Global $U(1)_Y \otimes$ BRST symmetry and the LSS theorem: Ward-Takahashi identities governing Green's functions, on-shell T-matrix elements, and the effective potential in the scalar sector of the spontaneously broken extended Abelian Higgs model

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The weak-scale $U(1)_Y$ Abelian Higgs model (AHM) is the simplest spontaneous symmetry breaking (SSB) gauge theory: a scalar $\phi = \frac{1}{\sqrt{2}}(H + i\pi) \equiv \frac{1}{\sqrt{2}}\tilde{H}e^{i\tilde{\pi}/\langle H\rangle}$ and a vector A^μ. The extended AHM (E-AHM) adds certain heavy ($M_\Phi^2, M_\psi^2 \sim M_{\rm Heavy}^2 \gg \langle H\rangle^2 \sim m_{\rm Weak}^2$) spin $S = 0$ scalars Φ and $S = \frac{1}{2}$ fermions ψ. In Lorenz gauge, $\partial_\mu A^\mu = 0$, the SSB AHM (and E-AHM) has a global $U(1)_Y$ conserved physical current, but no conserved charge. As shown by T. W. B. Kibble, the Goldstone theorem applies, so $\tilde{\pi}$ is a massless derivatively coupled Nambu-Goldstone boson (NGB).

Proof of all-loop-orders renormalizability and unitarity for the SSB case is tricky because the Becchi-Rouet-Stora-Tyutin (BRST)-invariant Lagrangian is not $U(1)_Y$ symmetric. Nevertheless, Slavnov-Taylor identities guarantee that on-shell T-matrix elements of physical states A^μ,ϕ, Φ, ψ (but not ghosts ω, $\bar{\eta}$) are independent of anomaly-free local $U(1)_Y$ gauge transformations. We observe here that they are therefore also independent of the usual anomaly-free $U(1)_Y$ global/rigid transformations. It follows that the associated global current, which is classically conserved only up to gauge-fixing terms, is exactly conserved for amplitudes of physical states in the AHM and E-AHM. We identify corresponding "undeformed" [i.e. with full global $U(1)_Y$ symmetry] Ward-Takahashi identities (WTI). The proof of renormalizability and unitarity, which relies on BRST invariance, is undisturbed.

In Lorenz gauge, two towers of "1-soft-pion" SSB global WTI govern the ϕ-sector, and represent a new global $U(1)_Y \otimes$ BRST symmetry not of the Lagrangian but of the physics. The first gives relations among off-shell Green's functions, yielding powerful constraints on the all-loop-orders ϕ-sector SSB E-AHM low-energy effective Lagrangian and an additional global shift symmetry for the NGB: $\tilde{\pi} \to \tilde{\pi} + \langle H\rangle\theta$. A second tower, governing on-shell T-matrix elements, replaces the old Adler self-consistency conditions with those for gauge theories, further severely constrains the effective potential, and guarantees infrared finiteness for zero NGB ($\tilde{\pi}$) mass. The on-shell WTI include a Lee-Stora-Symanzik theorem, also for gauge theories. This enforces the strong condition $m_\pi^2 = 0$ on the pseudoscalar π (not just the much weaker condition $m_{\tilde{\pi}}^2 = 0$ on the NGB $\tilde{\pi}$), and causes all relevant-operator contributions to the effective Lagrangian to vanish exactly.

In consequence, certain heavy CP-conserving Φ, ψ matter decouple completely in the $m_{Heavy}^2/m_{weak}^2 \to \infty$ limit. We prove four new low-energy heavy-particle decoupling theorems that are more powerful than the usual Appelquist-Carazzone decoupling theorem: including all virtual ϕ and ψ loop contributions, relevant operators operators vanish exactly due to the exact $U(1)_Y$ symmetry of 1-soft-π Adler-self-consistency relations governing on-shell T-matrix elements.

Underlying our results is that global $U(1)_Y$ transformations $\delta_{U(1)_Y}$, and nilpotent $s^2 = 0$ BRST transformations, commute: we prove $[\delta_{U(1)_Y}, s]$ in G. 't Hooft's R_ξ gauges. With its on-shell T-matrix constraints, SSB E-AHM physics therefore has more symmetry than does its BRST-invariant Lagrangian $L_{\text{E-AHM}}^{R_\xi}$: i.e. global $U(1)_Y \otimes$ BRST symmetry.

The NGB $\tilde{\pi}$ decouples from the observable particle spectrum B^μ,$\tilde{h}$, $\tilde{\Phi}$, $\tilde{\psi}$ in the usual way, when the observable vector $B_\mu \equiv A_\mu + \frac{1}{e\langle H\rangle}\partial_\mu\tilde{\pi}$ absorbs it, as if it were a gauge transformation, hiding both towers of $U(1)_Y$ WTI from observable particle physics.

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I. INTRODUCTION

What are the symmetries driving spontaneously broken Abelian Higgs model (AHM) physics [1]? Although the symmetries of the $U(1)_Y$ AHM Lagrangian are well known [2], local gauge invariance is lost in the AHM Lagrangian, broken by gauge-fixing terms, and replaced with global Becchi-Rouet-Stora-Tyutin (BRST) invariance [3–5].

In their seminal work, Elisabeth Kraus and Klaus Sibold [6] showed important new practicalities of the renormalizability and unitarity (to all loop orders) of the spontaneous symmetry breaking (SSB) AHM. They did this by deriving rigid invariance from BRST invariance. The SSB case is tricky because the globally Becchi-Rouet-Stora-Tyutin (BRST)-invariant Lagrangian is not $U(1)_Y$ symmetric. But they identified a set of "deformed" [i.e. with no remnant of the original $U(1)_Y$ group symmetry] *rigid/global* AHM transformations which, after inclusion of well-defined $U(1)_Y$ breaking by quantum loops (e.g. in scalar wave-function renormalization beyond the classical AHM), *are* compatible with BRST symmetry.

Kraus and Sibold then constructed deformed Ward-Takahashi identities (WTI) for quantum AHM Green's functions, showing them (with appropriate normalization conditions) to obey all-loop-orders renormalizability and unitarity. Because their renormalization relies only on deformed WTI, Kraus and Sibold's results are independent of the regularization scheme, for any acceptable scheme (i.e. if one exists). They did not construct WTI for on-shell T-matrix elements.

Nevertheless, Slavnov-Taylor identities [7] prove that the on-shell S-matrix elements of "physical states" A^μ, ϕ, Φ, ψ, (i.e. spin $S=0$ scalars h, π, Φ, $S=\frac{1}{2}$ (CP-conserving) fermions ψ, and $S=1$ gauge bosons A_μ, but not fermionic ghosts ω or antighosts $\bar{\eta}$) are independent, in the AHM, of the usual undeformed anomaly-free $U(1)_Y$ local/gauge transformations, even though these break the Lagrangian's BRST symmetry. We observe here that they are therefore also independent of anomaly-free undeformed $U(1)_Y$ global/rigid transformations, resulting in "new" global/rigid currents and appropriate undeformed $U(1)_Y$ Ward-Takahashi Identities.

We here distinguish carefully between off-shell Green's function WTI, which constrain the (unobservable) effective Lagrangian and action, and on-shell T-matrix WTI, which further severely constrain observable physics. We show here that, in the SSB Abelian Higgs model, a tower of WTI relates all relevant-operator contributions to AHM physical-scalar-sector physical observables to one another. An on-shell T-matrix WTI, i.e. the equivalent of an Adler self-consistency relation but for this gauge theory, then causes all such contributions to vanish. It does so through its insistence that the scalar mass squared vanishes exactly,

$$m_\pi^2 = 0, \tag{1}$$

in spontaneously broken ($\langle H\rangle \neq 0$) theories, which we term the Lee-Stora-Symanzik (LSS)[1] theorem after the three physicists who recognized its central role in the renormalization of global linear sigma models, and the one who was central to our understanding of its role in the renormalization of gauge theories.[2] In addition to constraining the parameters of the theory, the LSS theorem permits us to employ pion-pole dominance to compute the WTI.

The crucial advance over [18], which considered the global $SU(2)_L \times U(1)_Y$ linear sigma model, is a proof that the WTI remain in place in a SSB gauge theory, with the LSS theorem playing the same protective role as did the Goldstone theorem in the global theory [18].

Our new rigid $U(1)_Y$ WTI govern the scalar sector of the AHM and of the extensions we consider in Sec. IV. They are therefore independent of regularization scheme (assuming one exists). Although not a gauge-independent procedure, it may help the reader to imagine that loop integrals are cut off at a short-distance finite Euclidean UV scale, Λ, never taking the $\Lambda^2 \to \infty$ limit. Although that cutoff can be imagined to be near the Planck scale $\Lambda \simeq M_{\rm Pl}$, quantum gravitational loops are not included.

The structure of this paper is as follows:

Section II introduces $U(1)_Y \otimes$ BRST symmetry for the AHM and extended AHM (E-AHM) in a general 't Hooft R_ξ gauge, and explains why physical quantities obey that new symmetry.

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[1] Raymond Stora would never have named anything after himself, but we judge that, given the stature of B. W. Lee, R. Stora and K. Symanzik (now all deceased) in the history of the relevant physics, the community would refer to that result as the LSS theorem anyway.

[2] As first noted by Kibble [8], in Lorenz gauge a relation similar in appearance to (1), $m_{\tilde{\pi}}^2 = 0$, enforces the masslessness of a Nambu Goldstone boson (NGB) $\tilde{\pi}$, i.e. is a Goldstone theorem [9–11] for this gauge theory. This is regardless of the fact that the NGB is not a physical degree of freedom, but is absorbed ("eaten") by the gauge boson. However, as we describe in greater detail below [cf. Eq. (20)], $\tilde{\pi}$ is the angular degree of freedom in the Kibble representation of the complex scalar field, while π is the pseudoscalar degree of freedom in the linear representation. In global linear sigma models ($L\Sigma M$), the masslessness of the NGB and the LSS condition (III D) are equivalent. Indeed, B. Lee [12], K. Symanzik [13,14], A. Vassiliev [15] and classic texts [16] advocate that the spontaneously broken (Goldstone) mode of a $U(1)$ global $L\Sigma M$ is to be understood as the zero-explicit-breaking limit (i.e. $m_\pi^2 \to 0$) of the explicit $U(1)$-breaking partially conserved axial-vector current (PCAC) term, $L_{\rm PCAC} = \langle H\rangle m_\pi^2 H$, included in the $U(1)$ version of the Gell-Mann and Lévy $L\Sigma M$ [17]. The existence and masslessness of the purely derivatively coupled NGB is a result of and requires the vanishing of the explicit-symmetry-breaking pseudoscalar mass squared. In the $\mathbf{U(1)_Y}$ AHM gauge theory, the Goldstone theorem and the LSS theorem are not equivalent. To see this (or to at least suspect it) the reader should remember that one cannot incorporate explicit PCAC breaking of the local $U(1)_Y$ symmetry into the AHM gauge theory [7], without spoiling unitarity.

Section III concerns the correct renormalization of the spontaneously broken AHM in Lorenz gauge. We treat the AHM in isolation, as a stand-alone flat-space weak-scale quantum field theory, not embedded or integrated into any higher-scale "beyond-AHM" physics.

Section IV extends our AHM results to include the all-loop-orders virtual contributions of certain $M^2_{\rm Heavy} \gg m^2_{\rm Weak}$ heavy $U(1)_Y$ matter representations (which might arise in certain beyond-AHM models).

Section V reminds the reader [19] how the NGB $\tilde{\pi}$ disappears from the *observable* particle spectrum of the E-AHM.

Section VI discusses the exacting mathematical rigor that would have fully satisfied Raymond Stora.

Section VII reminds us that historically (with an important exception) the decoupling of heavy particles is the usual experience of physics.

Appendix A gives a complete and pedagogical derivation of the $U(1)_Y$ WTI governing the ϕ-sector of the AHM. Our renormalized WTI include all contributions from virtual transverse gauge bosons, ϕ scalars, and ghosts, A^μ, h and π, and $\bar{\eta}$ and ω, respectively.

Appendix B gives a complete and pedagogical derivation of $U(1)_Y$ (h,π)-sector WTI in the E-AHM, which now include the all-loop-orders contributions of certain additional $U(1)_Y$ matter representations: spin $S=0$ scalars Φ, and $S=\frac{1}{2}$ anomaly-canceling (CP-conserving) fermions ψ. They include all contributions from virtual transverse gauge bosons, ghosts, scalars, and fermions, $A^\mu; h,\pi;\bar{\eta},\omega;\Phi;\psi$.

II. $U(1)_Y \otimes$ BRST symmetry in 't Hooft R_ξ gauges

The BRST-invariant [3–5] Lagrangian of the $U(1)_Y$ AHM gauge theory may be written, in a general 't Hooft R_ξ gauge, in terms of a transverse vector A_μ, a complex scalar ϕ, a ghost ω, and an antighost $\bar{\eta}$,

$$L^{R_\xi}_{\rm AHM} = L^{\rm Gauge\ Invariant}_{\rm AHM} + L^{\rm Gauge\ Fix;R_\xi}_{\rm AHM} + L^{\rm Ghost;R_\xi}_{\rm AHM}, \quad (2)$$

where

$$L^{\rm Gauge\ Invariant}_{\rm AHM} = |D_\mu\phi|^2 - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - V_{\rm AHM}(\phi^\dagger\phi) \quad (3)$$

with

$$\begin{aligned} D_\mu\phi &= (\partial_\mu - ieY_\phi A_\mu)\phi, \\ A_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ V_{\rm AHM} &= \mu^2_\phi(\phi^\dagger\phi) + \lambda^2_\phi(\phi^\dagger\phi)^2, \end{aligned} \quad (4)$$

and

$$\phi = \frac{1}{\sqrt{2}}(H + i\pi), \qquad H = \langle H\rangle + h \quad \text{and} \quad Y_\phi = -1. \quad (5)$$

In G. 't Hooft's R_ξ gauges, gauge fixing and DeWitt-Fadeev-Popov ghost terms [20,21] are written in terms of a Nakanishi-Lautrup field b [22,23], and the SSB vector mass $m_A = eY_\phi\langle H\rangle = -e\langle H\rangle > 0$.

$$\begin{aligned} L^{\rm Gauge\ Fix;R_\xi}_{\rm AHM} + L^{\rm Ghost;R_\xi}_{\rm AHM} &= \frac{1}{2}\xi b^2 + b(\partial_\mu A^\mu + \xi m_A \pi) \\ &\quad - \bar{\eta}\left(\partial^2 + \xi\frac{m_A^2}{\langle H\rangle}H\right)\omega \\ &= s\left[\bar{\eta}\left(F_A + \frac{1}{2}\xi b\right)\right] \\ F_A &= \partial_\mu A^\mu + \xi m_A \pi \end{aligned} \quad (6)$$

with global BRST transformations [3–5,22–24] s,

$$\begin{aligned} sA_\mu &= \partial_\mu\omega, & s\bar{\eta} &= b, \\ sH &= -e\pi\omega, & sb &= 0, \\ s\pi &= eH\omega, & s\omega &= 0, \end{aligned} \quad (7)$$

so that the Lagrangian (2) is BRST invariant,

$$sL^{R_\xi}_{\rm AHM} = 0. \quad (8)$$

The *classical* equation of motion for the ghost is

$$sF_A = \left(\partial^2 + \xi\frac{m_A^2}{Y_\phi\langle H\rangle}H\right)\omega = 0. \quad (9)$$

Now define the properties of the various fields under the usual anomaly-free undeformed rigid/global $U(1)_Y$ transformation by a constant Ω,

$$\begin{aligned} \delta_{U(1)_Y}A_\mu &= 0, & \delta_{U(1)_Y}\bar{\eta} &= 0; \\ \delta_{U(1)_Y}H &= -e\pi\Omega, & \delta_{U(1)_Y}b &= 0; \\ \delta_{U(1)_Y}\pi &= eH\Omega, & \delta_{U(1)_Y}\omega &= 0. \end{aligned} \quad (10)$$

We discover that the R_ξ-gauge Lagrangian (2) is not invariant under such $U(1)_Y$ transformations,

$$\begin{aligned} \delta_{U(1)_Y}L^{R_\xi}_{\rm AHM} &= \delta_{U(1)_Y}\left(s\left[\bar{\eta}\left(F_A + \frac{1}{2}\xi b\right)\right]\right) \\ &= \xi e m_A(bH + e\bar{\eta}\pi\omega)\Omega \\ &= s\left(\delta_{U(1)_Y}\left[\bar{\eta}\left(F_A + \frac{1}{2}\xi b\right)\right]\right) \\ &\neq 0. \end{aligned} \quad (11)$$

Still, the actions of the BRST transformations (7) and the $U(1)_Y$ transformation (10) commute on all fields.

$$[\delta_{U(1)_Y}, s]A^\mu = 0; \qquad [\delta_{U(1)_Y}, s]\omega = 0;$$
$$[\delta_{U(1)_Y}, s]H = 0; \qquad [\delta_{U(1)_Y}, s]\bar{\eta} = 0;$$
$$[\delta_{U(1)_Y}, s]\pi = 0; \qquad [\delta_{U(1)_Y}, s]b = 0. \tag{12}$$

Thus, with the nilpotent property $s^2 = 0$ applied in (11),

$$[\delta_{U(1)_Y}, s]L_{\text{AHM}}^{R_\xi} = 0; \tag{13}$$

and the two separate global symmetries can therefore coexist in AHM physics.

Now add to (2) any $U(1)_Y$ local/gauge invariant, and therefore BRST invariant, Lagrangian $L_{\text{beyondAHM}}^{\text{Gauge Invariant}}(A_\mu,\phi;\Phi,\psi)$ involving new bosonic spin-zero fields Φ and new anomaly-canceling fermionic spin-$\frac{1}{2}$ fields ψ so as to form the E-AHM. Then

$$sL_{\text{E-AHM}}^{R_\xi} = 0,$$
$$\delta_{U(1)_Y} L_{\text{E-AHM}}^{R_\xi} = s(\xi e m_A \bar{\eta} H \Omega) \neq 0,$$
$$[\delta_{U(1)_Y}, s]L_{\text{E-AHM}}^{R_\xi} = 0. \tag{14}$$

We show in this paper that, due to (7), (10), (13), and (14), the AHM, and the E-AHM, simultaneously obey both the usual BRST symmetry and a global $U(1)_Y$ symmetry that controls Green's functions and on-shell T-matrix elements. We also show that our effective potential can be made gauge independent.

We reason as follows:

(i) All aspects of the SSB AHM and E-AHM obey BRST symmetry.

(ii) In both the special $\xi \to 0$ case of Landau gauge and in the closely related Lorenz gauge,

$$L_{\text{AHM}}^{\text{Landau}} = L_{\text{AHM}}^{\text{Gauge Invariant}} - \lim_{\xi\to 0}\frac{1}{2\xi}(\partial_\mu A^\mu + \xi m_A \pi)^2 - \bar{\eta}\partial^2\omega,$$
$$L_{\text{AHM}}^{\text{Lorenz}} = L_{\text{AHM}}^{\text{Gauge Invariant}} - \lim_{\xi\to 0}\frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \bar{\eta}\partial^2\omega, \tag{15}$$

global $U(1)_Y$ symmetry and the larger global $U(1)_Y \otimes$ BRST symmetry are preserved,

$$\delta_{U(1)_Y} L_{\text{AHM}}^{\text{Lorenz}} = 0,$$
$$\delta_{U(1)_Y} L_{\text{AHM}}^{R_\xi} \overset{\xi\to 0}{\Longrightarrow} \delta_{U(1)_Y} L_{\text{AHM}}^{\text{Landau}} = 0,$$
$$sL_{\text{AHM}}^{\text{Lorenz}} = 0,$$
$$sL_{\text{AHM}}^{\text{Landau}} = 0, \tag{16}$$

similarly for $L_{\text{E-AHM}}^{\text{Lorenz}}$ and $L_{\text{E-AHM}}^{\text{Landau}}$.

(iii) Physical states and time-ordered amplitudes of the exact renormalized scalar $\phi = \frac{1}{\sqrt{2}}(H + i\pi)$ and vector A_μ obey G. 't Hooft's gauge condition [25]

$$0 = \langle 0|T[(\partial_\mu A^\mu(z)) \times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}} \tag{17}$$

in Landau or Lorenz gauges. Here we have N external renormalized scalars $h = H - \langle H\rangle$ (coordinates x_i), and M external $(CP = -1)$ renormalized pseudoscalars π (coordinates y_i).

(iv) We prove in Appendix A for the AHM and in Appendix B for the E-AHM that, in Lorenz gauge $\partial_\mu A^\mu = 0$, scalar-sector connected amputated *on-shell* T-matrix elements obey (17) and the $U(1)_Y$ symmetry. Such on-shell WTI are gauge independent (i.e. true for general R_ξ gauges) even though (11) and (14) show that the BRST-invariant AHM (and E-AHM) Lagrangian is not invariant under the $U(1)_Y$ symmetry.

