PHYSICAL REVIEW D **96,** 065010 (2017)

Inevitable emergence of composite gauge bosons

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(Received 18 July 2017; published 15 September 2017)

A simple theorem is proved: When a gauge-invariant local field theory is written in terms of matter fields alone, a composite gauge boson or bosons must be formed dynamically. The theorem results from the fact that the Noether current vanishes in such theories. The proof is carried out by use of the charge-field algebra at equal time in the Heisenberg picture together with the well-established analyticity of the form factor of the current. While there is no need of diagram calculation for the proof, we demonstrate in the leading $1/N$ expansion of the existing models what the theorem means in diagrams and how the composite gauge boson emerges.

DOI: 10.1103/PhysRevD.96.065010

I. INTRODUCTION

Some theories possess a local gauge symmetry, yet do not contain a gauge field explicitly. The CP^N model [1] is one of the examples. It was shown in the leading $1/N$ expansion of the CP^{N-1} model that a U(1) gauge boson is indeed generated as a composite state of matter particles [2]. The U(1) gauge symmetry of the CP^N model was extended by Akhmedov [3] to the SU(2) symmetry. More recently, models were built with fermion matter alone [4]. Whether the symmetry is Abelian or non-Abelian, the models with fermion matter cannot be reproduced by extension of the CP^N model nor by means of the auxiliary field trick [5,6]. Nonetheless, it was explicitly shown by the large N expansion of the diagram calculation that these models indeed generate the composite gauge bosons as the massless bound states of the matter particles.

There is one peculiar feature common to the Lagrangian of composite gauge bosons. That is, the Noether current does not exist. This can be shown generally as a direct consequence of local gauge invariance without referring to specific binding forces [4]. In fact, in the case of the non-Abelian gauge theory, if the Noether current existed, formation of composite gauge bosons would contradict with the theorem of Weinberg and Witten [7].

The diagrammatic study of the composite gauge bosons has been limited to the leading order of the $1/N$ expansion which amounts to summing up an infinite series of loop diagrams of the matter particles [2,4]. Because of the complexity of perturbative computation, we cannot keep such calculation under control beyond the leading order of $1/N$. Nonetheless, it is natural to speculate that the composite gauge bosons are always formed irrespectively of specific details of the binding force when the total Lagrangian is gauge invariant with matter particles alone.

In this paper, we attempt to prove the formation of composite gauge bosons to all orders of binding interactions without recourse to diagrams. The proof is based on the equal-time algebra of charges and fields in the Heisenberg picture, which incorporates all orders of interactions. We show that a composite gauge boson must appear as a pole in the form factor of the current carrying its quantum numbers. Although a diagrammatic verification is redundant for the proof, it is reassuring and also visually helpful to understand the proof in terms of diagrams. After completing our proof, therefore, we demonstrate in the leading $1/N$ expansion of an existing model how the statement of our theorem is realized in diagrams.

We organize the paper as follows: First, the theorem is stated in Sec. II. After the necessary input of field theory is carefully reviewed in Sec. III, the theorem is proved in Sec. IV with the equal-time algebra of charges and fields for the non-Abelian gauge theories of the boson matter. In Sec. V, we demonstrate in diagrams how the statement of the theorem is realized in the leading $1/N$ order of a concrete non-Abelian model. It is shown in Sec. VI that the theorem holds just as well for the U(1) gauge theories. In order to apply our argument to the fermion matter, we discuss in Sec. VII an issue in the canonical quantization of the Dirac field, specifically, a problem related to the quantization of constrained systems and a possibility of justifying the charge-field algebra without relying on the canonical quantization. We conclude with some perspectives in theory and phenomenology in Sec. VIII.

II. THEOREM

The theorem is stated as follows:

If a gauge-invariant Lagrangian field theory is written in terms of matter fields alone, there must be a composite gauge boson or bosons made of the matter particles.

The gist of the theorem is that formation of the composite gauge boson(s) is not a possibility but the necessity. The input crucial to prove this theorem is the absence of the Noether current in this class of theories. We study the form factor of the current in the equal-time commutation relation of charges and fields by starting away from the gauge-symmetry limit. Then we approach the gauge symmetry by continuously varying a certain parameter and prove the theorem without referring to diagrams or details of binding forces.

The theorem holds in the flat space-time of $(3+1)$ dimensions for both the Abelian and non-Abelian theories with boson or fermion matters. It is not dual to the Weinberg-Witten theorem [7], which states that the non-Abelian massless gauge bosons cannot exist if the corresponding Lorentz-covariant conserved currents exist. Their theorem is mute as to whether the non-Abelian gauge bosons must exist or not when such currents are absent.

III. NON-ABELIAN SYMMETRY WITH BOSON MATTER

All that we use for the proof is the basic quantum field theory and its simple applications. To emphasize specific subtleties relevant to our proof, however, we give a brief review on elementary subjects, some of which may have fallen into oblivion by now.

A. Gauge variation of Lagrangian

The reason to discuss the spinless boson matter first is mainly the notational and technical simplicity related to the spins. But there is one complication in the canonical quantization of the Dirac field. Otherwise, no intrinsic difference exists between the boson matter and the fermion matter.

The Lagrangian is in the form of

$$L_{\text{tot}} = \partial^{\mu}\Phi^{\dagger}\partial_{\mu}\Phi - m^2\Phi^{\dagger}\Phi + L_{\text{int}}. \tag{1}$$

A set of the scalar fields $\Phi/\Phi^{\dagger}$ transform locally like an $n/\bar{n}$-dimensional representation of a Lie group;

$$\Phi \to U\Phi, \qquad \Phi^{\dagger} \to \Phi^{\dagger}U^{\dagger}, \tag{2}$$

where U is given in terms of the $n \times n$ generator matrices T_a as

$$U = \exp[iT_a\alpha_a(x)]. \tag{3}$$

The matrices T_a obey $[T_a, T_b] = if_{abc}T_c$ with the structure constants f_{abc}.

We introduce N *copies* of the n-component complex scalar pairs $\Phi_i/\Phi_i^{\dagger}$ $(i = 1, 2, 3 \cdots N)$ since, after completing the proof, we make the large N expansion in the diagram calculation to demonstrate how the theorem works in the explicit model.[1] However, we shall suppress the copy index i hereafter unless we need to recall it.

The interaction Lagrangian L_{int} is a functional of Φ, $\Phi^{\dagger}$ and their first derivatives in the known models. We assume that L_{int} does not contain time derivatives of fields higher than the first derivative. That is, L_{int} should be just as singular as the free Lagrangian L_0 in regard to the derivatives of the field. Otherwise, the gauge variation of L_0 cannot be compensated with that of L_{int}.[2]

Since the free Lagrangian L_0 is not invariant under the local gauge transformation, Eq. (2), the interaction Lagrangian L_{int} must counterbalance the gauge variation δL_0 of the free Lagrangian as

$$\delta L_{\text{int}} = -\delta L_0. \tag{4}$$

Since δL_0 is known from the free Lagrangian in Eq. (1) as

$$\delta L_0 = \partial^{\mu}\Phi^{\dagger}(U^{\dagger}\partial_{\mu}U)\Phi + \Phi^{\dagger}(\partial^{\mu}U^{\dagger}U)\partial_{\mu}\Phi + \Phi^{\dagger}(\partial^{\mu}U^{\dagger}\partial_{\mu}U)\Phi, \tag{5}$$

the relation of Eq. (4) determines the gauge variation δL_{int} uniquely even without knowing L_{int} itself. We place an emphasis on this trivial but powerful constraint of gauge invariance since it allows us to proceed in our proof without knowing an explicit form of L_{int}. We would need the form of L_{int} only when we carry out, as we shall do later, a diagrammatic demonstration of the theorem in the interaction picture.

