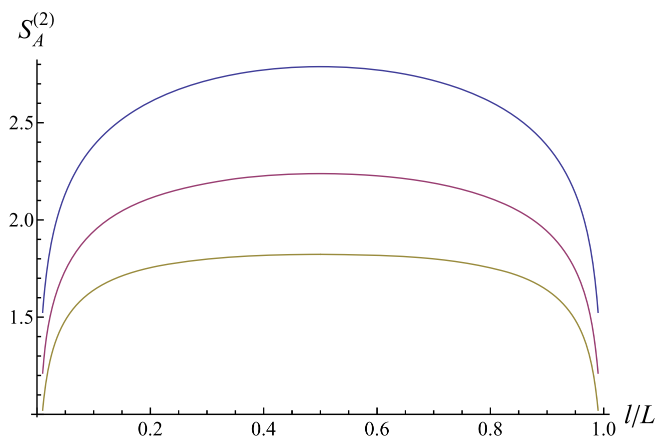

FIG. 4. The Rényi entropy $S_A^{(n=2)}$ as a function of l/L. From top to bottom: $\nu = 1/10, 1/5, 1/2$.

IV. CONCLUSION

We studied the dependence of entanglement entropy with the AB phase in $(1+1)$ dimensional conformal field theory on a one dimensional ring. We performed the gauge transformation (3) and the effect of AB phase is represented by the twisted boundary condition of the scalar field (4). We used the conformal map and the boundary condition was expressed by inserting twist operators σ_ν and $\sigma_{1-\nu}$ at $z = 0$ and $z = \infty$ in (9). We calculated exactly the Rényi entropy in charged free scalar field theory in (9). The Rényi entropy diverges when $\nu \to 0$. This divergence comes from the homogeneous mode and is similar to the infrared divergence of the entanglement entropy in a free massive scalar field. We gave the heuristic explanation of this divergence. We performed the analytical continuation in the limit $|x| \to 0$ and $\nu \to 0$ and obtained the entanglement entropy in (25).

We considered the ground state in the presence of the AB phase (i.e., the Wilson loop). This state is a kind of excited states in CFT without the AB phase. Entanglement entropy has been studied to quantify excited states in [36–39]. It is an interesting future problem to apply our method that the effect of AB phase is expressed by inserting twist operators to time dependent problems and excited states in the presence of the AB phase.

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APPENDIX A: THE CALCULATION OF THE INTEGRAL $I(a,b,c,x)$

We calculate the integral $I(a,b,c,x)$ in (17). The integral over the complex plane can be evaluated by splitting it into the sum of products of holomorphic and antiholomorphic contour integrals around cuts using a method used in Kawai *et al.* [40],

$$\begin{aligned}I(a,b,c,x) &= \int d^2z|z|^{2a}|z-x|^{2b}|z-1|^{2c}\\ &= \frac{\sin\pi c\sin\pi a}{\sin\pi(a+b+c)}\left[\int_0^x d\xi A\int_{\bar{x}}^1 d\eta B+\int_x^1 d\xi A\int_0^{\bar{x}} d\eta B\right]+\frac{\sin\pi(b+c)\sin\pi a}{\sin\pi(a+b+c)}\int_0^x d\xi A\int_0^{\bar{x}} d\eta B\\ &\quad+\frac{\sin\pi c\sin\pi(a+b)}{\sin\pi(a+b+c)}\int_x^1 d\xi A\int_{\bar{x}}^1 d\eta B=\frac{\sin\pi c\sin\pi a}{\sin\pi(a+b+c)}\\ &\quad\times\left[x^{a+b+1}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}F(-c,a+1,a+b+2,x)(1-\bar{x})^{1+b+c}\frac{\Gamma(1+c)\Gamma(1+b)}{\Gamma(2+b+c)}\right.\\ &\quad\left.\times F(-a,1+c,2+b+c,1-\bar{x})+\text{c.c}\right]+\frac{\sin\pi(b+c)\sin\pi a}{\sin\pi(a+b+c)}(x\bar{x})^{a+b+1}\left(\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}\right)^2\\ &\quad\times F(-c,a+1,a+b+2,x)F(-c,a+1,a+b+2,\bar{x})+\frac{\sin\pi c\sin\pi(a+b)}{\sin\pi(a+b+c)}((1-x)(1-\bar{x}))^{1+b+c}\\ &\quad\times\left(\frac{\Gamma(c+1)\Gamma(b+1)}{\Gamma(b+c+2)}\right)^2F(-a,c+1,b+c+2,1-x)F(-a,c+1,b+c+2,1-\bar{x}),\end{aligned} \tag{A1}$$