(v) We prove in Appendix A for the AHM and in Appendix B for the E-AHM that, in Lorenz gauge $\partial_\mu A^\mu = 0$, scalar-sector connected amputated gauge-dependent Green's functions also obey (17) and the $U(1)_Y$ symmetry.

(vi) We show that our AHM and E-AHM effective potentials can be made physical (i.e. gauge-independent) in Sec. V B 2, thus generalizing them to 't Hooft R_ξ (and all other well-behaved) gauges.

III. THE ABELIAN HIGGS MODEL IN LORENZ GAUGE

A. The Abelian Higgs model in Lorenz gauge

We form the AHM Lagrangian in Lorenz gauge

$$L_{\text{AHM}}^{\text{Lorenz}} = L_{\text{AHM}}^{\text{Gauge Invariant}} + L_{\text{AHM}}^{\text{Gauge Fix;Lorenz}} + L_{\text{AHM}}^{\text{Ghost;Lorenz}} \tag{18}$$

with (3), by writing the gauge-fixing and ghost terms,

$$L_{\text{AHM}}^{\text{Gauge Fix;Lorenz}} = -\lim_{\xi\to 0}\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$
$$L_{\text{AHM}}^{\text{Ghost;Lorenz}} = -\bar{\eta}\partial^2\omega. \tag{19}$$

The complex scalar ϕ is manifestly renormalizable in the linear representation (5). After SSB, $m_A^2 = e^2 Y_\phi^2 \langle H\rangle^2$.

This paper distinguishes carefully between the local BRST-invariant $U(1)_Y$ Lagrangian (18) and its three physical modes [12–16]: symmetric Wigner mode, the classically scale-invariant (SI) point and physical Goldstone mode.

(1) Symmetric Wigner mode $\langle H\rangle = 0, m_A^2 = 0, m_\pi^2 = m_{\rm BEH}^2 = \mu_\phi^2 \neq 0$:

This is QED with massless photons and massive charged scalars. Thankfully, nature is not in Wigner mode. Further analysis and renormalization of the Wigner mode lies outside the scope of this paper.

(2) Classically scale-invariant point $\langle H\rangle = 0, m_A^2 = 0, m_\pi^2 = m_{\rm BEH}^2 = 0$:

Analysis of the scale-invariant point is also outside the scope of this paper.

(3) Spontaneously broken Goldstone mode $\langle H\rangle \neq 0, m_A^2 = e^2\langle H\rangle^2 \neq 0, m_\pi^2 = 0, m_{\rm BEH}^2 \neq 0$:

The famous Abelian Higgs model, with its NGB eaten by the Brout-Englert-Higgs mechanism [and, as we see, WTI governed by the (Goldstone-like) LSS theorem] is the SSB Goldstone mode of the BRST-invariant local Lagrangian (18), and is the subject of this paper. We work in Lorenz gauge for many reasons.

(a) The $U(1)_Y$ ghosts $(\bar\eta, \omega)$ decouple from the quantum loop dynamics, and can (and will) be benevolently ignored going forward.

(b) After a subtlety concerning their mixing, π and A^μ are orthonormal species. A term $\sim A_\mu \partial^\mu \pi$ arises from $|D_\mu\phi|^2$ after SSB in (18); a term $\sim \pi\partial^\mu A_\mu$ is shown to vanish for physical states in (A4) and (B4). The resultant surface term $\partial^\mu(\pi A_\mu)$ vanishes (for physical states) because A_μ is massive.

(c) Only in the SSB Goldstone mode of the BRST-invariant Lagrangian (18), and only after first renomalizing in the linear ϕ representation, does the renormalized Kibble ϕ unitary representation

$$\phi = \frac{1}{\sqrt{2}}(H + i\pi) \equiv \frac{1}{\sqrt{2}}\tilde{H}e^{-iY_\phi\tilde\pi/\langle H\rangle}$$
$$H = \langle H\rangle + h; \qquad \tilde H = \langle H\rangle + \tilde h$$
$$\tilde\pi \equiv \langle H\rangle \vartheta \tag{20}$$

make sense. Here the ϕ-hypercharge $Y_\phi = -1$.

(d) We prove to all loop orders the AHM Lee-Stora-Symanzik theorem (50), (A27), a gauge theory analogue of an old theorem for global $L\Sigma M$ [12], which forces the π mass squared $m_\pi^2 = 0$.

(e) We use pion-pole dominance (i.e. $m_\pi^2 = 0$) arguments to derive $U(1)_Y$ SSB WTI (49), (A22), and (A30).

(f) We prove with $U(1)_Y$ WTI that, in SSB Goldstone mode, $\tilde\pi$ in (20) is a NGB, and that the resultant SSB gauge theory has a shift symmetry $\tilde\pi \to \tilde\pi + \langle H\rangle\theta$ for constant θ.

Analysis is done in terms of the exact renormalized interacting fields, which asymptotically become the in/out states, i.e. free fields for physical S-matrix elements.

An important issue is the classification and disposal of relevant operators, in this case the π, h and A_μ inverse propagators (together with tadpoles). Define the exact renormalized pseudoscalar propagator in terms of a massless π, the Källén-Lehmann [12,26] spectral density $\rho^\pi_{\rm AHM}$, and wave-function renormalization $Z^\phi_{\rm AHM}$. In Lorenz gauge,

$$\Delta^\pi_{\rm AHM}(q^2) = -i(2\pi)^2\langle 0|T[\pi(y)\pi(0)]|0\rangle|^{\rm Fourier}_{\rm Transform}$$
$$= \frac{1}{q^2 + i\epsilon} + \int dm^2 \frac{\rho^\pi_{\rm AHM}(m^2)}{q^2 - m^2 + i\epsilon}$$
$$[Z^\phi_{\rm AHM}]^{-1} = 1 + \int dm^2 \rho^\pi_{\rm AHM}(m^2). \tag{21}$$

Define also the Brout-Englert-Higgs (BEH) scalar propagator in terms of a BEH scalar pole and the (subtracted) spectral density $\rho_{\rm BEH}$, and the *same* wave-function renormalization. We assume h decays weakly, and resembles a resonance,

$$\Delta^{\rm BEH}_{\rm AHM}(q^2) = -i(2\pi)^2\langle 0|T[h(x)h(0)]|0\rangle|^{\rm Fourier}_{\rm Transform}$$
$$= \frac{1}{q^2 - m^2_{\rm BEH;Pole} + i\epsilon}$$
$$+ \int dm^2 \frac{\rho^{\rm BEH}_{\rm AHM}(m^2)}{q^2 - m^2 + i\epsilon}$$
$$[Z^\phi_{\rm AHM}]^{-1} = 1 + \int dm^2 \rho^{\rm BEH}_{\rm AHM}(m^2)$$
$$\int dm^2 \rho^\pi_{\rm AHM}(m^2) = \int dm^2 \rho^{\rm BEH}_{\rm AHM}(m^2). \tag{22}$$

The spectral density parts of the propagators are

$$\Delta^{\pi;{\rm Spectral}}_{\rm AHM}(q^2) \equiv \int dm^2 \frac{\rho^\pi_{\rm AHM}(m^2)}{q^2 - m^2 + i\epsilon},$$
$$\Delta^{\rm BEH;Spectral}_{\rm AHM}(q^2) \equiv \int dm^2 \frac{\rho^{\rm BEH}_{\rm AHM}(m^2)}{q^2 - m^2 + i\epsilon}.$$

Dimensional analysis of the wave-function renormalizations (21) and (22) shows that the contribution of a state of mass/energy $\sim M_{\rm Heavy}$ to the spectral densities $\rho^\pi_{\rm AHM}(M^2_{\rm Heavy})$ and $\rho^{\rm BEH}_{\rm AHM}(M^2_{\rm Heavy}) \sim \frac{1}{M^2_{\rm Heavy}}$, and similarly its contribution to $\Delta^{\pi;{\rm Spectral}}_{\rm AHM}$ and $\Delta^{\rm BEH;Spectral}_{\rm AHM}$ includes only irrelevant terms $\sim \frac{1}{M^2_{\rm Heavy}}$. The finite Euclidean cutoff contributes only irrelevant terms $\sim \frac{1}{\Lambda^2}$.

B. Rigid/global $U(1)_Y$ WTI and conserved rigid/global current, for the *physical states* of the SSB AHM, in Lorenz gauge. Rigid/global $U(1)_Y$ charge is *not* conserved

In their seminal work, E. Kraus and K. Sibold [6] identified, in the Abelian Higgs model, "rigid and current

Ward identity (*sic*) in accordance with … BRS[T] invariance." They are called deformed because they have no remnant of the original anomaly-free $U(1)_Y$ symmetry.

The SSB case is tricky because gauge-fixing terms explicitly break both local and global $U(1)_Y$ symmetry in the BRST-invariant Lagrangian. Still, Kraus and Sibold's construction allowed them to demonstrate (with appropriate normalization conditions) proof of all-loop-orders renormalizability and unitarity for the SSB Abelian Higgs model. Because their renormalization relies only on deformed WTI, Kraus and Sibold's results are independent of a regularization scheme, for any acceptable scheme (i.e. if one exists).[3]

Nevertheless, Slavnov-Taylor identities [7] prove that the on-shell S-matrix elements of "physical particles" [i.e. spin $S = 0$ scalars h, π, and $S = 1$ transverse gauge bosons A_μ, but not fermionic ghosts $(\bar{\eta}, \omega)$] are independent of the usual (undeformed) anomaly-free $U(1)_Y$ local/gauge transformations, even though these break the Lagrangian's BRST symmetry.

We observe here that SSB S-matrix elements are therefore also independent of anomaly-free undeformed $U(1)_Y$ global/rigid transformations, resulting in a new global/rigid current and appropriate undeformed $U(1)_Y$ Ward-Takahashi identities. All this is done without reference to the unbroken Wigner mode and scale-invariant point.

We are interested in rigid-symmetric relations among 1-(h, π)-irreducible (1-ϕ-I) connected amputated Green's functions $\Gamma_{N,M}$, and among 1-(h, π)-reducible (1-ϕ-R) connected amputated transition-matrix (T-matrix) elements $T_{N,M}$, with external ϕ scalars. Because these are 1-A_μ-R in the AHM, and also 1-Φ-R in the E-AHM (i.e. reducible by cutting an A_μ or Φ line), it is convenient to use the powerful old tools (e.g. canonical quantization) from vintage quantum field theory (vintage QFT), a name coined by Ergin Sezgin.

We focus on the rigid/global AHM current[4] constructed with (10),

$$J^\mu_{\mathrm{AHM}} = \pi\partial^\mu H - H\partial^\mu \pi - eA^\mu(\pi^2 + H^2). \tag{23}$$

Rigid/global transformations of the fields arise, as usual, from the equal-time commutators (A7),

$$\begin{aligned}
\delta_{U(1)_Y} H(t,\vec{y}) &= -i\int d^3z[J^0_{\mathrm{AHM}}(t,\vec{z}), H(t,\vec{y})]e\Omega \\
&= -\int d^3z\pi(t,\vec{z})\delta^3(\vec{z}-\vec{y})e\Omega \\
&= -\pi(t,\vec{y})e\Omega, \\
\delta_{U(1)_Y} \pi(t,\vec{y}) &= -i\int d^3z[J^0_{\mathrm{AHM}}(t,\vec{z}), \pi(t,\vec{y})]e\Omega \\
&= \int d^3zH(t,\vec{z})\delta^3(\vec{z}-\vec{y})e\Omega \\
&= H(t,\vec{y})e\Omega,
\end{aligned} \tag{24}$$

so $J^\mu_{\mathrm{AHM}}(t,\vec{z})$ serves as a "proper" local current for commutator purposes.

In contrast, we show below that, in Lorenz gauge, the $U(1)_Y$ AHM [and therefore also the $U(1)_Y$ E-AHM] has no associated proper global charge Q because $\frac{d}{dt}Q(t) \neq 0$. [See Eq. (32) below.]

The classical equations of motion reveal a crucial fact: due to gauge-fixing terms in the BRST-invariant Lagrangian (18), the classical current (23) is not conserved. In Lorenz gauge

$$\partial_\mu J^\mu_{\mathrm{AHM}} = H m_A F_A, \tag{25}$$

with

$$m_A = e\langle H\rangle \tag{26}$$

and F_A being the gauge-fixing condition,

$$F_A \equiv \partial_\beta A^\beta. \tag{27}$$

The global $U(1)_Y$ current (23) is, however, conserved by the physical states, and therefore still qualifies as a "real" current for commutator purposes (24). Strict quantum constraints must be imposed to force the relativistically covariant theory of gauge bosons to propagate *only* its true number of quantum spin $S = 1$ degrees of freedom. These constraints are implemented, in the modern literature, by use of spin $S = 0$ fermionic Fadeev-Popov ghosts $(\bar{\eta}, \omega)$. The physical states and their time-ordered products, but not the BRST-invariant Lagrangian (18), then obey G. 't Hooft's [25] Lorenz-gauge gauge-fixing condition (17).

Equations (17) and (A4) restore conservation of the rigid/global $U(1)_Y$ current for the ϕ-sector connected time-ordered products

$$\langle 0|T[(\partial_\mu J^\mu_{\mathrm{AHM}}(z)) \times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\mathrm{connected}} = 0. \tag{28}$$

[3]E. Kraus and K. Sibold also constructed, in terms of deformed WTI, all-loop-orders renormalized QED, QCD, and the electroweak standard model [27,29], independent of regularization scheme. From this grew the powerful technology of "algebraic renormalization," used by them, W. Hollik and others [30], to renormalize supersymmetry (SUSY) QED, SUSY QCD, and the minimal supersymmetric standard model.

[4]This is related to the rigid/global hypercharge current of the third-generation global Dirac neutrino standard model ($\nu_D\mathrm{SM}^{\mathrm{G}}_{tb\tau\nu_\tau}$) explored in [18]: replace $\pi \to \pi_3, \pi^2 \to \vec{\pi}^2$, ungauge A_μ, add a charged pion current $\pi_2\partial^\mu\pi_1 - \pi_1\partial^\mu\pi_2$, add the third generation of SM quarks (three colors and two flavors) and leptons (one charged flavor), add one ν_R with SSB Dirac mass m_ν, and change the overall sign $J^{\mu;\mathrm{So\ Modified}}_{\mathrm{AHM}} \to -J^\mu_{Y;\nu_D\mathrm{SM}}$.

It is in this physical connected-time-ordered-product sense that the rigid global $U(1)_Y$ physical current is conserved: the current-conservation equation (28) is obeyed only when the divergence of the current is projected in this way on the physical states. Current conservation is not a property of the abstract Noether-current operator derived from the BRST-invariant Lagrangian (18).

Appendix A derives two towers of quantum $U(1)_Y$ WTI that exhaust the information content of (28), severely constrain the dynamics (i.e. the connected time-ordered products) of the ϕ-sector physical states of the SSB AHM and realize the new $U(1)_Y \otimes$ BRST symmetry of Sec. II.

We might have hoped to also build a charge

$$Q_{\mathrm{AHM}}(t) = \int d^3z J^0_{\mathrm{AHM}}(t, \vec{z}) \tag{29}$$

which would be conserved when similarly restricted to physical connected time-ordered products,

$$\begin{aligned}
&\langle 0|T\left[\left(\frac{d}{dt}Q_{\mathrm{AHM}}(t)\right)\right.\\
&\qquad\left.\times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)\right]|0\rangle_{\mathrm{connected}}\\
&= \int d^3z\langle 0|T[(\vec{\nabla}\cdot\vec{J}_{\mathrm{AHM}}(t,\vec{z}))\\
&\qquad\times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\mathrm{connected}},\\
&= \int_{2-\mathrm{surface}} d^2z\hat{z}^{2-\mathrm{surface}}\cdot\langle 0|T[(\vec{J}_{\mathrm{AHM}}(t,\vec{z}))\\
&\qquad\times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\mathrm{connected}},
\end{aligned}\tag{30}$$

where we have used Stokes' theorem, and $\hat{z}_\mu{}^{2-\mathrm{surface}}$ is a unit vector normal to the 2-surface. The time-ordered product constrains the 2-surface to lie on or inside the light cone.

At a given point on the surface of a large enough 3-volume $\int d^3z$ (e.g. the volume of all space) that lies on or inside the light cone, all fields on the $z^{2-\mathrm{surface}}$ are asymptotic in states and out states; are properly quantized as free fields, with each field species orthogonal to the others; and are evaluated at equal times, so that time ordering is unnecessary.

Nevertheless, the time derivative of this charge does not vanish even in this restricted physical sense, because, with the symmetry spontaneously broken, a specific term in the surface integral of the right-hand side of (23) does not vanish,

$$\begin{aligned}
&\int_{\text{light cone}\to\infty} dz\hat{z}^{\text{light cone}}\cdot\langle 0|T[(-\langle H\rangle\vec{\nabla}\pi(z))\\
&\qquad\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle \neq 0.
\end{aligned}\tag{31}$$

In the SSB AHM, π is massless (in Lorenz gauge), and so capable of carrying (along the light cone) long-ranged pseudoscalar forces out to the very ends of the light cone ($z^{\text{light cone}} \to \infty$).

Equations (30) and (31) then show that the spontaneously broken $U(1)_Y$ AHM charge is not conserved, even for connected time-ordered products, in Lorenz gauge

$$\begin{aligned}
&\left\langle 0|T\left[\left(\frac{d}{dt}Q_{\mathrm{AHM}}(t)\right)\right.\right.\\
&\qquad\left.\left.\times h(x_1)\ldots h(x_N)\pi_{t_1}(y_1)\ldots\pi_{t_M}(y_M)\right]|0\right\rangle_{\mathrm{connected}} \neq 0,
\end{aligned}\tag{32}$$

dashing, at least for the authors, all further hope of a conserved charge.

The classic proof of the Goldstone theorem [8,10,11] requires a conserved charge $\frac{d}{dt}Q = 0$, so that proof fails for spontaneously broken gauge theories. This is a very famous result [8,19,31,32], and allows the spontaneously broken AHM to generate a mass gap m_A for the vector A^μ and to avoid massless particles in its observable physical spectrum. This is true even in Lorenz gauge, where there *is* a Goldstone theorem, and consequently $\tilde{\pi}$ is a derivatively coupled (hence massless) NGB [8,19], and where there is a LSS theorem, so π is massless.

Massless π (not $\tilde{\pi}$) is the basis of our pion-pole-dominance-based $U(1)_Y$ WTI, derived in Appendix A, which give relations among 1-ϕ-I connected amputated ϕ-sector Greens functions $\Gamma_{N,M}$ (33) and (A31); 1-soft-pion theorems (49), (A22), and (A30); infrared (IR) finiteness for $m_\pi^2 = 0$ (49) and (A22); a LSS (and Goldstone) theorem (50) and (A27); and vanishing 1-ϕ-R connected amputated on-shell ϕ-sector T-matrix elements $T_{N,M}$ (49) and (A30) that realize the full $U(1)_Y \otimes$ BRST symmetry of Sec. II.

C. Construction of the scalar-sector effective Lagrangian from those $U(1)_Y$ WTI that govern connected amputated 1-ϕ-I Greens functions

In Appendix A we derive $U(1)_Y$ pion-pole-dominance 1-ϕ-R connected amputated T-matrix WTI (A30) for the SSB AHM. Their solution is a tower of recursive $U(1)_Y$ WTI (A31) that govern 1-ϕ-I ϕ-sector connected amputated Greens functions $\Gamma_{N,M}$. For π with $CP = -1$, the result

$$\begin{aligned}
&\langle H\rangle\Gamma_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M)\\
&\quad=\sum_{m=1}^{M}\Gamma_{N+1,M-1}(q_m p_1\ldots p_N;q_1\ldots\widehat{q_m}\ldots q_M)\\
&\qquad-\sum_{n=1}^{N}\Gamma_{N-1,M+1}(p_1\ldots\widehat{p_n}\ldots p_N;p_n q_1\ldots q_M)
\end{aligned}\tag{33}$$

is valid for $N, M \geq 0$. On the left-hand side of (33) there are N renormalized h external legs (coordinates x, momenta p), M renormalized ($CP = -1$) π external legs (coordinates y, momenta q), and one renormalized soft external $\pi(k_\mu = 0)$ (coordinates z, momenta k). "Hatted" fields with momenta $(\widehat{p_n}, \widehat{q_m})$ are omitted.

The rigid $U(1)_Y$ WTI 1-soft-pion theorems (33) relate a 1-ϕ-I Green's function with $(N+M+1)$ external fields (which include a zero-momentum π) to two 1-ϕ-I Green's functions with $(N+M)$ external fields.[5] The Green's functions $\Gamma_{N,M}(p_1 \ldots p_N; q_1 \ldots q_M)$ are not themselves gauge independent. Furthermore, although 1-ϕ-I, they are 1-A^μ-reducible (1-A^μ-R) by cutting a transverse A_μ gauge boson line.