Whereas we are interested in the gauge-invariant Lagrangian of Eq. (1), we insert a parameter λ in front of L_{int} as

$$L^{\lambda}_{\text{tot}} = L_0 + \lambda L_{\text{int}}, \tag{6}$$

and study how physics varies as λ approaches unity. The purpose of this seemingly redundant procedure is the following: Since the composite gauge boson carries the same quantum numbers $J^{PC} = 1^{--}$ as the Noether current, we wish to study the gauge boson through the Noether current. However, if we stayed exactly in the gauge-symmetry limit $(\lambda = 1)$, we would not be able to do so since the Noether current vanishes there according to the general theorem. (cf. Appendix A.) In order to study the pole of a composite gauge boson in the form factor, therefore, we must approach the gauge-symmetry limit with L^{λ}_{tot} of Eq. (6) by continuously varying the value of parameter λ to 1. By doing so, we can study where the bound-state pole of $J^{PC} = 1^{--}$ is located off the gauge symmetry and how it moves to zero, turning into the

[1] In fact, there is another reason for considering a large N. In our proof one-particle states will be treated as the asymptotic states. If confinement occurs with the composite gauge bosons, the one-matter-particle states are, strictly speaking, not the asymptotic states of the S matrix. The simplest way to avoid this inconvenience is to consider the case that there exists a sufficient number of matter multiplets to counter the confinement.

[2] Higher derivatives would ruin causality in dynamics. Recall in classical physics that the solutions are acausal when the force contains a higher derivative, for instance, the radiation damping of a point charge. The same happens in classical field theory. In quantum theory we would not be able to quantize canonically in the Heisenberg picture if L_{int} is more singular.

massless gauge boson in the gauge limit. With $L^{\lambda}_{\rm tot}$ as given in Eq. (6), we approach the gauge limit along one special *path* in the functional space of the Lagrangian.[3]

B. Noether current

The Noether current vanishes in the gauge-symmetric field theories for which the Lagrangian consists only of matter fields. This is a simple inevitable consequence of gauge invariance, Abelian or non-Abelian. Since the Noether current due to the free Lagrangian cannot vanish by itself, this must happen such that the contribution from the interaction Lagrangian cancels that from the free Lagrangian. The proof is very simple, as is given in Appendix A for the non-Abelian boson matter. Extension to other cases is trivial.

In short, the gauge-symmetric Lagrangian $L_{\rm tot}$ varies under the infinitesimal local phase transformation by $\alpha_a(x)$ of Eqs. (2) and (3) as

$$\delta L_{\rm tot} = i(\partial^{\mu} J_{a\mu})\alpha_a + iJ_{a\mu}\partial^{\mu}\alpha_a + O(\alpha^2), \tag{7}$$

after use of the equations of motion for Φ and $\Phi^{\dagger}$ in the first term. Since $\alpha_a(x)$ is an arbitrary function of x, we can treat $\alpha_a(x)$ and $\partial_{\mu}\alpha_a(x)$ as independent of each other. Consequently, the first term of Eq. (7) leads to the definition of the Noether current and its conservation. The second term simply states that the Noether current must vanish.

Both L_0 and $L_{\rm int}$ contribute to $J_{a\mu}$ since both contain the first derivatives of Φ and $\Phi^{\dagger}$ in order to satisfy gauge invariance. When we modify $L_{\rm tot}$ into $L_0 + \lambda L_{\rm int}$, it is no longer gauge invariant away from $\lambda = 1$ and therefore the Noether current $J^{\lambda}_{a\mu}$ survives. It is simply given (cf. Appendix A) by

$$J^{\lambda}_{a\mu} = i(1-\lambda)(\Phi^{\dagger} T_a \overleftrightarrow{\partial}_{\mu} \Phi). \tag{8}$$

The factor $(1-\lambda)$ in front indicates the fact that the Noether current vanishes in the gauge limit. The Noether current thus takes the form identical with that of the free field theory up to the factor $(1-\lambda)$:

$$J^{\rm free}_{a\mu} = \lim_{\lambda\to 0}\left(\frac{1}{1-\lambda} J^{\lambda}_{a\mu}\right). \tag{9}$$

However, we make a trivial but important remainder about Eq. (8). That is,

$$J^{\lambda}_{a\mu} \neq (1-\lambda) J^{\rm free}_{a\mu}. \tag{10}$$

The reason is that when we use Eq. (8) the fields in right-hand side are in the Heisenberg picture; that is, the $\Phi/\Phi^{\dagger}$ fields in $J^{\lambda}_{a\mu}$ incorporate all the λ dependence through the interaction, while the $\Phi/\Phi^{\dagger}$ fields in $J^{\rm free}_{a\mu}$ are independent of $\lambda\,(=0)$ by definition. It would be clearer in this respect if we wrote the fields of the Heisenberg picture as $\Phi(x,\lambda)$ and $\Phi^{\dagger}(x,\lambda)$. The implicit λ dependence of Φ and $\Phi^{\dagger}$ in the Heisenberg picture incorporates all interactions and it is responsible for the formation of the bound states among others.

C. Equal-time algebra of charges and fields

We use the equal-time algebra of the charges and fields in the Heisenberg picture for our proof of the theorem. With the "canonical momentum" defined by $\Pi \equiv \partial L/\partial(\partial^0\Phi)$, the field Φ obeys the equal-time commutation relation,

$$[\Phi_r(\mathbf{x},t), \Pi_s(\mathbf{y},t)] = i\delta_{rs}\delta(\mathbf{x}-\mathbf{y}). \tag{11}$$

The subscripts (r,s) refer to components of the n-dimensional representation. Equation (11) holds separately for each of N copies. $\Phi^{\dagger}$ and $\Pi^{\dagger}$ obey the same form of commutation relation, and all other equal-time commutators among Φ, $\Phi^{\dagger}$, Π, and $\Pi^{\dagger}$ vanish. In terms of these canonical variables, the charge component of the Noether current is expressed as

$$\begin{aligned} J^{\lambda}_{a0} &= i(\Phi^{\dagger} T_a \Pi^{\dagger} - \Pi T_a \Phi) \\ &= i(1-\lambda)(\Phi^{\dagger} T_a \overleftrightarrow{\partial}_0 \Phi), \end{aligned} \tag{12}$$

where the summation over the N copies is understood. Notice that the factor $(1-\lambda)$ appears when J_{a0} is written in Φ, $\Phi^{\dagger}$ and their time derivatives. But Eq. (12) does not mean that Π and $\Pi^{\dagger}$ are proportional to $1-\lambda$ (cf. Appendix B). The Noether charge is defined by

$$Q^{\lambda}_a = \int d^3\mathbf{x} J^{\lambda}_{a0}(\mathbf{x},t). \tag{13}$$

It is independent of time since the Noether current is conserved. By use of the canonical commutation relations, one can show that the charges form the Lie algebra,

$$[Q^{\lambda}_a, Q^{\lambda}_b] = if_{abc} Q^{\lambda}_c. \tag{14}$$

The commutation relations of Q^{λ}_a with the fields $\Phi/\Phi^{\dagger}$ form the charge-field algebra,

$$[Q^{\lambda}_a, \Phi_r(x)] = -(T_a)_{rs}\Phi_s(x), \tag{15}$$

and the Hermitian conjugates. It should be emphasized that both Eqs. (14) and (15) are the direct consequences of the canonical commutation relations given by Eq. (11) and therefore valid irrespectively of $L_{\rm int}$. The peculiarity of the

[3] Obviously there are many different ways to approach the gauge limit. For instance, one may let $\lambda \to 1$ with the Lagrangian $L_{\rm tot} = L_0 + L_{\rm int} + (1-\lambda)L_{br}$ where L_{br} is some arbitrarily chosen gauge-breaking interaction. Instead, we have chosen here the specific form $L^{\lambda}_{\rm tot}$ for which the Noether current away from $\lambda = 1$ takes the simple form determined by the free Lagrangian L_0 alone.

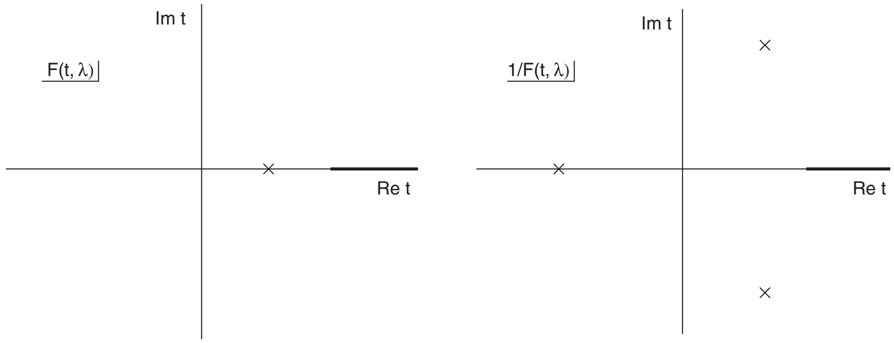

FIG. 1. Analyticity of $F(t,\lambda)$ and $1/F(t,\lambda)$ in the complex t plane. The cross in the left-side figure indicates the pole due to a bound state of $J^{PC}=1^{--}$ for $F(t,\lambda)$. The crosses in the right-side figure are due to possible poles of $1/F(t,\lambda)$, that is, zeros of $F(t,\lambda)$.

matter gauge theories to be emphasized here is that the Noether charge *operator* Q_a^λ vanishes in the gauge-symmetry limit according to Eq. (12).

