Global $SU(3)_C \times SU(2)_L \times U(1)_Y$ linear sigma model: Axial-vector Ward-Takahashi identities and decoupling of certain heavy BSM particles due to the Goldstone theorem

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In the $SU(2)_L \times SU(2)_R$ linear sigma model with partially conserved axial-vector currents, a tower of Ward-Takahashi identities (WTI) have long been known to give relations among 1-*scalar*-particle-irreducible (1-ϕ-I) Green's functions, and among I-*scalar*-particle-reducible (1-ϕ-R) transition-matrix (T-matrix) elements for external scalars [i.e. the Brout-Englert-Higgs (BEH) scalar H, and three pseudoscalars $\vec{\pi}$]. In this paper, we extend these WTI and the resulting relations to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ linear sigma model including the heaviest generation of Standard Model (SM) fermions—the ungauged (i.e. global) Standard Model $\mathrm{SM}^G_{tb\tau\nu_\tau}$—supplemented with the minimum necessary neutrino content—right-handed neutrinos and Yukawa-coupling-induced Dirac neutrino mass—to obtain the charge-parity (CP)-conserving $\nu_D\mathrm{SM}^G_{tb\tau\nu_\tau}$, and extract powerful constraints on the effective Lagrangian: e.g. showing that they make separate tadpole renormalization unnecessary, and guarantee infrared finiteness. The crucial observation is that ultraviolet quadratic divergences (UVQD), and all other relevant operators, contribute *only* to m^2_π, a *pseudo*-Nambu-Goldstone boson (NGB) mass-squared, which appears in intermediate steps of calculations. A WTI between T-matrix elements (or, in this global theory equivalently the Goldstone theorem) then enforces $m^2_\pi = 0$ exactly for the true NGB in the spontaneous symmetry breaking (SSB) mode of the theory. *The Goldstone theorem thus causes all relevant operator contributions*, originating to all-loop-orders from virtual scalars $H, \vec{\pi}$, quarks $q^c_L; t^c_R; b^c_R$ and leptons $l_L; \nu_{\tau R}; \tau_R$ with ($c = r, w, b$), *to vanish identically.*

We show that our regularization-scheme-independent, WTI-driven results are unchanged by the addition of certain $SU(3)_C \times SU(2)_L \times U(1)_Y$ heavy ($M^2_{\mathrm{Heavy}} \gg |q^2|, m^2_{\mathrm{Weak}}$) CP-conserving matter, such as originate in certain beyond the SM (BSM) models. The global axial-vector WTI again cause all UVQD and finite relevant operators to vanish, in the $\nu_D\mathrm{SM}^G_{tb\tau\nu_\tau+\mathrm{Heavy}}$ model. We demonstrate this with two examples: a singlet $M^2_S \gg m^2_{\mathrm{Weak}}$ real scalar field S with discrete Z_2 symmetry and $\langle S\rangle = 0$; and a singlet right-handed type I see-saw Majorana neutrino ν_R with $M^2_{\nu_R} \gg m^2_{\mathrm{Weak}}$. Specifically, we prove that these heavy degrees of freedom decouple completely from the low-energy $\nu_D\mathrm{SM}^G_{tb\tau\nu_\tau}$ effective Lagrangian, contributing only irrelevant operators after quartic-coupling renormalization.

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I. INTRODUCTION

Ward-Takahashi identities (WTI) are relations among Green's functions or amplitudes of field theories that result from the symmetries of the theory. They exist both in "unbroken" theories (in which the vacuum shares the symmetries of the Lagrangian) and in spontaneously broken theories (in which the vacuum does not share the symmetries of the Lagrangian). In this paper we are concerned specifically with the global $SU(2)_L \times U(1)_Y$ Schwinger [1] linear sigma model (LΣ M), the ungauged scalar sector of the Standard Model (SM), augmented by the third generation of SM fermions with their usual Yukawa couplings to the Higgs doublet (and so augmenting the symmetry with a global $SU(3)_C$ factor), as well as by a right-handed τ neutrino, with its allowed Yukawa couplings. For brevity, we call this global theory the $\nu_D\mathrm{SM}^G_{tb\tau\nu_\tau}$, with G for global and D indicating that the neutrinos have only Dirac masses. With SM isospin and hypercharge quantum numbers for fermions, the third-generation $\nu_D\mathrm{SM}^G_{tb\tau\nu_\tau}$ has zero axial anomaly. We prove here that the charge-parity (CP)-conserving $\nu_D\mathrm{SM}^G_{tb\tau\nu_\tau}$ is governed by axial-vector WTI directly analogous with those proved by B. W. Lee [2] for the $SU(2)_L \times SU(2)_R$ Gell-Mann-Lévy [3] with partially conserved axial-vector currents (PCAC). One of those axial-vector WTIs is equivalent in

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this global theory to the Goldstone theorem [4–6], which protects the mass of the Nambu-Goldstone bosons (NGB) from nonzero contributions.[1]

We also demonstrate that there exists a wide class of heavy matter $M^2_{\rm Heavy} \gg m^2_{\rm Weak}$ particles from which the low-energy effective $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ Lagrangian, fortified by the WTI, is protected. It may be no coincidence that this class includes heavy Majorana masses for right-handed neutrinos, as envisioned in the see-saw models of light neutrinos. Another theory might well have been less effectively protective.

Here we prove properties of the spontaneously broken mode of a quantum field theory with global symmetries that are the rigid versions of the local symmetries of the Standard Model, in anticipation of extending our arguments to the one-generation standard electroweak model itself [10,11]. We discover how the physics of the theory (as embodied in on-shell transition-matrix (T-matrix) elements) is more symmetric than the effective Lagrangian, because consistency conditions on the states constrain the physics. A particularly crucial role is played by a WTI among T-matrix elements, which is equivalent to the Goldstone theorem in this global theory. In upcoming papers we extend these results, first to a $U(1)_Y$ gauge theory, the CP-conserving Abelian Higgs model (AHM) [12], and then to the CP-conserving gauged electroweak SM, with the third generation of quarks, charged leptons, ν_L, ν_R, and Dirac-mass neutrinos, $\nu_D{\rm SM}_{tb\tau\nu_\tau}$ [10,11]. Along the way we discover that the important T-matrix WTI and the Goldstone theorem contain independent information.

The structure of the remainder of this paper is as follows.

Section II concerns the correct (i.e. axial-vector-WTI-obedient) renormalization of the scalar-sector effective $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ Lagrangian in its Goldstone (i.e. spontaneously broken) mode. In this section we treat the $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$, with its SM fermions, augmented by a right-handed neutrino, and consequently a Dirac neutrino mass, as a *stand-alone* flat-space quantum field theory, not embedded or integrated into any higher-scale "beyond the SM" (BSM) physics.

[1]In June 2011 [7] one of us (BWL) introduced these ideas. A December 2011 pedagogical companion paper [8] simplified the treatment of UVQDs in the context of the *global* Gell-Mann-Lévy model [3] with PCAC. In [9] we showed that, what we called the Goldstone theorem, but to be specific is really a WTI equivalent to the Goldstone theorem in this global theory, protects the weak-scale *global* spontaneous symmetry breaking (SSB) $SO(2)$ Schwinger model [1] (i.e. against 1-loop relevant operators $\sim M^2_{\rm Heavy} \gg m^2_{\rm Weak}$ which arise from virtual heavy particles) by way of 2 explicit 1-loop examples: a real singlet scalar S and a singlet Majorana neutrino ν_R with $M_S^2, M^2_{\nu_R} \gg |q^2|, \langle H\rangle^2$.

Section III extends our results to $M^2_{\rm Heavy} \gg m^2_{\rm BEH}$, the mass-squared of the Brout-Englert-Higgs particle a.k.a. the Higgs boson, heavy $SU(3)_{\rm color} \times SU(2)_L \times U(1)_Y$ matter representations, such as arise in certain BSM models.

Section IV draws a historical lesson.

In the Appendix extends the proof of B. W. Lee [i.e. for the WTI of $SU(2)_L \times SU(2)_R$ Gell-Mann-Lévy with PCAC], to WTI for $SU(3)_C \times SU(2)_L \times U(1)_Y$ $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$—the mathematical basis on which the results of this paper rest.

II. AXIAL-VECTOR-WTI-OBEDIENT RENORMALIZATION OF THE GLOBAL $SU(3)_C \times SU(2)_L \times U(1)_Y$ $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ EFFECTIVE LAGRANGIAN

The global $SU(3)_C \times SU(2)_L \times U(1)_Y$ Lagrangian of SM scalar and third generation fermion fields,[2] extended with a right-handed neutrino with Dirac mass[3] is

$$L_{\nu_D{\rm SM}^G_{tb\tau\nu_\tau}}(\phi; l_L, \tau_R, \nu_{\tau R}; q_L^c, b_R^c, t_R^c; \mu_\phi^2, \lambda_\phi^2; y_b, y_t, y_\tau, y_{\nu_\tau}). \tag{1}$$

$\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ parameters include quadratic and quartic scalar couplings $\mu_\phi^2, \lambda_\phi^2$,[4] and real Yukawa couplings. This Lagrangian conserves CP.

We define a complex BEH doublet representation for the scalars

$$\phi \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} H + i\pi_3 \\ -\pi_2 + i\pi_1 \end{bmatrix}. \tag{2}$$

[2]The ν_DSM matter fields are well-known: a spin $S = 0$ complex scalar doublet ϕ; $S = \frac{1}{2}$ left-handed and right-handed leptons $l_L^i = [\nu_L^i e_L^i]^T, e_R^i, \nu_R^i$, with i running over the three generations; $S = \frac{1}{2}$ left-handed and right-handed quarks $q_L^{i,c} = [d_L^{i,c} u_L^{i,c}]^T, d_R^{i,c}, u_R^{i,c}$, with i running over the three generations, and $c = r, w, b$ over the $SU(3)_C$ color index. Quarks, and separately leptons, have complex Yukawas, Dirac masses and mixings. The observable 3×3 Cabibbo-Kobayashi-Maskawa (CKM) and Pontecorvo—Maki—Nakagawa—Sakata (PMNS) matrices connect the weak eigenstates with mass eigenstates. In order to assure CP conservation, we limit ourselves to one generation of fermions, the third-generation of the Standard Model.

[3]Before the experimental observation of neutrino mixing, the SM was defined to include only left-handed neutrinos ν_L^i. The proof of our new axial-vector WTI's *requires* CP-conservation and that neutrinos be massive. We therefore pay homage to experimentally observed neutrino mixing, and study in this paper the CP-conserving $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$, here defined to include a right-handed neutrino ν_R; and a Dirac mass $m^\nu_{\rm Dirac} = y_\nu\langle H\rangle/\sqrt{2}$.

[4]We follow the language and power counting of the early literature [2], taking the quartic coupling constant to be λ_ϕ^2 rather than the modern [13] λ. Renormalized $\lambda_\phi^2 \geq 0$.

[This can trivially be mapped to an O(4) quartet of real scalars, since $\phi^\dagger\phi = \frac{1}{2}(H^2 + \vec{\pi}^2)$.] We use this manifestly renormalizable linear representation for the scalars in order to control relevant operators.

The Lagrangian (1) has three modes, which we characterize by the values of the renormalized BEH-vacuum-expectation-value (VEV) $\langle H\rangle$ and the renormalized (*pseudo-*)NGB mass-squared m_π^2: $\langle H\rangle = 0, m_\pi^2 > 0$, known as the "Wigner mode"; $\langle H\rangle = m_\pi^2 = 0$, the classically scale-invariant point; and $\langle H\rangle \neq 0, m_\pi^2 = 0$, the spontaneously broken or "Goldstone mode."

This paper distinguishes carefully between the global $SU(3)_{\rm color}\times SU(2)_L\times U(1)_Y$ Lagrangian of (a single generation of) SM matter fields [i.e. (1)] and the $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ itself: i.e. the $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ is the "Goldstone mode" of (1).

(1) *Symmetric* $\langle H\rangle = 0, m_\pi^2 \neq 0$ *Wigner mode*: This is analogous with the Schwinger-model x-axis Fig. 12-12 in the textbook by C. Itzykson and J-C. Zuber [14] and the similar Fig. 1 in [8]. The analysis and renormalization of Wigner mode (e.g. its infrared structure with massless fermions) is outside the scope of this paper. Thankfully, nature is not in Wigner mode. Our Universe is, instead, in the SSB Goldstone mode.

(2) *Classically scale-invariant point* $\langle H\rangle = 0, m_\pi^2 = 0$ is analogous with the Schwinger-model origin in Fig. 12-12 in the textbook by C. Itzykson and J-C. Zuber [14] and Fig. 1 in [8]. The analysis and renormalization of the classically scale-invariant point is outside the scope of this paper.

(3) *Spontaneously broken* $\langle H\rangle \neq 0, m_\pi^2 = 0$ *Goldstone mode*: The $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ is the Goldstone mode of $L_{\nu_d{\rm SM}^G}$ in (1). The "physics" is the spectrum of physical particles—$S = 0$ bosons, and the third generation of $S = \frac{1}{2}$ Dirac-massive quarks and leptons—and their associated dynamics.