The 1-ϕ-I π and h inverse propagators are

$$\Gamma_{0,2}(;q,-q) \equiv [\Delta_\pi(q^2)]^{-1},$$
$$\Gamma_{2,0}(q,-q;) \equiv [\Delta_{\rm BEH}(q^2)]^{-1}. \quad (34)$$

We can now form the ϕ-sector effective momentum-space Lagrangian in Lorenz gauge. All perturbative quantum loop corrections, to all loop orders and including all UVQD, log-divergent and finite contributions, are included in the ϕ-sector effective Lagrangian: 1-ϕ-I Green's functions $\Gamma_{N,M}(p_1 \ldots p_N; q_1 \ldots q_M)$, wave-function renormalizations, renormalized ϕ-scalar propagators (21) and (22), the BEH vacuum expectation value (VEV) $\langle H \rangle$ (A35), and all gauge boson and ghost propagators. This includes the full all-loop-orders renormalization of the AHM ϕ-sector, originating in quantum loops containing transverse virtual gauge bosons, ϕ scalars and ghosts: A^μ, $h, \pi, \bar{\eta}, \omega$, respectively. Because they arise entirely from global $U(1)_Y$ WTI, our results are independent of regularization scheme [6].

[5]The rigid $U(1)_Y$ WTI (33) for the $U(1)_Y$ AHM *gauge theory* are a generalization of the classic work of B. W. Lee [12], who constructed two all-loop-orders renormalized towers of WTI for the global $SU(2)_L \times SU(2)_R$ Gell-Mann Lévy (GML) model [17] with PCACs. We replace GML's strongly interacting linear sigma model (LΣ M) with a weakly interacting BEH LΣ M, with explicit PCAC breaking. Replace $\sigma \to H$, $\vec{\pi} \to \pi$, $m_\sigma \to m_{\rm BEH}$ and $f_\pi \to \langle H \rangle$, and add local gauge group $U(1)_Y$. This generates a set of global $U(1)_Y$ WTI governing relations among weak-interaction 1-ϕ-R T-matrix elements $T_{N,M}$. A solution set of those $U(1)_Y$ WTI then governs relations among $U(1)_Y$ 1-ϕ-I Green's functions $\Gamma_{N,M}$.
As observed by Lee for GML, one of those on-shell T-matrix WTI is equivalent to the Goldstone theorem. This equivalence relies on the ability to incorporate a PCAC term into the global theory, and then retrieve the spontaneously broken theory in the appropriate zero-explicit-breaking limit, namely $m_\pi^2 \to 0$. In the gauge theory, although explicit-breaking terms are allowed by power-counting, they violate the BRST symmetry and spoil unitarity [33]. Yet, the T-matrix WTI persists and forces $m_\pi^2 = 0$ in Lorenz gauge, which is now the new LSS theorem. The Goldstone theorem also persists in Lorenz gauge, and forces $m_{\tilde{\pi}}^2 = 0$.
Appendix A includes, in Table I, a translation between the WTI proofs in this paper (a gauge theory) and in B. W. Lee (a global theory).

We want to classify operators arising in AHM loops, and separate the finite operators from the divergent ones. We focus on finite relevant operators, as well as quadratic and logarithmically divergent operators.

There are three classes of finite operators.

(i) Finite $\mathcal{O}_{\rm AHM}^{1/\Lambda^2;{\rm Irrelevant}}$ vanish as $m_{\rm Weak}^2/\Lambda^2 \to 0$;

(ii) $\mathcal{O}_{\rm AHM}^{d>4;{\rm Light}}$ are finite-dimension $d>4$ operators, where only the light degrees of freedom A^μ, $h, \pi, \bar{\eta}, \omega$ contribute to all-loop-orders renormalization;

(iii) $\mathcal{O}_{\rm AHM}^{d\leq 4;{\rm Non\ Analytic}}$ are finite-dimension $d \leq 4$ operators that are nonanalytic in momenta or in a renormalization scale μ^2 (e.g. finite renormalization-group logarithms).

All such operators are ignored.

$$\mathcal{O}_{\rm AHM}^{\rm Ignore} = \mathcal{O}_{\rm AHM}^{1/\Lambda^2;{\rm Irrelevant}} + \mathcal{O}_{\rm AHM}^{d>4;{\rm Light}} + \mathcal{O}_{\rm AHM}^{d\leq 4;{\rm Non\ Analytic}}. \quad (35)$$

Such finite operators appear throughout the $U(1)_Y$ WTI (33),

(i) $N+M \geq 5$ is $\mathcal{O}_{\rm AHM}^{1/\Lambda^2;{\rm Irrelevant}}$ and $\mathcal{O}_{\rm AHM}^{d>4;{\rm Light}}$;

(ii) the left-hand side of (33) for $N+M=4$ is also $\mathcal{O}_{\rm AHM}^{1/\Lambda^2;{\rm Irrelevant}}$ and $\mathcal{O}_{\rm AHM}^{d>4;{\rm Light}}$;

(iii) $N+M \leq 4$ operators $\mathcal{O}_{\rm AHM}^{d\leq 4;{\rm Non\ Analytic}}$ appear in (33).

Finally, there are $N+M \leq 4$ operators that are analytic in momenta. We expand these in powers of momenta, count the resulting dimension of each term in the operator Taylor series, and ignore $\mathcal{O}_{\rm AHM}^{d>4;{\rm Light}}$ and $\mathcal{O}_{\rm AHM}^{1/\Lambda^2;{\rm Irrelevant}}$ terms in that series.

Suppressing gauge fields, the all-loop-orders renormalized scalar-sector effective Lagrangian with operator dimension less than or equal to 4 is then formed for (h, π) with $CP = (1,-1)$,

$$\begin{aligned} L_{{\rm AHM};\phi;{\rm Lorenz}}^{\rm Eff;Wigner,SI,Goldstone} = {} & \Gamma_{1,0}(0;)h + \frac{1}{2!}\Gamma_{2,0}(p,-p;)h^2 \\ & + \frac{1}{2!}\Gamma_{0,2}(;q,-q)\pi^2 + \frac{1}{3!}\Gamma_{3,0}(000;)h^3 \\ & + \frac{1}{2!}\Gamma_{1,2}(0;00)h\pi^2 + \frac{1}{4!}\Gamma_{4,0}(0000;)h^4 \\ & + \frac{1}{2!2!}\Gamma_{2,2}(00;00)h^2\pi^2 \\ & + \frac{1}{4!}\Gamma_{0,4}(;0000)\pi^4 + \mathcal{O}_{\rm Ignore}^{\rm AHM}. \end{aligned} \quad (36)$$

The WTI (33) for Green's functions severely constrain the effective Lagrangian (36).

(i) $N=0$, $M=1$ WTI,

$$\Gamma_{1,0}(0;) = \langle H \rangle \Gamma_{0,2}(;00), \quad (37)$$

since no momentum can run into the tadpoles.

(ii) $N=1$, $M=1$ WTI,

$$\begin{aligned}\Gamma_{2,0}(-q,q;)-\Gamma_{0,2}(;q,-q)\\ &=\langle H\rangle\Gamma_{1,2}(-q;q0)\\ &=\langle H\rangle\Gamma_{1,2}(0;00)+\mathcal{O}^{\rm AHM}_{\rm Ignore}\\ \Gamma_{2,0}(00;)=\Gamma_{0,2}(;00)+\langle H\rangle\Gamma_{1,2}(0;00).\end{aligned}\tag{38}$$

(iii) $N=2$, $M=1$ WTI,

$$\langle H\rangle\Gamma_{2,2}(00;00)=\Gamma_{3,0}(000;)-2\Gamma_{1,2}(0;00).\tag{39}$$

(iv) $N=0$, $M=3$ WTI,

$$\langle H\rangle\Gamma_{0,4}(;0000)=3\Gamma_{1,2}(0;00).\tag{40}$$

(v) $N=1$, $M=3$ WTI,

$$0=3\Gamma_{2,2}(00;00)-\Gamma_{0,4}(;0000).\tag{41}$$

(vi) $N=3$, $M=1$ WTI,

$$0=\Gamma_{4,0}(0000;)-3\Gamma_{2,2}(00;00).\tag{42}$$

(vii) The quadratic and quartic coupling constants are defined in terms of two-point and four-point 1-ϕ-I Green's function,

$$\begin{aligned}\Gamma_{0,2}(;00)&\equiv -m_\pi^2,\\ \Gamma_{0,4}(;0000)&\equiv -6\lambda_\phi^2.\end{aligned}\tag{43}$$

The all-loop-orders renormalized ϕ-sector momentum-space effective Lagrangian (36)—constrained only by those $U(1)_Y$ WTI governing Green's functions (33)—may be written

$$\begin{aligned}L^{\rm Eff;Wigner,SI,Goldstone}_{\rm AHM;\phi;Lorenz}&=L^{\rm Kinetic;Eff;Wigner,SI,Goldstone}_{\rm AHM;\phi;Lorenz}\\ &\quad -V^{\rm Eff;Wigner,SI,Goldstone}_{\rm AHM;\phi;Lorenz}+\mathcal{O}^{\rm AHM}_{\rm Ignore},\end{aligned}\tag{44}$$

with

$$\begin{aligned}L^{\rm Kinetic;Eff;Wigner,SI,Goldstone}_{\rm AHM;\phi;Lorenz}\\ &=\frac{1}{2}(\Gamma_{0,2}(;p,-p)-\Gamma_{0,2}(;00))h^2\\ &\quad+\frac{1}{2}(\Gamma_{0,2}(;q,-q)-\Gamma_{0,2}(;00))\pi^2,\end{aligned}\tag{45}$$

incorporating finite nontrivial wave-function renormalization

$$\Gamma_{0,2}(;q,-q)-\Gamma_{0,2}(;00)\sim q^2,\tag{46}$$

and

$$\begin{aligned}V^{\rm Eff;Wigner,SI,Goldstone}_{\rm AHM;\phi;Lorenz}&=m_\pi^2\left[\frac{h^2+\pi^2}{2}+\langle H\rangle h\right]\\ &\quad+\lambda_\phi^2\left[\frac{h^2+\pi^2}{2}+\langle H\rangle h\right]^2.\end{aligned}\tag{47}$$

The ϕ-sector effective Lagrangian (44) has insufficient boundary conditions to distinguish among the three modes [12–15] of the BRST-invariant Lagrangian $L_{\rm AHM}$ in (18). For example, the effective potential $V^{\rm Eff;Wigner;SI;Goldstone}_{\rm AHM;\phi;Lorenz}$ becomes in various limits[6] the AHM Wigner mode ($m_A^2=0,\langle H\rangle=0,m_\pi^2=m_{\rm BEH}^2\neq 0$), the AHM SI point ($m_A^2=0,\langle H\rangle=0,m_\pi^2=m_{\rm BEH}^2=0$), or AHM Goldstone mode ($m_A^2\neq 0,\langle H\rangle\neq 0,m_\pi^2=0,m_{\rm BEH}^2\neq 0$), with

$$\begin{aligned}V^{\rm Eff;Wigner}_{\rm AHM;\phi;Lorenz}&=m_\pi^2\left[\frac{h^2+\pi^2}{2}\right]+\lambda_\phi^2\left[\frac{h^2+\pi^2}{2}\right]^2\\ V^{\rm Eff;Scale\ Invariant}_{\rm AHM;\phi;Lorenz}&=\lambda_\phi^2\left[\frac{h^2+\pi^2}{2}\right]^2,\\ V^{\rm Eff;Goldstone}_{\rm AHM;\phi;Lorenz}&=\lambda_\phi^2\left[\frac{h^2+\pi^2}{2}+\langle H\rangle h\right]^2.\end{aligned}\tag{48}$$

Equation (44) has exhausted the constraints (on the allowed terms in the ϕ-sector effective Lagrangian) due to those $U(1)_Y$ WTI that govern 1-ϕ-I ϕ-sector Green's functions $\Gamma_{N,M}$ (33), (A31). In order to provide boundary conditions that distinguish among the effective potentials in (48), we must turn to the $U(1)_Y$ WTI that govern ϕ-sector 1-ϕ-R T-matrix elements $T_{N,M}$.

D. The LSS theorem: IR finiteness and automatic tadpole renormalization

"Whether you like it or not, you have to include in the Lagrangian all possible terms consistent with locality and power counting, unless otherwise constrained by Ward identities" (Kurt Symanzik, in a private letter to Raymond Stora [36].)

In strict obedience to K. Symanzik's edict, we now further constrain the allowed terms in the ϕ-sector effective Lagrangian, using those $U(1)_Y$ Ward-Takahashi identities that govern 1-ϕ-R T-matrix elements $T_{N,M}$.

In Appendix A, we extend Adler's self-consistency condition [originally written for the global $SU(2)_L\times SU(2)_R$ Gell-Mann-Lévy linear sigma model with PCAC

[6]The inclusive Gell-Mann Lévy [17] effective potential derived [34] from B. W. Lee's WTI [12] reduces to the three different effective potentials of the global $SU(2)_L\times SU(2)_R$ Schwinger model [35]: Schwinger Wigner mode ($\langle H\rangle=0$, $m_\pi^2=m_{\rm BEH}^2\neq 0$), Schwinger scale-invariant point ($\langle H\rangle=0$, $m_\pi^2=m_{\rm BEH}^2=0$), or Schwinger Goldstone mode ($\langle H\rangle\neq 0$, $m_\pi^2=0,m_{\rm BEH}^2\neq 0$).

[37,38]], but now derived for the AHM gauge theory in Lorenz gauge (A22)

$$\langle H\rangle T_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M)\times(2\pi)^4\delta^4\left(\sum_{n=1}^{N}p_n+\sum_{m=1}^{M}q_m\right)\Bigg|_{q_1^2=q_2^2\ldots=q_M^2=0}^{p_1^2=p_2^2\ldots=p_N^2=m_{\rm BEH}^2}=0. \tag{49}$$

The T-matrix elements vanish as one of the pion momenta goes to 0 provided all other physical scalar particles are on mass shell. In other words, these are new 1-soft-pion theorems. Equation (49) also

> *"asserts the absence of [IR] divergences in the scalar-sector (of AHM) Goldstone mode (in Lorenz gauge). Although individual Feynman diagrams are IR divergent, those IR divergent parts cancel exactly in each order of perturbation theory. Furthermore, the Goldstone mode amplitude must vanish in the soft-pion limit" (B. W. Lee [12]).*

It is crucial to note that the external states in $\mathbf{T_{N,M}}$ are N h's and M π's, not $\tilde{\pi}$'s. We are working in the soft-π, not the soft-$\tilde{\pi}$ limit.

The $N=0$, $M=1$ case of (49) is the LSS theorem (A27),

$$\langle H\rangle T_{0,2}(;00)=0. \tag{50}$$

This looks like the Goldstone theorem[7] but, since it involves π not $\tilde{\pi}$, it is quite distinct.

We write the LSS theorem (50) as a further constraint on the 1-ϕ-I Green's function,[8]

[7]B. W. Lee [12] proves two towers of WTI for the global $SU(2)_L\times SU(2)_R$ GML model [17] in the presence of the PCAC hypothesis. The PCAC conserves the vector current $\partial_\mu\vec{J}^{\mu;\rm GML}_{L+R}=0$, but explicitly breaks the axial-vector current, $\partial_\mu\vec{J}^{\mu;\rm GML}_{L-R}=\gamma^{\rm GML}_{\rm PCAC}\vec{\pi}$. Lee identifies the all-loop-orders GML WTI

$$\gamma^{\rm GML}_{\rm PCAC}=-\langle H\rangle\Gamma^{\rm GML}_{0,2}(;00) \tag{51}$$

as the "Goldstone theorem in the presence of PCAC." Exact conservation of $\vec{J}^{\mu;\rm GML}_{L-R}$, i.e. $\gamma^{\rm GML}_{\rm PCAC}=0$, is restored for both GML's Wigner mode ($\langle H\rangle\equiv 0,\Gamma^{\rm GML}_{0,2}(;00)\neq 0$) and its Goldstone mode ($\langle H\rangle\neq 0,\Gamma^{\rm GML}_{0,2}(;00)\equiv 0$).
The PCAC analogy for the Lorenz-gauge AHM would have been

$$\partial_\mu J^{\mu;\rm AHM}_L=\gamma^{\rm AHM}_{\rm PCAC}\pi+\langle H\rangle\times(\text{a gauge-fixing term})$$
$$\gamma^{\rm AHM}_{\rm PCAC}=-\langle H\rangle\Gamma^{\rm AHM}_{0,2}(;00), \tag{52}$$

but the AHM is a local/gauge theory. This requires that $\gamma^{\rm AHM}_{\rm PCAC}\equiv 0$ exactly. SSB current conservation can be broken only softly by gauge-fixing terms as in (25), in order to preserve renormalizability and unitarity [7]. The Lorenz-gauge AHM LSS theorem therefore reads

$$\boldsymbol{\gamma^{\rm AHM}_{\rm PCAC}=-\langle H\rangle\Gamma^{\rm AHM}_{0,2}(;00)\equiv 0}, \tag{53}$$

as in (56). The crucial fact here is that, in the SSB Goldstone mode of the AHM (and SSB E-AHM, $\mathrm{SM}^{\rm Bosons}_{\rm Ghosts}$, ν_DSM and $E-\nu_D$SM [39]) with $\langle H\rangle\neq 0$,

$$0\equiv\Gamma^{\rm AHM}_{0,2}(;00)=[\Delta^{\rm AHM}_\pi(0)]^{-1}=-m_\pi^2. \tag{54}$$

This condition that the mass squared of the pseudoscalar π is exactly 0 is distinct from, and more powerful than, the more familiar condition $m^2_{\tilde{\pi}}=0$, i.e. the masslessness of the NGB $\tilde{\pi}$.
We see that (49) adds information to that contained in Green's function WTI (33), (A31). Beyond IR finiteness [12], on-shell T-matrix WTI (49), (A27), and (A22) provide absolutely crucial constraints on the gauge theory by insisting that $\gamma^{\rm AHM}_{\rm PCAC}\equiv 0$ as in (53) and (54), that the $U(1)_Y$ current is softly broken or conserved as in (25), (17), and (28), and that unitarity and renormalizability of the AHM gauge theory is preserved [7].

[8]A SSB 1-ϕ-R T-matrix element $T_{N,M}$ consists of a sum of many possible diagrams, $T^i_{N,M}$, where i indexes all the possibilities. We can represent each such diagram as a set of 1-ϕ-I vertices $\Gamma_{n,m}$ (which we term beads) attached by ϕ propagators, in such a way as to leave N external h lines and M external π lines.
Consider in particular $T_{0,2}(;q,-q)$. For any diagram $T^i_{0,2}(;q,-q)$ contributing to $T_{0,2}(;q,-q)$, there is a unique "string" of ϕ propagators that threads from end to end through the diagram. Each bead on this string has two ϕ legs, with equal and opposite 4-momenta q and $-q$. Since $\Gamma_{0,0}=\Gamma_{0,1}=\Gamma_{1,0}=0$, one cannot have additional ϕ legs connecting off this main ϕ line to another "side bead" unless they connect in groups of two or more. But in this case, the main bead and the secondary bead cannot be separated by cutting one ϕ line, and so are part of the same bead. Since $CP=(+1,-1)$ for (h,π), and is conserved in this paper, the 1-h-reducible contribution vanishes, and so the beads must be connected only by π s, and each bead is just a $\Gamma_{0,2}(;q,-q)$.
Thus the diagram corresponding to $T^i_{0,2}(;q,-q)$ would appear to consist of $i+1$ copies of $\Gamma_{0,2}(;q,-q)$ irreducible vertices connected by π propagators $\Delta_\pi(q^2)$, and so $T^i_{0,2}(;q,-q)=\Gamma_{0,2}(;q,-q)[\Gamma_{0,2}(;q,-q)\Delta_\pi(q^2)]^i$. $T_{0,2}(;q,-q)$ would then consist of the sum over all such strings.
However, $\Gamma_{0,2}(;q,-q)\Delta_\pi(q^2)=1$, and so, in fact, one should not separately count each $T^i_{0,2}(;q,-q)$, but rather

$$T_{0,2}(;q,-q)=\Gamma_{0,2}(;q,-q)=[\Delta_\pi(q^2)]^{-1}. \tag{55}$$

$$\langle H\rangle\Gamma_{0,2}(;00) = \langle H\rangle[\Delta_\pi(0)]^{-1} = 0, \quad (56)$$

or in terms of the π mass

$$\langle H\rangle m_\pi^2 = 0. \quad (57)$$

Evaluating the effective potential[9] in (44) with $\langle H\rangle \neq 0$, and then in the Kibble representation

$$\begin{aligned} V^{\mathrm{Eff;PreLSSGoldstone\ Mode}}_{\mathrm{AHM};\phi;\mathrm{Lorenz}} &= m_\pi^2\left[\frac{h^2+\pi^2}{2} + \langle H\rangle h\right] + \lambda_\phi^2\left[\frac{h^2+\pi^2}{2} + \langle H\rangle h\right]^2 \\ &= m_\pi^2\left[\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right] + \lambda_\phi^2\left[\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right]^2 \\ &= \frac{m_\pi^2}{2}[\tilde{H}^2 - \langle H\rangle^2] + \frac{\lambda_\phi^2}{4}[\tilde{H}^2 - \langle H\rangle^2]^2. \end{aligned} \quad (58)$$

As expected, the NGB $\tilde{\pi}$ has disappeared from the effective potential, has purely derivative couplings through its kinetic term, and obeys the shift symmetry $\tilde{\pi} \to \tilde{\pi} + \langle H\rangle\theta$ for constant θ. In other words, the Goldstone theorem is, on the face of it, already properly enforced.