where

$$A\equiv|\xi|^a|\xi-x|^b|\xi-1|^c,\qquad B\equiv|\eta|^a|\eta-\bar{x}|^b|\eta-1|^c. \tag{A2}$$

Note that the expression of $I(a,b,c,x)$ in the Appendix in [30] is not useful when $a+b=-1$ and we gave the different expression which is useful when $a+b=-1$ in (A1). We can see that (A1) is the same as the result in the Appendix in [30] by using the following identity which is obtained by a contour integral around cuts;

$$\sin\pi a\int_0^x d\xi A+\sin\pi(a+b)\times\int_x^1 d\xi A+\sin\pi(a+b+c)\int_1^\infty d\xi A=0. \tag{A3}$$

From (A1), we obtain the necessary integral for the Rényi entropy (13),

$$\begin{aligned}I(a-1,-a,c-1,x) &= \pi\frac{\Gamma(1-c)\Gamma(a)}{\Gamma(1+a-c)}[F(1-c,a,1,x)F(1-c,a,1+a-c,1-\bar{x})\\ &\quad+F(1-c,a,1,\bar{x})F(1-c,a,1+a-c,1-x)]\\ &\quad+\pi^2\frac{\sin\pi(a-c)}{\sin\pi a\sin\pi c}F(1-c,a,1,x)F(1-c,a,1,\bar{x}),\end{aligned} \tag{A4}$$

$$I(-a,a-1,-c,x)=|1-x|^{2(a-c)}I(a-1,-a,c-1,x). \tag{A5}$$

APPENDIX B: DERIVATION OF THE FOUR POINT FUNCTION OF TWIST OPERATORS BY ANOTHER METHOD

We calculate the four point function of twist operators by the method in [31,32]. We consider a complex field $X(z,\bar{z})$ and the action for $X(z,\bar{z})$ is given by

$$S[X,\bar{X}]=\frac{1}{4\pi}\int(\partial_z X\partial_{\bar{z}}\bar{X}+\partial_{\bar{z}}X\partial_z\bar{X})d^2z. \tag{B1}$$

We consider the following four point function

$$Z(z_i,\bar{z}_i)\equiv\langle\sigma_{k_1/n}(z_1)\sigma_{1-k_1/n}(z_2)\sigma_{k_3/n}(z_3)\sigma_{1-k_3/n}(z_4)\rangle. \tag{B2}$$

From [31,32], we consider the Green function in the presence of four twist operators,

$$g(z,w,z_i) \equiv \frac{-\frac{1}{2}\langle \partial_z X(z)\partial_w \bar{X}(w)\sigma_{k_1/n}(z_1)\sigma_{1-k_1/n}(z_2)\sigma_{k_3/n}(z_3)\sigma_{1-k_3/n}(z_4)\rangle}{\langle \sigma_{k_1/n}(z_1)\sigma_{1-k_1/n}(z_2)\sigma_{k_3/n}(z_3)\sigma_{1-k_3/n}(z_4)\rangle}. \tag{B3}$$