The one-generation *global* $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ is invariant under global $SU(3)_C \times SU(2)_L \times U(1)_Y$ transformations and conserves *CP*. Global *axial-vector* WTI therefore govern the dynamics of the ϕ-sector (i.e. BEH scalar H and three pseudoscalars $\vec{\pi}$) of the all-loop-orders renormalized $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ effective Lagrangian.[5] There are two sets of such *CP*-conserving axial-vector WTI. One set governs relations among connected amputated 1-$(h,\vec{\pi})$ scalar-particle-irreducible (1-ϕ-I) Green's functions. A separate set governs relations among connected amputated 1-$(h,\vec{\pi})$ scalar-particle-reducible (1-ϕ-R) T-matrix elements. As observed by Lee for the Gell-Mann-Lévy model,[6] one of those T-matrix WTI is equivalent to the Goldstone theorem in this global theory.

We use "pion-pole dominance" arguments to derive these axial-vector WTIs for the SSB $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ in the Appendix, and so rely on the masslessness of the NGB in Goldstone mode. In this global theory (although not in the gauge theories that we consider in [10–12]) that masslessness translates precisely into the masslessness of the pseudoscalar boson $m_\pi^2 = 0$ when $\langle H\rangle \neq 0$. $L_{\nu_d{\rm SM}^G_{tb\tau\nu_\tau}}|_{\langle H\rangle\neq 0}$ is the subject of the remainder of this section.

These global axial-vector WTI for the $\nu_D{\rm SM}^{\rm G}_{tb\tau\nu_\tau}$ are a generalization of the classic work of B. W. Lee [2], who constructed the all-loop-orders renormalized tower of quantum WTIs for the $SU(2)_L \times SU(2)_R$ Gell-Mann-Lévy (GML) model [3] with partially conserved axial-vector currents (PCAC). We replace GML's strongly interacting LΣM with a weakly interacting BEH LΣM: $\sigma \to H, \vec{\pi} \to \vec{\pi}, m_\sigma \to m_h, f_\pi \to \langle H\rangle$; we eliminate the explicit symmetry breaking of PCAC ($\gamma = 0$), and reduce the symmetry from $SU(2)_L \times SU(2)_R$ to $SU(2)_L \times U(1)_Y$ when we add SM fermions and their attendant Yukawa couplings. We also introduce a quark $SU(3)_C$, so that the resultant generation has SM couplings, which ensures that our WTI have zero axial anomaly.

[5]Our interest in axial-vector WTI may be surprising, given that while $SU(2)_L$ is a symmetry of the Lagrangian, $SU(2)_{L-R}$ is not. This interest is justified by our insistence on *CP* conservation, as described in detail in the Appendix. In future work, we will consider the interesting consequences for our WTI, and for physics, of small amounts of *CP* violation.

[6]Reference [8] used B. W. Lee's Gell-Mann-Lévy (GML) WTI to construct the all-loop-orders renormalized low-energy $(|q^2|, \langle H\rangle^2, m^2_{\rm Weak} \ll$Euclidean UV cutoff Λ^2), effective GML Lagrangian including UV quadratic divergences (UVQD). The $SU(2)_L \times SU(2)_R$ $(\frac{1}{2},\frac{1}{2})$ representation is $\Phi \equiv \frac{1}{\sqrt{2}}[H + i\vec{\sigma}\cdot\vec{\pi}]$, while the $SU(2)_L \times U(1)_Y$ doublet in this paper is $\phi \equiv \Phi[\begin{smallmatrix}1\\0\end{smallmatrix}]$. Including all $\mathcal{O}(\Lambda^2), \mathcal{O}(\ln\Lambda^2)$ divergences,

$$\begin{aligned} L^{\rm Eff;All\text{-}loops;}_{\rm GML} &= \frac{1}{2}{\rm Tr}|\partial_\mu\Phi|^2 - V^{\rm Eff}_{\rm GML} V^{\rm Eff;All\text{-}loops}_{\rm GML} \\ &= \frac{\lambda_\phi^2}{4}\left[H^2 + \vec{\pi}^2 - \left(\langle H\rangle^2 - \frac{m_\pi^2}{\lambda_\phi^2}\right)\right]^2 - \langle H\rangle m_\pi^2 H \\ &\quad + \mathcal{O}^{\rm GML}_{\rm Ignore}, \end{aligned} \tag{3}$$

causes tadpoles to vanish identically, so that separate tadpole renormalization is unnecessary. $\mathcal{O}^{\rm GML}_{\rm Ignore}$ denotes finite operators that do not contribute to UVQD,

$$\mathcal{O}^{\rm GML}_{\rm Ignore} = \mathcal{O}^{\rm GML}_{D>4} + \mathcal{O}^{\rm GML}_{D\leq 4;\rm Non\ Analytic} + \mathcal{O}^{\rm GML}_{1/\Lambda^2;\rm Irrelevant}.$$

The effective potential (3) reduces to the three effective potentials of the Schwinger model [1] as: $\langle H\rangle \to 0, m_\pi^2 \neq 0$ (Schwinger Wigner mode); $\langle H\rangle \to 0, m_\pi^2 \to 0$ (Schwinger scale-invariant point); or $\langle H\rangle \neq 0, m_\pi^2 \to 0$ (Schwinger Goldstone mode). Reference [8] extended (3) to include SM quarks and leptons, but possible IR divergences, due to massless SM neutrinos, were out of scope and ignored.

A. Axial-vector Ward-Takahashi identities in $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$

We focus on the global isospin axial-vector current $\vec{J}^{\mu}_{L-R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}$. The global color $SU(3)_C$, $SU(2)_{L+R}$ and electromagnetic currents are vector currents and are not spontaneously broken, so they do not yield further WTI information of interest to this paper. In the Appendix, we describe how CP conservation enables us to consider amplitudes of the axial-vector current, and derive towers of WTI. In future work, we will consider the generalization to the case where CP is violated.

Because we are interested in global-symmetric relations among 1-scalar-particle-irreducible (1-ϕ-I) connected amputated Green's functions (GF) with external ϕ scalars, it is convenient to use tools (e.g. canonical quantization) from vintage quantum field theory (V-QFT), a name coined by Ergin Sezgin. Analysis is done in terms of the exact renormalized interacting $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ fields, which asymptotically become the in/out states, i.e. free fields for physical scattering-matrix (S-matrix) elements. In the Appendix gives details of the derivation of our rigid axial-vector WTIs, some highlights of which we present in this section.

For $\langle H\rangle \neq 0$, the pseudoscalars $\vec{\pi}$ are massless.[7] We therefore solve/obey the axial-vector "pion-pole-dominance" T matrix (which recall is related to the better-known S matrix by $S = 1 + iT$) identity proved in the Appendix,

$$\langle H\rangle T_2^{t_1\ldots t_M t}(p_1\ldots p_N; q_1\ldots q_M 0) = \sum_{m=1}^{M} \delta^{t t_m} T^{t_1\ldots\widehat{t_m}\ldots t_M}(p_1\ldots p_N q_m; q_1\ldots\widehat{q_m}\ldots q_M) - \sum_{n=1}^{N} T^{t_1\ldots t_M t}(p_1\ldots\hat{p}_n\ldots p_N; q_1\ldots q_M p_n) \quad (5)$$

with N renormalized $h = H - \langle H\rangle$ external legs (coordinates x, momenta p), and M renormalized $(CP = -1)$ $\vec{\pi}$ external legs (coordinates y, momenta q, isospin t).

Equation (5) relates either T-matrix elements all with even numbers of $\vec{\pi}$ (if M is odd), or T-matrix elements all with odd numbers of $\vec{\pi}$ (if M is even). Because CP is conserved, T-matrix elements with odd numbers of $\vec{\pi}$ vanish, hence (5) is of interest only for M odd.

Here $T \equiv T_1 + T_2$. T_1 includes only diagrams with an extra zero-momentum external leg $\vec{\pi}$, attached directly to an external h or another external $\vec{\pi}$ leg as in Fig. 1. The notation $\widehat{p_n}, \widehat{q_m}, \widehat{t_m}$ indicates that "hatted" external fields and momenta are to be omitted.

A tower of quantum WTI recursion relations, among renormalized connected amputated 1-scalar-particle-irreducible (1-ϕ-I) Green's functions (GF) $\Gamma^{t_1\ldots t_M}_{N,M}(p_1\ldots p_N; q_1\ldots q_M)$, with N external renormalized $h = H - \langle H\rangle$ (coordinates x, momenta p), and M external $(CP = -1)$ renormalized $\vec{\pi}$ (coordinates y, momenta q, isospin t), is shown in the Appendix to be a solution to the T-matrix identity (5). The resulting WTI relate a 1-ϕ-I connected amputated GF with $(N + M + 1)$ external fields, including an extra zero-momentum $\vec{\pi}$, to two 1-ϕ-I amputated GFs with $(N + M)$ external fields. For $\vec{\pi}$ with $CP = -1$, the result

$$\langle H\rangle \Gamma^{t_1\ldots t_M t}_{N,M+1}(p_1\ldots p_N; q_1\ldots q_M 0) = \sum_{m=1}^{M} \delta^{t t_m} \Gamma^{t_1\ldots\hat{t}_m\ldots t_M}_{N+1,M-1}(p_1\ldots p_N q_m; q_1\ldots\hat{q}_m\ldots q_M) - \sum_{n=1}^{N} \Gamma^{t_1\ldots t_M t}_{N-1,M+1}(p_1\ldots\hat{p}_n\ldots p_N; q_1\ldots q_M p_n) \quad (6)$$

is valid for $N, M \geq 0$, though nontrivial only for odd M. (Hatted quantities are again omitted.)

We form the ϕ-sector effective Lagrangian as a sum

$$L_{\phi}^{\mathrm{Eff};\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} = \sum_{N,M} L_{\phi;N,M}^{\mathrm{Eff};\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} \quad (7)$$

over all possible numbers of external scalars h and pseudoscalars π^i. Each term, $L_{\phi;N,M}^{\mathrm{Eff};\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}$ is obtained by attaching to $\Gamma^{t_1\ldots t_M}_{N,M}$: N appropriate external scalar wave functions; M appropriate external pseudoscalar wave

[7]The masslessness of $\vec{\pi}$, $m_\pi^2 = 0$, in Goldstone mode, is closely related to the masslessness of the Nambu-Goldstone bosons of the broken global symmetry in this ungauged theory. To identify the NGBs, we must pass from the linear representation (2) to the unitary Kibble representation [13,15], with transformed fields $\tilde{H}$ and $\tilde{\vec{\pi}}$, and VEVs $\langle\tilde{H}\rangle = \langle H\rangle$ and $\langle\tilde{\vec{\pi}}\rangle = 0$,

$$\phi = \frac{1}{\sqrt{2}}\tilde{H}U; \qquad U \equiv \exp\left[i\frac{\vec{\sigma}\cdot\tilde{\vec{\pi}}}{\langle H\rangle}\right]\begin{bmatrix}1\\0\end{bmatrix}. \quad (4)$$

$\tilde{\vec{\pi}}$ (not $\vec{\pi}$) are the purely derivatively coupled NGBs. We note that it is the ability to transform to the unitary representation that makes $\tilde{\vec{\pi}}$ derivatively coupled. The transformation (4) is not possible in Wigner mode nor at the scale-invariant point. In the ungauged theory it is also possible to add a Polkinghorne PCAC term γH that explicitly violates the axial symmetry. In [8] we follow [2] in considering the $|\langle H\rangle|$ vs. m_π^2 quarter plane, in which the Wigner mode is the x-axis ($\langle H\rangle = 0$), the Goldstone mode is the y-axis ($m_\pi^2 = 0$), the scale-invariant point is the origin, and the symmetry is explicitly broken off these axes. Lee [2] points out a remarkable WTI: $\gamma = \langle H\rangle m_\pi^2$. The Goldstone mode and the Wigner mode are thus just the $m_\pi^2 \to 0$ and $\langle H\rangle \to 0$ limits of the explicitly broken theory. In the quarter plane, the transformation to the Kibble representation finds that $\tilde{\vec{\pi}}$ has nonderivative couplings, including a mass, proportional to m_π^2. These all vanish in the $m_\pi^2 \to 0$ limit, and the Goldstone theorem (among other WTIs) is recovered. This connection between $m_\pi^2 = 0$ and $m_{\mathrm{NGB}}^2 = 0$ appears to be severed in the gauge theory where the explicit breaking term is thought to be forbidden by unitarity.

functions, with sums over isospins; and combinatoric factors for identical external boson fields $h, \vec{\pi}$.

It is worth emphasizing that all perturbative quantum loop corrections, to all-loop-orders, are included in the ϕ-sector effective Lagrangian: 1-ϕ-I connected amputated GF $\Gamma^{t_1 \ldots t_M}_{N,M}(p_1 \ldots p_N; q_1 \ldots q_M)$ in (6); wave-function renormalizations; renormalized scalar propagators (9), (10); the BEH VEV $\langle H \rangle$. Equation (6) includes the full set of quantum all-loop-orders from the global $SU(3)_C \times SU(2)_L \times U(1)_Y$ theory, originating in loops containing virtual $\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$: quarks q^c_L, t^c_R, b^c_R and leptons $l_L, \nu_{\tau R}, \tau_R$, with colors $c = r, w, b$; and scalars $h, \vec{\pi}$. Because they arise entirely from global axial-vector WTI, our results are independent of regularization-scheme [16].