Equation (58) appears at first sight to embrace a disaster: the term linear in $\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2$ [a remnant of Wigner mode in (48)] persists, destroying the symmetry of the famous "Mexican hat," and the AHM is not actually in Goldstone mode. To the rescue, the LSS theorem, (50), (56) or (57) (and not the Goldstone theorem) forces the AHM gauge theory fully into its true Goldstone $\langle H\rangle \neq 0$ mode,[10]

$$\begin{aligned} V^{\mathrm{Eff;LSSGoldstone\ Mode}}_{\mathrm{AHM};\phi;\mathrm{Lorenz}} &= \frac{\lambda_\phi^2}{4}[\tilde{H}^2 - \langle H\rangle^2]^2 \\ &= \lambda_\phi^2\left[\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right]^2. \end{aligned} \quad (60)$$

[9]In the AHM-forbidden case of $\langle H\rangle m_\pi^2 \neq 0$ imagined in (58), $\lim_{k_\lambda\to 0} k^2\Delta_\pi(k^2, m_\pi^2 \neq 0) = 0$ in (A17), so (A20), (33), and (49) are still true for all three modes: these include Wigner mode and the scale-invariant point where $\langle H\rangle = 0$, and where the LSS theorem $\langle H\rangle T_{0,2}(;00) = 0$, and all the Adler self-consistency conditions, are satisfied trivially.

[10]Reference [53] shows that, including $d > 4$ operators, the SSB AHM scalar potential may be written, from symmetry and WTI alone, in the form

$$V^{\mathrm{eff}}_{\phi;\mathrm{AHM}} = -\sum_{n=2}^{\infty}\frac{1}{(2n)!}\Gamma_{0,2n}(;0...0)(\tilde{H}^2 - \langle H\rangle^2)^n. \quad (59)$$

So can the E-AHM.

A central result of this paper is to recognize that, in order to force Eq. (58) to Eq. (60), the LSS theorem incorporates a new on-shell T-matrix symmetry, which is not a full symmetry of the BRST-invariant AHM Lagrangian. AHM physics, but not its Lagrangian, has the $U(1)_Y \otimes$ BRST symmetry of Sec. II, a conserved current (23) and (28), undeformed WTI governing connected amputated Green's functions (33), and undeformed WTI governing connected amputated on-shell T-matrix elements (49).

A crucial effect of the LSS theorem (57), together with the $N = 0$, $M = 1$ $U(1)_Y$ Ward-Takahashi Green's function identity (33), is to automatically eliminate tadpoles in (36)

$$\Gamma_{1,0}(0;) = \langle H\rangle\Gamma_{0,2}(;00) = 0, \quad (61)$$

so that separate tadpole renormalization is unnecessary.

The proof of the Lee-Stora-Symanzik theorem for the AHM (in Appendix A) is extended to the E-AHM (which includes certain beyond-the-AHM scalars Φ and CP-conserving fermions ψ) in Appendix B. The AHM LSS considerations in Sec. III therefore have their direct corresponding analogs, for the E-AHM, in Secs. IV and V. We do not needlessly repeat ourselves there.

E. Further constraints on the ϕ-sector effective Lagrangian: $m^2_{\mathrm{BEH}} = 2\lambda_\phi^2\langle H\rangle^2$

We rewrite the Goldstone-mode effective Lagrangian (44) and effective potential (58), but now including the constraint from the LSS theorem, (50), (56), and (57),

$$\begin{aligned} L^{\mathrm{Eff;Goldstone}}_{\mathrm{AHM};\phi;\mathrm{Lorenz}} &= L^{\mathrm{Kinetic;Eff;Goldstone}}_{\mathrm{AHM};\phi;\mathrm{Lorenz}} - V^{\mathrm{Eff;Goldstone}}_{\mathrm{AHM};\phi;\mathrm{Lorenz}} + \mathcal{O}^{\mathrm{AHM}}_{\mathrm{Ignore}} \\ V^{\mathrm{Eff;Goldstone}}_{\mathrm{AHM};\phi;\mathrm{Lorenz}} &= \lambda_\phi^2\left[\frac{h^2+\pi^2}{2} + \langle H\rangle h\right]^2, \end{aligned} \quad (62)$$

with wave-function renormalization

$$\Gamma_{0,2}(;q,-q) - \Gamma_{0,2}(;00) = q^2 + \mathcal{O}^{\mathrm{AHM}}_{\mathrm{Ignore}}, \quad (63)$$

so the ϕ-sector Goldstone-mode effective coordinate-space Lagrangian becomes

$$L^{\mathrm{Eff;Goldstone}}_{\mathrm{AHM};\phi;\mathrm{Lorenz}} = |D_\mu\phi|^2 - \lambda_\phi^2\left[\frac{h^2+\pi^2}{2} + \langle H\rangle h\right]^2 + \mathcal{O}^{\mathrm{AHM}}_{\mathrm{Ignore}}. \quad (64)$$

Equation (64) is the ϕ-sector effective Lagrangian of the *spontaneously broken* Abelian Higgs model, in Lorenz gauge, constrained by the LSS theorem.[11]

(i) It is derived from the local BRST-invariant Lagrangian L_{AHM} (18).

(ii) It includes all divergent $\mathcal{O}(\Lambda^2), \mathcal{O}(\ln\Lambda^2)$ and finite terms that arise to all perturbative loop orders in the full $U(1)_Y$ gauge theory, due to virtual transverse gauge bosons, ϕ scalars and ghosts (A^μ, $h, \pi, \bar{\eta}, \omega$, respectively).

(iii) It obeys the LSS theorem (50) and (56) and all other $U(1)_Y$ Ward-Takahashi Green's function and T-matrix identities.

(iv) It obeys the Goldstone theorem in the Lorenz gauge, having a massless derivatively coupled NGB, $\tilde{\pi}$.

(v) It is minimized at $(H = \langle H\rangle, \pi = 0)$, and obeys stationarity [16] of that true minimum.

(vi) It preserves the theory's renormalizability and unitarity, which require that wave-function renormalization, $\langle H\rangle_{\mathrm{Bare}} = [Z^{\phi}_{\mathrm{AHM}}]^{1/2}\langle H\rangle$ [16,18,26], forbid UVQD, relevant, or any other dimension-2 operator corrections to $\langle H\rangle$.

(vii) The LSS theorem (50) has caused all relevant operators in the spontaneously broken Abelian Higgs model to vanish.

In order to make manifest that $\tilde{\pi}$ is a true NGB [7,41] in Lorenz gauge, rewrite (64) in the Kibble representation [2,41], with $Y_\phi = -1$ being the ϕ hypercharge. In coordinate space,

$$L^{\mathrm{Eff;Goldstone}}_{\mathrm{AHM};\phi;\mathrm{Lorenz}} = \frac{1}{2}(\partial_\mu \tilde{h})^2 + \frac{1}{2}e^2(\langle H\rangle + \tilde{h})^2\left(A_\mu + \frac{1}{e\langle H\rangle}\partial_\mu\tilde{\pi}\right)^2 - \frac{\lambda_\phi^2}{4}(\tilde{h}^2 + 2\langle H\rangle\tilde{h})^2 + \mathcal{O}^{\mathrm{AHM}}_{\mathrm{Ignore}} \tag{68}$$

[11]Imagine we suspected that π is not all-loop-orders massless in Lorenz gauge SSB AHM, and simply/naively wrote a mass-squared $m^2_{\pi:\mathrm{Pole}}$ into the π inverse propagator

$$[\Delta_\pi(0)]^{-1} \equiv -m_\pi^2 = -m^2_{\pi;\mathrm{Pole}}\left[1 + m^2_{\pi;\mathrm{Pole}}\int dm^2 \frac{\rho_\pi(m^2)}{m^2}\right]^{-1}. \tag{65}$$

However, the LSS theorem (56) insists instead that

$$\langle H\rangle[\Delta_\pi(0)]^{-1} \equiv -\langle H\rangle m_\pi^2 = \langle H\rangle\Gamma_{0,2}(;00) = 0. \tag{66}$$

The π pole-mass vanishes *exactly.*

$$m^2_{\pi;\mathrm{Pole}} = m_\pi^2\left[1 - m_\pi^2\int dm^2\frac{\rho_\pi(m^2)}{m^2}\right]^{-1} = 0. \tag{67}$$

shows that $\tilde{\pi}$ has only derivative couplings and, for constant θ, a shift symmetry

$$\tilde{\pi} \to \tilde{\pi} + \langle H\rangle\theta. \tag{69}$$

The Green's function WTI (33) for $N = 1$, $M = 1$, constrained by the LSS theorem (56), relates the BEH mass to the coefficient of the $h\pi^2$ vertex

$$\Gamma_{2,0}(00;) = \langle H\rangle\Gamma_{1,2}(0;00). \tag{70}$$

Therefore, the BEH mass squared in (68),

$$m^2_{\mathrm{BEH}} = 2\lambda_\phi^2\langle H\rangle^2, \tag{71}$$

arises entirely from SSB, as does (together with its AHM decays) the resonance pole-mass squared,

$$m^2_{\mathrm{BEH;Pole}} = 2\lambda_\phi^2\langle H\rangle^2\left[1 - 2\lambda_\phi^2\langle H\rangle^2\int dm^2\frac{\rho^{\mathrm{BEH}}_{\mathrm{AHM}}(m^2)}{m^2 - i\epsilon}\right]^{-1} + \mathcal{O}^{\mathrm{Ignore}}_{\mathrm{AHM};\phi}. \tag{72}$$

IV. EXTENDED ABELIAN HIGGS MODEL: WTI-ENFORCED DECOUPLING OF CERTAIN HEAVY MATTER REPRESENTATIONS

If the Euclidean cutoff Λ^2 were a true proxy for very heavy $M^2_{\mathrm{Heavy}} \gg m^2_{\mathrm{Weak}}$ spin $S = 0$ scalars Φ, and $S = \frac{1}{2}$ fermions ψ, we would already be in a position to comment on their decoupling. Unfortunately, although the literature seems to cite such a proxy, it is simply not true. "In order to prove theorems that reveal symmetry-driven results in gauge theories, one must keep *all* of the terms arising from *all* Feynman graphs, not just a selection of interesting terms from a representative subset of Feynman graphs" (Ergin Sezgin's dictum).

A. ϕ-sector effective Lagrangian for the E-AHM

1. 1-ϕ-I *connected amputated* ϕ*-sector Green's functions* $\Gamma^{\mathrm{E\text{-}AHM}}_{N,M}$

In Appendix B we derive a tower of recursive $U(1)_Y$ WTI (B18) that govern connected amputated 1-ϕ-I Green's functions for the E-AHM,

$$\langle H\rangle\Gamma^{\mathrm{E\text{-}AHM}}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) = \sum_{m=1}^{M}\Gamma^{\mathrm{E\text{-}AHM}}_{N+1,M-1}(q_m p_1\ldots p_N;q_1\ldots\widehat{q_m}\ldots q_M) - \sum_{n=1}^{N}\Gamma^{\mathrm{E\text{-}AHM}}_{N-1,M+1}(p_1\ldots\widehat{p_n}\ldots p_N;p_n q_1\ldots q_M), \tag{73}$$

valid for $N, M \geq 0$.

$\Gamma^{\text{E-AHM}}_{N,M}$ includes the all-loop-orders renormalization of the ϕ-sector SSB E-AHM, including virtual transverse gauge bosons, ϕ scalars, ghosts, and new CP-conserving scalars and fermions: A^μ, h, π, $\bar{\eta}$, ω, Φ and ψ, respectively.

In the full SSB E-AHM gauge theory, there are four classes of *finite* operators that cannot spoil the decoupling of heavy particles.

(i) Finite $\mathcal{O}^{1/\Lambda^2;\text{Irrelevant}}_{\text{E-AHM};\phi}$ vanish as $m^2_{\text{Weak}}/\Lambda^2 \to 0$ or $M^2_{\text{Heavy}}/\Lambda^2 \to 0$.

(ii) Finite $\mathcal{O}^{d>4;\text{Light}}_{\text{E-AHM};\phi}$ are dimension $d>4$ operators, where only the light degrees of freedom (A^μ, $h, \pi, \bar{\eta}, \omega$ and also Φ_{Light} and ψ_{Light}) contribute to all-loop-orders renormalization.

(iii) $\mathcal{O}^{d\leq 4;\text{Non Analytic;Light}}_{\text{E-AHM};\phi}$ are finite-dimension $d \leq 4$ operators that are nonanalytic in momenta or in a renormalization scale μ^2, where only the light degrees of freedom contribute to all-loop-orders renormalization.

(iv) $\mathcal{O}^{1/M^2_{\text{Heavy}};\text{Irrelevant}}_{\text{E-AHM};\phi}$ vanish as $m^2_{\text{Weak}}/M^2_{\text{Heavy}} \to 0$.

In addition $\mathcal{O}^{d\leq 4;\text{Non Analytic;Heavy}}_{\text{E-AHM};\phi}$ are finite-dimension $d \leq 4$ operators that are nonanalytic in momenta or in a renormalization scale μ^2, where the heavy degrees of freedom $\Phi_{\text{Heavy}};\psi_{\text{Heavy}}$ contribute to all-loop-orders renormalization. Analysis of these operators lies outside the scope of this paper.

All such operators are ignored,

$$\begin{aligned}\mathcal{O}^{\text{Ignore}}_{\text{E-AHM};\phi} = \mathcal{O}^{1/\Lambda^2;\text{Irrelevant}}_{\text{E-AHM};\phi} + \mathcal{O}^{d>4;\text{Light}}_{\text{E-AHM};\phi} \\ + \mathcal{O}^{d\leq 4;\text{Non Analytic;Light}}_{\text{E-AHM}\phi} \\ + \mathcal{O}^{d\leq 4;\text{Non Analytic;Heavy}}_{\text{E-AHM}\phi} \\ + \mathcal{O}^{1/M^2_{\text{Heavy}};\text{Irrelevant}}_{\text{E-AHM};\phi}.\end{aligned} \quad (74)$$

Such finite operators appear throughout the extended $U(1)_Y$ WTI (73),

(i) $N+M \geq 5$ is $\mathcal{O}^{1/\Lambda^2;\text{Irrelevant}}_{\text{E-AHM};\phi}$, $\mathcal{O}^{d>4;\text{Light}}_{\text{E-AHM};\phi}$, and $\mathcal{O}^{1/M^2_{\text{Heavy}};\text{Irrelevant}}_{\text{E-AHM};\phi}$;

(ii) The left-hand side of (73) for $N+M=4$ is also $\mathcal{O}^{1/\Lambda^2;\text{Irrelevant}}_{\text{E-AHM};\phi}$, $\mathcal{O}^{d>4;\text{Light}}_{\text{E-AHM};\phi}$ and $\mathcal{O}^{1/M^2_{\text{Heavy}};\text{Irrelevant}}_{\text{E-AHM};\phi}$.

(iii) $N+M \leq 4$ operators $\mathcal{O}^{d\leq 4;\text{Non Analytic;Light}}_{\text{E-AHM};\phi}$ also appear in (73).

Finally, there are $N+M \leq 4$ operators that are analytic in momenta. We expand these in powers of momenta, count the resulting dimension of each term in the operator Taylor series, and then ignore $\mathcal{O}^{d>4;\text{Light}}_{\text{E-AHM};\phi}$, $\mathcal{O}^{1/\Lambda^2;\text{Irrelevant}}_{\text{E-AHM};\phi}$ and $\mathcal{O}^{1/M^2_{\text{Heavy}};\text{Irrelevant}}_{\text{E-AHM};\phi}$ in that series.

Suppressing gauge fields, the all-loop-orders renormalized ϕ-sector effective momentum-space Lagrangian, with operator dimensions ≤ 4, for E-AHM is then formed for (h, π) external particles with $CP=(1,-1)$,

$$\begin{aligned}L^{\text{Eff;Wigner,SI,Goldstone}}_{\text{E-AHM};\phi} &= \Gamma^{\text{E-AHM}}_{1,0}(0;)h + \frac{1}{2!}\Gamma^{\text{E-AHM}}_{2,0}(p,-p;)h^2 \\ &+ \frac{1}{2!}\Gamma^{\text{E-AHM}}_{0,2}(;q,-q)\pi^2 + \frac{1}{3!}\Gamma^{\text{E-AHM}}_{3,0}(000;)h^3 \\ &+ \frac{1}{2!}\Gamma^{\text{E-AHM}}_{1,2}(0;00)h\pi^2 + \frac{1}{4!}\Gamma^{\text{E-AHM}}_{4,0}(0000;)h^4 \\ &+ \frac{1}{2!2!}\Gamma^{\text{E-AHM}}_{2,2}(00;00)h^2\pi^2 \\ &+ \frac{1}{4!}\Gamma^{\text{E-AHM}}_{0,4}(;0000)\pi^4 + \mathcal{O}^{\text{E-AHM}}_{\text{Ignore}}.\end{aligned} \quad (75)$$

The $U(1)_Y$ WTI (73) severely constrain the effective Lagrangian of the E-AHM,

(i) $N=0$, $M=1$ WTI:

$$\Gamma^{\text{E-AHM}}_{1,0}(0;) = \langle H\rangle \Gamma^{\text{E-AHM}}_{0,2}(;00) \quad (76)$$

since no momentum can run into the tadpoles.

(ii) $N=1$, $M=1$ WTI[12]:

$$\begin{aligned}&\Gamma^{\text{E-AHM}}_{2,0}(-q,q;) - \Gamma^{\text{E-AHM}}_{0,2}(;q,-q) \\ &\quad = \langle H\rangle \Gamma^{\text{E-AHM}}_{1,2}(-q;q0) \\ &\quad = \langle H\rangle \Gamma^{\text{E-AHM}}_{1,2}(0;00) + \mathcal{O}^{\text{E-AHM}}_{\text{Ignore}} \\ &\Gamma^{\text{E-AHM}}_{2,0}(00;) \\ &\quad = \Gamma^{\text{E-AHM}}_{0,2}(;00) + \langle H\rangle \Gamma^{\text{E-AHM}}_{1,2}(0;00).\end{aligned} \quad (79)$$

(iii) $N=2$, $M=1$ WTI:

$$\langle H\rangle \Gamma^{\text{E-AHM}}_{2,2}(00;00) = \Gamma^{\text{E-AHM}}_{3,0}(000;) - 2\Gamma^{\text{E-AHM}}_{1,2}(0;00). \quad (80)$$

[12]In previous papers on the $SU(2)_L \times SU(2)_R$ Gell-Mann-Lévy $L\Sigma M$ [17], we have written the $N=1$, $M=1$ WTI as a mass relation between the BEH h scalar and the *pseudo*-Nambu-Goldstone boson π pseudoscalar. In the Källén-Lehmann representation

$$\begin{aligned}m^2_{\text{BEH}} &= m^2_\pi + 2\lambda^2_\phi \langle H\rangle^2 \\ m^2_\pi &= \left[\frac{1}{m^2_{\pi;\text{Pole}}} + \int dm^2 \frac{\rho_\pi(m^2)}{m^2}\right]^{-1} \\ m^2_{\text{BEH}} &= \left[\frac{1}{m^2_{\text{BEH;Pole}}} + \int dm^2 \frac{\rho_{\text{BEH}}(m^2)}{m^2}\right]^{-1}\end{aligned} \quad (77)$$

so that

$$m^2_{\text{BEH}} \xrightarrow{m^2_\pi, m^2_{\pi;\text{Pole}} \to 0} 2\lambda^2_\phi \langle H\rangle^2 \quad (78)$$

arises entirely from spontaneous symmetry breaking, in obedience to the $U(1)_Y$ on-shell T-matrix WTI, i.e. the LSS theorem.