The Green function obeys the following asymptotic conditions;

$$\begin{aligned} g(z,w,z_i) &\sim (z-w)^{-2} + \text{finite} \quad \text{as } z\to w \sim \text{const}\times(z-z_{1,3})^{-k_{1,3}/n} \quad \text{as } z\to z_{1,3} \sim \text{const}\times(z-z_{2,4})^{-(1-k_{1,3}/n)} \quad \text{as } z\to z_{2,4} \\ &\sim \text{const}\times(w-z_{1,3})^{-(1-k_{1,3}/n)} \quad \text{as } w\to z_{1,3} \sim \text{const}\times(w-z_{2,4})^{-k_{1,3}/n} \quad \text{as } w\to z_{2,4}. \end{aligned} \tag{B4}$$

Thus, we can write $g(z,w,z_i)$ as

$$g(z,w,z_i) = \omega_k(z)\omega_{n-k}(w)\frac{1}{(z-w)^2}[A_0(w) + A_1(w)(z-w) + A_2(w)(z-w)^2], \tag{B5}$$

where

$$\begin{aligned} \omega_k(z) &\equiv (z-z_1)^{-k_1/n}(z-z_2)^{-(1-k_1/n)}(z-z_3)^{-k_3/n}(z-z_4)^{-(1-k_3/n)} \\ \omega_{n-k}(z) &\equiv (z-z_1)^{-(1-k_1/n)}(z-z_2)^{-k_1/n}(z-z_3)^{-(1-k_3/n)}(z-z_4)^{-k_3/n} \end{aligned} \tag{B6}$$

and

$$A_0(w) \equiv \prod_{j=1}^{4}(w-z_j), \qquad A_1(w) \equiv A_0(w)\sum_{j=1}^{4}\frac{k_j}{n}\frac{1}{w-z_j}, \tag{B7}$$

here we defined $k_{2,4} \equiv n - k_{1,3}$ and $A_2(w)$ will be determined by the global monodromy condition. The global monodromy condition is

$$\Delta_{\mathcal{C}}X = \oint_{\mathcal{C}} dz\partial_z X + \oint_{\mathcal{C}} d\bar{z}\partial_{\bar{z}}X = 0 \tag{B8}$$

for all closed loops.

Before determining A_2, we will extract the differential equation of $Z(z_i,\bar{z}_i)$. Let us consider the limit $w\to z$

$$\begin{aligned} \lim_{w\to z}[g(z,w,z_i) - (z-w)^{-2}] &= \frac{\langle T(z)\sigma_{k_1/n}(z_1)\sigma_{1-k_1/n}(z_2)\sigma_{k_3/n}(z_3)\sigma_{1-k_3/n}(z_4)\rangle}{\langle \sigma_{k_1/n}(z_1)\sigma_{1-k_1/n}(z_2)\sigma_{k_3/n}(z_3)\sigma_{1-k_3/n}(z_4)\rangle} \\ &= A_2(z)\sum_j \frac{1}{z-z_j}\prod_{l\neq j}(z_j-z_l)^{-1} + \sum_{j=1}^{4} h_j(z-z_j)^{-2} - \sum_j \frac{1}{z-z_j}\sum_{l\neq j}\frac{k_j}{n}\frac{k_l}{n}\frac{1}{z_j-z_l}, \end{aligned} \tag{B9}$$

where $T(z)$ is the stress tensor and $h_j \equiv \frac{1}{2}(k_j/n)(1-k_j/n)$ is the conformal weight of the twist operator $\sigma_{k_j/n}$. We apply the operator product

$$T(z)\sigma_{k_2}(z_2) \sim \frac{h_2\sigma_{k_2}(z_2)}{(z-z_2)^2} + \frac{\partial_{z_2}\sigma_{k_2}(z_2)}{z-z_2} + \ldots \tag{B10}$$

to (B9) and obtain the differential equation

$$\partial_{z_2}\ln Z(z_i,\bar{z}_i) = \frac{A_2(z_2)}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} - \frac{k_2}{n}\left(\frac{k_1}{n}\frac{1}{z_2-z_1} + \frac{k_3}{n}\frac{1}{z_2-z_3} + \frac{k_4}{n}\frac{1}{z_2-z_4}\right). \tag{B11}$$