Now here comes the key point. One might notice that something does not look quite right about Eqs. (14) and (15) at least superficially. Let us take the matrix elements of both sides of Eq. (15), for instance. When the charge Q_a^λ is expressed with the Noether current as written in the second line of Eq. (12), it looks as if its matrix elements were always proportional to $(1-\lambda)$. If so, when they are substituted in Eq. (15), the left-hand side would be infinitesimally small like $(1-\lambda)$ near $\lambda=1$. On the other hand, the matrix element of the right-hand side does not vanish at $\lambda=1$. The same superficial inconsistency appears as $(1-\lambda)^2$ vs $(1-\lambda)$ from Eq. (14) too. How should we answer to this question?

There is no computational error here. The fact that charge operator Q^λ is proportional to $(1-\lambda)$ is a manifestation of the absence of the Noether current in the gauge-invariant theories that consist only of matter fields. Then, how can the charge-field commutation relation of Eq. (15) hold near $\lambda=1$?

We shall find that this is the place where the formation of the composite gauge bosons enters and solves the puzzle. By examining the form factor of the Noether current in the following section, we shall find that a composite vector bound state is formed in the channel of $J_{a\mu}^\lambda$, and therefore that the matrix element of $i(\Phi^\dagger T_a \overleftrightarrow{\partial}_\mu \Phi)$ at zero momentum transfer turns out to be proportional to $1/(1-\lambda)$ and compensates the factor $(1-\lambda)$ in front of the operator $(\Phi^\dagger T_a \overleftrightarrow{\partial}_\mu \Phi)$.

D. Dispersion relation for the form factor of the Noether current

To study the consistency of the powers of $(1-\lambda)$, we need to examine the matrix elements for both sides of Eq. (15) between the vacuum $\langle 0|$ and the one-particle state $|\mathbf{p}\rangle$, in particular, the one-particle matrix element of $J_{a\mu}^\lambda$ near the zero momentum-transfer limit.

We define the Lorentz-scalar form factor $F(t,\lambda)$ by separating $(1-\lambda)$ from $J_{a\mu}^\lambda$ as

$$\begin{aligned}\frac{1}{1-\lambda}\langle \mathbf{p}',s|J_{a\mu}^\lambda(0)|\mathbf{p},r\rangle &= \langle \mathbf{p}',s|i(\Phi^\dagger T_a \overleftrightarrow{\partial}_\mu \Phi)|\mathbf{p},r\rangle \\ &= \sqrt{\frac{1}{4E_{\mathbf{p}'}E_{\mathbf{p}}}}(p'+p)_\mu (T_a)_{sr} F(t,\lambda), \qquad (16)\end{aligned}$$

where the variable t is the invariant momentum transfer $t=(p'-p)^2$. Even after the factor $(1-\lambda)$ is removed from the Noether current, the form factor $F(t,\lambda)$ still depends on λ. This λ dependence comes from the multiple interactions of $L_{\rm int}^\lambda$ of Eq. (6), which is implicit in the Heisenberg operator $i(\Phi^\dagger T_a \overleftrightarrow{\partial}_\mu \Phi)$, as we have already pointed out.

Analyticity of the function $F(t,\lambda)$ is well known. $F(t,\lambda)$ is analytic in the variable t with the branch points on the positive real axis of the complex t plane. The lowest branch point t_0 is located at the invariant mass squared of the lowest two-particle threshold. If there is a bound state of $J^{PC}=1^{--}$ with mass $m_{\rm bound}$, the function $F(t,\lambda)$ has a simple pole at $m_{\rm bound}^2$ below t_0 and, barring a tachyon, above $t=0$ for $\lambda\neq 1$. (See the left-side figure in Fig. 1.)

The inverse of the form factor $1/F(t,\lambda)$ possesses the cuts at the same locations as $F(t,\lambda)$, but a bound-state pole of $F(t,\lambda)$ becomes a zero of $1/F(t,\lambda)$ and therefore does not generate a singularity. The dispersion relation for $1/F(t,\lambda)$ therefore takes the form of[4]

[4]If $F(t,\lambda)$ has a zero, it turns into a pole of $1/F(t,\lambda)$, which would have to be taken into account in writing the dispersion relation for $1/F(t,\lambda)$. Such zeros can appear, in general, on the real axis of t and/or pairwise symmetrically above and below the real axis because of the relation $F(t,\lambda)^*=F(t^*,\lambda)$, where the asterisk indicates a complex conjugate. But a zero does not appear for $F(t,\lambda)$ at $t=0$. The reason for $F(0,\lambda)\neq 0$ is that $(1-\lambda)F(0,\lambda)$ is equal to the nonvanishing charge of the global symmetry for $\lambda\neq 1$, which must be nonzero.

$$\frac{1}{F(t,\lambda)} = \frac{1}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{Im}(1/F(t',\lambda))}{t'-t-i\epsilon}dt' + \sum_i \frac{c_i(\lambda)}{t_i(\lambda)-t} + c_0(\lambda), \tag{17}$$

where $t_i(\lambda)$'s $(i = 1, 2, \cdots)$ are the locations of zeros of $F(t,\lambda)$ and $c_i(\lambda)$'s are constants independent of t with $c_0(\lambda) = 1/F(\infty,\lambda)$. We are interested in the formation of a composite vector boson with small mass ($\rightarrow 0$ as $\lambda \rightarrow 1$), that is, a zero of $1/F(t,\lambda)$ on the positive real axis in the neighborhood of $t = 0$. Given Eq. (17), we can expand $1/F(t,\lambda)$ in the Taylor series in t in the neighborhood of $t = 0$ as

$$\frac{1}{F(t,\lambda)} = a_0(\lambda) + a_1(\lambda)t + O(t^2), \qquad (\lambda \neq 1), \tag{18}$$

where $a_0(\lambda)$ and $a_1(\lambda)$ are some real finite constants that may depend on λ. Having expressed the behavior of $1/F(t,\lambda)$ in the form of Eq. (18), we are ready to prove the theorem.

IV. PROOF OF THEOREM

We take the matrix element of Eq. (15) between the vacuum $\langle 0|$ and the one-matter-particle state $|\mathbf{p}, s\rangle$, and insert a complete set of states $\sum |n\rangle\langle n|$ between Q_a^λ and $\Phi(x)$. Since Q_a^λ is a generator of a Lie group, only the one-particle state that belongs to the same representation as $|\mathbf{p}, s\rangle$ survives in the sum. We use Eq. (16) to express $\langle \mathbf{p}, s|Q_a^\lambda|\mathbf{p}, r\rangle$ in terms of the form factor. We also use the relations,

$$\langle 0|\Phi_r(x)|\mathbf{p}, s\rangle = \sqrt{\frac{1}{2E_\mathbf{p}}}\sqrt{Z_2}\delta_{rs}e^{-ipx},$$
$$\langle 0|Q_a^\lambda = 0, \tag{19}$$

where Z_2 is the wave-function renormalization of the matter particle ($0 < Z_2 < 1$). It should be emphasized that Eq. (19) is valid to all orders of interaction. After factoring out the group-theory coefficients and $\sqrt{Z_2}$, we are simply left with

$$(1-\lambda)F(0,\lambda) = 1, \tag{20}$$

or

$$F(0,\lambda) = \frac{1}{1-\lambda}. \tag{21}$$

This is what the charge-field algebra imposes on the form factor $F(t,\lambda)$ at $t = 0$. Since the charge-field algebra is just as fundamental as quantum field theory itself, the form factor $F(t,\lambda)$ must obey Eq. (21) no matter what the interaction of matter particles may be.