(iv) $N = 0$, $M = 3$ WTI:

$$\langle H\rangle \Gamma_{0,4}^{\text{E-AHM}}(;0000) = 3\Gamma_{1,2}^{\text{E-AHM}}(0;00). \tag{81}$$

(v) $N = 1$, $M = 3$ WTI:

$$0 = 3\Gamma_{2,2}^{\text{E-AHM}}(00;00) - \Gamma_{0,4}^{\text{E-AHM}}(;0000). \tag{82}$$

(vi) $N = 3$, $M = 1$ WTI:

$$0 = \Gamma_{4,0}^{\text{E-AHM}}(0000;) - 3\Gamma_{2,2}^{\text{E-AHM}}(00;00). \tag{83}$$

(vii) The quadratic and quartic coupling constants are defined in terms of two-point and four-point 1-scalar-particle-irreducible connected amputated Green's functions,

$$\Gamma_{0,2}^{\text{E-AHM}}(;00) \equiv -m_\pi^2, \qquad \Gamma_{0,4}^{\text{E-AHM}}(;0000) \equiv -6\lambda_\phi^2. \tag{84}$$

Still suppressing gauge fields, the all-loop-orders renormalized ϕ-sector effective Lagrangian (75), severely constrained only by the $U(1)_Y$ WTI governing connected amputated Green's functions (73), may be written

$$\begin{aligned}
L_{\text{E-AHM};\phi}^{\text{Eff;Wigner,SI,Goldstone}} &= L_{\text{E-AHM};\phi}^{\text{Kinetic}} - V_{\text{E-AHM};\phi}^{\text{Wigner,SI,Goldstone}} + \mathcal{O}_{\text{E-AHM};\phi}^{\text{Ignore}}\\
L_{\text{E-AHM};\phi}^{\text{Kinetic}} &= \frac{1}{2}(\Gamma_{0,2}^{\text{E-AHM}}(;p,-p) - \Gamma_{0,2}^{\text{E-AHM}}(;00))h^2 + \frac{1}{2}(\Gamma_{0,2}^{\text{E-AHM}}(;q,-q) - \Gamma_{0,2}^{\text{E-AHM}}(;00))\pi^2\\
V_{\text{E-AHM};\phi}^{\text{Wigner,SI,Goldstone}} &= m_\pi^2\left[\frac{h^2+\pi^2}{2} + \langle H\rangle h\right] + \lambda_\phi^2\left[\frac{h^2+\pi^2}{2} + \langle H\rangle h\right]^2
\end{aligned} \tag{85}$$

with finite nontrivial wave-function renormalization

$$\Gamma_{0,2}^{\text{E-AHM}}(;q,-q) - \Gamma_{0,2}^{\text{E-AHM}}(;00) \sim q^2. \tag{86}$$

The ϕ-sector effective Lagrangian (85) for the E-AHM has insufficient boundary conditions to distinguish among the three modes of the BRST-invariant Lagrangian $L_{\text{E-AHM}}$.[13] The effective potential $V_{\text{E-AHM};\phi}^{\text{Wigner,SI,Goldstone}}$ becomes in various limits the E-AHM Wigner mode ($m_A^2 = 0, \langle H\rangle = 0, m_\pi^2 = m_{\text{BEH}}^2 \neq 0$), E-AHM scale-invariant point ($m_A^2 = 0, \langle H\rangle = 0, m_\pi^2 = m_{\text{BEH}}^2 = 0$), or E-AHM

[13]It is instructive, and we argue dangerous, to ignore vacuum energy and rewrite the potential in (85) as

$$V_{\text{E-AHM}}^{\text{Wigner;SI;Goldstone}} = \lambda_\phi^2\left[\phi^\dagger\phi - \frac{1}{2}\left(\langle H\rangle^2 - \frac{m_\pi^2}{\lambda_\phi^2}\right)\right]^2 \tag{87}$$

using $\frac{h^2+\pi^2}{2} + \langle H\rangle h = \phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2$. If one then minimizes $V_{\text{E-AHM}}^{\text{Wigner;SI;Goldstone}}$ while ignoring the crucial constraint imposed by the LSS theorem, the resultant (incorrect and unphysical) minimum $\langle H\rangle_{\text{unphysical}}^2 \equiv (\langle H\rangle^2 - \frac{m_\pi^2}{\lambda_\phi^2})$ does not distinguish properly among the three modes of (87).
At issue is renormalized

$$m_\pi^2 = \mu_{\phi;\text{Bare}}^2 + C_\Lambda\Lambda^2 + C_{\text{BEH}}m_{\text{BEH}}^2 + \delta m_{\pi;\text{Miscellaneous}}^2 + M_{\text{Heavy}}^2[C_{\text{Heavy}} + C_{\text{Heavy};ln}\ln(M_{\text{Heavy}}^2) + C_{\text{Heavy};\ln\Lambda}\ln(\Lambda^2) + ++] + \lambda_\phi^2\langle H\rangle^2 \tag{88}$$

where the C's are constants, $\delta m_{\pi;\text{Miscellaneous}}^2$ sweeps up the remaining loop corrections, and $m_{\text{BEH}}^2 = m_\pi^2 + 2\lambda^2\langle H\rangle^2$. For pedagogical clarity, we display the linearized approximation to contributions $\sim M_{\text{Heavy}}^2$ explicitly. It is fashionable to simply drop the UVQD term $C_\Lambda\Lambda^2$ in (88), and argue that it is somehow an artifact of dimensional regularization (DR), even though M. J. G. Veltman [42] showed that UVQD appear at one loop in the SM and are properly handled by DR's poles at dimension $d = 2$. We keep UVQD. For pedagogical efficiency, we have included in (88) terms with $M_{\text{Heavy}}^2 \gg m_{\text{Weak}}^2$, such as might arise in Majorana neutrino or beyond-AHM physics (cf. Sec. IV D or IV B).
In the spontaneously broken (Goldstone) mode, where $\langle H\rangle \neq 0$, as in AHM, so too in the E-AHM, in obedience to the LSS theorem (93) the bare counterterm $\mu_{\phi;\text{Bare}}^2$ in (88) is defined by

$$m_\pi^2 \equiv 0. \tag{89}$$

We show below that, for constant θ, the zero value in (89) is protected by the LSS theorem and a NGB shift symmetry

$$\tilde{\pi} \to \tilde{\pi} + \langle H\rangle\theta. \tag{90}$$

Minimization of (87) violates stationarity of the true minimum at $\langle H\rangle$ [16] and destroys the theory's renormalizability and unitarity, which require that dimensionless wave-function renormalization $\langle H\rangle_{\text{Bare}} = [Z^\phi]^{1/2}\langle H\rangle$ contain no relevant operators [16,26,43]. The crucial observation is that, in obedience to the LSS theorem, Renormalized$(\langle H\rangle_{\text{Bare}}^2) \neq \langle H\rangle_{\text{unphysical}}^2$.

Goldstone mode $(m_A^2 \neq 0, \langle H\rangle \neq 0, m_\pi^2 = 0, m_{\rm BEH}^2 \neq 0)$.

$$V_{\text{E-AHM};\phi}^{\rm Wigner} = m_\pi^2\left[\frac{h^2+\pi^2}{2}\right] + \lambda_\phi^2\left[\frac{h^2+\pi^2}{2}\right]^2,$$
$$V_{\text{E-AHM};\phi}^{\rm Scale\ Invariant} = \lambda_\phi^2\left[\frac{h^2+\pi^2}{2}\right]^2,$$
$$V_{\text{E-AHM};\phi}^{\rm Goldstone} = \lambda_\phi^2\left[\frac{h^2+\pi^2}{2} + \langle H\rangle h\right]^2. \tag{91}$$

Equation (85) has exhausted the constraints on the allowed terms in the ϕ-sector effective E-AHM Lagrangian due to those $U(1)_Y$ WTI that govern 1-ϕ-I connected amputated Green's functions $\Gamma_{N,M}^{\text{E-AHM}}$.

2. 1-ϕ-R *connected amputated ϕ-sector T-matrix elements* $T_{N,M}^{\text{E-AHM}}$

In order to provide such boundary conditions [which distinguish among the effective potentials in (91)], we turn to the off-shell T matrix and strict obedience to the wisdom of K. Symanzik's edict at the top of Sec. III E: "[…] unless otherwise constrained by Ward identities." We can further constrain the allowed terms in the ϕ-sector effective E-AHM Lagrangian with those $U(1)_Y$ Ward-Takahashi identities that govern 1-ϕ-R T-matrix elements.

In Appendix B, we derive three such identities governing 1-ϕ-R connected amputated T-matrix elements $T_{N,M}^{\text{E-AHM}}$ in the ϕ-sector of the E-AHM.

(i) Adler self-consistency conditions [originally written for the *global* $SU(2)_L \times SU(2)_R$ Gell-Mann-Lévy model with PCAC [37,38]] constrain the E-AHM *gauge theory's* effective ϕ-sector Lagrangian in Lorenz gauge (B10)

$$\langle H\rangle T_{N,M+1}^{\text{E-AHM}}(p_1\ldots p_N;0q_1\ldots q_M) \times (2\pi)^4\delta^4\left(\sum_{n=1}^N p_n + \sum_{m=1}^M q_m\right)\Bigg|_{q_1^2=q_2^2\ldots=q_M^2=0}^{p_1^2=p_2^2\ldots=p_N^2=m_{\rm BEH}^2} = 0. \tag{92}$$

The E-AHM T matrix vanishes as one of the pion momenta goes to 0 (i.e. 1-soft-pion theorems), provided all other physical scalar particles are on mass shell. Equation (92) also shows that there are no IR divergences in the (ϕ-sector E-AHM) Goldstone mode (in Lorenz gauge) [12].

(ii) The $N=0$, $M=1$ case of (92) comprises the LSS theorem (B15) [12],

$$\langle H\rangle T_{0,2}^{\text{E-AHM}}(;00) = 0,$$
$$\langle H\rangle \Gamma_{0,2}^{\text{E-AHM}}(;00) \equiv -\langle H\rangle m_\pi^2 = 0. \tag{93}$$

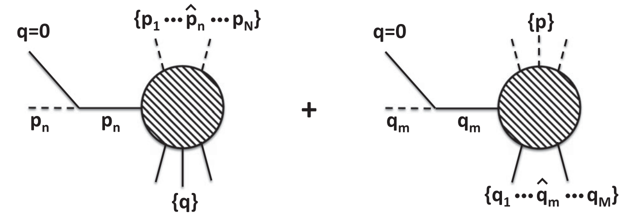

FIG. 1. $T_{N,M+1}^{\text{E-AHM;External}}$: Hashed circles are 1-$\phi$-R $T_{N,M}^{\text{E-AHM}}$, solid lines π, dashed lines h. One (zero-momentum) soft pion is attached to an external leg in all possible ways. $T_{N,M}^{\text{E-AHM}}$ is 1-A^μ-R by cutting an A^μ line, and also 1-Φ-R by cutting a Φ line. Figure 1 is the E-AHM analogy of B. W. Lee's Fig. 10 [12]. The same graph topologies, but without internal beyond-AHM $\Phi\psi$ heavy matter, are used in the proof of (A30) for the (unextended) AHM.

(iii) Define $T_{N,M+1}^{\text{E-AHM;External}}$ as the 1-ϕ-R ϕ-sector T-matrix with one soft $\pi(q_\mu = 0)$ attached to an external leg, as in Fig. 1. Now separate

$$T_{N,M+1}^{\text{E-AHM}}(p_1\ldots p_N;0q_1\ldots q_M) = T_{N,M+1}^{\text{E-AHM;External}}(p_1\ldots p_N;0q_1\ldots q_M) + T_{N,M+1}^{\text{E-AHM;Internal}}(p_1\ldots p_N;0q_1\ldots q_M). \tag{94}$$

Appendix B (B17) proves that

$$\langle H\rangle T_{N,M+1}^{\text{E-AHM;Internal}}(p_1\ldots p_N;0q_1\ldots q_M) = \sum_{m=1}^M T_{N+1,M-1}^{\text{E-AHM}}(q_m p_1\ldots p_N;q_1\ldots\widehat{q_m}\ldots q_M) - \sum_{n=1}^N T_{N-1,M+1}^{\text{E-AHM}}(p_1\ldots\widehat{p_n}\ldots p_N;p_n q_1\ldots q_M). \tag{95}$$

The $U(1)_Y$ WTI (73), (B18) governing 1-ϕ-I connected amputated Greens functions $\Gamma_{N,M}^{\text{E-AHM}}$ are solutions to (95) and (B17).

We rewrite the E-AHM effective ϕ-sector Lagrangian (85) but now include the constraint from the LSS theorem (93) and (B15), in the SSB $\langle H\rangle \neq 0$ case, $m_\pi^2 = 0$,

$$L_{\text{E-AHM};\phi}^{\rm Eff;Goldstone} = L_{\text{E-AHM};\phi}^{\rm Kinetic} + \mathcal{O}_{\rm Ignore}^{\text{E-AHM}} - V_{\text{E-AHM};\phi}^{\rm Eff;Goldstone}$$
$$V_{\text{E-AHM};\phi}^{\rm Eff;Goldstone} = \lambda_\phi^2\left[\frac{h^2+\pi^2}{2} + \langle H\rangle h\right]^2 \tag{96}$$

and wave-function renormalization

$$\Gamma_{0,2}^{\text{E-AHM}}(;q,-q) - \Gamma_{0,2}^{\text{E-AHM}}(;00) = q^2 + \mathcal{O}_{\text{Ignore}}^{\text{E-AHM}}. \quad (97)$$

A crucial effect of the LSS theorem, together with the $N=0$, $M=1$ Ward-Takahashi Green's function identity (73), is to automatically eliminate tadpoles in (96)

$$\Gamma_{1,0}^{\text{E-AHM}}(0;) = \langle H\rangle \Gamma_{0,2}^{\text{E-AHM}}(;00) = 0, \quad (98)$$

so that separate tadpole renormalization is unnecessary.

We form the effective Goldstone-mode Lagrangian governing low-energy ϕ-sector physics in coordinate space[14]

$$L_{\text{E-AHM};\phi}^{\text{Eff;Goldstone}} = |D_\mu \phi|^2 - V_{\text{E-AHM};\phi}^{\text{Eff;Goldstone}} + \mathcal{O}_{\text{E-AHM};\phi}^{\text{Ignore}}$$
$$V_{\text{E-AHM};\phi}^{\text{Eff;Goldstone}} = \lambda_\phi^2 \left[\frac{h^2+\pi^2}{2} + \langle H\rangle h\right]^2. \quad (99)$$

Equation (99) is the ϕ-sector effective Lagrangian of the *spontaneously broken* E-AHM in Lorenz gauge.

(i) It obeys the LSS theorem (93) and (B15) and all other $U(1)_Y$ WTI (73), (92), (93), (95), (B10), (B15), (B17), and (B18).

(ii) It obeys the Goldstone theorem in the Lorenz gauge, having a massless derivatively coupled NGB, $\tilde{\pi}$.

(iii) It is minimized at $(H=\langle H\rangle, \pi=0)$, and obeys stationarity [16] of that true minimum.

(iv) It preserves the theory's renormalizability and unitarity, which require that wave-function renormalization, $\langle H\rangle_{\text{Bare}} = [Z_{\text{E-AHM}}^{\phi}]^{1/2}\langle H\rangle$ [16,18,26], forbid *any* relevant operator corrections to $\langle H\rangle$.

(v) It includes all divergent $\mathcal{O}(\Lambda^2), \mathcal{O}(\ln\Lambda^2)$ and finite terms that arise to all perturbative loop orders in the full $U(1)_Y$ theory, due to virtual transverse gauge bosons, AHM scalars, ghosts, and new CP-conserving scalars and fermions (A^μ, h, π,$\bar{\eta}$, ω, and Φ, ψ respectively).

(vi) The LSS theorem (93) and (B15) has caused all relevant operators in (99) to vanish.

3. The LSS theorem comes from exact $U(1)_Y$ symmetry. Minimization of the effective potential does not

It is important to compare the results of our LSS theorem to those of the mainstream literature. For pedagogical simplicity, in this subsection we suppress mention of vacuum energy and $\mathcal{O}_{\text{E-AHM};\phi}^{\text{Ignore}}$. After renormalization, but before application of the LSS theorem, the effective potential (85), which is derived entirely from Green's function WTI, can be written

$$\begin{aligned} V_{\text{E-AHM};\phi}^{\text{Eff;Wigner,SI,Goldstone}} &= \mu_\phi^2(\phi^\dagger\phi) + \lambda_\phi^2(\phi^\dagger\phi)^2 \\ &= (\mu_\phi^2 + \lambda_\phi^2\langle H\rangle^2)\left(\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right) \\ &\quad + \lambda_\phi^2\left(\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right)^2 \\ &= (\mu_\phi^2 + \lambda_\phi^2\langle H\rangle^2)\left(\frac{h^2+\pi^2}{2} + \langle H\rangle h\right) \\ &\quad + \lambda_\phi^2\left(\frac{h^2+\pi^2}{2} + \langle H\rangle h\right)^2, \end{aligned} \quad (100)$$

where $V_{\text{E-AHM};\phi}^{\text{Eff;Wigner,SI,Goldstone}}$, ϕ, μ_ϕ^2, λ_ϕ^2 and $\langle H\rangle^2$ in (100) are all renormalized quantities.

The vanishing of relevant operators due to heavy Φ, ψ in the effective E-AHM theory is therefore not itself controversial. The mainstream literature minimizes (100) to find the vacuum,

$$\frac{\partial}{\partial h} V_{\text{E-AHM};\phi}^{\text{Eff;Wigner,SI,Goldstone}}|_{h=\pi=0} = \langle H\rangle(\mu_\phi^2 + \lambda_\phi^2\langle H\rangle^2) = 0 \quad (101)$$

which, for the SSB case, gives

$$\frac{\partial}{\partial h} V_{\text{E-AHM};\phi}^{\text{Eff;Goldstone}}|_{h=\pi=0} = 0$$
$$\mu_\phi^2 + \lambda_\phi^2\langle H\rangle^2 = 0. \quad (102)$$

This is conventionally interpreted as a calculation of $\langle H\rangle^2$,

$$\langle H\rangle^2 = -\frac{\mu_\phi^2}{\lambda_\phi^2}, \quad (103)$$

where, in renormalized μ_ϕ^2, UVQD and all other relevant contributions, such as those due to Φ, ψ in loops, are regarded as having canceled against a bare counterterm $\delta\mu_{\phi;\text{Bare}}^2$.

In contrast, we have derived a tower of Adler self-consistency conditions (92) in Lorenz gauge in Appendix B, i.e. derived directly from the exact $U(1)_Y$ symmetry obeyed by gauge-independent *on-shell* T-matrix elements. One of these, the $N=0$, $M=1$ case, is the LSS theorem,

$$\langle H\rangle m_\pi^2 = \langle H\rangle(\mu_\phi^2 + \lambda_\phi^2\langle H\rangle^2) = 0, \quad (104)$$

which, for the SSB case, gives

$$m_\pi^2 = \mu_\phi^2 + \lambda_\phi^2\langle H\rangle^2 = 0 \quad (105)$$

whose practical effect is the same as minimization of the effective potential, as captured in (102).

[14]It is not lost on the authors that, since we derived it from connected amputated Green's functions (where all vacuum energy and disconnected vacuum bubbles are absorbed into an overall phase, which cancels exactly in the S matrix [16,26]), the vacuum energy in $V_{\text{E-AHM};\phi}^{\text{Eff;Goldstone}}$ in (99) is exactly 0.

So, we agree with the mainstream literature that all relevant operators vanish in the effective low-energy E-AHM theory.

4. Decoupling of heavy matter representations

Adding a $U(1)_Y$ local/gauge invariant Lagrangian $L^{\text{Gauge Invariant}}_{\text{beyondAHM}}(A_\mu, \phi, \Phi, \psi_L, \psi_R)$ to (18) forms the E-AHM.

In order to force renormalized connected amplitudes with an odd number of π s to vanish, the new particles Φ, ψ_L, ψ_R are taken in this paper to conserve CP.

In Secs. IV A 4 through IV A 7, we take all of the new scalars Φ, left-handed fermions ψ_L and right-handed fermions ψ_R to be very heavy,

$$M^2_{\psi_L}, M^2_{\psi_R}, M^2_\Phi \sim M^2_{\text{Heavy}} \gg (|q^2|, m^2_A, m^2_{\text{BEH}}) \sim m^2_{\text{Weak}} \sim (100\ \text{GeV})^2, \tag{106}$$

with q_μ being typical for a studied low-energy process. Fermion $U(1)_Y$ hypercharges are chosen so that the axial anomaly is 0. To remain perturbative, we keep the Yukawa couplings $y_{\phi\psi}, y_{\Phi\psi} \lesssim 1$, but take the Majorana masses squared

$$L^{\text{Majorana}}_{\text{beyondAHM};\psi} = -\frac{1}{2}M_{\psi_L}(\psi_L^{\text{Weyl}}\psi_L^{\text{Weyl}} + \bar{\psi}_L^{\text{Weyl}}\bar{\psi}_L^{\text{Weyl}}) - \frac{1}{2}M_{\psi_R}(\psi_R^{\text{Weyl}}\psi_R^{\text{Weyl}} + \bar{\psi}_R^{\text{Weyl}}\bar{\psi}_R^{\text{Weyl}})$$

to be heavy. We keep all Yukawas and masses real for pedagogical simplicity.

Some comments are in order.

(i) We have ignored finite $\mathcal{O}^{1/M^2_{\text{Heavy}};\text{Irrelevant}}_{\text{E-AHM};\phi}$ that decouple and vanish as $m^2_{\text{Weak}}/M^2_{\text{Heavy}} \to 0$.

(ii) Among the terms included in (99) are finite relevant operators dependent on the heavy matter representations,

$$\mathcal{O}(M^2_{\text{Heavy}}), \mathcal{O}(M^2_{\text{Heavy}} \ln(M^2_{\text{Heavy}})), \mathcal{O}(M^2_{\text{Heavy}} \ln(m^2_{\text{Weak}})), \mathcal{O}(m^2_{\text{Weak}} \ln(M^2_{\text{Heavy}})), \tag{107}$$

but they have become invisible to us because of the LSS theorem (93) and (B15). That fact is one of the central results of this paper.