It is useful to use the following conformal map

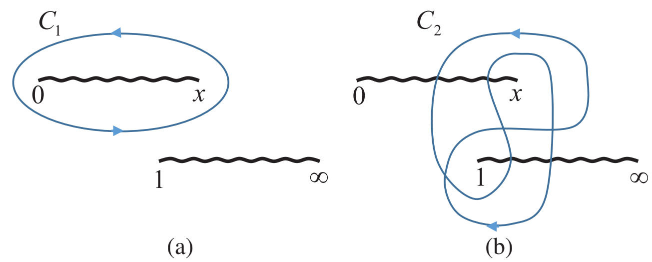

FIG. 5. The two closed loops $\mathcal{C}_1$ and $\mathcal{C}_2$ we consider as a basis of the loops in the complex plane.

$$z \to \frac{(z_1 - z)(z_3 - z_4)}{(z_1 - z_3)(z - z_4)}, \tag{B12}$$

which sends z_1, z_2, z_3, and z_4 into $0, x, 1$ and ∞ respectively, where x is the cross ratio ($z_{ij} \equiv z_i - z_j$)

$$x \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad x(1-x) = \frac{z_{12}z_{34}z_{14}z_{23}}{z_{13}^2 z_{24}^2}. \tag{B13}$$

Thus (B11) becomes

$$\partial_x \ln Z(x,\bar{x}) = -\frac{\tilde{A_2}}{x(1-x)} - \frac{k_2}{n}\left(\frac{k_1}{n}\frac{1}{x} - \frac{k_3}{n}\frac{1}{1-x}\right). \tag{B14}$$

where

$$\begin{aligned} Z(x,\bar{x}) &= \lim_{z_\infty \to \infty} \langle \sigma_{k_1/n}(0)\sigma_{1-k_1/n}(x)\sigma_{k_3/n}(1)\sigma_{1-k_3/n}(z_\infty)\rangle \tilde{A_2} \\ &= \lim_{z_\infty \to \infty} -z_\infty^{-1} A_2(z_1 = 0, z_2 = x, z_3 = 1, z_4 = z_\infty; w = x). \end{aligned} \tag{B15}$$

In order to determine A_2 by the global monodromy condition, we introduce the auxiliary correlation function

$$h(z,w,z_i) \equiv \frac{-\frac{1}{2}\langle \partial_{\bar{z}} X(\bar{z}) \partial_w \bar{X}(w) \sigma_{k_1/n}(z_1)\sigma_{1-k_1/n}(z_2)\sigma_{k_3/n}(z_3)\sigma_{1-k_3/n}(z_4)\rangle}{\langle \sigma_{k_1/n}(z_1)\sigma_{1-k_1/n}(z_2)\sigma_{k_3/n}(z_3)\sigma_{1-k_3/n}(z_4)\rangle} = B(z_i,\bar{z}_i)\bar{\omega}_{n-k}(\bar{z})\omega_{n-k}(w), \tag{B16}$$

where we determined h in the same way as g was determined. From the global monodromy condition (B8), we obtain

$$\oint_{\mathcal{C}_i} dz g(z,w,z_i) + \oint_{\mathcal{C}_i} d\bar{z} h(z,w,z_i) = 0, i = 1,2 \tag{B17}$$

where we chose the two loops $\mathcal{C}_1$ and $\mathcal{C}_2$ shown in Fig. 5 as a basis of the loops.

We divide (B17) by $\omega_{n-k}(w)$ and set $w = z_2$, $z_1 = 0$, $z_2 = x$, $z_3 = 1$ and $z_4 \to \infty$ and obtain

$$\left(-x(1-x)\frac{d}{dx} + \tilde{A_2}\right)\oint_{\mathcal{C}_i} dz \omega_k'(z) + \tilde{B}\oint_{\mathcal{C}_i} d\bar{z}\bar{\omega}_{n-k}{}'(\bar{z}) = 0, \qquad i = 1,2 \tag{B18}$$

where $\tilde{B} \equiv \lim_{z_\infty \to \infty} z_\infty^{-2} B(z_1 = 0, z_2 = x, z_3 = 1, z_4 = z_\infty)$ and

$$\begin{aligned} \omega_k'(z) &\equiv z^{-k_1/n}(z-x)^{-(1-k_1/n)}(z-1)^{-k_3/n} \\ \bar{\omega}_{n-k}'(\bar{z}) &\equiv \bar{z}^{-(1-k_1/n)}(\bar{z}-\bar{x})^{-k_1/n}(\bar{z}-1)^{-(1-k_3/n)}. \end{aligned} \tag{B19}$$