How can the form factor of $i(\Phi^\dagger T_a \overleftrightarrow{\partial}_\mu \Phi)$ satisfy Eq. (21)? There must be some dynamical reason for it. The only possibility allowed by analyticity is that a bound state is present in this channel with the mass square proportional to $(1-\lambda)$ so that $F(t,\lambda) \sim 1/(m^2_{\mathrm{bound}} - t)$ near $t = 0$. No other possibility exists according to the behavior of the form factor allowed by analyticity.

When we compare Eq. (21) with Eq. (18), namely, the expansion of $1/F(t,\lambda)$ near $t = 0$, we obtain

$$a_0(\lambda) = 1-\lambda, \tag{22}$$

therefore,

$$\frac{1}{F(t,\lambda)} = (1-\lambda) + a_1(\lambda)t + O(t^2). \tag{23}$$

The coefficient $a_1(\lambda)$ cannot be determined by the group theory alone. Equation (23) means that $F(t,\lambda)$ has a dynamical pole at

$$t = -\frac{1-\lambda}{a_1(\lambda)}. \tag{24}$$

We call this pole *dynamical* since it is not an artifact due to a definition or a kinematical choice of amplitude. The value of $a_1(\lambda)$ that determines the location of the pole depends not only on λ but also on details of the binding force. Therefore, this pole in t possesses all the properties of a physical bound state. It ought to be a composite vector meson.

Analyticity of the form factor follows from local field theory. With the help of analyticity, the charge-field algebra thus requires that a bound state be formed in the channel of $J^{PC} = 1^{--}$ with the mass squared proportional to $(1-\lambda)$. When this happens, the multiplicative factor $1-\lambda$ of the charge operator Q_a^λ coming from the Noether current is canceled by the dynamical factor $1/(1-\lambda)$ due to the bound-state pole $\sim 1/(m^2_{\mathrm{bound}} - t)$ in $F(t,\lambda)$, where $m^2_{\mathrm{bound}} \propto (1-\lambda)$. There is no other possibility. The puzzle is thus solved and the proof has been completed.

It should be pointed out that the crucial relation Eq. (21) for our proof can also be obtained in the form of $[(1-\lambda)F(0,\lambda)]^2 = (1-\lambda)F(0,\lambda)$ by taking the one-particle expectation value for both sides of the charge algebra Eq. (14).

We add a few remarks before closing this short section.

The preceding argument gives us one interesting by-product: Although the local Noether current vanishes in the gauge limit, the conserved Noether charge can still be defined for the matter particles through the limiting value $\lim_{\lambda\rightarrow 1}(1-\lambda)F(0,\lambda)$. The value of this charge is equal to what we would naively assign as the global charge to the matter particle. It is reassuring that we still have the global Noether charge as the conserved quantum number in the gauge-symmetry limit even though the Noether current operator itself disappears.

Existence of the non-Abelian Noether charges as the limiting values has no conflict with the Weinberg-Witten theorem. To rule out the non-Abelian gauge-boson formation by the Weinberg-Witten theorem, we must have a Lorentz-covariant conserved current *density* that is capable of transferring spatial momentum [7]. In the gauge theories that consist only of matter fields, such a local current density does not exist in the gauge-symmetry limit. Therefore the global charge as defined above does not interfere with the Weinberg-Witten theorem.

Once a set of massless vector-bound states are formed in a gauge-invariant theory, these bosons ought to be the gauge bosons of the underlying Lie group. The argument leading to this conclusion is, in short, that there is no other way known in field theory to accommodate such massless vector bosons in conformity with the gauge symmetry built in the total Lagrangian. When the couplings of higher dimension are included, perturbative renormalizability does not hold in the space-time dimension of four. Nonetheless, when they are written in terms of effective gauge fields, all interactions up to the dimension four are exactly the same as in the standard renormalizable gauge theory. The couplings of the higher dimension for the matter fields can be combined and cast into gauge-invariant combinations with the effective vector gauge fields. The explicit demonstration was given through diagram computation of the higher-dimensional couplings up to the dimension six in the $1/N$ expansion of the known Abelian and non-Abelian models [4].

V. DIAGRAMMATIC STUDY

The proof of our theorem is complete in the preceding section. Nothing needs to be added mathematically. Since the proof does not refer to any specific group property of the matter fields or their interactions, the theorem should hold for all non-Abelian gauge theories of boson matter. Nonetheless, it is reassuring to see that the bound-state pole is indeed generated in the form factor and that the pole migrates with the value of parameter λ in the way as we have asserted. It will help us to envision the theorem in terms of diagrams since the diagrams often give us better or more intuitive understanding of physics.

For diagrammatic demonstration, we choose the SU(2)-doublet model and make the large N expansion. Except for keeping the leading $1/N$ terms, the diagrammatic calculation below makes no approximation. To work in the large N expansion, we introduce the N doublets of matter. The interaction Lagrangian of the SU(2) gauge symmetry is given by [3,4]

$$L^{\lambda}_{\rm int} = \lambda \frac{(\sum_i \Phi_i^\dagger \tau_a \overleftrightarrow{\partial}_\mu \Phi_i)(\sum_j \Phi_j^\dagger \tau_a \overleftrightarrow{\partial}^\mu \Phi_j)}{4\sum_k \Phi_k^\dagger \Phi_k}, \tag{25}$$

where the summations over i, j, and k run from 1 to N. When the free Lagrangian of Φ and $\Phi^\dagger$ is added to this $L^{\lambda}_{\rm int}$,

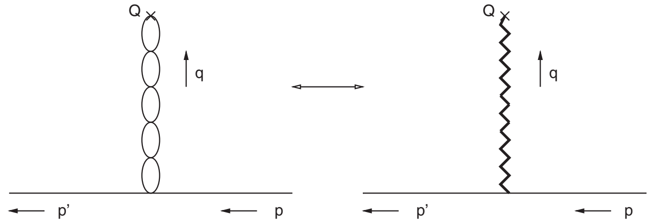

FIG. 2. The form factor $F(t,\lambda)$ in the leading $1/N$ order ($t = q^2$). Each bubble in the left-side figure gives the function $K(t)$ in Eq. (27) and its iteration generates a vector-bound state in the right-side figure.

the total Lagrangian $L_0 + L^{\lambda}_{\rm int}$ is SU(2) gauge invariant at $\lambda = 1$. When the value of λ is in a right range, this interaction generates an SU(2) triplet of bound states in the channel of $J^{PC} = 1^{--}$. In the gauge-symmetry limit, the force is just right to make the bound states exactly massless in the leading $1/N$ order.[5]

When we perform the diagram calculation, we express the denominator of Eq. (25) in sum of its vacuum expectation value and normal-ordered product and expand it around the vacuum expectation value in the power series of the normal-ordered terms [4],

$$L^{\lambda}_{\rm int} = \lambda \frac{(\sum_i \Phi_i^\dagger \tau_a \overleftrightarrow{\partial}_\mu \Phi_i)(\sum_j \Phi_j^\dagger \tau_a \overleftrightarrow{\partial}^\mu \Phi_j)}{4\sum_k \langle 0|\Phi_k^\dagger \Phi_k|0\rangle} \times \sum_{l=0} (-1)^l \left(\frac{\sum_k :\Phi_k^\dagger \Phi_k:}{\sum_k \langle 0|\Phi_k^\dagger \Phi_k|0\rangle} \right)^l, \tag{26}$$

where $:\Phi^\dagger\Phi:$ denotes the normal-ordered product of $\Phi^\dagger\Phi$. To obtain the form factor $F(t,\lambda)$ of $i(\Phi^\dagger \frac{1}{2}\tau_a \overleftrightarrow{\partial}_\mu \Phi)$ defined in Eq. (16), we follow the leading $1/N$ computation of the two-body scattering amplitude performed in Ref. [4]. It amounts to an iteration of the bubble diagrams, as shown in Fig. 2.