There remains, however, one more crucial step. *We must impose all those symmetries of the* 1-ϕ-*R T matrix that are not symmetries of the* 1-ϕ-I *Green's functions* (6) *nor of the complete effective Lagrangian (7).*[8] In particular the $\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$-analogue of the Adler self-consistency conditions [17,18] (see for example [2] p. 37), derived in the Appendix, of which the Goldstone theorem is a special case, ensures the infrared finiteness of the theory for exactly zero pseudoscalar masses, $m^2_\pi = 0$.

B. Construction of SM scalar-sector effective Lagrangian from axial-vector Ward-Takahashi IDs

We want to classify operators arising from $\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ degrees of freedom, and separate the finite operators from the divergent ones. There are finite operators that arise entirely from $\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ degrees of freedom. Although important for computing "physical observables" in the $\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ (e.g. the analogy of the successful 1-loop high-precision Standard Model predictions for the top-quark from Z-pole physics [19,20] in 1984 and the $W^\pm$ mass [20,21] in 1980, as well as the 2-loop BEH mass from Z-pole physics [19,22,23] and the $W^\pm$ mass [21–23]), they are not the point of this paper. We want instead to focus on UVQD, logarithmic UV divergences, and finite relevant operators, to see how they are related by the WTIs. The reader might imagine $\mathcal{O}(\Lambda^2)$ and $\mathcal{O}(\ln \Lambda^2)$ divergences, never taking the limit $\Lambda^2 \to \infty$.

In actuality, there are three classes of *finite* operators in the $\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ that we will ignore:

(i) finite $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{1/\Lambda^2;\mathrm{Irrelevant}}$ vanish as $m^2_{\mathrm{Weak}}/\Lambda^2 \to 0$;

(ii) $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Dim}>4}$ are finite dimension > 4 operators;

(iii) $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Dim}\leq 4;\mathrm{Non\ Analytic}}$ are finite dimension ≤ 4 operators that are nonanalytic in momenta or in a renormalization scale μ^2.

[8] Failing to impose those T-matrix symmetries (e.g. crucially the one that is equivalent to the Goldstone theorem) results in a *mistake*.

Such finite operators appear throughout the axial-vector Ward-Takahashi IDs (6):

(i) $N + M \geq 5$ gives relations among $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{1/\Lambda^2;\mathrm{Irrelevant}}$ and $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Dim}>4}$;

(ii) the left-hand side of (6) for $N + M = 4$ is also $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Dim}>4}$ or $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{1/\Lambda^2;\mathrm{Irrelevant}}$;

(iii) $N + M \leq 4$ operators $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Dim}\leq 4;\mathrm{Non\ Analytic}}$ also appear in those WTI.

All such operators will be ignored below,

$$\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Ignore}} = \mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{1/\Lambda^2;\mathrm{Irrelevant}} + \mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Dim}>4} + \mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Dim}\leq 4;\mathrm{Non\ Analytic}}. \tag{8}$$

Finally, there are $N + M \leq 4$ operators that are analytic in momenta. We expand these in powers of momenta, count the resulting dimension of each term in the operator Taylor series, and ignore $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Dim}>4}$ and $\mathcal{O}^{\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{1/\Lambda^2;\mathrm{Irrelevant}}$ terms.

We seek next to classify the relevant operators, in this case the $\vec{\pi}$ and h inverse propagators (together with tadpoles).

Define the exact renormalized pseudoscalar propagator (no sum on j) in terms of a $\vec{\pi}$ pole, the K*ä*ll*é*n-Lehmann spectral density ρ_π [14,24], and wave-function renormalization. We assume $\vec{\pi}$ decays weakly,

$$\begin{aligned}
\Delta_\pi(q^2) &= -i(2\pi)^2 \langle 0|T[\pi_j(y)\pi_j(0)]|0\rangle|^{\mathrm{Fourier}}_{\mathrm{Transform}} \\
&= \frac{1}{q^2 - m^2_{\pi;\mathrm{Pole}} + i\epsilon} + \int dm^2 \frac{\rho_\pi(m^2)}{q^2 - m^2 + i\epsilon} \\
Z^{-1}_\phi &= 1 + \int dm^2 \rho_\pi(m^2).
\end{aligned} \tag{9}$$

Define similarly the BEH scalar propagator in terms of a BEH scalar pole, the spectral density ρ_{BEH}, and the *same* wave-function renormalization. We assume h also decays weakly and resembles a resonance,

$$\begin{aligned}
\Delta_{\mathrm{BEH}}(q^2) &= -i(2\pi)^2 \langle 0|T[h(x)h(0)]|0\rangle|^{\mathrm{Fourier}}_{\mathrm{Transform}} \\
&= \frac{1}{q^2 - m^2_{h;\mathrm{Pole}} + i\epsilon} + \int dm^2 \frac{\rho_{\mathrm{BEH}}(m^2)}{q^2 - m^2 + i\epsilon} \\
Z^{-1}_\phi &= 1 + \int dm^2 \rho_{\mathrm{BEH}}(m^2) \\
\int dm^2 \rho_\pi(m^2) &= \int dm^2 \rho_{\mathrm{BEH}}(m^2).
\end{aligned} \tag{10}$$

The connected amputated 1-ϕ-I $\vec{\pi}$ and h inverse propagators are

$$\Gamma_{0,2}^{t_1 t_2}(;q,-q) \equiv \delta^{t_1 t_2}\Gamma_{0,2}(;q,-q)$$
$$\Gamma_{0,2}(;q,-q) \equiv [\Delta_\pi(q^2)]^{-1}$$
$$\Gamma_{2,0}(q,-q;) \equiv [\Delta_{\rm BEH}(q^2)]^{-1}. \quad (11)$$

The spectral density parts of the propagators

$$\Delta^{\rm BEH}_{\rm Spectral\,Density}(q^2) = \int dm^2 \frac{\rho_{\rm BEH}(m^2)}{q^2 - m^2 + i\epsilon}$$
$$\Delta^{\vec\pi}_{\rm Spectral\,Density}(q^2) = \int dm^2 \frac{\rho_\pi(m^2)}{q^2 - m^2 + i\epsilon} \quad (12)$$

are clearly finite. From dimensional analysis of (9), (10), the contribution of a state of mass/energy $\sim M_{\rm Heavy}$ to the spectral densities $\rho_\pi(M^2_{\rm Heavy})$ and $\rho_{\rm BEH}(M^2_{\rm Heavy})$, and to $\Delta^{\rm BEH}_{\rm Spectral\,Density}, \Delta^{\vec\pi}_{\rm Spectral\,Density}$, scale as $M^{-2}_{\rm Heavy}$. The Euclidean cutoff therefore contributes only $\sim\frac{1}{\Lambda^2}$.

We now form the all-loop-orders renormalized scalar-sector effective Lagrangian for $(h,\vec\pi)$ with CP $= (1,-1)$

$$\begin{aligned} L^{\rm Eff}_{\phi;\nu_D{\rm SM}^G_{tb\tau\nu_\tau}} &= \Gamma_{1,0}(0;)h + \frac{1}{2!}\Gamma_{2,0}(p,-p;)h^2 \\ &+ \frac{1}{2!}\Gamma^{t_1t_2}_{0,2}(;q,-q)\pi_{t_1}\pi_{t_2} + \frac{1}{3!}\Gamma_{3,0}(000;)h^3 \\ &+ \frac{1}{2!}\Gamma^{t_1t_2}_{1,2}(0;00)h\pi_{t_1}\pi_{t_2} + \frac{1}{4!}\Gamma_{4,0}(0000;)h^4 \\ &+ \frac{1}{2!2!}\Gamma^{t_1t_2}_{2,2}(00;00)h^2\pi_{t_1}\pi_{t_2} \\ &+ \frac{1}{4!}\Gamma^{t_1t_2t_3t_4}_{0,4}(;0000)\pi_{t_1}\pi_{t_2}\pi_{t_3}\pi_{t_4} + \mathcal{O}^{\rm Ignore}_{\nu_D{\rm SM}^G_{tb\tau\nu_\tau}}. \end{aligned} \quad (13)$$

The connected amputated Green's function identities (6) severely constrain the effective Lagrangian (13). For pedagogical clarity, we first separate out the isospin indices

$$\begin{aligned} \Gamma^{t_1t_2}_{0,2}(;q,-q) &\equiv \delta^{t_1t_2}\Gamma_{0,2}(;q,-q), \\ \Gamma^{t_1t_2}_{1,2}(-q;q0) &\equiv \delta^{t_1t_2}\Gamma_{1,2}(-q;q0), \\ \Gamma^{t_1t_2}_{2,2}(00;00) &\equiv \delta^{t_1t_2}\Gamma_{2,2}(00;00), \\ \Gamma^{t_1t_2t_3t_4}_{0,4}(;0000) &\equiv \Gamma_{0,4}(;0000)[\delta^{t_1t_2}\delta^{t_3t_4} + \delta^{t_1t_3}\delta^{t_2t_4} + \delta^{t_1t_4}\delta^{t_2t_3}]. \end{aligned} \quad (14)$$

Itemizing the relevant WTI and their effects on (13), setting momenta to zero except where needed, suppressing the isospin indices, and indicating the finite operators as simply $\mathcal{O}_{\rm Ignore}$:

(i) WTI $N = 0$, $M = 1$

$$\delta^{t_1t_2}\Gamma_{1,0}(q;) = \langle H\rangle\Gamma^{t_1t_2}_{0,2}(;q,-q),$$
$$\Gamma_{1,0}(0;) = \langle H\rangle\Gamma_{0,2}(;00), \quad (15)$$

since no momentum can run into the tadpoles.

(ii) WTI $N = 1$, $M = 1$

$$\begin{aligned} &\delta^{t_1t_2}\Gamma_{2,0}(-q,q;) - \Gamma^{t_1t_2}_{0,2}(;q,-q) \\ &\quad = \langle H\rangle\Gamma^{t_1t_2}_{1,2}(-q;q0), \\ &\Gamma_{2,0}(-q,q;) - \Gamma_{0,2}(;q,-q) \\ &\quad = \langle H\rangle\Gamma_{1,2}(-q;q0) = \langle H\rangle\Gamma_{1,2}(0;00) + \mathcal{O}^{\nu_D{\rm SM}^G}_{\rm Ignore}, \\ &\Gamma_{2,0}(00;) = \Gamma_{0,2}(;00) + \langle H\rangle\Gamma_{1,2}(0;00) \end{aligned} \quad (16)$$

(iii) WTI $N = 2$, $M = 1$

$$\langle H\rangle\Gamma^{t_1t_2}_{2,2}(00;00) = \delta^{t_1t_2}\Gamma_{3,0}(000;) - 2\Gamma^{t_1t_2}_{1,2}(0;00),$$
$$\langle H\rangle\Gamma_{2,2}(00;00) = \Gamma_{3,0}(000;) - 2\Gamma_{1,2}(0;00). \quad (17)$$

(iv) WTI $N = 0$, $M = 3$

$$\begin{aligned} -\langle H\rangle\Gamma^{t_1t_2t_3t_4}_{0,4}(;0000) &= \delta^{t_1t_2}\Gamma^{t_3t_4}_{1,2}(0;00) \\ &\quad + \delta^{t_1t_3}\Gamma^{t_2t_4}_{1,2}(0;00) \\ &\quad + \delta^{t_1t_4}\Gamma^{t_2t_3}_{1,2}(0;00), \\ -\langle H\rangle\Gamma_{0,4}(;0000) &= \Gamma_{1,2}(0;00). \end{aligned} \quad (18)$$

(v) WTI $N = 1$, $M = 3$

$$\begin{aligned} &\delta^{t_1t_2}\Gamma^{t_3t_4}_{2,2}(00;00) + \delta^{t_1t_3}\Gamma^{t_2t_4}_{2,2}(00;00) + \delta^{t_1t_4}\Gamma^{t_2t_3}_{2,2}(00;00) \\ &\quad -\Gamma^{t_1t_2t_3t_4}_{0,4}(;0000) = 0, \\ &\Gamma_{2,2}(00;00) = \Gamma_{0,4}(;0000). \end{aligned} \quad (19)$$

(vi) WTI $N = 3$, $M = 1$

$$-\delta^{t_1t_2}\Gamma_{4,0}(0000;) + 3\Gamma^{t_1t_2}_{2,2}(00;00) = 0,$$
$$-\Gamma_{4,0}(0000;) + 3\Gamma_{2,2}(00;00) = 0. \quad (20)$$

The quadratic and quartic coupling constants are defined in terms of 2-point and 4-point 1-ϕ-I connected amputated GF

$$\Gamma_{0,2}(;00) \equiv -m^2_\pi$$
$$\Gamma_{0,4}(;0000) \equiv -2\lambda^2_\phi. \quad (21)$$

The pseudoscalar and h (BEH) scalar masses are most usefully defined as

$$m_\pi^2 \equiv -\Gamma_{0,2}(;00) = \left[\frac{1}{m^2_{\pi;\text{Pole}}} + \int dm^2 \frac{\rho_\pi(m^2)}{m^2}\right]^{-1}$$

$$m_h^2 \equiv -\Gamma_{2,0}(00;) = \left[\frac{1}{m^2_{h;\text{Pole}}} + \int dm^2 \frac{\rho_{\text{BEH}}(m^2)}{m^2}\right]^{-1}. \quad (22)$$

The third $N=1$, $M=1$ WTI of Eq. (16) can then be rewritten instructively as a mass relation between the BEH h scalar and the three pseudoscalar bosons $\vec{\pi}$

$$m_h^2 = m_\pi^2 + 2\lambda_\phi^2 \langle H\rangle^2, \quad (23)$$

a more familiar form which we have employed in previous papers [7–9].