We calculate all integrals in (B18) and obtain

$$\oint_{\mathcal{C}_1} dz \omega_k'(z) = (-1 + \alpha^{-k_1})\alpha^{-\frac{1}{2}(k_2+k_3)}\Gamma\left(1 - \frac{k_1}{n}\right)\Gamma\left(\frac{k_1}{n}\right)F\left(\frac{k_3}{n}, 1 - \frac{k_1}{n}, 1, x\right), \tag{B20}$$

$$\oint_{\mathcal{C}_1} d\bar{z}\bar{\omega}_{n-k}{}'(\bar{z}) = (-1 + \alpha^{-k_1})\alpha^{-\frac{1}{2}(k_2+k_3)}\Gamma\left(1 - \frac{k_1}{n}\right)\Gamma\left(\frac{k_1}{n}\right)F\left(1 - \frac{k_3}{n}, \frac{k_1}{n}, 1, \bar{x}\right), \tag{B21}$$

$$\oint_{\mathcal{C}_2} dz \omega_k'(z) = (-1 + \alpha^{-k_2} - \alpha^{-(k_2+k_3)} + \alpha^{-k_3})\alpha^{-\frac{1}{2}k_3}(1-x)^{\frac{(k_1-k_3)}{n}} \times F\left(\frac{k_1}{n}, 1 - \frac{k_3}{n}, 1 - \frac{k_3}{n} + \frac{k_1}{n}, 1 - x\right)\frac{\Gamma(1 - \frac{k_3}{n})\Gamma(\frac{k_1}{n})}{\Gamma(1 - \frac{k_3}{n} + \frac{k_1}{n})}, \tag{B22}$$

$$\oint_{\mathcal{C}_2} d\bar{z}\bar{\omega}_{n-k}{}'(\bar{z}) = -(-1+\alpha^{-k_2}-\alpha^{-(k_2+k_3)}+\alpha^{-k_3})\alpha^{-\frac{1}{2}k_3}(1-\bar{x})^{\frac{(k_3-k_1)}{n}} \times F\left(1-\frac{k_1}{n},\frac{k_3}{n},1+\frac{k_3}{n}-\frac{k_1}{n},1-\bar{x}\right)\frac{\Gamma(1-\frac{k_1}{n})\Gamma(\frac{k_3}{n})}{\Gamma(1+\frac{k_3}{n}-\frac{k_1}{n})}, \tag{B23}$$

where $\alpha \equiv e^{i2\pi/n}$. Solving Eq. (B18) for $\tilde{A}_2$, we obtain

$$\frac{\tilde{A}_2}{x(1-x)} = \partial_x \ln\left[\oint_{\mathcal{C}_1} dz\omega_k'(z)\oint_{\mathcal{C}_2} d\bar{z}\bar{\omega}_{n-k}{}'(\bar{z}) - \oint_{\mathcal{C}_2} dz\omega_k'(z)\oint_{\mathcal{C}_1} d\bar{z}\bar{\omega}_{n-k}{}'(\bar{z})\right]. \tag{B24}$$