(iii) Marginal operators $\sim \ln(M^2_{\text{Heavy}})$ have been absorbed in (99), i.e. in the renormalization of gauge-independent observables (i.e. the quartic-coupling constant λ^2_ϕ calculated in the Kibble representation, and the BEH VEV $\langle H \rangle$), and in unobservable wave-function renormalization (97).

No trace of M_{Heavy}-scale Φ, ψ survives in (99). All the heavy beyond-AHM matter representations have completely decoupled.

5. First decoupling theorem: SSB ϕ-sector connected amputated 1-ϕ-I Green's functions

We take $\mathcal{O}^{1/\Lambda^2;\text{Irrelevant}}_{\text{E-AHM};\phi} \to 0$ (to unencumber our notation) and work in the $m^2_{\text{Weak}}/M^2_{\text{Heavy}} \to 0$ limit.

In the SSB E-AHM, $\Gamma^{\text{E-AHM}}_{N,M}$ with

(i) $N + M \geq 5$ obey the Appelquist-Carazzone decoupling theorem [44];

(ii) $N + M = 3$, 4 are absorbed by coupling constant renormalization;

(iii) $N + M = 2$ are absorbed by wave-function renormalization, vanish due to the LSS theorem $m^2_\pi = 0$, or contribute to SSB origination of $m^2_{\text{BEH}} = 2\lambda^2_\phi \langle H \rangle^2$ (see below).

Therefore, including the contributions to relevant operators from heavy CP-conserving Φ, ψ matter in virtual loops,

$$\Gamma^{\text{E-AHM}}_{N,M} \xrightarrow{m^2_{\text{Weak}}/M^2_{\text{Heavy}} \to 0} \Gamma^{AMH}_{N,M}. \tag{108}$$

6. Second decoupling theorem: SSB ϕ-sector connected amputated 1-ϕ-R T matrices

In the limit $m^2_{\text{Weak}}/M^2_{\text{Heavy}} \to 0$

$$T^{\text{Extended}}_{N,M} \xrightarrow{m^2_{\text{Weak}}/M^2_{\text{Heavy}} \to 0} T_{N,M}, \tag{109}$$

including heavy CP-conserving Φ, ψ matter contributions to relevant operators.

7. Third decoupling theorem: SSB ϕ-sector BEH pole-mass squared

The $N = 1$, $M = 1$ connected amputated Green's function $U(1)_Y$ WTI (73), augmented by the LSS theorem (93), reads

$$\Gamma^{\text{E-AHM}}_{2,0}(00;) = \langle H \rangle \Gamma^{\text{E-AHM}}_{1,2}(0;00) = -2\lambda^2_\phi \langle H \rangle^2$$
$$\lim_{\langle H \rangle \to 0} \Gamma^{\text{E-AHM}}_{2,0}(00;) = 0, \tag{110}$$

showing that the BEH pole-mass squared arises entirely from SSB. Defining

$$\Delta^{\text{BEH}}_{\text{E-AHM}}(q^2) = \frac{1}{q^2 - m^2_{\text{BEH;Pole}} + i\epsilon} + \int dm^2 \frac{\rho^{\text{BEH}}_{\text{E-AHM}}(m^2)}{q^2 - m^2 + i\epsilon}, \tag{111}$$

$m^2_{\text{BEH;Pole}}$ is the BEH resonance pole-mass squared. In analogy with (23), the spectral density $\rho^{\text{BEH}}_{\text{E-AHM}}(M^2_{\text{Heavy}}) \sim 1/M^2_{\text{Heavy}}$. Thus

$$\rho^{\rm BEH}_{\rm E\text{-}AHM}(m^2) = \rho^{\rm BEH}_{\rm AHM}(m^2) + \mathcal{O}^{1/M^2_{\rm Heavy};{\rm Irrelevant}}_{\rm E\text{-}AHM;\phi}$$

$$\begin{aligned}\Gamma^{\rm E\text{-}AHM}_{2,0}(00;) &\equiv [\Delta^{\rm BEH}_{\rm E\text{-}AHM}(0)]^{-1} \\ &= -2\lambda^2_\phi \langle H\rangle^2 \\ &= -m^2_{\rm BEH;Pole}\left[1 + m^2_{\rm BEH;Pole}\int dm^2 \frac{\rho^{\rm BEH}_{\rm AHM}(m^2)}{m^2 - i\epsilon}\right]^{-1} \\ &\quad + \mathcal{O}^{1/M^2_{\rm Heavy};{\rm Irrelevant}}_{\rm E\text{-}AHM;\phi}\end{aligned} \tag{112}$$

and we have

$$m^2_{\rm BEH;Pole} = 2\lambda^2_\phi\langle H\rangle^2\left[1 - 2\lambda^2_\phi\langle H\rangle^2 \int dm^2 \frac{\rho^{\rm BEH}_{\rm AHM}(m^2)}{m^2 - i\epsilon}\right]^{-1} + \mathcal{O}^{1/M^2_{\rm Heavy};{\rm Irrelevant}}_{\rm E\text{-}AHM;\phi}. \tag{113}$$

Because $\lambda^2_\phi, Z^\phi_{\rm ExrendedAHM}$ are dimensionless, λ^2_ϕ and

$$\langle H\rangle = [Z^\phi_{\rm ExrendedAHM}]^{-\frac{1}{2}}\langle H\rangle_{\rm Bare} \tag{114}$$

absorb no relevant operators, Eq. (113) shows that the BEH pole-mass squared $m^2_{\rm BEH;Pole}$ also absorbs no relevant operators.

No trace of $M_{\rm Heavy}$-scale Φ, ψ, including their contributions to relevant operators, survives in (113). All the heavy beyond-AHM matter representations have completely decoupled, and the BEH pole-masses squared

$$m^{2;\rm E\text{-}AHM}_{\rm BEH;Pole} \xrightarrow{m^2_{\rm Weak}/M^2_{\rm Heavy}\to 0} m^{2;\rm AHM}_{\rm BEH;Pole} \tag{115}$$

become equal in the limit $m^2_{\rm Weak}/M^2_{\rm Heavy} \to 0$. We call (115) the "SSB BEH-mass decoupling theorem."

By dimensional analysis, heavy Φ, ψ also decouple from the π spectral functions

$$\Delta^{\pi;\rm Spectral}_{\rm E\text{-}AHM}(q^2) = \Delta^{\pi;\rm Spectral}_{\rm AHM}(q^2) + \mathcal{O}(1/M^2_{\rm Heavy}). \tag{116}$$

B. Example: Decoupling of gauge-singlet $M_S^2 \gg m^2_{\rm Weak}$ real scalar field S with discrete Z_2 symmetry and $\langle S\rangle = 0$

We consider a $U(1)_Y$ gauge-singlet real scalar S, with $(S \to -S)$ Z_2 symmetry, $M_S^2 \gg m^2_{\rm Weak}$, and $\langle S\rangle = 0$. We add to the renormalized theory

$$\begin{aligned}L_S &= \frac{1}{2}(\partial_\mu S)^2 - V_{\phi S}, \\ V_{\phi S} &= \frac{1}{2}M_S^2 S^2 + \frac{\lambda_S^2}{4}S^4 + \frac{1}{2}\lambda^2_{\phi S}S^2\left[\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right], \\ \phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2 &= \frac{h^2+\pi^2}{2} + \langle H\rangle h.\end{aligned} \tag{117}$$

Since S is a gauge singlet, it is also a rigid/global singlet. Its $U(1)_Y$ hypercharge, transformation and current

$$Y_S = 0; \qquad \delta_{U(1)_Y} S(t,\vec{y}) = 0 \qquad J^{\mu;S}_{\rm beyondAHM} = 0 \tag{118}$$

therefore satisfy all of the decoupling criteria in Appendix B.

(i) Since it is massive, S cannot carry information to the surface $z^{3-\rm surface} \to \infty$ of the (all-space-time) 4-volume $\int d^4z$, and so satisfies (B8).

(ii) The equal-time commutators satisfy (B6)

$$\begin{aligned}\delta(z_0 - y_0)[J^{0;S}_{\rm beyondAHM}(z), H(y)] &= 0, \\ \delta(z_0 - y_0)[J^{0;S}_{\rm beyondAHM}(z), \pi(y)] &= 0.\end{aligned} \tag{119}$$

(iii) The classical equation of motion

$$\partial_\mu(J^{\mu;S}_{\rm beyondAHM} + J^\mu_{\rm AHM}) = \partial_\mu J^\mu_{\rm AHM} = m_A H\partial_\beta A^\beta \tag{120}$$

restores conservation of the rigid/global $U(1)_Y$ extended current for ϕ-sector physical states, and satisfies (B5)

$$\langle 0|T[\partial_\mu(J^{\mu;S}_{\rm beyondAHM} + J^\mu_{\rm AHM})(z) \times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected} = 0. \tag{121}$$

(iv) The zero VEV $\langle S\rangle = 0$ satisfies (B7).

The $U(1)_Y$ WTI governing the extended ϕ-sector transition matrix $T^{\rm E\text{-}AHM;S}_{N,M}$ are therefore true: namely, the extended Adler self-consistency conditions (92) and (B10), together with their proof of infrared finiteness in the presence of massless NGB, and the extended 1-soft-π theorems (95) and (B17); the extended $U(1)_Y$ WTI (73) and (B18) governing connected amputated ϕ-sector Green's functions $\Gamma^{\rm E\text{-}AHM;S}_{N,M}$ are also true. The $U(1)_Y \otimes$ BRST symmetry of Sec. II is faithfully represented by these, and the tower of on-shell T-matrix extended WTI (92) and (B10) $T^{\rm E\text{-}AHM;S}_{N,M}|_{\rm on\text{-}shell} = 0$, and its extended LSS theorem (93) and (B15).

The three decoupling theorems (109), (108), and (115) therefore follow, so that no trace of the $M_S^2 \sim M^2_{\rm Heavy}$ scalar S survives the $m^2_{\rm Weak}/M^2_{\rm Heavy} \to 0$ limit: i.e. it has completely decoupled. The ϕ-sector connected amputated T matrices and Green's functions, and the BEH pole masses squared

$$T^{\text{E-AHM};S}_{N,M} \xrightarrow{m^2_{\text{Weak}}/M^2_S \to 0} T_{N,M}$$
$$\Gamma^{\text{E-AHM};S}_{N,M} \xrightarrow{m^2_{\text{Weak}}/M^2_S \to 0} \Gamma_{N,M}$$
$$m^{2;\text{E-AHM};S}_{\text{BEH;Pole};\phi} \xrightarrow{m^2_{\text{Weak}}/M^2_{\text{Heavy}} \to 0} m^{2;\text{AHM}}_{\text{BEH;Pole};\phi} \tag{122}$$

become equal in the limit $m^2_{\text{Weak}}/M^2_{\text{Heavy}} \to 0$, including all contributions to relevant operators from heavy S in virtual loops.

C. One generation of standard-model quarks and leptons, augmented by a right-handed neutrino ν_R with Dirac mass, gauged hypercharge and global colors

We consider the addition of one standard model generation of spin $S = \frac{1}{2}$ fermions, t_L, b_L, t_R, b_R, τe_L, ν_{τ_L}, τ_R, augmented by one right-handed neutrino ν_{τ_R}, with global $SU(3)$ colors $c =$ red, white, blue, and gauged $U(1)_Y$ hypercharge. These are regarded here as E-AHM matter representations.

Baryon-number and lepton-number-conserving Dirac masses squared arise entirely from SSB and are light, in the sense that $m^2_{\text{Quark}}, m^2_{\text{Lepton}} \lesssim m^2_{\text{Weak}}$. The so-extended $U(1)_Y$ AHM gauge theory has zero axial anomaly because quark/lepton AHM quantum numbers are chosen to be their SM hypercharges (including $Y_{\nu_R} = 0$). This addition also retains the CP conservation of the AHM. We choose the third generation mostly for definiteness, but also slightly to emphasize that we are not relying in any way on the smallness of quark Yukawas.

Adding beyond-AHM Dirac quarks augments $L^{\text{Lorenz}}_{\text{AHM}}$ of (18) with

$$L^{\text{Global Invariant}}_{\text{beyondAHM};q} = L^{\text{Kinetic}}_{\text{beyondAHM};q} + L^{\text{Yukawa}}_{\text{beyondAHM};q},$$
$$L^{\text{Kinetic}}_{\text{beyondAHM};q} = i \sum_{\text{color}}^{r,w,b} \sum_{\text{flavor}}^{t,b} (\bar{q}^c_L \gamma^\mu D_\mu q^c_L + \bar{q}^c_R \gamma^\mu D_\mu q^c_R),$$
$$L^{\text{Yukawa}}_{\text{beyondAHM};q} = -\sum_{\text{color}}^{r,w,b} \sum_{\text{flavor}}^{t,b} y_q (\bar{q}^c_L \phi q^c_R + \bar{q}^c_R \phi^\dagger q^c_L). \tag{123}$$

The $U(1)_Y$ quark current and transformation properties are

$$J^{\mu;\text{Dirac}}_{\text{beyondAHM};q} = -\sum_{\text{color}}^{r,w,b} \sum_{\text{flavor}}^{t,b} (Y_{q_L} \bar{q}^c_L \gamma^\mu q^c_L + Y_{q_R} \bar{q}^c_R \gamma^\mu q^c_R),$$
$$\delta_{U(1)_Y} q^c_L(t,\vec{x}) = -i Y_{q_L} q^c_L(t,\vec{x})\theta,$$
$$\delta_{U(1)_Y} q^c_R(t,\vec{x}) = -i Y_{q_R} q^c_R(t,\vec{x})\theta,$$
$$Y_{t_L} = \frac{1}{3}; \quad Y_{b_L} = \frac{1}{3}; \quad Y_{t_R} = \frac{4}{3}; \quad Y_{b_R} = -\frac{2}{3}. \tag{124}$$

Adding beyond-AHM Dirac leptons further adds to $L^{\text{Lorenz}}_{\text{AHM}}$,

$$L^{\text{Global Invariant}}_{\text{beyondAHM};l} = L^{\text{Kinetic}}_{\text{beyondAHM};l} + L^{\text{Yukawa}}_{\text{beyondAHM};l},$$
$$L^{\text{Kinetic}}_{\text{beyondAHM};l} = i \sum_{\text{flavor}}^{\nu_\tau,\tau} (\bar{l}_L \gamma^\mu D_\mu l_L + \bar{l}_R \gamma^\mu D_\mu l_R),$$
$$L^{\text{Yukawa}}_{\text{beyondAHM};l} = -\sum_{\text{flavor}}^{\nu_\tau,\tau} y_l (\bar{l}_L \phi l_R + \bar{l}_R \phi^\dagger l_L). \tag{125}$$

The lepton $U(1)_Y$ current and transformation properties are

$$J^{\mu;\text{Dirac}}_{\text{beyondAHM};l} = -\sum_{\text{flavor}}^{\nu,e} (Y_{l_L} \bar{l}_L \gamma^\mu l_L + Y_{l_R} \bar{l}_R \gamma^\mu l_R),$$
$$\delta_{U(1)_Y} l_L(t,\vec{x}) = -i Y_{l_L} l_L(t,\vec{x})\theta,$$
$$\delta_{U(1)_Y} l_R(t,\vec{x}) = -i Y_{l_R} l_R(t,\vec{x})\theta,$$
$$Y_{\nu_{\tau,L}} = -1; \quad Y_{\tau_L} = -1; \quad Y_{\nu_{\tau,R}} = 0; \quad Y_{e_R} = -2; \tag{126}$$

with these standard-model quark and lepton hypercharges Y_i, our $U(1)_Y$ WTI have zero axial anomaly.

We now prove applicability of our $U(1)_Y$ WTI for connected amputated ϕ-sector Green's functions $\Gamma^{\text{E-AHM}}_{N,M}$ and for on-shell T-matrix elements $T^{\text{E-AHM}}_{N,M}$.

(i) The equal-time quantum commutators satisfy (B6)

$$\delta(z_0 - y_0)[J^{0;\text{Dirac}}_{\text{beyondAHM};q}(z), H(y)] = 0,$$
$$\delta(z_0 - y_0)[J^{0;\text{Dirac}}_{\text{beyondAHM};q}, \pi(y)] = 0,$$
$$\delta(z_0 - y_0)[J^{0;\text{Dirac}}_{\text{beyondAHM};l}(z), H(y)] = 0,$$
$$\delta(z_0 - y_0)[J^{0;\text{Dirac}}_{\text{beyondAHM};l}, \pi(y)] = 0. \tag{127}$$

(ii) The classical equation of motion

$$\partial_\mu (J^{\mu;\text{Dirac}}_{\text{beyondAHM};l} + J^{\mu;\text{Dirac}}_{\text{beyondAHM};q} + J^\mu_{\text{AHM}}) = m_A H \partial_\beta A^\beta \tag{128}$$

restores conservation of the rigid/global $U(1)_Y$ extended current for ϕ-sector physical states, and satisfies (B5)

$$\langle 0|T[\partial_\mu (J^{\mu;\text{Dirac}}_{\text{beyondAHM};l} + J^{\mu;\text{Dirac}}_{\text{beyondAHM};q} + J^\mu_{\text{AHM}})(z) \times h(x_1)\ldots h(x_N)\pi_{t_1}(y_1)\ldots\pi_{t_M}(y_M)]|0\rangle_{\text{connected}} = 0. \tag{129}$$

(iii) Dirac-mass-quark surface terms vanish. Since the quarks t and b are taken to have Dirac masses, $m_t = \frac{1}{\sqrt{2}} y_u \langle H \rangle$ and $m_b = \frac{1}{\sqrt{2}} y_d \langle H \rangle$, and since we need only *connected* graphs, the quarks cannot carry information to the 3-surface at timelike infinity of the 4-volume of space-time, and so do not spoil Eq. (B8). In contrast, massless quarks could carry $U(1)_Y$ information on the light cone to this surface;

they would therefore violate (B8), and so destroy the spirit, results and essence of our $U(1)_Y$-WTI-based heavy particle decoupling results here in Sec. IV.

(iv) Charged-lepton surface terms also vanish. Since τ is massive, $m_\tau = \frac{1}{\sqrt{2}} y_e \langle H \rangle$, and we need only connected graphs; the charged lepton τ also cannot carry information to the 3-surface at timelike infinity of the 4-volume of spacetime, and so satisfies (B8).

(v) Dirac-neutrino surface terms: Since ν_τ is taken to be massive in deference to observed SM neutrino mixing, $m_\nu^{\mathrm{Dirac}} = \frac{1}{\sqrt{2}} y_\nu \langle H \rangle$, ν_τ also satisfies (B8). In contrast, a massless neutrino *could* carry $U(1)_Y$ information on the light cone to the 3-surface at infinity and would violate (B8),[15] and so destroy the spirit, results and essence of our $U(1)_Y$-WTI-based heavy particle decoupling results here in Sec. IV.

Having satisfied all of the criteria in Appendix B, the $U(1)_Y$ WTI governing the extended ϕ-sector transition matrix $T_{N,M}^{\text{E-AHM};q,l}$ are therefore true: namely, the extended Adler self-consistency conditions (92) and (B10), together with their proof of infrared finiteness in the presence of massless NGB; the extended 1-soft-π theorems (95) and (B17); and the extended $U(1)_Y$ WTI (73), (B18) governing connected amputated ϕ-sector Green's functions $\Gamma_{N,M}^{\text{E-AHM};q,l}$. The $U(1)_Y \otimes$ BRST symmetry of Sec. II is faithfully represented by these, and the tower of on-shell T-matrix extended WTI (92) and (B10) $T_{N,M}^{\text{E-AHM};q,l}|_{\text{on-shell}} = 0$, and its extended LSS theorem (93) and (B15).

[15]Our proof of axial-vector WTI in Appendix B requires that neutrinos be incapable of carrying information to the 3-surface at timeline infinity of the 4-volume of spacetime. We have worked here within SSB E-AHM, with its explicit Dirac neutrino mass, for this purely mathematical reason.
Imagine, however, that we are able to extend this work to the *CP*-conserving standard electroweak model with *two generations* of quarks, charged leptons, and ν_L, ν_R, with neutrino Dirac masses, but zero Majorana masses. [Reference [39] analyzes local $SU(2) \otimes U(1)_Y$ with one such generation and nonzero Majorana ν_R mass.] With its gauge group $SU(2)_L \times U(1)_Y$, we would build two sets of *rigid/global* WTI: unbroken electromagnetic $U(1)_{\mathrm{QED}}$, and spontaneously broken $SU(2)_L$. It is then amusing to elevate such rigid/global WTI to a principle of nature, so as to give them predictive power for actual experiments and observations. The $U(1)_{\mathrm{QED}}$ WTI would be unbroken vector-current identities. Focus instead on the spontaneously broken $SU(2)_L$. Start with Yukawa couplings which generate, after SSB, masses and mixings among weak-eigenstate neutrinos. The observable 2×2 Pontecorvo-Maki-Nakagawa-Sakata matrix would then rotate those to mass eigenstates $m_{\nu_1}^{\mathrm{Dirac}}, m_{\nu_2}^{\mathrm{Dirac}}$.
The axial-vector current WTI from the spontaneously broken $SU(2)_L$ require and demand a neutrino Dirac mass for each and every one of the mass eigenstates $m_{\nu_1}^{\mathrm{Dirac}}, m_{\nu_2}^{\mathrm{Dirac}} \neq 0$. Would we then claim that SSB $SU(2)_L$ WTI *predict* neutrino oscillations? To make a possible connection with nature, although current experimental neutrino-mixing data cannot rule out an exactly zero mass for the lightest neutrino [45], the mathematical self-consistency of $SU(2)_L$ WTI would.