The all-loop-orders renormalized ϕ-sector effective Lagrangian (13), constrained *only* by those axial-vector WTIs governing Green's functions (6), may be written

$$L^{\text{Wigner;SI;Goldstone}} = L^{\text{Kinetic}} - V^{\text{Wigner;SI;Goldstone}} + \mathcal{O}_{\text{Ignore}}^{\nu_D\text{SM}^{\text{G}}_{tb\tau\nu_\tau}}. \quad (24)$$

The kinetic term incorporates the nontrivial (but finite) wave-function renormalization

$$L^{\text{Kinetic}} = \frac{1}{2}(\Gamma_{0,2}(;p,-p) - \Gamma_{0,2}(;00))h^2 + \frac{1}{2}(\Gamma_{0,2}(;q,-q) - \Gamma_{0,2}(;00))\vec{\pi}^2, \quad (25)$$

with

$$\Gamma_{0,2}(;q,-q) - \Gamma_{0,2}(;00) \sim q^2, \quad (26)$$

while the effective potential

$$V^{\text{Wigner;SI;Goldstone}} = m_\pi^2\left[\frac{h^2+\vec{\pi}^2}{2} + \langle H\rangle h\right] + \lambda_\phi^2\left[\frac{h^2+\vec{\pi}^2}{2} + \langle H\rangle h\right]^2 \quad (27)$$

incorporates all three modes (i.e. Wigner mode, the scale-invariant point (SI) and Goldstone mode) of the Lagrangian (1).[9] The effective potential in (27) becomes in various limits: Wigner mode ($\langle H\rangle = 0; m_\pi^2 = m_h^2 \neq 0$); scale-invariant point

[9]It is instructive, and we argue [9] dangerous, to ignore vacuum energy and rewrite the potential in (27) as

$$V^{\text{Wigner;SI;Goldstone}}_{\nu_D\text{SM}^G} = \lambda_\phi^2\left[\phi^\dagger\phi - \frac{1}{2}\left(\langle H\rangle^2 - \frac{m_\pi^2}{\lambda_\phi^2}\right)\right]^2 \quad (28)$$

using $\frac{h^2+\vec{\pi}^2}{2} + \langle H\rangle h = \phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2$. If one then minimizes $V_\phi^{\text{Wigner;SI;Goldstone}}$ while ignoring the crucial constraint imposed by the Goldstone theorem, (or more precisely by the WTI that is equivalent to the Goldstone theorem in this ungauged theory: see Sec. II C), the resultant (incorrect and unphysical) minimum $\langle H\rangle^2_{FT} \equiv (\langle H\rangle^2 - \frac{m_\pi^2}{\lambda_\phi^2})$ does not distinguish properly between the three modes (32) of (28). At issue is the renormalized

$$m_\pi^2 = \mu^2_{\phi;\text{Bare}} + C_\Lambda\Lambda^2 + C_{\text{BEH}}m^2_{\text{BEH}} + \delta m_\pi^2 + C_{\text{Heavy}}M^2_{\text{Heavy}} + C_{\text{Heavy;ln}}M^2_{\text{Heavy}}\ln(M^2_{\text{Heavy}}) + C_{\text{Heavy;}\Lambda}M^2_{\text{Heavy}}\ln(\Lambda^2) + \lambda_\phi^2\langle H\rangle^2, \quad (29)$$

where the C's are constants, and $m_h^2 = m_\pi^2 + 2\lambda^2\langle H\rangle^2$. It is "fashionable" to simply drop the UVQD term $C_\Lambda\Lambda^2$ in (29), and argue that it is somehow an artifact of dimensional regularization (DR), even though M. J. G. Veltman [25] showed that UVQD *do* appear at 1-loop in the SM and are properly handled by DR's poles at dimension Dim = 2. We keep UVQD. For pedagogical efficiency, we have included in (29) terms with $M^2_{\text{Heavy}} \gg m^2_{\text{Weak}}$, such as might arise in BSM physics (cf. Sec. III). In Wigner mode, where $\langle H\rangle = 0$,

$$m_h^2 = m_\pi^2 \sim \Lambda^2, \qquad M^2_{\text{Heavy}} \gg m^2_{\text{Weak}}. \quad (30)$$

During renormalization of a tree-level weak-scale BEH mass-squared $m^2_{h;\text{Bare}} \sim m^2_{\text{Weak}}$, relevant operators originating in quantum loops appear to "naturally" force the renormalized value up to the heavy scale (30). Wigner mode is therefore quantum-loop unstable, because the heavy scale cannot decouple from the weak scale. Equation (30) is the motivation for much BSM physics, even though our Universe is not in Wigner mode.

In the spontaneously broken Goldstone mode, where $\langle H\rangle \neq 0$, in obedience to a WTI (equivalent to the Goldstone theorem) in Sec. II C below, the bare counterterm $\mu^2_{\phi;\text{Bare}}$ in (29) is defined by

$$m_\pi^2 \equiv 0. \quad (31)$$

We show in Sec. II C that, for constant $\vec{\theta}$, the zero-value in (31) is protected by the NGB shift symmetry

$$\tilde{\vec{\pi}} \to \tilde{\vec{\pi}} + \tilde{\vec{\pi}} \times \vec{\theta} + \langle H\rangle\vec{\theta} + \mathcal{O}(\theta^2). \quad (32)$$

Minimization of (28) violates stationarity of the true minimum at $\langle H\rangle$ [14] and destroys the theory's renormalizability and unitarity, which require that dimensionless wave function-renormalization $\langle H\rangle_{\text{Bare}} = [Z^\phi]^{1/2}\langle H\rangle$ contain no relevant operators [7,14,24]. The crucial observation is that, in obedience to the Goldstone theorem, Renormalized$(\langle H\rangle^2_{\text{Bare}}) \neq \langle H\rangle^2_{FT}$.

($\langle H\rangle=0; m_\pi^2=m_h^2=0$); or Goldstone mode $\langle H\rangle \neq 0$; $m_\pi^2=0; m_h^2\neq 0$);

$$V^{\text{Wigner}} = m_\pi^2\left[\frac{h^2+\vec{\pi}^2}{2}\right] + \lambda_\phi^2\left[\frac{h^2+\vec{\pi}^2}{2}\right]^2$$
$$V^{\text{Scale Invariant}} = \lambda_\phi^2\left[\frac{h^2+\vec{\pi}^2}{2}\right]^2$$
$$V^{\text{Goldstone}} = \lambda_\phi^2\left[\frac{h^2+\vec{\pi}^2}{2}+\langle H\rangle h\right]^2. \tag{33}$$

Equation (33) has exhausted the constraints, on the allowed terms in the ϕ-sector effective Lagrangian, due to those axial-vector WTIs which govern 1-ϕ-I connected amputated Green's functions $\Gamma_{N,M}$. In order to distinguish among the effective potentials in (33), we must turn to those axial-vector WTIs that govern 1-ϕ-R connected amputated T-matrix elements.

C. Infrared finiteness, Goldstone theorem, and automatic tadpole renormalization

"Whether you like it or not, you have to include in the Lagrangian all possible terms consistent with locality and power counting, unless otherwise constrained by Ward identities." Kurt Symanzik, in a 1970 private letter to Raymond Stora [26].

In the Appendix we extend Adler's self-consistency condition [17,18] [originally written for the $SU(2)_L\times SU(2)_R$ Gell-Mann-Lévy model [3]], to the case of the $\nu_D\text{SM}^{\text{G}}_{tb\tau\nu_\tau}$ Lagrangian (1)

$$\lim_{q_\mu\to 0}\langle H\rangle T^{tt_1\ldots t_M}(p_1\ldots p_N;qq_1\ldots q_M)|^{p_1^2=\ldots p_N^2=m_h^2}_{q_1^2=\ldots q_M^2=0}$$
$$\equiv \langle H\rangle T^{tt_1\ldots t_M}(p_1\ldots p_N;0q_1\ldots q_M)|^{p_1^2=\ldots p_N^2=m_h^2}_{q_1^2=\ldots q_M^2=0}$$
$$=0, \tag{34}$$

where, for pedagogical simplicity, we will suppress $M+1$ isospin indices in $T^{tt_1\ldots t_M}(p_1\ldots p_N;qq_1\ldots q_M)$ going forward. The T matrix vanishes as one of the pion momenta goes to zero, provided all other physical scalar particles are on mass-shell. These are "1-soft-pion" theorems [18]. Equation (34) asserts the absence of infrared divergences in the physical-scalar sector in Goldstone mode $\nu_D\text{SM}^{\text{G}}_{tb\tau\nu_\tau}$. "Although individual Feynman diagrams may be IR divergent, those IR divergent parts cancel exactly in each order of perturbation theory. Furthermore, the Goldstone mode amplitude must vanish in the soft-pion limit [2]".

A special case of (34) is the Goldstone theorem itself, or at least equivalent to it—the $N=0$, $M=1$ case of (34) reads

$$\langle H\rangle T_{0,2}(;00)=0, \tag{35}$$

where momentum conservation forces $q_1=0$ (so that $q_1^2=0$). We may write[10] (35) as a further constraint on the 1-ϕ-I connected amputated Green's functions

$$\langle H\rangle\Gamma_{0,2}(;00)\equiv -\langle H\rangle m_\pi^2=0. \tag{36}$$

As described in footnote 7 above, the actual Goldstone theorem states that the mass of the NGB $\tilde{\pi}$ vanishes, where $\tilde{\pi}$ are the angular degrees of freedom in the unitary representation of the Φ field. However, $m_{\tilde{\pi}}^2=0$ if and only if $m_\pi^2=0$ in this global theory.

A crucial effect of the Adler relation (35), together with the $N=0$, $M=1$ Ward-Takahashi Green's function identity (15), is to automatically eliminate tadpoles in (40)

$$\Gamma_{1,0}(0;)=\langle H\rangle\Gamma_{0,2}(;00)=0, \tag{37}$$

so that separate tadpole renormalization is unnecessary.

With $\langle H\rangle\neq 0$, (35) and (36) may be written,

$$-\Gamma_{0,2}(;00)\equiv -[\Delta_\pi(0)]^{-1}\equiv m_\pi^2$$
$$= m_{\pi;\text{Pole}}^2\left[1+m_{\pi;\text{Pole}}^2\int dm^2\frac{\rho_\pi(m^2)}{m^2}\right]^{-1}$$
$$=0. \tag{38}$$

The pole mass of the pseudoscalars $\vec{\pi}$ in the $\nu_D\text{SM}^{\text{G}}_{tb\tau\nu_\tau}$ therefore vanishes exactly

$$m_{\pi;\text{Pole}}^2 = m_\pi^2\left[1-m_\pi^2\int dm^2\frac{\rho_\pi(m^2)}{m^2}\right]^{-1}=0, \tag{39}$$

which is the reason that $\tilde{\vec{\pi}}$ are Nambu-Goldstone bosons.

D. $\nu_D\text{SM}^{\text{G}}_{tb\tau\nu_\tau}$ scalar-sector effective Lagrangian obedient to Goldstone theorem $\langle H\rangle\neq 0, m_\pi^2\equiv 0$

We now rewrite the effective Lagrangian (24) including the constraint from (36), i.e. $m_\pi^2=0$,

$$L^{\text{Eff}}_{\nu_D\text{SM}^{\text{G}}_{tb\tau\nu_\tau}} = L^{\text{Kinetic}}_{\nu_D\text{SM}^{\text{G}}_{tb\tau\nu_\tau}} - V^{\text{Eff;Goldstone}}_{\nu_D\text{SM}^{\text{G}}_{tb\tau\nu_\tau}} + \mathcal{O}^{\nu_D\text{SM}^{\text{G}}_{tb\tau\nu_\tau}}_{\text{Ignore}} \tag{40}$$

[10]Recall that $T_{0,2}$ is 1-P-R, while $\Gamma_{0,2}$ is 1-P-I. Consider the sum of all diagrams contributing to $T_{0,2}(;00)$ to all loops. Each of these diagrams has exactly two (amputated) external legs, both zero four-momentum π's. Attach to either of these external legs a π propagator (at zero four-momentum) $\Delta_\pi(0)$ and a $\Gamma_{0,2}(;00)$. This diagram is also a contribution to $T_{0,2}(;00)$. [Indeed, we can repeat this procedure an arbitrary number of times, and each resulting diagram must again be a contribution to $T_{0,2}(;00)$.] Now $\Delta_\pi(0)$ has a pole at zero four-momentum when $m_\pi^2=0$. so if these contributions to $T_{0,2}(;00)$ are not to diverge, $\Gamma_{0,2}$ must vanish as the external momenta go to zero, so that the product $\Delta_\pi(0)\Gamma_{0,2}(;00)$ does not diverge.

with the all-loop-orders renormalized Goldstone-mode SSB $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ effective potential[11]

$$V^{\mathrm{Eff;Goldstone}}_{\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} = \lambda_\phi^2\left[\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right]^2. \tag{41}$$

Equations (40) and (41) are the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ effective SSB Lagrangian, derived from the global $SU(3)_{\mathrm{color}} \times SU(2)_L \times U(1)_Y$ Lagrangian $L_{\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}$ in (1). It obeys the Goldstone theorem; is minimized at $(H = \langle H\rangle, \vec{\pi} = 0)$; obeys stationarity of that true minimum [14] at $\langle H\rangle$; and preserves the theory's renormalizability and unitarity, which require [2,14,27–29] that dimensionless wave-function renormalization $\langle H\rangle_{\mathrm{Bare}} = Z_\phi^{1/2}\langle H\rangle$ not attract any relevant operators.