We substitute (B24) for (B14) and obtain

$$Z(x,\bar{x}) = C(\bar{x})x^{-k_1k_2/n^2}(1-x)^{-k_2k_3/n^2}(W(k_1/n,k_3/n,x,\bar{x}))^{-1}, \tag{B25}$$

where $C(\bar{x})$ is an arbitrary function of $\bar{x}$ and

$$\begin{aligned} W(k_1/n,k_3/n,x,\bar{x}) = {} & \frac{\Gamma(1-\frac{k_1}{n})\Gamma(\frac{k_3}{n})}{\Gamma(1+\frac{k_3}{n}-\frac{k_1}{n})}(1-\bar{x})^{\frac{(k_3-k_1)}{n}}F\left(1-\frac{k_1}{n},\frac{k_3}{n},1+\frac{k_3}{n}-\frac{k_1}{n},1-\bar{x}\right)F\left(\frac{k_3}{n},1-\frac{k_1}{n},1,x\right) \\ & +\frac{\Gamma(1-\frac{k_3}{n})\Gamma(\frac{k_1}{n})}{\Gamma(1+\frac{k_1}{n}-\frac{k_3}{n})}(1-x)^{\frac{(k_1-k_3)}{n}}F\left(\frac{k_1}{n},1-\frac{k_3}{n},1+\frac{k_1}{n}-\frac{k_3}{n},1-x\right)F\left(1-\frac{k_3}{n},\frac{k_1}{n},1,\bar{x}\right). \end{aligned} \tag{B26}$$

In order to fix the $\bar{x}$-dependence of $Z(x,\bar{x})$, we consider $\partial_{\bar{x}}\ln Z(x,\bar{x})$ in the same way as $\partial_x \ln Z(x,\bar{x})$. The differential equation for $\partial_{\bar{x}}\ln Z(x,\bar{x})$ is obtained by replacing $x\to\bar{x}$, $\bar{x}\to x$ and $k_i\to n-k_i$ in that for $\partial_x\ln Z(x,\bar{x})$ and we obtain

$$Z(x,\bar{x}) = D(x)\bar{x}^{-k_1k_2/n^2}(1-\bar{x})^{-(n-k_2)(n-k_3)/n^2}(W(1-k_1/n,1-k_3/n,\bar{x},x))^{-1}, \tag{B27}$$

where $D(x)$ is an arbitrary function of x. Finally, from (B25) and (B27), we obtain

$$Z(x,\bar{x}) = f\cdot|x|^{-2k_1(n-k_1)/n^2}|1-x|^{2k_1k_3/n^2-(k_1+k_3)/n}(\tilde{W}(k_1/n,k_3/n,x,\bar{x}))^{-1}, \tag{B28}$$

where f is an integration constant and

$$\begin{aligned} \tilde{W}(k_1/n,k_3/n,x,\bar{x}) = {} & \frac{\Gamma(1-\frac{k_1}{n})\Gamma(\frac{k_3}{n})}{\Gamma(1+\frac{k_3}{n}-\frac{k_1}{n})}|1-x|^{\frac{(k_3-k_1)}{n}}F\left(1-\frac{k_1}{n},\frac{k_3}{n},1+\frac{k_3}{n}-\frac{k_1}{n},1-\bar{x}\right)F\left(\frac{k_3}{n},1-\frac{k_1}{n},1,x\right) \\ & +\frac{\Gamma(1-\frac{k_3}{n})\Gamma(\frac{k_1}{n})}{\Gamma(1+\frac{k_1}{n}-\frac{k_3}{n})}|1-x|^{\frac{(k_1-k_3)}{n}}F\left(1-\frac{k_3}{n},\frac{k_1}{n},1+\frac{k_1}{n}-\frac{k_3}{n},1-x\right)F\left(1-\frac{k_3}{n},\frac{k_1}{n},1,\bar{x}\right). \end{aligned} \tag{B29}$$

We can show that the integration constant f is independent of k_1 and k_3 by taking the limit $|x|\to 0$ and considering the operator product expansion (OPE) of $\sigma_{k_1/n}(0)\sigma_{1-k_1/n}(x)$. After some calculation, we obtain the following identity;

$$(\tilde{W}(a,c,x,\bar{x}))^2 = \frac{1}{\pi^2}I(a-1,-a,c-1,x)I(-a,a-1,-c,x). \tag{B30}$$

Thus (B28) is equal to (14) for $k_2 = n-k_1$ when we set $f=\kappa^2/\pi$.

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