After the group-theory coefficients have been factored out, the form factor $F(t,\lambda)$ is obtained as the solution of the simple algebraic equation

$$F(t,\lambda) = 1 + K(t)F(t,\lambda), \tag{27}$$

where $K(t)$ comes from the single bubble in the left-side figure of Fig. 2. Since we are interested in $F(t,\lambda)$ near $t = 0$, we need $K(t)$ also near $t = 0$ in Eq. (27). We carry out the loop integral of the bubble with the dimensional regularization to preserve gauge invariance. The result is

[5]We should remark here that the form of $L_{\rm int}$ appears to be unique up to the addition of terms that are gauge invariant by themselves, e.g., globally invariant nonderivative interactions. It is easy to show that such nonderivative interactions do not affect the composite gauge-boson mass nor coupling in the leading $1/N$ order [4].

$$K(t) = \lambda\left(1 + (1 - D/2)\frac{t}{6m^2}\right) + O(t^2), \tag{28}$$

where m is the matter-particle mass and D is the space-time dimension. With this function $K(t)$, the inverse form factor is given by

$$\frac{1}{F(t,\lambda)} = (1-\lambda) - \lambda\frac{(1-D/2)t}{6m^2} + O(t^2). \tag{29}$$

This form of $1/F(t,\lambda)$ clearly shows that a vector-boson pole exists in $F(t,\lambda)$ and that the pole goes to zero as $\lambda \to 1$. By comparing Eq. (29) with the coefficients defined in Eq. (18) in the preceding section, we find

$$a_0(\lambda) = 1 - \lambda,$$
$$a_1(\lambda) = -\lambda(1 - D/2)/6m^2. \tag{30}$$

The coefficient $a_0(\lambda) = 1 - \lambda$ agrees with what we have obtained in Eq. (22) in the preceding section. This is no surprise since it is a requirement of the Noether charge being the generator of the global symmetry group off $\lambda = 1$. The coefficient $a_1(\lambda)$ determines the location of the bound-state pole m^2_{bound} as a function of λ and the matter-particle mass m. As we expect, the location of the pole reaches zero as we approach the gauge-symmetry limit, $\lambda \to 1$:

$$m^2_{\mathrm{bound}} = \frac{6(1-\lambda)}{\lambda(1-D/2)}m^2. \tag{31}$$

This exercise in the SU(2) model illustrates how our theorem works. While the Noether current operator disappears like $(1-\lambda)$ as we approach the gauge limit, the location of the bound-state pole converges to zero so as to cancel this $(1-\lambda)$ factor with $1/m^2_{\mathrm{bound}} \propto 1/(1-\lambda)$ at $t = 0$.

The diagrammatic exercise presented here indicates that up to a proportionality constant the Noether current acts like a composite vector-boson field V_μ whose mass turns to zero in the gauge limit. This may recall some theorists of the field-current identity of Kroll, Lee, and Zumino [8] that identified the gauge current of hadrons with the (massive) gauge field. They attempted to equate the electromagnetic current J^{EM}_μ to the $\rho^\circ - \omega$ or $\rho^\circ - \omega - \phi$ field up to a scale factor; $J^{\mathrm{EM}}_\mu = fV^{\rho-\omega}_\mu$. But there is a fundamental difference. Being massive, the ρ°/ω mesons are not gauge bosons of the flavor $\mathrm{SU}(2) \times \mathrm{U}(1)$. The photon being composite was not their option. Our passing remark here is only that if one lets m^2_ρ, $m^2_\omega \to 0$ in the field-current identity, such a limit has some resemblance to our matter gauge models.

Although the SU(2) matter model was shown to produce the gauge bosons as bound states in the leading order of $1/N$ expansion [4], going beyond this order in the diagram calculation is nearly impossible because of the complexity of the nonleading orders. However, now that our theorem has been proved, the gauge-boson generation is correct to all orders of the $1/N$ expansion; that is, there is no need to do higher-order diagram calculation. This is one place where the power of our theorem should be appreciated.

We make one closing remark for this section. Our proof turns out to be extremely simple primarily because the charge operator Q^λ_a connects a one-particle state only to another one-particle state that belongs to the same multiplet. This would not be the case if the momentum transfer $\mathbf{q}$ is nonvanishing across the current. The spatial Fourier components $Q^\lambda_a(\mathbf{q},t)$ of the charge density $J^\lambda_{a0}(\mathbf{x},t)$ do not form a finite algebra:

$$[Q^\lambda_a(\mathbf{q},t), \Phi(\mathbf{q}',t)] = -T_a\Phi(\mathbf{q}+\mathbf{q}',t). \tag{32}$$

When we insert a complete set of states $\sum|n\rangle\langle n|$ between $Q^\lambda_a(\mathbf{q},t)$ and $\Phi(\mathbf{q}',t)$, all multiparticle states also contribute as long as their quantum numbers are right. In this case, the one-particle matrix element $\langle\mathbf{p}'|Q^\lambda_a(\mathbf{q},t)|\mathbf{p}\rangle \sim 1/(m^2_{\mathrm{bound}} + |\mathbf{q}|^2)$ vanishes like $(1-\lambda)$ as $\lambda \to 1$ since $\mathbf{q}^2 \neq 0$. Then, comparing the matrix elements on both sides of Eq. (32), it may look as if our power dependence argument of $(1-\lambda)$ would fail like $(1-\lambda)$ vs 1 since the one-particle state no longer provides $1/(1-\lambda)$ in the left-hand side. In this case, however, multiparticle states in $\sum|n\rangle\langle n|$ contribute as well without a constraint of energy conservation.[6] In particular, the composite vector boson enters the continuum and its polarization sum generates the mass singularity $\sim(-g_{\mu\nu} + k_\mu k_\nu/m^2_{\mathrm{bound}})$ through its longitudinal polarization [11]. This mass singularity would be canceled out if the vector-boson mass is generated by spontaneous symmetry breaking [12,13] and if the matrix elements are a set of physically observable scattering amplitudes. Since our matrix elements satisfy neither conditions, it ought to happen that the mass singularity proportional to $1/(1-\lambda)$ of the light vector composite survives and restores consistency in the $(1-\lambda)$ powers. We do not attempt a computation of the mass singularities here.

VI. U(1) GAUGE THEORIES

We can repeat our argument made for the non-Abelian theories and show that the theorem works for the U(1) gauge theories as well. Since the U(1) Noether current also vanishes in the gauge limit, we approach the U(1) gauge-symmetry limit by multiplying the same parameter λ on L_{int} as we have done. To avoid arbitrariness in the overall U(1) charge scale, we define the Noether current as

[6]We end up with a sum rule which involves a continuum of states all the way up to infinite energies. Some examples using the charge density algebra are found in Ref. [9]. See also Ref. [10].

$$J^\lambda_\mu = -i\frac{\partial L^\lambda}{\partial^\mu\Phi}\Phi + i\Phi^\dagger\frac{\partial L^\lambda}{\partial^\mu\Phi^\dagger},$$
$$= i(1-\lambda)(\Phi^\dagger\overset{\leftrightarrow}{\partial}_\mu\Phi),$$
$$Q^\lambda = \int J^\lambda_0(\mathbf{x},t)d^3\mathbf{x}. \tag{33}$$

Just as in the non-Abelian case, the factor $(1-\lambda)$ does not appear in J^λ_0 when we express it by use of $\Pi/\Pi^\dagger$;

$$J^\lambda_0 = i(\Phi^\dagger\Pi^\dagger - \Pi\Phi). \tag{34}$$

Consequently the charge-field commutation relation does not have an explicit dependence on $(1-\lambda)$;

$$[Q^\lambda, \Phi(\mathbf{x},t)] = -\Phi(\mathbf{x},t), \tag{35}$$

in spite that $Q^\lambda = i(1-\lambda)\int(\Phi^\dagger\overset{\leftrightarrow}{\partial}_0\Phi)d^3\mathbf{x}$.

We take the matrix element between the vacuum $\langle 0|$ and the one-particle state $|\mathbf{p}\rangle$ for both sides of Eq. (35). When we insert a complete set of states $\sum|n\rangle\langle n|$ between the Q^λ and $\Phi(\mathbf{x},t)$, we are immediately led to

$$\langle\mathbf{p}|Q^\lambda|\mathbf{p}\rangle = 1. \tag{36}$$

The reasoning goes from here exactly as in the non-Abelian case: When $\langle\mathbf{p}'|Q^\lambda|\mathbf{p}\rangle$ is written as $(1-\lambda)F(t,\lambda)$ with the form factor $F(t,\lambda)$ of the Heisenberg operator $i(\Phi^\dagger\overset{\leftrightarrow}{\partial}_\mu\Phi)$, Eq. (36) requires that the function $F(t,\lambda)$ must behave like

$$F(t,\lambda) \to \frac{1}{1-\lambda} + O(t) \tag{37}$$

near $\lambda = 1$ in the neighborhood of $t = 0$. This is realized only if $F(t,\lambda)$ has a bound-state pole, $\mu^2/(m^2_{\rm bound} - t)$, on the real axis in the complex t plane and if $m^2_{\rm bound}$ reaches zero at $\lambda \to 1$ as $m^2_{\rm bound} = \mu^2(1-\lambda)$.