With Goldstone mode wave-function renormalization

$$\Gamma_{0,2}(;q,-q) - \Gamma_{0,2}(;00) = q^2 + \mathcal{O}^{\nu_D\mathrm{SM}^G}_{\mathrm{Ignore}}, \tag{42}$$

the *coordinate-space* effective Lagrangian reads

$$\begin{aligned} L^{\mathrm{Eff}}_{\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} &= |\partial_\mu\phi|^2 - \lambda_\phi^2\left[\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right]^2 + \mathcal{O}^{\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Ignore}} \\ &= |\partial_\mu\phi|^2 - \lambda_\phi^2\left[\frac{(h^2+\vec{\pi}^2)}{2} + \langle H\rangle h\right]^2 + \mathcal{O}^{\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}_{\mathrm{Ignore}}. \end{aligned} \tag{43}$$

We conclude Sec. II with a few observations about Eq. (43):

(i) It includes all $\mathcal{O}(\Lambda^2)$, $\mathcal{O}(\ln\Lambda^2)$ and finite terms that arise, to all perturbative loop-orders, in the full $SU(3)_{\mathrm{Color}} \times SU(2)_L \times U(1)_Y$ theory, i.e. due to virtual fermions and scalars.

(ii) $SU(2)_L \times U(1)_Y$ is spontaneously broken.

(iii) The ultraviolet properties of the Goldstone-mode $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ effective potential are analogous with those of the Goldstone mode of the global Schwinger LΣM [1] corresponding to the y-axis of the quarter plane characterizing the Gell-Mann-Lévy LΣM with PCAC, as in Fig. 1 in [8] and Fig. 12-12 in the textbook by C. Itzykson and J-C. Zuber [14].

(iv) All relevant operators [e.g. UVQD $\sim \mathcal{O}(\Lambda^2)$ in the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$] have vanished identically due to the Goldstone theorem.

(v) The $N = M = 1$ WTI

$$\begin{aligned} \Gamma_{2,0}(00;) &= \langle H\rangle\Gamma_{1,2}(0;00) + \Gamma_{0,2}(;00) \\ &= \langle H\rangle\Gamma_{1,2}(0;00) \end{aligned} \tag{44}$$

relates the BEH mass-squared from (43) to the coefficient of the $h\vec{\pi}^2$ vertex, so that

$$m_h^2 \equiv m^2_{\mathrm{BEH}} = 2\lambda_\phi^2\langle H\rangle^2 \tag{45}$$

arises entirely from SSB.

(vi) The observable BEH resonance pole-mass-squared,

$$m^2_{h;\mathrm{Pole}} = 2\lambda_\phi^2\langle H\rangle^2\left[1 - 2\lambda_\phi^2\langle H\rangle^2\int dm^2\frac{\rho_h(m^2)}{m^2 - i\epsilon}\right]^{-1} + \mathcal{O}^{\nu_D SM^G}_{\mathrm{Ignore}}. \tag{46}$$

(vii) $\langle H\rangle = Z_\phi^{-\frac{1}{2}}\langle H\rangle_{\mathrm{Bare}}$ absorbs no relevant operators (i.e. at worst $\sim\ln\Lambda^2$).

(viii) As promised, $\tilde{\vec{\pi}}$ are true NGB. In the unitary Kibble representation [13,15],

$$\begin{aligned} L^{\mathrm{Eff}}_{\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} &= \frac{1}{2}(\partial_\mu\tilde{H})^2 + \frac{1}{4}\tilde{H}^2\mathrm{Tr}[\partial_\mu U^\dagger\partial_\mu U] \\ &\quad - \frac{\lambda_\phi^2}{4}[\tilde{H}^2 - \langle H\rangle^2]^2 + \mathcal{O}^{\nu_D SM^G}_{\mathrm{Ignore}} \\ U &= e^{i\vec{\sigma}\cdot\tilde{\vec{\pi}}/\langle H\rangle}. \end{aligned} \tag{47}$$

(ix) The vanishing of m_π^2 allows $\tilde{\vec{\pi}}$ to have only derivative couplings and therefore possess the required shift symmetry for constant $\vec{\theta}$

$$\tilde{\vec{\pi}} \to \tilde{\vec{\pi}} + \tilde{\vec{\pi}} \times \vec{\theta} + \langle H\rangle\vec{\theta} + \mathcal{O}(\theta^2). \tag{48}$$

III. THE GOLDSTONE THEOREM AND AXIAL-VECTOR WTI CAUSE CERTAIN HEAVY BSM PARTICLES TO DECOUPLE FROM THE LOW-ENERGY $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ SCALAR-SECTOR EFFECTIVE LAGRANGIAN

If the Euclidean cutoff Λ^2 were a true proxy for very heavy BSM particles, we would already be in a position to comment on their decoupling. Unfortunately, although the literature often cites such proxy, it is simply not true. To quote Ergin Sezgin "In order to prove theorems that reveal symmetry-driven results in field theories, one must keep *all* of the terms arising from *all* Feynman graphs, not just a selection of interesting terms from a representative subset of Feynman graphs."

[11]It is not lost on the authors that, since we derived it from *connected* amputated Green's functions (where all vacuum energy and disconnected vacuum bubbles are absorbed into an overall phase, which cancels exactly in the S matrix [14,24]), the vacuum energy in $L_\phi^{\mathrm{Eff};\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}$ in (40) is exactly zero.

A. Criteria for the extension of axial-vector WTIs to include the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$, extended with certain heavy BSM particles

In the Appendix, we derive the T-matrix and Green's function WTI for the case of the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ Lagrangian, extended to include neutrino Dirac masses.

We here derive criteria that anomaly-free beyond the BSM spin $S=0$ scalars Φ, and $S=\frac{1}{2}$ fermions ψ, must obey in order that the axial-vector WTI remain true.

(1) *Begin by focusing on the global $SU(2)_L$ $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ isospin current* and dividing it into vector and axial-vector parts,

$$\begin{aligned}2\vec{J}^{\mu}_{L+R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} &= \vec{\pi}\times\partial^{\mu}\vec{\pi}+\sum_c \bar{q}^c\gamma^{\mu}\vec{t}q^c+\bar{l}\gamma^{\mu}\vec{t}l\\ 2\vec{J}^{\mu}_{L-R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} &= \vec{\pi}\partial^{\mu}H-H\partial^{\mu}\vec{\pi}\\ &\quad+\sum_c \bar{q}^c\gamma^{\mu}\gamma^5\vec{t}q^c+\bar{l}\gamma^{\mu}\gamma^5\vec{t}l\\ \vec{J}^{\mu}_{L;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} &= \vec{J}^{\mu}_{L+R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}+\vec{J}^{\mu}_{L-R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} \end{aligned}\tag{49}$$

with colors $c=r,\ w,\ b$, isospin $\vec{t}=\frac{1}{2}\vec{\sigma}$, Pauli matrices $\vec{\sigma}$.

The classical equations of motion show only that the $SU(2)_L$ isospin current is conserved

$$\partial_{\mu}\vec{J}^{\mu}_{L;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}=0.\tag{50}$$

But in the $\nu\mathrm{SM}^{\mathrm{G}}_{ude\nu}$ studied here, CP is conserved, so that on-shell and off-shell connected amputated T-matrix elements and Green's functions of an odd number of $\vec{\pi}$s and their derivatives are zero. They also vanish for an odd number of $\vec{\pi}$s and fermion bilinears with the isospin quantum numbers of $\vec{\pi}$.

$SU(2)_{L-R}$ is not a subgroup of the $SU(2)_L$ symmetry group, but CP conservation ensures that the global vector current transforms as an even number of $\vec{\pi}$s, while the global axial-vector current transforms as an odd number of $\vec{\pi}$s. Thus, for M even,

$$\begin{aligned}&\langle 0|T[(\vec{J}^{\mu}_{L-R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}(z))\\ &\quad\times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle^{M\mathrm{even}}_{\mathrm{Connected}}=0\\ &\langle 0|T[(\partial_{\mu}\vec{J}^{\mu}_{L-R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}(z))\\ &\quad\times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle^{M\mathrm{even}}_{\mathrm{Connected}}=0.\end{aligned}\tag{51}$$

Meanwhile, (49) and (50) show that, for M odd,

$$\begin{aligned}&\langle 0|T[\partial_{\mu}(\vec{J}^{\mu}_{L-R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}(z))\\ &\quad\times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle^{M\mathrm{odd}}_{\mathrm{Connected}}\\ &=\langle 0|T[\partial_{\mu}(\vec{J}^{\mu}_{L;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}-\vec{J}^{\mu}_{L+R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}})(z)\\ &\quad\times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle^{M\mathrm{odd}}_{\mathrm{Connected}}\\ &=\langle 0|T[(\partial_{\mu}\vec{J}^{\mu}_{L;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}(z))\\ &\quad\times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle^{M\mathrm{odd}}_{\mathrm{Connected}}\\ &=0.\end{aligned}\tag{52}$$

Thus the $SU(2)_{L-R}$ current is "effectively conserved."

Similarly, although $SU(2)_{L+R}$ is not a subgroup of $SU(2)_L$, its current is also effectively conserved for Green's functions and T-matrix elements for all M,

$$\begin{aligned}&\langle 0|T[(\partial_{\mu}\vec{J}^{\mu}_{L+R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}(z))\\ &\quad\times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle_{\mathrm{Connected}}=0.\end{aligned}\tag{53}$$

This paper is based on the effective conservation of $\vec{J}^{\mu}_{L-R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}$ for on-shell and off-shell connected amputated Green's functions and T-matrix elements, in (51) and (52).[12]

(2) *We extend the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ with certain BSM matter particles.* These must carry zero anomaly.

In order to force renormalized connected amplitudes with an odd number of πs to vanish, the new particles Φ,ψ are taken in this paper to conserve CP. Divide the conserved isospin current into axial-vector (i.e. transforming as an odd number of $\vec{\pi}$s under isospin) and vector (i.e. transforming as an even number of $\vec{\pi}$s under isospin) parts.

$$\begin{aligned}&\partial_{\mu}\vec{J}^{\mu}_{L;\mathrm{Total}}\equiv\partial_{\mu}(\vec{J}^{\mu}_{L;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}+\vec{J}^{\mu}_{L;BSM})=0\\ &\vec{J}^{\mu}_{L+R;BSM}+\vec{J}^{\mu}_{L-R;BSM}\equiv\vec{J}^{\mu}_{L;BSM}\\ &\vec{J}^{\mu}_{L-R;\nu_D SM^G}+\vec{J}^{\mu}_{L-R;BSM}\equiv\vec{J}^{\mu}_{L-R;\mathrm{Total}}\\ &\langle 0|T[(\partial_{\mu}\vec{J}^{\mu}_{L-R;\mathrm{Total}}(z))h(x_1)\ldots h(x_N)\\ &\quad\times\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle_{\mathrm{Connected}}=0.\end{aligned}\tag{54}$$

[12] Had CP not been conserved, the vector and axial-vector currents would not have separately been conserved. We look forward to future works to exploring the consequences of soft CP violation on WTIs.

(3) *Canonical quantization is imposed on the exact renormalized fields,* yielding equal-time quantum commutators at space-time points y, z. The BSM axial-vector currents must commute with H and $\vec{\pi}$,

$$\delta(z_0 - y_0)[\vec{J}^0_{L-R;BSM}(z), H(y)] = 0$$
$$\delta(z_0 - y_0)[J^{0;i}_{L-R;BSM}(z), \pi^j(y)] = 0. \tag{55}$$

Only certain BSM matter will obey this condition.

(4) *BSM scalars must have zero VEV.* Only certain BSM matter will obey this condition. Note that Green's functions are then usually 1-BSM scalar-reducible, by cutting a BSM-scalar line.

(5) *Certain surface integrals must vanish:* the Appendix used pion-pole dominance to derive 1-soft-pion theorems, which require that the connected surface integral (56) vanish. In (56) we have N external renormalized $h = H - \langle H \rangle$ (coordinates x, momenta p), M external ($CP = -1$) renormalized $\vec{\pi}$ (coordinates y, momenta q, isospin t). Because CP is conserved, only axial-vector WTI are needed to put the effective Lagrangian into the desired form. We form the surface integral

$$\begin{aligned}
&\lim_{k_\lambda \to 0} \int d^4z e^{ikz} \partial_\mu \langle 0|T[(2\vec{J}^\mu_{L-R;\mathrm{Total}} + \langle H \rangle \partial^\mu \vec{\pi})(z) \\
&\quad \times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle_{\mathrm{Connected}} \\
&= \int d^4z \partial_\mu \langle 0|T[(2\vec{J}^\mu_{L-R;\mathrm{Total}} + \langle H \rangle \partial^\mu \vec{\pi})(z) \\
&\quad \times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle_{\mathrm{Connected}} \\
&= \int_{3-\mathrm{surface}} d^3z \hat{z}_\mu{}^{3-\mathrm{surface}} \\
&\quad \times \langle 0|T[(2\vec{J}^\mu_{L-R;\mathrm{Total}} + \langle H \rangle \partial^\mu \vec{\pi})(z^{3-\mathrm{surface}} \to \infty) \\
&\quad \times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle_{\mathrm{Connected}} \\
&= 0,
\end{aligned} \tag{56}$$

where we have used Stokes theorem, and $\hat{z}_\mu{}^{3-\mathrm{surface}}$ is a unit vector normal to the $3-$surface. The time-ordered-product constrains the $3-$surface to lie on, or inside, the light cone.