D. (Practical) decoupling of a gauge-singlet right-handed type-I-seesaw Majorana neutrino with $M_{\nu_R}^2 \gg m_{\mathrm{BEH}}^2 \sim m_{\mathrm{Weak}}^2$ (as in the νAHM)

We consider here the addition to the AHM of a heavy $U(1)_Y$ gauge-singlet right-handed Majorana neutrino ν_R, with $M_{\nu_R}^2 \gg m_{\mathrm{Weak}}^2$, involved in a type-1 seesaw with a left-handed neutrino ν_L, through a Yukawa coupling y_ν, with resulting Dirac mass $m_\nu^{\mathrm{Dirac}} = y_\nu \langle H \rangle / \sqrt{2}$.

We add to the renormalized theory in Sec. IV C a Majorana mass

$$L_{\nu_R}^{\mathrm{Majorana}} = -\frac{1}{2} M_{\nu_R} (\nu_R^{\mathrm{Weyl}} \nu_R^{\mathrm{Weyl}} + \bar{\nu}_R^{\mathrm{Weyl}} \bar{\nu}_R^{\mathrm{Weyl}}). \tag{130}$$

Since ν_R is a gauge singlet, it is also a rigid/global singlet. Its hypercharge $U(1)_Y$ transformation and current

$$Y_{\nu_R} = 0; \qquad \delta_{U(1)_Y} \nu_R(t, \vec{y}) = 0$$
$$J_{\mathrm{beyondAHM};\nu_R}^{\mu;\mathrm{Majorana}} = 0 \tag{131}$$

therefore satisfy all of the decoupling criteria in Appendix B.

(i) Since it has a Dirac mass, the neutrino ν cannot carry information to the surface $z^{3-\mathrm{surface}} \to \infty$ of the (all-space-time) 4-volume $\int d^4 z$, and so satisfies (B8).

(ii) The equal-time quantum commutators satisfy (B6)

$$\delta(z_0 - y_0)[J_{\mathrm{beyondAHM};\nu_R}^{0;\mathrm{Majorana}}(z), H(y)] = 0,$$
$$\delta(z_0 - y_0)[J_{\mathrm{beyondAHM};\nu_R}^{0;\mathrm{Majorana}}(z), \pi(y)] = 0. \tag{132}$$

(iii) The classical equation of motion

$$\begin{aligned} &\partial_\mu (J_{\mathrm{beyondAHM};\nu_R}^{\mu;\mathrm{Majorana}} + J_{\mathrm{beyondAHM};l}^{\mu;\mathrm{Dirac}} \\ &\quad + J_{\mathrm{beyondAHM};q}^{\mu;\mathrm{Dirac}} + J_{\mathrm{AHM}}^{\mu}) \\ &\qquad = \partial_\mu (J_{\mathrm{beyondAHM};l}^{\mu;\mathrm{Dirac}} + J_{\mathrm{beyondAHM};q}^{\mu;\mathrm{Dirac}} + J_{\mathrm{AHM}}^{\mu}) \\ &\qquad = m_A H \partial_\beta A^\beta \end{aligned} \tag{133}$$

restores conservation of the extended rigid/global $U(1)_Y$ current for ϕ-sector physical states, and satisfies (B5),

$$\begin{aligned} &\langle 0 | T[\partial_\mu (J_{\mathrm{beyondAHM};\nu_R}^{\mu;\mathrm{Majorana}} + J_{\mathrm{beyondAHM};l}^{\mu;\mathrm{Dirac}} \\ &\quad + J_{\mathrm{beyondAHM};q}^{\mu;\mathrm{Dirac}} + J_{\mathrm{AHM}}^{\mu})(z) \\ &\quad \times h(x_1)\ldots h(x_N) \pi_{t_1}(y_1)\ldots \pi_{t_M}(y_M)] | 0 \rangle_{\mathrm{connected}} \\ &\qquad = 0. \end{aligned} \tag{134}$$

Having satisfied all of the criteria in Appendix B, the $U(1)_Y$ WTI governing the extended ϕ-sector transition matrix $T_{N,M}^{\text{E-AHM};q,l,M_{\nu_R}}$ are therefore true: namely, the extended Adler self-consistency conditions (92) and (B10), together with their proof of infrared finiteness in the presence of massless NGB, and the extended 1-soft-π theorems (95) and (B17); the extended $U(1)_Y$ WTI (73) and (B18) governing connected amputated ϕ-sector Green's functions $\Gamma_{N,M}^{\text{E-AHM};q,l,M_{\nu_R}}$ are also true. The $U(1)_Y \otimes$ BRST symmetry of Sec. II is faithfully represented by these, and the tower of on-shell T-matrix extended WTI (92) and (B10) $T_{N,M}^{\text{E-AHM};q,l,M_{\nu_R}}|_{\text{on-shell}} = 0$, and its extended LSS theorem (93) and (B15).

The three decoupling theorems (108), (109), and (115) follow, but there is a "nondecoupling subtlety." The vanishing of the ν_L surface terms requires a nonzero neutrino Dirac mass

$$m_{\nu;\text{Dirac}} = \frac{1}{\sqrt{2}} y_\nu \langle H \rangle \neq 0. \tag{135}$$

The light and heavy type-I-seesaw ν masses are

$$m_{\nu;\text{Light}} \sim m_{\nu;\text{Dirac}}^2 / M_{\nu_R}, \qquad m_{\nu;\text{Heavy}} \sim M_{\nu_R}, \tag{136}$$

but, in obedience to our proof of $U(1)_Y$ WTI, m_{Light} must not vanish. Therefore type-I-seesaw ν's do not allow the $M_{\nu_R} \to \infty$ limit. For the decoupling theorems, we instead imagine huge, but finite, M_{ν_R} with

$$1 \gg m_{\nu;\text{Dirac}}^2 / M_{\nu_R}^2 \neq 0. \tag{137}$$

No practical trace of the $M_{\nu_R}^2 \sim M_{\text{Heavy}}^2$ right-handed neutrino ν_R survives.

Still, our $U(1)_Y$ WTI insist that, in principle, a very heavy Majorana mass M_{ν_R} cannot completely decouple. It may still have some measurable or observational effect that we have not identified.

V. SSB E-AHM'S PHYSICAL PARTICLE SPECTRUM EXCLUDES THE NGB $\tilde{\pi}$

G. S. Guralnik, C. R. Hagan and T. W. B. Kibble [19] first showed in the spontaneously broken Abelian Higgs model that, although there are no massless particles in the $(A^0 = 0, \vec{\nabla} \cdot \vec{A} = 0)$ "radiation gauge," there is a Goldstone theorem, and a true massless NGB, in the covariant $\partial_\mu A^\mu = 0$ Lorenz gauge. T. W. B. Kibble then showed [8] that the results of experimental measurements are nevertheless the same in radiation and Lorenz gauges, and that the spectrum and dynamics of the *observable particle states* are gauge independent.

A. SSB E-AHM's physical particle spectrum excludes the NGB $\tilde{\pi}$, whose S-matrix elements all vanish

The BRST-invariant Lagrangian for the E-AHM in Lorenz gauge is

$$L_{\text{E-AHM}}^{\text{Lorenz}} = L_{\text{AHM}}^{\text{Lorenz}} + L_{\text{beyondAHM}}^{\text{Gauge Invariant}}(A_\mu, \phi; \Phi, \psi) \tag{138}$$

with $L_{\text{AHM}}^{\text{Lorenz}}$ in (18).

1. Lagrangian governing dynamics of observable particles

We now identify the observable particle spectrum of Lorenz gauge E-AHM by rewriting (138) in terms of a new gauge field

$$B_\mu \equiv A_\mu + \frac{1}{e\langle H \rangle} \partial_\mu \tilde{\pi} \tag{139}$$

and transforming to the Kibble representation [2]:

(i) Gauge field

$$\begin{aligned} A_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \partial_\mu B_\nu - \partial_\nu B_\mu \equiv B_{\mu\nu} \end{aligned} \tag{140}$$

(ii) AHM scalar

$$\begin{aligned} \tilde{\pi} &= \langle H \rangle \vartheta \\ \phi &= \frac{1}{\sqrt{2}} \tilde{H} e^{-iY_\phi \vartheta}; \qquad \tilde{H} = \tilde{h} + \langle H \rangle \\ D_\mu \phi &= \frac{1}{\sqrt{2}} [\partial_\mu - ieY_\phi A_\mu] \tilde{H} e^{-iY_\phi \vartheta} \\ &= \frac{1}{\sqrt{2}} \left[\partial_\mu \tilde{H} - ieY_\phi \tilde{H} \left(A_\mu + \frac{1}{e} \partial_\mu \vartheta \right) \right] e^{-iY_\phi \vartheta} \\ &= \frac{1}{\sqrt{2}} [\partial_\mu \tilde{H} - ieY_\phi \tilde{H} B_\mu] e^{-iY_\phi \vartheta} \end{aligned} \tag{141}$$

(iii) Beyond-AHM scalar

$$\begin{aligned} \Phi &= \tilde{\Phi} e^{-iY_\Phi \vartheta} \\ \langle \tilde{\Phi} \rangle &= 0 \\ D_\mu \Phi &= [\partial_\mu - ieY_\Phi A_\mu] \tilde{\Phi} e^{-iY_\Phi \vartheta} \\ &= \left[\partial_\mu \tilde{\Phi} - ieY_\Phi \tilde{\Phi} \left(A_\mu + \frac{1}{e} \partial_\mu \vartheta \right) \right] e^{-iY_\Phi \vartheta} \\ &= [\partial_\mu \tilde{\Phi} - ieY_\Phi \tilde{\Phi} B_\mu] e^{-iY_\Phi \vartheta} \end{aligned} \tag{142}$$

(iv) Beyond-AHM fermion(s)

$$\begin{aligned} \psi &= \tilde{\psi} e^{-iY_\psi \vartheta} \\ D_\mu \psi &= [\partial_\mu - ieY_\psi A_\mu] \tilde{\psi} e^{-iY_\psi \vartheta} \\ &= \left[\partial_\mu \tilde{\psi} - ieY_\psi \tilde{\psi} \left(A_\mu + \frac{1}{e} \partial_\mu \vartheta \right) \right] e^{-iY_\Phi \vartheta} \\ &= [\partial_\mu \tilde{\psi} - ieY_\psi \tilde{\psi} B_\mu] e^{-iY_\Phi \vartheta}. \end{aligned} \tag{143}$$

The E-AHM Lagrangian, which governs the spectrum and dynamics of *particle physics*, is

$$L^{\text{Particle Physics}}_{\text{E-AHM}}(B_\mu;\tilde{H};\tilde{\Phi};\tilde{\psi}) = L^{\text{Lorenz}}_{\text{AHM};\tilde{H},B_\mu}(B_\mu;\tilde{H};\bar{\eta},\omega) + L^{\text{Gauge Invariant}}_{\text{beyondAHM};\tilde{\Phi}} + L^{\text{Gauge Invariant}}_{\text{beyondAHM};\tilde{\psi}} \tag{144}$$

where the spin $S = 1$ field B_μ

$$\begin{aligned} L^{\text{Lorenz}}_{\text{AHM}}(B_\mu;\tilde{H};\bar{\eta},\omega) &= L^{\text{Gauge Invariant}}_{\text{AHM};\tilde{H},B_\mu} + L^{\text{Gauge Fix;Lorenz}}_{\text{AHM};B_\mu} + L^{\text{Ghost;Lorenz}}_{\text{AHM};B_\mu} \\ L^{\text{Gauge Invariant}}_{\text{AHM};\tilde{H},B_\mu} &= -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{2}e^2Y_\phi^2\langle H\rangle^2 B_\mu B^\mu + \frac{1}{2}(\partial_\mu\tilde{H})^2 + \frac{1}{2}e^2Y_\phi^2(\tilde{H}^2 - \langle H\rangle^2)B_\mu B^\mu - V_{\text{AHM}} \\ L^{\text{Gauge Fix;Lorenz}}_{\text{AHM};B_\mu} &= -\lim_{\xi\to 0}\frac{1}{2\xi}(\partial_\mu B^\mu)^2 \\ L^{\text{Ghost;Lorenz}}_{\text{AHM};B_\mu} &= -\bar{\eta}\partial^2\omega \\ V_{\text{AHM}} &= \frac{1}{4}\lambda_\phi^2(\tilde{H}^2 - \langle H\rangle^2). \end{aligned} \tag{145}$$

For the beyond-AHM scalar(s)

$$\begin{aligned} L^{\text{Gauge Invariant}}_{\text{beyondAHM};\tilde{\Phi}} &= |D_\mu\tilde{\Phi}|^2 - V_{\tilde{\Phi}} - V_{\tilde{\phi}\tilde{\Phi}} \\ D_\mu\tilde{\Phi} &= [\partial_\mu - ieY_\Phi B_\mu]\tilde{\Phi} \\ V_{\tilde{\Phi}} &= M_\Phi^2(\tilde{\Phi}^\dagger\tilde{\Phi}) + \lambda_\Phi^2(\tilde{\Phi}^\dagger\tilde{\Phi})^2 \\ V_{\tilde{\phi}\tilde{\Phi}} &= \frac{1}{2}\lambda_{\phi\Phi}^2(\tilde{H}^2)(\tilde{\Phi}^\dagger\tilde{\Phi}) \end{aligned} \tag{146}$$

while, for beyond-AHM fermions, we take a standard model generation of fermions with anomaly-canceling hypercharges

$$\begin{aligned} L^{\text{Gauge Invariant}}_{\text{beyondAHM};\tilde{\psi}} &= i\bar{\tilde{\psi}}_L D_\mu\tilde{\psi}_L + i\bar{\tilde{\psi}}_R D_\mu\tilde{\psi}_R + L^{\text{Yukawa}}_{\text{beyondAHM};\tilde{\psi}} + L^{\text{Majorana}}_{\text{beyondAHM};\tilde{\nu}_R} \\ D_\mu\tilde{\psi}_L &= [\partial_\mu - ieY_{\psi_L}B_\mu]\tilde{\psi}_L \\ D_\mu\tilde{\psi}_R &= [\partial_\mu - ieY_{\psi_R}B_\mu]\tilde{\psi}_R \\ L^{\text{Yukawa}}_{\text{beyondAHM};\tilde{\psi}} &= -\frac{1}{\sqrt{2}}y_{\phi\psi}(\bar{\tilde{\psi}}_L\tilde{\psi}_R + \bar{\tilde{\psi}}_R\tilde{\psi}_L)\tilde{H} - y_{\Phi\psi}(\bar{\tilde{\psi}}_L\tilde{\Phi}\tilde{\psi}_R + \bar{\tilde{\psi}}_R\tilde{\Phi}^\dagger\tilde{\psi}_L) \\ L^{\text{Majorana}}_{\text{beyondAHM};\tilde{\nu}_R} &= -\frac{1}{2}M_{\nu_R}(\tilde{\nu}_R^{\text{Weyl}}\tilde{\nu}_R^{\text{Weyl}} + \bar{\tilde{\nu}}_R^{\text{Weyl}}\bar{\tilde{\nu}}_R^{\text{Weyl}}). \end{aligned} \tag{147}$$

For $y_{\Phi\psi} \neq 0$, the heavy scalar hypercharge $Y_\Phi = -1$.

The B_μ mass squared in (145) arises entirely from SSB,

$$m_B^2 = m_A^2 = e^2\langle H\rangle^2. \tag{148}$$

Dimensional analysis shows that the contribution of a state of mass/energy $\sim M_{\text{Heavy}}$ to the spectral function $\Delta^{B;\text{Spectral}}_{\text{E-AHM}}$ gives terms $\sim 1/M^2_{\text{Heavy}}$, so that

$$\begin{aligned} \Delta^B_{\text{E-AHM}}(q^2) &= \Delta^B_{\text{AHM}}(q^2) + \mathcal{O}(1/M^2_{\text{Heavy}}), \\ \Delta^B_{\text{AHM}}(q^2) &= \frac{1}{q^2 - m^2_{B;\text{Pole}} + i\epsilon} + \int dm^2\frac{\rho^B_{\text{AHM}}(m^2)}{q^2 - m^2 + i\epsilon}, \\ Z^B_{\text{E-AHM}} &= Z^B_{\text{AHM}} + \mathcal{O}(1/M^2_{\text{Heavy}}). \end{aligned} \tag{149}$$

Therefore the B_μ pole-mass squared is

$$[\Delta^B_{\text{E-AHM}}(0)]^{-1} = -m_B^2 = -e^2\langle H\rangle^2$$

with

$$m^2_{B;\text{Pole}} = e^2\langle H\rangle^2\left[1 - e^2\langle H\rangle^2\int dm^2\frac{\rho^B_{\text{AHM}}(m^2)}{m^2 - i\epsilon}\right]^{-1} + \mathcal{O}(1/M^2_{\text{Heavy}}). \tag{150}$$

2. Decoupling of NGB $\tilde{\pi}$, particle spectrum and dynamics

The Lagrangian (144) is guaranteed to generate all of the results in Secs. III and IV, and Appendixes A and B. In practice, this is done via the manifestly renormalizeable E-AHM Lagrangian (138).

G. Guralnik *et al.* [19], and T. W. B. Kibble [8], showed that, in the Kibble representation in Lorenz gauge, the $U(1)_Y$ AHM quantum states factorize. In the analogous $U(1)_Y$ E-AHM, and in the $m^2_{\text{Weak}}/M^2_{\text{Heavy}} \to 0$ limit the analogous $U(1)_Y$ E-AHM also factorizes,

$$|\Psi(A^\mu;\phi;\bar{\eta},\omega;\Phi;\psi)\rangle \to |\Psi^{\text{Particles}}(B^\mu;\tilde{H})\rangle \times |\Psi^{\text{Ghost}}(\bar{\eta},\omega)\rangle|\Psi^{\text{Goldstone}}(\tilde{\pi})\rangle|\Psi^{B-\text{AHM}}(\tilde{\Phi};\tilde{\psi})\rangle. \tag{151}$$

With $\partial^2\omega = 0; \partial^2\bar{\eta} = 0$, the ghost ω and antighost $\bar{\eta}$ are free and massless and decouple in Lorenz gauge.

It is crucial for SSB gauge theories [8,19] to remember the additional gauge-fixing term inside (138). The E-AHM Lorenz gauge condition is rewritten as

$$\begin{aligned} L^{\text{Gauge Fix;Lorenz}}_{\text{E-AHM}} &= -\lim_{\xi\to 0}\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ &= -\lim_{\xi\to 0}\frac{1}{2\xi}(\partial_\mu B^\mu)^2 - \lim_{\xi\to 0}\frac{1}{2\xi}\left(\frac{1}{e\langle H\rangle}\partial^2\tilde{\pi}\right) \times\left(\frac{1}{e\langle H\rangle}\partial^2\tilde{\pi} - 2\partial_\mu B^\mu\right). \end{aligned} \tag{152}$$

Besides enforcing the new Lorenz gauge-fixing constraint $\partial_\mu B^\mu = 0$ in (145), the auxiliary solution to the gauge-fixing condition (152) is $\partial^2\tilde{\pi} = 0$, which forces $\tilde{\pi}$ to be a free massless particle. The NGB $\tilde{\pi}$ therefore completely decouples from, and disappears from, the observable particle spectrum and its dynamics [8,19], whose states factorize as in (151).

In the $m^2_{\rm Weak}/M^2_{\rm Heavy} \to 0$ limit, all physical measurements and observations are then entirely predicted by the AHM Lagrangian (145) and its states in (151),

$$L^{\rm Lorenz}_{{\rm AHM};B_\mu}(\tilde{H};B_\mu;\bar{\eta},\omega);$$
$$|\Psi^{\rm Particle\ Physics}(B^\mu;\tilde{H};\bar{\eta},\omega)\rangle \to |\Psi^{\rm Particles}(B^\mu;\tilde{H})\rangle|\Psi^{\rm Ghost}(\bar{\eta},\omega)\rangle. \tag{153}$$

What has become of our SSB $U(1)_Y$ Ward-Takahashi identities? Although the NGB $\tilde{\pi}$ has decoupled, it still governs the SSB dynamics and particle spectrum of (153); it is simply *hidden* from explicit view. Still, that decoupling NGB causes powerful hidden constraints on (153) to arise from its hidden shift symmetry

$$\tilde{\pi} \to \tilde{\pi} + \langle H\rangle\theta \tag{154}$$

for constant θ.