VII. FERMION MATTER

The Noether theorem is based on the invariance of the Lagrangian under the phase rotation of fields. Therefore, whether fields are canonically independent or not, the conserved Noether current consists of all the fields that enter the Lagrangian,

$$J_{a\mu} = -i\frac{\partial L}{\partial(\partial^\mu\Psi)}T_a\Psi + i\Psi^\dagger T_a\frac{\partial L}{\partial(\partial^\mu\Psi^\dagger)}. \tag{38}$$

If we want to treat Ψ and $\Psi^\dagger$ on the equal footing, we may choose the free Lagrangian in the form

$$L_0 = \frac{i}{2}\overline{\Psi}\overset{\leftrightarrow}{\partial\!\!\!/}\Psi - m\overline{\Psi}\Psi, \tag{39}$$

by adding a total divergence term. With $L^\lambda_{\rm int}$ added to this L_0, it may look trivial to repeat our proof for the boson matter to prove the theorem for the fermion matter. But it is not the case.

If we formally defined the conjugate momentum by $\Pi = \partial L/\partial(\partial_0\Psi)$ with $L_0 + L^\lambda_{\rm int}$ and similarly for $\Pi^\dagger$, the Noether charge density would take the form of

$$J^\lambda_{a0} = i(\Psi^\dagger T_a\Pi^\dagger - \Pi T_a\Psi), \tag{40}$$

where $T_a = \frac{1}{2}\tau_a$ for the SU(2) doublet and $T_a \to 1$ for a unit U(1) charge. If we blindly imposed the canonical anticommutation relations by treating (Ψ, Π, $\Psi^\dagger$, $\Pi^\dagger$) as all independent of each other, it looks that we would obtain the charge-field algebra at equal time,

$$[Q^\lambda_a, \Psi] = -T_a\Psi \tag{41}$$

and its Hermitian conjugate just as in the case of bosons. Then, with Eq. (41), our proof for the boson models would apply to the fermion models with no modification. However, we encounter one problem: This naive derivation of Eq. (41) is incorrect although the final result is most likely correct. There is a subtlety special to the canonical formalism of the Dirac field [14–18].

The problem arises from the fact that the Lagrangian of the Dirac field is linear in the time derivative and therefore that only two of those four variables above can be treated as canonically independent. For instance, if one chooses Ψ and Π as independent variables, $\Psi^\dagger$ and $\Pi^\dagger$ are functions of Ψ and Π. This turns the equal-time anticommutator $\{\Psi, \Psi^\dagger\}_+$ nontrivial and dependent on the interaction, in general.

In the matter gauge theories, the interaction $L_{\rm int}$ contains the derivatives of field in order to counterbalance the gauge variation of the free Lagrangian L_0. In a such case, unlike the Dirac field interacting with a nonderivative interaction, we do not have an option of setting $\Pi^\dagger = 0$ by choosing L_0 asymmetric in Φ and $\Phi^\dagger$. Consequently the equal-time anticommutator between Ψ and $\Psi^\dagger$ may become dependent on $L_{\rm int}$ in general. Although the prescription to determine the anticommutators has been known when this happens, one has to go through cumbersome steps. The canonical quantization is thus not best suited for our purpose in the case of the Dirac field since we would have to check each model one by one to make sure that the algebra of Eq. (41) is indeed valid for a given interaction.

In some cases we can circumvent this procedure. For instance, in the known model of the U(1) symmetry [4], we can remove the time derivative of $\Psi^\dagger$ entirely and realize $\Pi^\dagger = 0$ by an appropriate rewriting of the Lagrangian. Then the independent canonical variables are only Ψ and Π, and they obey the simple equal-time anticommutator $\{\Psi, \Pi\}_+ = i\delta(\mathbf{x}-\mathbf{y})$. It is interesting to note that in this case Π turns out to be twice as large as what we would obtain formally by ignoring the interdependency of the variables. Since the Noether charge is given by a single term $J_0^\lambda = -i\Pi\Psi$ in the case of $\Pi^\dagger = 0$, the correct charge-field algebra $[Q^\lambda, \Psi] = -\Psi$ immediately follows in the same form as that for the bosons. We shall describe in Appendix C how it works for the U(1) model.

In the case of the boson matter the charge-field algebra is an immediate consequence of the canonical quantization. In contrast, its derivation through the canonical quantization requires some knowledge of the interaction in advance in the case of the Dirac field. Our goal is to prove the theorem as generally as possible without referring to specific properties of the interaction or without knowing the interaction at all. For this purpose, it is desirable to derive the charge-field algebra of Eq. (41) in a way that does not rely on the canonical quantization.

In fact, a line of argument can be made to advocate the validity of the charge-field algebra irrespectively of the interaction. It goes as follows: The charge-field algebra of Eq. (41) is obtained as the $O(\alpha)$ terms of the global symmetry rotation of the fields by angle α,

$$e^{-iQ\alpha}\Psi(x)e^{iQ\alpha} = e^{i\alpha}\Psi(x) \tag{42}$$

for the field of a unit U(1) charge. For non-Abelian symmetries, Q and α should be modified appropriately by attaching relevant group-component indices. Then, going from Eq. (42) backward, ask what kind of operator the Q can be. The operator Q must be a space-time independent Lorentz-scalar since the symmetry at $\lambda \neq 1$ is global but unbroken. The operator Q is dimensionless and has a negative charge parity since it generates a phase of the opposite sign for $\Psi^\dagger$ as $\Psi^\dagger e^{-i\alpha}$. The only possible candidate for Q is a charge of some conserved vector current J_μ. Up to an overall proportionality constant, therefore, this current ought to be the Noether current that arises from the phase rotation of the fields. It is the only candidate that we have at hand. The Noether current has the right scale of proportionality constant since its scale is fixed by Eq. (42), which corresponds to the rotation per a unit angle of α. This argument is a little wordy, but it is almost equally as good as the derivation based on the canonical quantization. It works for the boson matter too.

Once Eq. (41) has been accepted in one way or another, we can repeat what we have done for the boson matter. We define the electric and magnetic form factors in the standard way as

$$\begin{aligned}\frac{1}{1-\lambda}&\langle\mathbf{p}'|J^\lambda_{a\mu}(0)|\mathbf{p}\rangle\\ &= \langle\mathbf{p}'|\overline{\Psi}T_a\gamma_\mu\Psi|\mathbf{p}\rangle\\ &= \sqrt{\frac{m^2}{E_{\mathbf{p}'}E_{\mathbf{p}}}}\bar{u}_{p'}T_a\left(\gamma_\mu F_1(t,\lambda) + \frac{i\sigma_{\mu\nu}q^\nu}{2m}F_2(t,\lambda)\right)u_p,\end{aligned} \tag{43}$$

where we have suppressed the indices for spins, copies, and multiplet components of the fermion. Compare the one-particle matrix elements for both sides of the charge algebra equation (41) near $\lambda = 1$. The consistency in the power of $(1-\lambda)$ on both sides requires that the electric form factor $F_1(t,\lambda)$ must obey

$$F_1(0,\lambda) = \frac{1}{1-\lambda}. \tag{44}$$

It means the existence of a pole of the composite gauge boson in $F_1(t,\lambda)$ at $t = m^2_{\mathrm{bound}} \propto (1-\lambda)$. The magnetic form factor $F_2(t,\lambda)$ does not enter the $(q_\mu = 0)$ limit because of the kinematical factor $i\sigma_{\mu\nu}q^\nu$.

Our proof ought to hold for any SU(2) multiplet other than the doublet and for any group higher than SU(2) as well, *if such a model is built*.

The diagrammatic demonstration is a little less simple for the fermion matter since two channels $^3S_1$ and $^3D_1$ couple to form the vector bound state [4]. But it is no more than a small technical complication.