At a given point on the surface of a large enough 4-volume $\int d^4z$ (i.e. the volume of all space-time): all fields are asymptotic in-states and out-states, properly quantized as free fields, with each field species orthogonal to the others, and they are evaluated at equal times, making time-ordering unnecessary at ($z^{3-\mathrm{surface}} \to \infty$). Input the global axial-vector current (49) to (56), using $\partial_\mu \langle H \rangle = 0$. The contribution to (56) from $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ vanishes

$$\begin{aligned}
&\int_{3-\mathrm{surface}} d^3z \hat{z}_\mu{}^{3-\mathrm{surface}} \langle 0|T\Bigg[\Bigg(\vec{\pi}\partial^\mu h - h\partial^\mu \vec{\pi} \\
&\quad + \sum_c \bar{q}^c \gamma^\mu \gamma^5 \vec{t} q^c + \bar{l}\gamma^\mu \gamma^5 \vec{t} l\Bigg)(z^{3-\mathrm{surface}} \to \infty) \\
&\quad \times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)\Bigg]|0\rangle_{\mathrm{Connected}} \\
&= 0.
\end{aligned} \tag{57}$$

The first and second terms vanish because the BEH boson h is massive. The third term vanishes because all quarks have nonzero Dirac masses. The 4th term vanishes because all leptons in the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$, *including neutrinos* [30], have nonzero Dirac masses. Propagators connecting massive h, q^c_L, l_L from points on $z^{3-\mathrm{surface}} \to \infty$ to the localized interaction points $(x_1\ldots x_N; y_1\ldots y_M)$, must stay inside the light cone, die off exponentially with mass, and are incapable of carrying information that far.

It is the central observation for "pion-pole-dominance" and this paper, that *this argument fails* for the remaining term in the axial-vector current $2\vec{J}^\mu_{L-R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}}$ in (49).

$$\begin{aligned}
&\int_{2-\mathrm{surface}} d^2z \hat{z}_\mu{}^{2-\mathrm{surface}} \\
&\quad \times \langle 0|T[(-\langle H \rangle \partial^\mu \vec{\pi})(z^{2-\mathrm{surface}} \to \infty) \\
&\quad \times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle_{\mathrm{Connected}} \neq 0.
\end{aligned} \tag{58}$$

$\vec{\pi}$ is massless, capable of carrying (along the light cone) long-ranged pseudoscalar forces out to the $2-$surface ($z^{2-\mathrm{surface}} \to \infty$), i.e. the very ends of the light cone (but not inside it). That masslessness is the basis of our pion-pole-dominance-based axial-vector WTIs which, as derived in the Appendix, give 1-soft-pion theorems, infrared finiteness for $m^2_\pi = 0$, and a "Goldstone theorem."

(6) *In order to include spin $S = 0$ scalar, and $S = \frac{1}{2}$ fermionic, BSM matter representations* in our axial-vector Ward-Takahashi identities, a certain surface integral must vanish.

$$\begin{aligned}
&\int d^4z \partial_\mu \langle 0|T[(\vec{J}^\mu_{L-R;BSM}(z)) \\
&\quad \times h(x_1)\ldots h(x_N)\pi_{t_1}(y_1)\ldots\pi_{t_M}(y_M)]|0\rangle = 0.
\end{aligned} \tag{59}$$

Additional BSM particles must generically be massive, and thus incapable of carrying information to the surface at infinity. They must also have zero vacuum expectation values. Only certain BSM matter will obey this condition.

We examine below the consequences of extending the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ to include certain high-mass-scale $M^2_{\mathrm{Heavy}} \gg m^2_{\mathrm{Weak}}$ BSM matter, especially the relevant operator contributions $\mathcal{O}(\Lambda^2)$, $\mathcal{O}(M^2_{\mathrm{Heavy}} \ln \Lambda^2)$, $\mathcal{O}(M^2_{\mathrm{Heavy}} \ln M^2_{\mathrm{Heavy}})$, $\mathcal{O}(M^2_{\mathrm{Heavy}})$, $\mathcal{O}(M^2_{\mathrm{Heavy}} \ln m^2_{\mathrm{Weak}})$ and $\mathcal{O}(m^2_{\mathrm{Weak}} \ln M^2_{\mathrm{Heavy}})$, to the effective Lagrangian of weak-scale $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ scalars ϕ. We show that, for low-energy $|q^2|, m^2_{\mathrm{Weak}} \ll M^2_{\mathrm{Heavy}}$ physics, the heavy degrees of freedom decouple completely,[13] including marginal operators $\sim\mathcal{O}(\ln M^2_{\mathrm{Heavy}})$, leaving only irrelevant operators, at worst $\sim\mathcal{O}(M^{-2}_{\mathrm{Heavy}})$. We demonstrate this below for two heavy BSM-particle examples, a heavy fermion and a heavy scalar.

B. $\nu\mathrm{MSM}^{\mathrm{G}}_{tb\tau\nu_\tau}$: Singlet right-handed type I see-saw Majorana neutrino ν_R with $M^2_{\nu_R} \gg m^2_{\mathrm{BEH}}$

For the heavy fermion we consider a global $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet right-handed Majorana neutrino ν_R, with $M^2_{\nu_R} \gg m^2_{\mathrm{Weak}}$, such as might be involved in a type 1 see-saw with a left-handed neutrino ν_L, with Yukawa coupling y_ν and resulting Dirac mass $m_D = y_\nu \langle H\rangle/\sqrt{2}$. We add to the renormalized theory

$$L^{Majorana}_{\nu_R} = -M_{\nu_R}(\nu_R\nu_R + \bar{\nu}_R\bar{\nu}_R)/2. \tag{60}$$

Since ν_R is a $SU(2)_L$ singlet, its currents

$$\vec{J}^{\mu;Majorana}_{L;\nu_R} = \vec{J}^{\mu;Majorana}_{L+R;\nu_R} = \vec{J}^{\mu;Majorana}_{L-R;\nu_R} = 0 \tag{61}$$

satisfy all of the criteria, in Sec. III A, for the extension of our axial-vector Ward-Takahashi IDs,

$$\begin{aligned}
&\langle 0|T[\partial_\mu(\vec{J}^\mu_{L-R;\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}} + \vec{J}^{\mu;Majorana}_{L-R;\nu_R})(z) \\
&\quad\times h(x_1)\ldots h(x_N)\pi^{t_1}(y_1)\ldots\pi^{t_M}(y_M)]|0\rangle_{\mathrm{Connected}} = 0 \\
&\quad\times \delta(z_0 - y_0)[\vec{J}^{0;Majorana}_{L-R;\nu_R}(z), H(y)] = 0 \\
&\quad\times \delta(z_0 - y_0)[\vec{J}^{0;Majorana}_{L-R;\nu_R}(z), \vec{\pi}(y)] = 0 \\
&\quad\times \int d^4z\,\partial_\mu\langle 0|T[(\vec{J}^{\mu;Majorana}_{L-R;\nu_R}(z)) \\
&\quad\cdot h(x_1)\ldots h(x_N)\pi_{t_1}(y_1)\ldots\pi_{t_M}(y_M)]|0\rangle = 0.
\end{aligned} \tag{62}$$

Since it is massive ν_R cannot carry information to the surface of the 4-volume $\int d^4z$, nor can it induce any "neutrino-pole-dominance" terms. It follows that the WTI for T-matrix elements (5), Green's functions (6), Adler's self-consistency and IR finiteness (34), and Goldstone theorem (36), are still true for the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ with a nonzero Majorana neutrino mass.

In order that total neutrino masses $\sim m^{2;Dirac}_\nu/M_{\nu_R}$ remain nonzero in this type I see-saw, we can not take the strict $M_{\nu_R} \to \infty$ limit: that would destroy our axial-vector WTIs. Instead we take $1 \gg m^2_{\mathrm{Weak}}/M^2_{\nu_R} > 0$, so that ν_R decouples *in practice*.

The weak-scale effective Lagrangian therefore remains (40) with (41).

C. Singlet $M^2_S \gg m^2_{\mathrm{BEH}}$ real scalar field S with discrete Z_2 symmetry and $\langle S\rangle = 0$

Consider an $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet real scalar S, with $(S \to -S)$ Z_2 symmetry, $M^2_S \gg m^2_h$, and $\langle S\rangle = 0$. We add to the renormalized theory

$$\begin{aligned}
L_S &= \frac{1}{2}(\partial_\mu S)^2 - V_{\phi S} \\
V_{\phi S} &= \frac{1}{2}M^2_S S^2 + \frac{\lambda^2_S}{4}S^4 + \frac{1}{2}\lambda^2_{\phi S}S^2\left[\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right]
\end{aligned} \tag{63}$$

with $M^2_S > 0$. Again,

$$\vec{J}^\mu_{L;S} = \vec{J}^\mu_{L+R;S} = \vec{J}^\mu_{L-R;S} = 0 \tag{64}$$

and all the analogues of Eq. (62) follow.

Since it is massive, S cannot carry information to the surface of the 4-volume $\int d^4z$. $S - h$ mixing, which might well have spoiled the protection that the WTI provide to m^2_h, is forbidden by the Z_2 symmetry $S \to -S$. It follows that the axial-vector WTI for T-matrix elements (5), Green's functions (6), Adler's self-consistency and IR finiteness (34), including the Goldstone theorem, are still true for the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ extended to include this scalar singlet. Note that Green's functions are usually $1 - S$-Reducible, by cutting an S line.

In the $m^2_{\mathrm{Weak}}/M^2_S \to 0$ limit, the weak-scale effective Lagrangian therefore again remains (40) with (41).

IV. CONCLUSION: HISTORICALLY, COMPLETE DECOUPLING OF HEAVY INVISIBLE PARTICLES IS THE USUAL PHYSICS EXPERIENCE

We defined the $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ as the global $SU(3)_C \times SU(2)_L \times U(1)_Y$ model of a complex Higgs doublet and third-generation SM quarks and leptons, augmented by a right-handed neutrino with Dirac mass. With SM isospin and hypercharge assignments for fermions, $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ has zero axial anomaly. We showed that, in the presence of CP conservation, the weak-scale low-energy effective Lagrangian of the spontaneously broken $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ is

[13] Except for high-precision electroweak T and U [13,31,32].

severely constrained by, and protected by, new rigid/global SSB axial-vector Ward-Takahashi identities (WTI) including an equivalent of the Goldstone theorem. In particular, the weak-scale SSB $\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ has an $SU(2)_L$ shift symmetry for constant $\vec{\theta}$

$$\vec{\tilde{\pi}} \to \vec{\tilde{\pi}} + \vec{\tilde{\pi}} \times \vec{\theta} + \langle H \rangle \vec{\theta} + \mathcal{O}(\theta^2). \qquad (65)$$

This protects it, and causes the complete decoupling of certain heavy $M^2_{\mathrm{Heavy}} \gg m^2_{\mathrm{Weak}}$ BSM matter-particles. (Note that such decoupling is modulo special cases: e.g. heavy Majorana ν_R, and possibly $\mathcal{O}^{\mathrm{Dim}\leq 4;\mathrm{nonanalytic;heavy}}_{\nu_D SM^G}$, which are dimension ≤ 4 operators, nonanalytic in momenta or a renormalization scale μ^2, involve heavy particles, and are beyond the scope of this paper.)

Such heavy-particle decoupling is historically the usual physics experience at each energy scale as experiments probed smaller and smaller distances. After all, Willis Lamb did not need to know the top-quark or BEH mass [33] in order to interpret theoretically the experimentally observed $\mathcal{O}(m_e \alpha^5 \ln \alpha)$ splitting in the spectrum of hydrogen.

Such heavy-particle decoupling may be the reason why the Standard Model, viewed as an effective low-energy weak-scale theory, is the most experimentally and observationally successful and accurate theory of nature known to humans, i.e. when augmented by classical general relativity and neutrino mixing, that "core theory" [34] has no known experimental or observational counterexamples.

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APPENDIX: PROOF OF THE WARD-TAKAHASHI IDENTITIES FOR $SU(2)_L \times SU(2)_R$ and $SU(2)_L \times U(1)_Y$

In 1970, B. Lee presented a series of lectures at the Cargese Summer School on Chiral Dynamics, with detailed results on the renormalization of the Gell Mann-Lévy model—the $SU(2)_L \times SU(2)_R$ linear sigma model (LΣM) with an approximate $SU(2)_{L-R}$ chiral symmetry, but with an explicit breaking term [known as the partially conserved axial current (PCAC), or Polkinghorne term]. Of specific interest to us, in Sec. V of those lectures, he proved a tower of $SU(2)_{L-R}$ Ward-Takahashi identities (WTI) among $(h, \vec{\pi})$ scalar-sector (Φ-sector) connected amputated Green's functions, Adler's self-consistency conditions, and the Goldstone theorem.

Those WTI, and the proof thereof, are of immense value to us in this and companion papers. Unfortunately, the volume in which the lecture appears [2] is difficult to obtain. We therefore present in this Appendix, for the benefit of the reader, those WTI and Lee's proof of them for the case of conserved axial-vector currents. We hew closely to Lee's presentation, language, notation and pedagogy. Although we sometimes comment/elaborate on specific details, we mostly just let Lee explain. Because we are interested in weak interactions rather than strong interactions, we set the explicit PCAC $SU(2)_{L-R}$ breaking term (parametrized by γ in Lee's notation) to zero: the result is the $SU(2)_L \times SU(2)_R$ Schwinger model [1]. Because we are interested in including SM fermions, whose Yukawa couplings break global $SU(2)_L \times SU(2)_R$ explicitly to global $SU(2)_L \times U(1)_Y$, we derive $SU(3)_C \times SU(2)_L \times U(1)_Y$ WTIs, analogous with those of Lee.

The conserved vector and axial-vector currents of the Φ-sector $SU(2)_L \times SU(2)_R$ LΣM with the Lagrangian (3) are

$$\vec{V}_\mu = \vec{\pi} \times \partial_\mu \vec{\pi}; \qquad \partial^\mu \vec{V}_\mu(x) = 0$$
$$\vec{A}_\mu = \vec{\pi} \partial_\mu H - H \partial_\mu \vec{\pi}; \qquad \partial^\mu \vec{A}_\mu(x) = 0. \qquad (A1)$$

In Lee's lectures, there is an explicit PCAC breaking of the chiral symmetry, $\partial_\mu \vec{A}^\mu(x) = \gamma \vec{\pi}(x)$. In this paper we take $\gamma \equiv 0$.

In $SU(2)_L \times U(1)_Y$, we are interested in the left-handed combination of $\vec{V}_\mu$ and $\vec{A}_\mu$: $2\vec{J}^\mu_L \equiv \vec{V}^\mu + \vec{A}^\mu$. The addition of fermions in $SU(3)_C \times SU(2)_L \times U(1)_Y$ representations adds contributions to both $\vec{V}_\mu$ and $\vec{A}_\mu$: The various $SU(2)_L, SU(2)_{L+R}, SU(2)_{L-R}$ currents in the $\nu_D \mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$, together with their conservation laws, are given in Eqs. (49) through (53). We focus the remainder of our attention on the $SU(3)_C \times SU(2)_L \times U(1)_Y$ case, which has some additional subtleties compared to Lee's $SU(2)_L \times SU(2)_R$.

We examine time-ordered amplitudes of products of the *axial-vector current* $\vec{J}^{L-R}_\mu$, with N scalars (coordinates x, momenta p), and M pseudoscalars (coordinated y, momenta q, isospin t): $\langle 0 | T[\vec{J}^{L-R}_\mu(z) h(x_1) \cdots h(x_N) \pi^{t_1}(y_1) \cdots \pi^{t_M}(y_M)] | 0 \rangle$. Here $h = H - \langle H \rangle$ and $\vec{\pi}$ are all-loop-orders renormalized fields, normalized so that $\langle 0 | h(0) | h \rangle = 1$ and $\langle 0 | \pi^i(0) | \pi^j \rangle = \delta^{ij}$. We want the divergence of such amplitudes.

We are reminded that, as discussed in Sec. III A, such axial-vector-current amplitudes (as distinct from left-handed current amplitudes) are of interest, and can be considered separately from vector-current amplitudes, solely because of CP conservation. Making use of axial-vector-current conservation [Eqs. (49) through (53)], and the equal-time commutation relations (again assuming CP conservation)

$$\delta(z_0 - x_0)[2J_0^{L-R;i}(z), h(x)] = -i\pi^i(x)\delta^{(4)}(x-z)$$
$$\delta(z_0 - y_0)[2J_0^{L-R;i}(z), \pi^j(y)] = +i\delta^{ij}h(y)\delta^{(4)}(y-z). \tag{A2}$$

A short calculation reveals

$$\begin{aligned}&\partial^\mu \langle 0|T[2J_\mu^{L-R;t}(z)h(x_1)\cdots h(x_N)\pi^{t_1}(y_1)\cdots\pi^{t_M}(y_M)]|0\rangle \\ &= i\sum_m^M \langle 0|T[h(x_1)\cdots h(x_N)h(z)\pi^{t_1}(y_1)\cdots\widehat{\pi^{t_m}(y_m)}\cdots\pi^{t_M}(y_M)]|0\rangle\delta^{(4)}(y_m - z)\delta^{t,t_m} \\ &\quad + i\langle H\rangle\sum_m^M \langle 0|T[h(x_1)\cdots h(x_N)\pi^{t_1}(y_1)\cdots\widehat{\pi^{t_m}(y_m)}\cdots\pi^{t_M}(y_M)]|0\rangle\delta^{(4)}(y_m - z)\delta^{i,i_j} \\ &\quad - i\sum_n^N \langle 0|T[h(x_1)\cdots\widehat{h(x_n)}\cdots h(x_N)\pi^t(x_n)\pi^{t_1}(y_1)\cdots\pi^{t_M}(y_M)]|0\rangle\delta^{(4)}(x_n - z),\end{aligned} \tag{A3}$$

where $\widehat{\pi^{t_m}(y_m)}$ or $\widehat{h(x_n)}$ indicates that copy of π or h is to be omitted from the product of fields. The lhs of (A3) has M pion fields. On the rhs, the first two terms have $M-1$, and the third term $M+1$, pions.

The fermion contributions to $\vec{J}_\mu^{L-R}$ commute with $h(x)$ and $\pi(y)$ and so do not contribute to the rhs of (A3). Fermion contributions to the lhs remain.

We define the Fourier transform of these amplitudes in the usual way

$$\begin{aligned}&iG_\mu^{t;t_1\cdots t_M}(k;p_1\cdots p_N;q_1\cdots q_M)(2\pi)^2\delta^{(4)}\left(k+\sum_n^N p_n + \sum_m^M q_m\right) \\ &\equiv \int d^4z e^{ik\cdot z}\Pi_{n=1}^N \int d^4x_n e^{ip_n\cdot x_n}\Pi_{m=1}^M \int d^4y_m e^{iq_m\cdot y_m}\langle 0|T[2J_\mu^{L-R;t}(z)h(x_1)\cdots h(x_N)\pi^{t_1}(y_1)\cdots\pi^{t_M}(y_M)].|0\rangle\end{aligned} \tag{A4}$$

To economize on notation, going forward we will omit all isospin indices, letting momenta stand in for the isospin indices as well.

The reader is warned that, in this Appendix (and in Lee), it is assumed that $\vec{\pi}$ are the only massless fields in the theory, so that certain surface integrals, which are discussed in the body of this paper, vanish.

Taking the Fourier transform of the divergence of the amplitude, and applying Stokes theorem,

$$\begin{aligned}&\int d^4z e^{ik\cdot z}\Pi_{n=1}^N \int d^4x_i e^{ip_n\cdot x_n}\Pi_{m=1}^M \int d^4y_m e^{iq_j\cdot y_m}\partial_z^\mu\langle 0|T[2J_\mu^L(z)h(x_1)\cdots h(x_N)\pi(y_1)\cdots\pi(y_M)]|0\rangle \\ &= k^\mu G_\mu(k;p_1\cdots p_N;q_1\cdots q_M)(2\pi)^4\delta^{(4)}\left(k+\sum_n^N p_n + \sum_m^N q_m\right)\end{aligned} \tag{A5}$$

$$
\begin{aligned}
&k^\mu G_\mu(k; p_1 \cdots p_N; q_1 \cdots q_M) \\
&\quad = \sum_n^M G(p_1 \cdots \widehat{p_m} \cdots p_N; k + p_m, q_1 \cdots q_M) \\
&\qquad - \sum_m^M G(k + q_m, p_1 \cdots p_N; q_1 \cdots \widehat{q_m} \cdots q_M)\delta_{t,t_m} \\
&\qquad - \langle H \rangle \sum_m^M G(p_1 \cdots p_N; q_1 \cdots \widehat{q_m} \cdots q_M) \\
&\qquad \times \delta_{t,t_m}(2\pi)^4 \delta^{(4)}(k + q_m). \qquad \text{(A6)}
\end{aligned}
$$

This holds for all $N, M \geq 1$. For $N = 0$ and $M = 1$,

$$k^\mu G_\mu(k; ; q) = i\langle H \rangle. \qquad \text{(A7)}$$

The T matrix restricts us to *connected* graphs. The last term on the rhs of (A6) corresponds entirely to disconnected graphs. Denote by H_μ and H the connected parts of the amplitudes G_μ and G defined above. Then

$$
\begin{aligned}
&k^\mu H_\mu(k; p_1 \cdots p_N; q_1 \cdots q_M) \\
&\quad = \sum_n^N H(p_1 \cdots \widehat{p_n} \cdots p_N; k + p_n, q_1 \cdots q_M) \\
&\qquad - \sum_m^M H(k + q_m, p_1 \cdots p_N; q_1 \cdots \widehat{q_m} \cdots q_M)\delta_{t,t_m} \qquad \text{(A8)}
\end{aligned}
$$

for all $N, M \geq 1$. For $N = 0$ and $M = 1$,

$$k^\mu H_\mu(k; ; q) = i\langle H \rangle. \qquad \text{(A9)}$$

In order to derive "1-soft-pion theorems", the limit $k \to 0$ is taken in Eqs. (A8) and (A9). Here we must be careful. Because of the pion pole in H_μ at $k^2 = 0$, $k^\mu H_\mu \to$ constant. We therefore isolate the pion-pole contribution to H_μ by writing

$$
\begin{aligned}
&H_\mu(k; p_1 \cdots p_N; q_1 \cdots q_M) \\
&\quad \equiv i\langle H \rangle k_\mu H(k; p_1 \cdots p_N; q_1 \cdots q_M) \\
&\qquad + \bar{H}_\mu(k; p_1 \cdots p_N; q_1 \cdots q_M). \qquad \text{(A10)}
\end{aligned}
$$

The first term contains the pion-pole contribution; the second term is nonsingular at $k^2 = 0$.

With this new decomposition, the lhs of Eq. (A8) is

$$i\langle H \rangle k^2 H(p_1 \cdots p_N; kq_1 \cdots q_M) + k^\mu \bar{H}_\mu. \qquad \text{(A11)}$$

As $k \to 0$, the second term vanishes, and the first term goes to a limit, since H has a pole at $k^2 = 0$. Therefore

$$
\begin{aligned}
&i\langle H \rangle \lim_{k\to 0} k^2 H(p_1 \cdots p_N; kq_1 \cdots q_M) \\
&\quad = \sum_n^N H(p_1 \cdots \widehat{p_n} \cdots p_N; p_n, q_1 \cdots q_M) \\
&\qquad - \sum_m^M H(p_1 \cdots p_N q_m; q_1 \cdots \widehat{q_m} \cdots q_M)\delta_{t,t_m} \qquad \text{(A12)}
\end{aligned}
$$

for all $N, M \geq 1$. For $N = 0$, and $M = 1$,

$$\langle H \rangle \lim_{k^2 \to 0} (k^2 \Delta_\pi(k^2)) = \langle H \rangle, \qquad \text{(A13)}$$

where

$$
\begin{aligned}
iH(; q, -q) &= \int d^4x e^{iq\cdot x} \langle 0 | T[\pi(x)\pi(0)] | \rangle \\
&\equiv i\Delta_\pi(q^2), \qquad \text{(A14)}
\end{aligned}
$$

and $\Delta_\pi(q^2)$ is the pion propagator. The relation (A13) looks like the Goldstone theorem and is indeed equivalent to it in this ungauged theory, where the masslessness of the NGB $\tilde{\pi}$ is equivalent to the masslessness of π, as discussed in the body of the paper.

Equations (A12) and (A13) are the Ward-Takahashi identities (WTI) of the theory. They are the fundamental identities upon which the arguments of this paper are based. We can combine them to write (for $N \geq 0, M \geq 1$),

$$
\begin{aligned}
&-\langle H \rangle [i\Delta_\pi(0)]^{-1} H(p_1 \cdots p_N; 0q_1 \cdots q_M) \\
&\quad = \sum_n^M H(p_1 \cdots \widehat{p_n} \cdots p_N; p_n q_1 \cdots q_M) \\
&\qquad - \sum_m^M H(p_1 \cdots p_n q_m; q_1 \cdots \widehat{q_m} \cdots q_M). \qquad \text{(A15)}
\end{aligned}
$$

Equation (A15) is of the form

$$
\begin{aligned}
&\langle H \rangle (N + M + 1) - \text{point function} \\
&\quad \propto \sum [(N + M) - \text{point functions}]. \qquad \text{(A16)}
\end{aligned}
$$

B. W. Lee develops perturbation theory as an expansion in his λ, the square-root of his all-loop-orders renormalized 4-point coupling λ^2 (written less compactly λ_ϕ^2 in the body of this paper). Treating $\langle H \rangle$ as $\mathcal{O}(\lambda^{-1})$, Lee points out that Eq. (A15), being of the form (A16), "is satisfied in each order of perturbation theory." That is, if the $N + M + 1$-point function on the lhs is computed in the l-loop approximation, that is to order $\lambda^{2(l-1)+N+M+1}$, and the $N + M$-point functions on the rhs are computed in the same l-loop approximation, that is to order $\lambda^{2(l-1)+N+M}$, then the equation is identically satisfied.