VIII. SUMMARY AND DISCUSSION

We can realize gauge invariance without introducing a fundamental vector gauge field of any kind. In order to connect between the matter fields at separate space-time points in such theories, the interaction Lagrangian must be carefully concocted by including the derivatives of matter fields. In this paper we have proved that such matter interactions inevitably generate composite gauge bosons.

The proof is based on the following three properties:

(1) Most importantly, the Noether current vanishes in the gauge-symmetry limit of such theories.
(2) The equal-time charge-field algebra holds in the Heisenberg picture.
(3) The form factor of current obeys the well-established analyticity.

In our proof we have started with a globally invariant but not locally invariant theory ($\lambda \neq 1$) and then have approached the gauge symmetry by continuously varying the value of parameter λ. When we follow this path to the gauge symmetry, consistency of the charge-field algebra requires that a bound state must be present in the channel of $J^{PC} = 1^{--}$ and turn massless in the gauge-symmetry limit. The proof has been given step by step in detail for the non-Abelian gauge theories of the boson matter. The proof has been trivially extended to the Abelian theories. The theorem holds for the fermion matter as well. But we have cautioned about the issue that we encounter if we rely on the canonical quantization of the Dirac field. Our proof is valid to all orders of interactions since the theorem has been proved in the Heisenberg picture.

This theorem gives us another way to understand why the composite state of $J^{CP} = 1^{--}$ cannot be massless if the

Noether current exists: Because, if a massless bound state were formed in the presence of the nonvanishing Noether current, it would lead to the inconsistency $O(1/(1-\lambda)) = O(1)$ as $\lambda \to 1$ in the charge-field algebra. This observation applies to the Abelian theories equally well, while the theorem of Weinberg and Witten [7] is limited to the non-Abelian theories.

The gauge-boson formation was proved in the past only in the leading $1/N$ order of the perturbative diagram calculation [2,4]. Now we have no need to attempt the higher-order perturbative calculation. With our theorem, the gauge-boson formation is valid to all orders. This is certainly one significant advancement. If someone succeeds in writing a matter gauge Lagrangian with a higher symmetry or with a multiplet of a higher representation within SU(2), our theorem guarantees that such a theory must have composite gauge bosons before they are shown by diagrammatic computation. This is the main advancement.

Looking forward, some may ask how useful or relevant our theorem will be to phenomenology of particle physics. It is natural to wonder whether one can introduce in one way or another the idea of the composite gauge bosons into the standard model in the flat space-time of dimension four. At present, we have one obvious problem of group theory in doing so. That is, the non-Abelian models have been built only with the SU(2)-doublet matter particles. This is sufficient for the minimal electroweak interaction of $\mathrm{SU}(2)\times\mathrm{U}(1)$. But what shall we do about the composite gluons? Is the so-far unsuccessful attempt to build a matter gauge theory beyond the SU(2) doublet only for a technical reason or for a more fundamental reason? In the past we saw a few cases in which physics cannot be extended beyond SU(2). One is the G parity ($G = C\exp[iT_2\pi]$) of low-energy hadron physics. We know why it cannot. Another is the instanton solution of the non-Abelian gauge theory [19]. This is because of the winding number arising from mapping of the SU(2) solution onto the sphere $\mathbf{S}^3$ of the four-dimensional space-time. Recall that the QCD instanton is no more than the SU(2) instantons embedded into the SU(3) parameter space. In our case, unlike the instanton, there seems to be nothing topological. In the no-Abelian models so far invented, the special property of $\frac{1}{2}\tau_a$ for the SU(2) doublet plays a crucial role. If an extension is possible beyond the SU(2) doublet, it appears that we shall need a very different approach to model building.

Once we have proved the formation of composite gauge bosons, it is not necessary every time to go back to the original matter Lagrangian as far as the gauge-boson interactions of dimension four are concerned. An obvious question is how to handle the effective interactions of dimensions higher than four. This is the place where we expect to see difference between the elementary gauge bosons and the composite ones phenomenologically. It is too early to speculate on it.

ACKNOWLEDGMENTS

The author thanks Professor K. Bardakci for helpful discussions on issues related to the quantization of the constrained system. This work was supported by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

APPENDIX A: NOETHER CURRENT

We show that the Noether current is identically zero in gauge theories which consist only of matter fields [4]. Since this is the basis of our theorem, we reiterate it in the simplest way. We choose the non-Abelian gauge theory of boson matter as an example. An extension to fermion matter involves only minor modifications due to spins and anticommutativity.

Gauge invariance of the action of the total Lagrangian L_{tot} requires to the first order in $\alpha_a(x)$,

$$\partial^\mu\left(\frac{\partial L}{\partial(\partial^\mu\Phi)}T_a\Phi - \Phi^\dagger T_a\frac{\partial L}{\partial(\partial^\mu\Phi^\dagger)}\right)\alpha_a + \left(\frac{\partial L}{\partial(\partial^\mu\Phi)}T_a\Phi - \Phi^\dagger T_a\frac{\partial L}{\partial(\partial^\mu\Phi^\dagger)}\right)\partial^\mu\alpha_a + 0(\alpha^2) = 0, \tag{A1}$$

where the equation of motion has been used in the first term as usual. Since α_a are arbitrary functions of x^μ, the terms proportional to α_a and $\partial_\mu\alpha_a$ must vanish separately in Eq. (A1). The terms proportional to α_a allow us to define the Noether current J^μ_a and lead us to its conservation:

$$J_{a\mu} \equiv -i\frac{\partial L}{\partial(\partial^\mu\Phi)}T_a\Phi + i\Phi^\dagger T_a\frac{\partial L}{\partial(\partial^\mu\Phi^\dagger)}, \tag{A2}$$

$$\partial^\mu J_{a\mu} = 0. \tag{A3}$$

Then the requirement that the terms proportional to $\partial^\mu\alpha_a$ be zero in Eq. (A1) is nothing other than the vanishing of the Noether current:

$$J_{a\mu} = 0. \tag{A4}$$

When L_{int} is multiplied with λ and turned into L^λ_{int},

$$L_{\mathrm{int}} \to \lambda L_{\mathrm{int}} \equiv L^\lambda_{\mathrm{int}}, \tag{A5}$$

it breaks gauge invariance of the total Lagrangian $L^\lambda_{\mathrm{tot}} \equiv L_0 + \lambda L_{\mathrm{int}}$ so that the Noether current $J_{a\mu}$ no longer vanishes for $\lambda \neq 1$. However, we do not need an explicit form of L_{int} to obtain the Noether current for $\lambda \neq 1$ since the variation of L_{int} is determined by that of the free Lagrangian

L_0 alone through gauge invariance of $L_0 + L_{\rm int}$. To obtain the Noether current in this case, split the Lagrangian as

$$L^{\lambda}_{\rm tot} = (1-\lambda)L_0 + \lambda(L_0 + L_{\rm int}). \tag{A6}$$

The second term does not contribute to the Noether current since it is gauge invariant. The Noether current arises only from the first term and takes the form of $(1-\lambda)$ times the Noether current due to L_0;

$$J^{\lambda}_{a\mu} = i(1-\lambda)(\Phi^{\dagger}T_a\overset{\leftrightarrow}{\partial}_{\mu}\Phi). \tag{A7}$$

APPENDIX B: EFFECT OF INTERACTION IN EQUAL-TIME ALGEBRAS

The equal-time algebras of the charge Q^{λ}_a are free of an explicit dependence on the factor $(1-\lambda)$. It is because this factor does not appear in Q^{λ}_a when it is written in terms of Π and $\Pi^{\dagger}$ instead of $\partial_0\Phi$ and $\partial_0\Phi^{\dagger}$. The purposes of Appendix B is to show how the charge density acquires the factor $(1-\lambda)$ when we switch from Π and $\Pi^{\dagger}$ to $\partial_0\Phi$ and $\partial_0\Phi^{\dagger}$, but that neither Π nor $\Pi^{\dagger}$ vanishes individually as $\lambda\to 1$.

We go back to the canonical quantization rule of quantum mechanics in the Heisenberg picture, $[q_i, p_j] = i\delta_{ij}$, and make the correspondence $q_i(t)\to\Phi(\mathbf{x},t)$ and $p_i(t)\to\Pi(\mathbf{x},t) = \partial L_{\rm tot}/\partial(\partial_0\Phi(\mathbf{x},t))$. According to the standard quantization rule, a pair of the canonical "coordinate" and "momentum" obeys the equal-time commutation relation,

$$[\Phi(\mathbf{x},t), \Pi(\mathbf{y},t)] = i\delta(\mathbf{x}-\mathbf{y}), \tag{B1}$$

and so forth. The unit matrices are to be understood in the right-hand side of Eq. (B1) with respect to the components of the group indices, the copies and so forth.

According to Eq. (A2), the charge density can be expressed as

$$J^{\lambda}_{a0} = i(\Phi^{\dagger}T_a\Pi^{\dagger} - \Pi T_a\Phi). \tag{B2}$$

A factor of $(1-\lambda)$ does not appear in the right-hand side of Eq. (B2). Consequently, the celebrated equal-time algebra of the charge densities results [9] as

$$[J^{\lambda}_{a0}(\mathbf{x},t), J^{\lambda}_{b0}(\mathbf{y},t)] = if_{abc}J^{\lambda}_{c0}(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y}) \tag{B3}$$

without $(1-\lambda)$. Similarly

$$[J^{\lambda}_{a0}(\mathbf{x},t), \Phi(\mathbf{y},t)] = -T_a\Phi(\mathbf{y},t)\delta(\mathbf{x}-\mathbf{y}). \tag{B4}$$

When the Noether charge is written with $\partial_0\Phi$ and $\partial_0\Phi^{\dagger}$ instead of Π and $\Pi^{\dagger}$, the factor of $(1-\lambda)$ appears. But this does not mean that Π and $\Pi^{\dagger}$ are proportional to $(1-\lambda)$. It is interesting to see in the known model how the factor $(1-\lambda)$ appears in the charge density upon switching from Π and $\Pi^{\dagger}$ to $\partial_0\Phi$ and $\partial_0\Phi^{\dagger}$.

Take the SU(2)-doublet model [4] as an example. The interaction is given by

$$L_{\rm int} = \lambda\frac{(\Phi^{\dagger}\tau_a\overset{\leftrightarrow}{\partial}{}^{\mu}\Phi)(\Phi^{\dagger}\tau_a\overset{\leftrightarrow}{\partial}_{\mu}\Phi)}{4(\Phi^{\dagger}\Phi)}. \tag{B5}$$

The momenta conjugate to Φ and $\Phi^{\dagger}$ are given by

$$\Pi = \frac{\partial L^{\lambda}_{\rm tot}}{\partial(\partial_0\Phi)} = \partial_0\Phi^{\dagger} + \lambda\frac{(\Phi^{\dagger}\tau_a\overset{\leftrightarrow}{\partial}_0\Phi)}{2(\Phi^{\dagger}\Phi)}\Phi^{\dagger}\tau_a, \tag{B6}$$

and its Hermitian conjugate, respectively. Notice that neither Π nor $\Pi^{\dagger}$ vanishes as $\lambda\to 1$. However, taking the combination of $\Phi^{\dagger}\tau_a\Pi^{\dagger} - \Pi\tau_a\Phi$ and using $[\tau_a, \tau_b] = 2\delta_{ab}$, we obtain

$$i\left(\Phi^{\dagger}\frac{\tau_a}{2}\Pi^{\dagger} - \Pi\frac{\tau_a}{2}\Phi\right) = (1-\lambda)\left(\Phi^{\dagger}\frac{\tau_a}{2}\overset{\leftrightarrow}{\partial}_0\Phi\right). \tag{B7}$$

Dependence on the interaction enters the Noether current through Π and $\Pi^{\dagger}$. However, in the combination of $(\Phi^{\dagger}\frac{1}{2}\tau_a\Pi^{\dagger} - \Pi\frac{1}{2}\tau_a\Phi)$, the contribution of the interaction turns out to be simply λ times $(\Phi^{\dagger}\frac{1}{2}\tau_a\overset{\leftrightarrow}{\partial}_0\Phi)$ with a minus sign.

APPENDIX C: CANONICAL QUANTIZATION OF THE DIRAC FIELD

The complication in the canonical quantization of the Dirac field is due to the fact that the Lagrangian is linear in the time derivative and therefore the Hermitian conjugate field $\Psi^{\dagger}$ is no longer canonically independent of (Ψ, Π) after Ψ and Π are chosen as the canonical variables. This is an example of the so-called constrained dynamical systems [14–18,20].

Let us first recall the free Dirac field. When we choose the Lagrangian in the asymmetric form,

$$L_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi, \tag{C1}$$

we obtain $\Pi = \partial L_0/\partial(\partial_0\Psi) = i\Psi^{\dagger}$ and impose $\{\Psi, \Pi\}_+ = i\delta(\mathbf{x}-\mathbf{y})$ at equal time. The canonical quantization is complete with this condition since $\Pi^{\dagger} = \partial L/\partial(\partial_0\Psi^{\dagger}) = 0$.

We may add a total divergence term to L_0 and antisymmetrize it with respect to $\partial_{\mu}\Psi$ and $\partial_{\mu}\Psi^{\dagger}$ as

$$L_0 = \frac{i}{2}\overline{\Psi}\overset{\leftrightarrow}{\partial\!\!\!/}\Psi - m\overline{\Psi}\Psi. \tag{C2}$$

In this case we cannot proceed with the naive rule of quantization by treating both Ψ and $\Psi^{\dagger}$ as independent coordinates.

Let us consider the interacting Dirac fields. We can sometimes circumvent the difficulty by modifying $L_{\rm int}$ without changing physics. Consider the U(1) matter model [4] as an example. The interaction is given by

$$L^{\lambda}_{\rm int} = -\frac{i\lambda}{2}\frac{(\overline{\Psi}\gamma_\mu\Psi)(\overline{\Psi}\overleftrightarrow{\partial}^{\mu}\Psi)}{(\overline{\Psi}\Psi)}. \tag{C3}$$

We add a total derivative term,

$$\Delta L^{\lambda}_{\rm int} = -\frac{i\lambda}{2}\partial^\mu((\bar{\Psi}\gamma_\mu\Psi)\log(\bar{\Psi}\Psi)), \tag{C4}$$

to the interaction of Eq. (C3) and turn it into

$$L^{\lambda}_{\rm int} + \Delta L^{\lambda}_{\rm int} = -i\lambda\frac{(\overline{\Psi}\gamma_\mu\Psi)(\overline{\Psi}\partial^{\mu}\Psi)}{(\overline{\Psi}\Psi)}. \tag{C5}$$

Here we have used $\partial^\mu(\bar{\Psi}\gamma_\mu\Psi) = 0$. The purpose of adding $\Delta L^{\lambda}_{\rm int}$ is to remove the term $\partial_0\Psi^\dagger$ from the interaction. Now the total Lagrangian reads

$$L^{\lambda}_{\rm tot} = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - i\lambda\frac{(\overline{\Psi}\gamma_\mu\Psi)(\overline{\Psi}\partial^{\mu}\Psi)}{(\overline{\Psi}\Psi)}. \tag{C6}$$

Since $\Pi^\dagger = \partial L/\partial(\partial_0\Psi^\dagger) = 0$ for this Lagrangian, we can now choose Ψ and Π as canonically independent variables and treat $\Psi^\dagger$ as a trivial dependent variable, i.e., the constraint variable. The variable Π defined by $\Pi = \partial L/\partial(\partial_0\Psi)$ with the Lagrangian of Eq. (C6) turns out to be twice as large as what we would obtain for Π by pretending (Ψ, Π, $\Psi^\dagger$, $\Pi^\dagger$) as all independent in the original Lagrangian. Since the simple canonical quantization relation

$$\{\Psi(\mathbf{x},t), \Pi(\mathbf{y},t)\}_+ = i\delta(\mathbf{x}-\mathbf{y}) \tag{C7}$$

holds, we are led to the desired result, Eq. (41) for $[Q,\Psi]$. Its Hermitian conjugate correctly gives what we want for $[Q,\Psi^\dagger]$.

Alternatively we can choose Ψ and $\Psi^\dagger$, instead of Ψ and Π, as the canonical variables for the original $L^{\lambda}_{\rm tot}$. To do so, we must take account of the interdependency of the variables by making sure that Hamilton's equation of motion should hold correctly. The general prescriptions of this procedure have been discussed in length, but the case of the Lagrangian linear in the time derivative can be presented in a compact mathematical form, which is found, for instance, in Ref. [20].

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