The off-mass-shell T matrix for the N scalar, M pseudoscalar process is obtained from the connected amplitude $H(p_1\cdots p_N;q_1\cdots q_M)$ by "amputating" the propagators of the external lines,

$$H(p_1\cdots p_N;q_1\cdots q_M) = \Pi_{n=1}^{N}[i\Delta_h(p_n^2)]\Pi_{m=1}^{M}[i\Delta_\pi(q_m^2)]T(p_1\cdots p_N;q_1\cdots q_M). \tag{A17}$$

The off-shell 1-(h,π) scalar-particle-reducible (1-ϕ-R) connected amputated T-matrix elements are expressed in terms of the all-loop-orders renormalized h and π propagators,

$$i\Delta_h(p^2) = \int d^4x e^{ip\cdot x}\langle 0|T[h(x)h(0)]|0\rangle$$
$$i\delta_{ij}\Delta_\pi(q^2) = \int d^4x e^{iq\cdot x}\langle 0|T[\pi_i(x)\pi_j(0)]|0\rangle, \tag{A18}$$

and 1-(h,π)-Scalar-Particle-Irreducible (1-ϕ-I) connected amputated Green's functions $\Gamma_{N,M}(p_1\cdots p_N;q_1\cdots q_M)$, which cannot be disconnected by cutting a h or π propagator line.

$$T(p_1\cdots p_N;q_1\cdots q_M) = \Gamma_{N,M}(p_1\cdots p_N;q_1\cdots q_M) + \text{reducible part}. \tag{A19}$$

$\Gamma_{N,M}(p_1\cdots p_N;q_1\cdots q_M)$ is the 1-ϕ-I vertex for N h's and M π's. The "reducible part" can be written in terms of irreducible vertices of lower order and full propagators. Expressed in terms of the full propagators and the irreducible vertices, the T matrix has a tree structure—i.e. it can be represented by graphs without loops.

Because the T matrix contains only connected graphs, and our WTI concern only the axial-vector current, and CP is conserved,

$$\Gamma_{0,0} = \Gamma_{0,1} = \Gamma_{1,1} = 0. \tag{A20}$$

In terms of connected amputated Green's functions, the propagators

$$\Gamma_{2,0}(p,-p;) \equiv [\Delta_h(p^2)]^{-1}$$
$$\Gamma_{0,2}(;q,-q) \equiv [\Delta_\pi(q^2)]^{-1}. \tag{A21}$$

We now examine $\Gamma_{1,2}$. By definition

$$H(p;0,-p) = [i\Delta_h(p^2)][i\Delta_\pi(0)] \times [i\Delta_\pi(p^2)]\Gamma_{1,2}(p;0,-p) \tag{A22}$$

so with (A15) we have

$$\langle H\rangle\Gamma_{1,2}(p;0,-p) = [\Delta_h(p^2)]^{-1} - [\Delta_\pi(p^2)]^{-1} = \Gamma_{2,0}(p,-p;) - \Gamma_{0,2}(;p,-p). \tag{A23}$$

We can easily verify that this holds to lowest order where $\Gamma_{1,2}(p;0,-p) = -2\lambda^2\langle H\rangle$, $[\Delta_h(p^2)]^{-1} = p^2 - m_h^2 = p^2 - 2\lambda^2\langle H\rangle^2$ and $[\Delta_\pi(p^2)]^{-1} = p^2$.

To proceed further, we must express the WTI in terms of the connected amputated T matrix rather than the Green's functions. We can rewrite Eq. (A15) for $N \geq 0$, $M \geq 1$,

$$\langle H\rangle T(p_1\cdots p_N;0q_1\cdots q_M) = \sum_m^M i\Delta_h(q_m^2)[i\Delta_\pi(q_m^2)]^{-1} \times T(p_1\cdots p_N q_m;q_1\cdots \widehat{q_m}\cdots q_M) - \sum_n^N i\Delta_\pi(p_n^2)[i\Delta_h(p_n^2)]^{-1} \times T(p_1\cdots \hat{p}_t\cdots p_N;q_1\cdots q_M p_t). \tag{A24}$$

A corollary of (A24) are Adler's self-consistency conditions for global $SU(2)_L \times U(1)_Y$,

$$\langle H\rangle \lim_{q_1\to 0} T(p_1\cdots p_N;q_1\cdots q_M)|^{q_2^2=\cdots q_M^2=0}_{p_1^2=\cdots=p_N^2=m_h^2} = 0, \tag{A25}$$

which shows that, for $\langle H\rangle \neq 0$, the T matrix vanishes as one of the pion momenta goes to zero, provided all the external particles are on the mass shell. Equation (A25) asserts the absence of infrared divergences in Goldstone mode. "Individual Feynman diagrams are IR divergent, but the divergent parts must cancel in every order of perturbation theory. Furthermore, the amplitude must vanish in the soft-pion limit [2]."

In (A24), the zero-momentum pion in $T(p_1\cdots p_N;0q_1\cdots q_M)$ can either come off a "branch" (Lee's word for an external ϕ line) or off the "body" (our word) of the diagram. Let T_1 be the sum of the subset of the tree graphs belonging to $T(p_1\cdots p_N;0q_1\cdots q_M)$ in which the zero-momentum pion comes off a branch, as in Fig. 1. The branch is either a π branch (left-hand graph of Fig. 1), with finite-momentum q_m, written

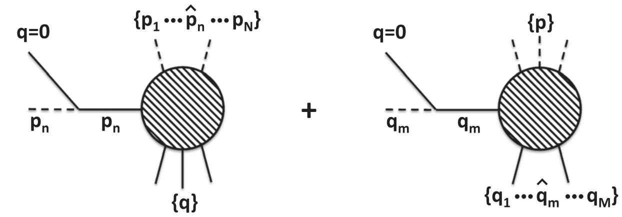

FIG. 1. $T_1^{t_1\ldots t_M t}(p_1\ldots p_N;q_1\ldots q_M 0)$: Hashed circles are 1-$\phi$-$R$ $T^{t_1\ldots t_M}(p_1\ldots p_N;q_1\ldots q_M)$, solid lines $\vec{\pi}$, dashed lines h. One (zero-momentum) soft pion is attached to an external leg (i.e. a branch) in all possible ways. Figure 1 is the $SU(3)_C \times SU(2)_L \times U(1)_Y$ $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ analogy of B.W. Lee's Fig. 10 [2].

$$i\Gamma_{1,2}(p_n;0,-p_n)i\Delta_\pi(p_n^2)T(p_1\cdots\widehat{p_n}\cdots p_N;p_n,q_1\cdots q_M)$$

or a h branch (right-hand graph of Fig. 1), with momentum q_m, written

$$i\Gamma_{1,2}(q_m;0,-q_m)i\Delta_\pi(q_m^2)T(p_1\cdots p_N;q_1\cdots\widehat{q_m}\cdots q_M).$$

Forming T_1 from these, and using (A23)

$$\begin{aligned}\langle H\rangle T_1 = &\sum_m^M T(p_1\cdots p_N q_m;q_1\cdots\widehat{q_m}\cdots q_M)\\ &\times(1-[i\Delta_\pi(q_m^2)]^{-1}[i\Delta_h(q_m^2)])\\ &-\sum_n^N T(p_1\cdots\widehat{p_n}\cdots p_N;q_1\cdots q_M p_n)\\ &\times(1-[i\Delta_h(p_n^2)]^{-1}[i\Delta_\pi(p_n^2)]).\end{aligned} \tag{A26}$$

Having accounted for T_1, we define

$$T_1+T_2\equiv T(p_1\cdots p_N;0q_1\cdots q_M) \tag{A27}$$

so that, combining (A24), (A26), (A27), the WTI for T_2 are simply

$$\begin{aligned}&\langle H\rangle T_2(p_1\cdots p_N;0q_1\cdots q_M)\\ &\quad=\sum_m^M T(p_1\cdots p_N q_m;q_1\cdots\widehat{q_m}\cdots q_M)\\ &\qquad-\sum_n^N T(p_1\cdots\widehat{p_n}\cdots p_N;q_1\cdots q_M p_n)\end{aligned} \tag{A28}$$

for $N\geq 0$, $M\geq 1$

These are identities for T-matrix elements. How do they translate into relations among the irreducible vertices? Lee shows that Eq. (A28) is satisfied for $N\geq 0$, $M\geq 1$ if

$$\begin{aligned}&\langle H\rangle\Gamma_{N,M+1}(p_1\cdots p_N;0q_1\cdots q_M)\\ &\quad=\sum_m^M \Gamma_{N+1,M-1}(p_1\cdots p_N q_m;q_1\cdots\widehat{q_m}\cdots q_M)\\ &\qquad-\sum_n^N \Gamma_{N-1,M+1}(p_1\cdots\widehat{p_n}\cdots p_N;q_1\cdots q_M p_n).\end{aligned} \tag{A29}$$

The proof of (A29) is by induction on $N+M$, starting from $N=M=1$, which is just Eq. (A23). Assume then that (A29) holds for $N+M<n+m$. Let $N=n$, $M=m$. T_2 in (A29) contains two classes of graphs, shown in Fig. 2:

(i) *Figure 2, top graphs* are reducible graphs in which the zero-momentum pion comes out of an irreducible vertex. However, this does not include graphs in which the zero-momentum pion comes out of a three-prong irreducible vertex of which two prongs are external lines, since those belong to T_1, not T_2. For the sum of the 1-ϕ-R graphs, we may use (A29), for $N+M<n+m$, to show that the 1-ϕ-I contributions from both sides of (A28) are identical and cancel. This leaves only 1-ϕ-I vertices on both sides of Eq. (A29), giving us (A29) for $N=n$, $M=m$, as desired.

(ii) *Figure 2, bottom graph* is 1-ϕ-I, and already satisfies (A29).

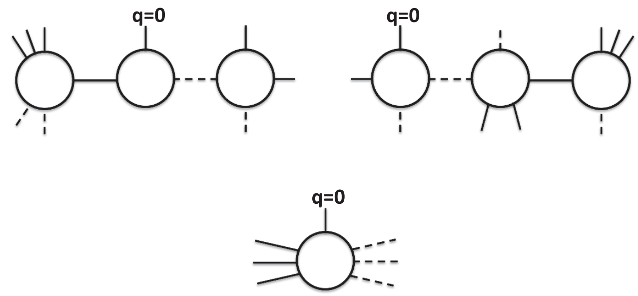

FIG. 2. Circles are 1-ϕ-I $\Gamma_{n,m}$, solid lines $\vec{\pi}$, dashed lines h, with $n+m<N+M$. One (zero-momentum) soft pion emerges in all possible ways from the connected amputated Green's functions. Figure 2 is the $SU(3)_C\times SU(2)_L\times U(1)_Y$ $\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$ analogy of B. W. Lee's Fig. 11 [2].

Having proved Eq. (A29), we can now restore all the isospin indices and display it in its full glory,

$$\begin{aligned}&\langle H\rangle\Gamma^{t,t_1\cdots t_M}_{N,M+1}(p_1\cdots p_N;0q_1\cdots q_M)\\ &\quad=\sum_m^M \delta^{t,t_m}\Gamma^{t_1\cdots\widehat{t_m}\cdots t_M}_{N+1,M-1}(p_1\cdots p_N q_m;q_1\cdots\widehat{q_m}\cdots q_M)\\ &\qquad-\sum_n^N \Gamma^{t_1\cdots t_M t}_{N-1,M+1}(p_1\cdots\widehat{p_n}\cdots p_N;q_1\cdots q_M p_n);\end{aligned} \tag{A30}$$

valid for all $N,M\geq 0$, and nontrivial for M odd.

As Lee emphasizes for the $SU(2)_L\times SU(2)_R$ symmetric theory, "*the identities (A30) are valid in any renormalizable theory in which the divergence [of the axial vector current vanishes] … and the $H,\vec{\pi}$ fields transform as the $[\frac{1}{2},\frac{1}{2}]$ representation under chiral $SU(2)\times SU(2)$ transformations. Whether … other fields are included is irrelevant, so long as the chiral symmetry is broken in a way to ensure the divergence remains zero. When there are other fields present, the irreducible vertices we have defined here may still be reducible with respect to these [new] fields.*". To Lee's statement, we add the strong constraint: *as long as the new fields are massive.*

With the addition of massive Standard Model fermions to the $SU(2)_L\times SU(2)_R$ symmetric scalar theory, the symmetry is explicitly broken down to $SU(2)_L\times U(1)_Y$.

The addition of certain other new massive BSM particles that do not contribute to the divergence of the $SU(2)_L \times U(1)_Y$ current will again leave the form (A25), (A30) of the WTI identities, and the Goldstone theorem, unchanged. This is discussed more explicitly in Sec. III.

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