Our SSB $U(1)_Y$ WTI, and all of the results of Secs. III and IV and Appendixes A and B are also hidden but still in force: connected amputated Green's functions $\Gamma_{N,M}$ (73) and (B18); connected amputated T-matrix elements $T_{N,M}$ (95) and (B17); Adler self-consistency conditions (92) and (B10) together with their proof of IR finiteness; LSS theorem (93) and (B15); 1-soft-π theorems (95), (B10), and (B17); decoupling theorems for Green's functions and T-matrix elements (109) and (108); and the decoupling theorem for the BEH pole-mass squared $m^2_{\rm BEH;Pole}$ (113). These still govern the SSB dynamics and particle spectrum of (153): they are simply hidden from explicit view. We call this "the hidden $\mathbf{U}(1)_Y \otimes$BRST symmetry of the SSB AHM."

B. SSB causes decoupling of heavy $M^2_{\rm Heavy} \gg m^2_{\rm Weak}$ particles. This fact is hidden, from the observable particle spectrum of the $U(1)_Y$ E-AHM and its dynamics, by the decoupling of the NGB $\tilde{\pi}$

We now take all of the new scalars $\tilde{\Phi}$ and fermions $\tilde{\psi}$ in the E-AHM to be very heavy, and are only interested in low-energy processes,

$$M^2_{\tilde{\Phi}}, M^2_{\tilde{\psi}} \sim M^2_{\rm Heavy} \gg m^2_{\rm Weak}$$
$$|q^2| \lesssim m^2_{\rm Weak}, \tag{155}$$

where q_μ is a typical momentum transfer. In the limit $m^2_{\rm Weak}/M^2_{\rm Heavy} \to 0$ the effective Lagrangian of the spontaneously broken E-AHM gauge theory obeys the Appelquist-Carazzone decoupling theorem [44]

$$L^{\rm Eff;SSB}_{\rm E\text{-}AHM}(k_\mu;B_\nu;\tilde{H};\tilde{\Phi};\tilde{\psi}) \to L^{\rm Eff;SSB}_{\rm AHM}(k_\mu;B_\nu;\tilde{H}) + \mathcal{O}(m^2_{\rm Weak}/M^2_{\rm Heavy}). \tag{156}$$

1. Fourth decoupling theorem: SSB Abelian Higgs model

The ϕ-sector of the extended theory is subject to all of the results of Secs. III and IV and Appendixes A and B. Therefore we know that the BEH pole-mass squared (113) arises entirely from SSB and (unextended) AHM decays. We also know that

$$\begin{aligned} V^{\rm Eff}_{\rm E\text{-}AHM} &= \lambda^2_\phi\left(\phi^\dagger\phi - \frac{1}{2}\langle H\rangle^2\right)^2 + \mathcal{O}^{\rm Ignore}_{\rm E\text{-}AHM} \\ &= \frac{\lambda^2_\phi}{4}(\tilde{H}^2 - \langle H\rangle^2)^2 + \mathcal{O}^{\rm Ignore}_{\rm E\text{-}AHM} \\ &= \frac{\lambda^2_\phi}{4}(\tilde{h}^2 + 2\langle H\rangle\tilde{h})^2 + \mathcal{O}^{\rm Ignore}_{\rm E\text{-}AHM}. \end{aligned} \tag{157}$$

(i) In (113) and (157) finite $\mathcal{O}^{1/M^2_{\rm Heavy};{\rm Irrelevant}}_{{\rm E\text{-}AHM};\phi}$ decouple and vanish as $m^2_{\rm Weak}/M^2_{\rm Heavy} \to 0$.

(ii) Among the terms included in (157) are finite relevant operators dependent on the heavy matter representations,

$$M^2_{\rm Heavy}, \qquad M^2_{\rm Heavy}\ln(M^2_{\rm Heavy}),$$
$$M^2_{\rm Heavy}\ln(m^2_{\rm Weak}), \qquad m^2_{\rm Weak}\ln(M^2_{\rm Heavy}), \tag{158}$$

but the LSS theorem (93) has made them vanish. That fact is a central point of this paper.

(iii) Marginal operators $\sim\ln(M^2_{\rm Heavy})$ have been absorbed in (157): i.e. in the renormalization of gauge-independent observables (i.e. the quartic-coupling constant λ^2_ϕ calculated in the Kibble representation, and the BEH VEV $\langle H\rangle$), and in the unobservable wave-function renormalization $Z^\phi_{\rm E\text{-}AHM}$ (97).

Therefore, no trace of $M_{\rm Heavy}$-scale Φ, ψ, including their virtual loop-contributions to relevant operators, survives in (113) and (157). All the heavy beyond-AHM matter representations have completely decoupled, and the two SSB gauge theories

$${\rm E\text{-}AHM} \xrightarrow{m^2_{\rm Weak}/M^2_{\rm Heavy}\to 0} {\rm AHM} \tag{159}$$

become equivalent in the limit $m^2_{\rm Weak}/M^2_{\rm Heavy} \to 0$, a central result of this paper.

2. Gauge independence of our results

S.-H. Henry Tye and Y. Vtorov-Karevsky [46] show that, calculated in the Kibble representation of Lorenz gauge (i.e. their "polar gauge" [46]), the effective potential is gauge independent. Nielsen [47] went on to prove that any gauge dependence of the effective potential can be reabsorbed by a field redefinition. (For more details see [48].) With λ_ϕ^2 calculated in the Kibble representation, e.g. taken from experiment, the dimension-4 AHM effective potential

$$V_{\mathrm{AHM}}^{\mathrm{Eff}} = \frac{\lambda_\phi^2}{4}(\tilde{h}^2 + 2\langle H\rangle\tilde{h})^2 \tag{160}$$

is therefore all-loop-orders gauge independent. The renormalized experimentally measured gauge-coupling-constant squared at zero momentum $e^2 \equiv e^2(0)$ is also gauge independent. With our four decoupling theorems (109), (108), (113), and (159), so are $\lambda_\phi^2, \langle H\rangle^2$ and $V_{\mathrm{E\text{-}AHM}}^{\mathrm{Eff}}$ in (157), and the B_μ pole-mass squared (150), when calculated in the polar gauge. These all appear in the decoupled particle physics (153) of E-AHM.

After the $\tilde{\pi}$ NGB decouples, the all-loop-orders effective (dimension ≤ 4 operator) Lagrangian that governs low-energy scalar-sector E-AHM physics becomes, in the $m_{\mathrm{Weak}}^2/M_{\mathrm{Heavy}}^2 \to 0$ decoupling limit,

$$\begin{aligned} L_{\phi;\mathrm{E\text{-}AHM}}^{\mathrm{Eff}} &\to \frac{1}{2}|(\partial_\mu + ieB_\mu)\tilde{H}|^2 - V_{\phi;\mathrm{E\text{-}AHM}}^{\mathrm{eff}} \\ V_{\phi;\mathrm{E\text{-}AHM}}^{\mathrm{eff}} &= \frac{\lambda_\phi^2}{4}(\tilde{H}^2 - \langle H\rangle^2)^2 \\ \phi^\dagger\phi - \frac{\langle H\rangle^2}{2} &= \frac{1}{2}(\tilde{H}^2 - \langle H\rangle^2) \\ \tilde{H} &= \tilde{h} + \langle H\rangle; \qquad \langle\tilde{h}\rangle = 0. \end{aligned} \tag{161}$$

Equation (161) is proved gauge independent by extension of the work of Tye and Vtorov-Karevsky [46] and of Nielsen [47] to the E-AHM.

VI. BWL AND GDS: THIS RESEARCH, VIEWED THROUGH THE PRISM OF MATHEMATICAL RIGOR DEMANDED BY RAYMOND STORA

Raymond Stora regarded vintage QFT as incomplete, fuzzy in its definitions, and primitive in technology. For example, he worried about whether the off-shell T matrix could be mathematically rigorously defined to exist in Lorenz gauge: e.g. without running into some IR subtlety. The Adler self-consistency conditions proved here guarantee the IR finiteness of the ϕ-sector on-shell T matrix.

Although he agreed on the correctness of the results presented here, Raymond might complain that we fall short of a strict mathematically rigorous proof (according to his exacting mathematical standards). He reminded us that much has been learned about quantum field theory, via modern path integrals, in the recent ~45 years. In the time up to his passing, he was intent on improving this work by focusing on the following three issues:

(i) properly defining and proving the Lorenz-gauge results presented here with modern path integrals;
(ii) tracking our central results directly to SSB, via BRST methods, in an arbitrary manifestly IR finite 't Hooft R_ξ gauge, i.e. proving to his satisfaction that they are not an artifact of Lorenz gauge;
(iii) tracking our central results directly to those Slavnov-Taylor identities governing the SSB Goldstone mode of the BRST-invariant E-AHM Lagrangian.

Any errors, wrong-headedness, misunderstanding, or misrepresentation appearing in this paper are solely our fault.

VII. CONCLUSION

AHM and E-AHM physics (e.g. on-shell T-matrix elements) have more symmetry than their BRST-invariant Lagrangians. We introduced global $U(1)_Y \otimes$ BRST symmetry in Sec. II, and showed in Secs. IV and V and Appendix B that the low-energy weak-scale effective SSB E-AHM Lagrangian is protected (i.e. against loop contributions from certain heavy $M_{\mathrm{Heavy}}^2 \gg m_{\mathrm{Weak}}^2$ beyond-AHM particles Φ, ψ) by the following hidden 1-soft-π theorems for gauge theories:

(i) A tower of rigid SSB $U(1)_Y$ WTI governing relations among Green's functions.
(ii) A new tower of rigid SSB $U(1)_Y$ WTI which force on-shell T-matrix elements to vanish, and represent the new on-shell behavior of the $U(1)_Y \otimes$ BRST symmetry.
(iii) A new Lee-Stora-Symanzik theorem.
(iv) Four new decoupling theorems (109), (108), (113) and (159).

What is remarkable is that heavy-particle decoupling is obscured/hidden from the physical particle spectrum (153) and its dynamics Once in a while you get shown the light, in the strangest of places, if you look at it right. [52]. The decoupling of the NGB $\tilde{\pi}$ has famously spared the AHM an observable massless particle [19,31,32]. It has also hidden from that physical particle spectrum and dynamics our $U(1)_Y$ WTI (73), (92), (93), (95), (B10), (B15), (B17) and (B18) and their severe constraints on the effective low-energy E-AHM Lagrangian. In particular, the weak-scale E-AHM SSB gauge theory has a hidden $U(1)_Y$ shift symmetry, for constant θ

$$\tilde{\pi} \to \tilde{\pi} + \langle H\rangle\theta \tag{162}$$

which, together with the LSS theorem, has caused the complete[16] decoupling of certain heavy $M_{\mathrm{Heavy}}^2 \gg m_{\mathrm{Weak}}^2$ $U(1)_Y$ matter particles.

[16]Modulo special cases: e.g. heavy Majorana ν_R in Sec. IV D, and possibly $\mathcal{O}_{\mathrm{E\text{-}AHM}\phi}^{d\leq 4;\mathrm{Non\ Analytic;Heavy}}$ in (74).

Such heavy-particle decoupling is historically (i.e. except for high-precision electroweak S,T and U [2,49,50]) the usual physics experience, at each energy scale, as experiments probed smaller and smaller distances. After all, Willis Lamb did not need to know the top-quark or BEH mass in order to interpret theoretically the experimentally observed $\mathcal{O}(m_e\alpha^5 \ln\alpha)$ 2S-2P splitting in the spectrum of hydrogen.

Such heavy-particle decoupling may be the reason why the standard model [39], viewed as an effective low-energy weak-scale theory, is the most experimentally and observationally successful and accurate theory of nature known to humans (when augmented by classical general relativity and neutrino mixing). That "core theory" [51] has no known experimental or observational counterexamples.

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We thank Raymond Stora as an active coresearcher in this work during 2014 and 2015, even increasing his effort on this specific paper in the last months of his life. Although he assured us that he intended to join the paper as a full coauthor, and his contributions fully merit his inclusion, we have been unable to persuade the editors to allow Prof. Stora to co-author posthumously. In particular, we acknowledge his specific contribution of much of the original thinking behind Sec. II's discussion of the role of BRST symmetry; his codiscovery that BRST commutes with spontaneously broken rigid U(1) symmetry in R_ξ gauges, and that there exists a conserved rigid $U(1)_Y$ current for T-matrix elements, together with two towers of rigid $U(1)_Y$ WTI; and his codiscovery that this behavior is ubiquitous to spontaneously broken rigid, gauge and structure groups G, and therefore to SSB gauge theories.

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APPENDIX A: $U(1)_Y$ WARD-TAKAHASHI IDENTITIES IN THE SSB ABELIAN HIGGS MODEL

We present here the full self-contained and detailed derivation of our $U(1)_Y$ WTI for the SSB AHM. We begin by focusing on the rigid/global current $J^\mu_{\rm AHM}$ of the Abelian Higgs model, the spontaneously broken gauge theory of a complex scalar $\phi = \frac{1}{\sqrt{2}}(H + i\pi) = \frac{1}{\sqrt{2}}\tilde{H}e^{i\tilde{\pi}/\langle H\rangle}$, and a massive $U(1)_Y$ gauge field A_μ.

Construct the rigid/global $U(1)_Y$ current with (10)

$$J^\mu_{\rm AHM} = \pi\partial^\mu H - H\partial^\mu\pi - eA^\mu(\pi^2 + H^2). \tag{A1}$$

The classical equations of motion reveal the following crucial fact: due to gauge-fixing terms in the BRST-invariant Lagrangian, the classical axial-vector current (A1) is not conserved. Lorenz gauge

$$\begin{aligned} \partial_\mu J^\mu_{\rm AHM} &= H m_A F_A \\ m_A &= eY_\phi\langle H\rangle \\ F_A &= \partial_\beta A^\beta \end{aligned} \tag{A2}$$

with F_A being the gauge-fixing function. Still, the *physical states* A_μ, h, π of the theory (but not the BRST-invariant Lagrangian) obey $F_A = 0$. In Lorenz gauge, A_μ is transverse and $\tilde{\pi}$ is a massless NGB.

The purpose of Appendix A is to derive a tower of quantum $U(1)_Y$ Ward-Takahashi identities that exhausts the information content of (A2) and severely constrains the dynamics (i.e. the connected time-ordered products) of the physical states of the spontaneously broken Abelian Higgs model.

(1) We study a total differential of a certain connected time-ordered product

$$\partial_\mu\langle 0|T[J^\mu_{\rm AHM}(z) \times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected} \tag{A3}$$

written in terms of the *physical states* of the complex scalar ϕ. Here we have N external renormalized scalars $h = H - \langle H\rangle$ (coordinates x, momenta p), and M external $(CP = -1)$ renormalized pseudoscalars π (coordinates y, momenta q).

(2) Conservation of the global $U(1)_Y$ current for the *physical states*: Strict quantum constraints are imposed that force the relativistically covariant theory of gauge bosons to propagate *only* its true number of quantum spin $S = 1$ degrees of freedom. These constraints are implemented by use of spin $S = 0$ fermionic Fadeev-Popov ghosts $(\bar{\eta}, \omega)$ and, in Lorenz gauge, $S = 0$ massless π. Physical states and their connected time-ordered products, but not the BRST-invariant Lagrangian, obey [25] the gauge-fixing condition $F_A = \partial_\beta A^\beta = 0$ in Lorenz gauge,

$$\langle 0|T[(\partial_\beta A^\beta(z)) \times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected} = 0. \tag{A4}$$

This restores conservation of the rigid/global $U(1)_Y$ current for physical states

$$\langle 0|T[(\partial_\mu J^\mu_{\rm AHM}(z)) \tag{A5}$$

$$\times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected}$$
$$= 0. \tag{A6}$$

It is in this time-ordered-product sense that the physical rigid global $U(1)_Y$ current $J^\mu_{\rm AHM}$ is conserved, and it is this conserved current that generates two towers of quantum $U(1)_Y$ WTI. These WTI severely constrain the dynamics of the ϕ-sector.

(3) Vintage QFT and canonical quantization: Equal-time commutators are imposed on the exact renormalized fields, yielding equal-time quantum commutators at space-time points y, z.

$$\delta(z_0 - y_0)[J^0_{\rm AHM}(z), H(y)] = -i\pi(y)\delta^4(z-y),$$
$$\delta(z_0 - y_0)[J^0_{\rm AHM}(z), \pi(y)] = iH(y)\delta^4(z-y),$$
$$\delta(z_0 - y_0)[J^0_{\rm AHM}(z), A^\mu(y)] = 0,$$
$$\delta(z_0 - y_0)[J^0_{\rm AHM}(z), \omega(y)] = 0,$$
$$\delta(z_0 - y_0)[J^0_{\rm AHM}(z), \bar\eta(y)] = 0. \tag{A7}$$

Nontrivial commutators include

$$\delta(z_0 - y_0)[\partial^0 H(z), H(y)] = -i\delta^4(z-y),$$
$$\delta(z_0 - y_0)[\partial^0 \pi(z), \pi(y)] = -i\delta^4(z-y). \tag{A8}$$

(4) Certain surface integrals vanish: As appropriate to our study of the massless π, we use pion-pole dominance to derive 1-soft-pion theorems, and form the surface integral

$$\lim_{k_\lambda\to 0}\int d^4z e^{ikz}\partial_\mu\langle 0|T[(J^\mu_{\rm AHM} + \langle H\rangle\partial^\mu\pi)(z)$$
$$\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected}$$
$$= \int d^4z\partial_\mu\langle 0|T[(J^\mu_{\rm AHM} + \langle H\rangle\partial^\mu\pi)(z)$$
$$\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected}$$
$$= \int_{3-\rm surface\to\infty} d^3z\hat z_\mu{}^{3-\rm surface}$$
$$\times\langle 0|T[(J^\mu_{\rm AHM} + \langle H\rangle\partial^\mu\pi)(z)$$
$$\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected}$$
$$= 0, \tag{A9}$$

where we have used Stokes' theorem, and $\hat z_\mu{}^{3-\rm surface}$ is a unit vector normal to the 3-surface. The time-ordered product constrains the 3-surface to lie on, or inside, the light cone.

At a given point on the surface of a large enough 4-volume $\int d^4z$ (i.e. the volume of all space-time), all fields are asymptotic in states and out states, properly quantized as free fields, with each field species orthogonal to the others, and they are evaluated at equal times, making time ordering unnecessary at ($z^{3-\rm surface}\to\infty$). Input the global AHM current (A1) to (A9), using $\partial_\mu\langle H\rangle = 0$

$$\int_{3-\rm surface\to\infty} d^3z\hat z_\mu{}^{3-\rm surface}\langle 0|T[$$
$$\times(\pi\partial^\mu h - h\partial^\mu\pi - eA^\mu(\pi^2 + H^2))(z)$$
$$\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected} = 0. \tag{A10}$$

The surface integral (A10) vanishes because both (h, A^μ) are massive in the spontaneously broken $U(1)_Y$ AHM, with $(m^2_{\rm BEH}\neq 0, m^2_A = e^2\langle H\rangle^2)$ respectively. Propagators connecting (h, A^μ), from points on $z^{3-\rm surface}\to\infty$ to the localized interaction points $(x_1\ldots x_N; y_1\ldots y_M)$, must stay inside the light cone, die off exponentially with mass, and are incapable of carrying information that far.

It is very important for pion-pole dominance and this paper that this argument fails for the remaining term in $J^\mu_{\rm AHM}$ in (A1),

$$\int_{3-\rm Surface\to\infty} d^3z\hat z_\mu{}^{3-\rm surface}\times\langle 0|T[(-\langle H\rangle\partial^\mu\pi(z))$$
$$\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected}\neq 0. \tag{A11}$$

π is massless in the SSB AHM, capable of carrying (along the light cone) long-ranged pseudoscalar forces out to the 3-surface ($z^{2-\rm surface}\to\infty$): i.e. the very ends of the light cone (but not inside it). That masslessness is the basis of our pion-pole-dominance-based $U(1)_Y$ WTI, which give 1-soft-pion theorems (A18), infrared finiteness for $m^2_\pi = 0$ (A22), and the LSS theorem (A27).

(5) Master equation: Using (A5) and (A8) in (A3) to form the right-hand side, and (A10) in (A3) to form the left-hand side, we write the master equation

$$\lim_{k_\lambda\to 0}\int d^4z e^{ikz}\times\Big\{-\langle H\rangle\partial^z_\mu\langle 0|T[(\partial^\mu\pi(z))$$
$$\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected}$$
$$-\sum_{m=1}^M i\delta^4(z-y_m)\langle 0|T[h(z)h(x_1)\ldots h(x_N)$$
$$\times\pi(y_1)\ldots\widehat{\pi(y_m)}\ldots\pi(y_M)]|0\rangle_{\rm connected}$$
$$+\sum_{n=1}^N i\delta^4(z-x_n)\langle 0|T[h(x_1)\ldots\widehat{h(x_n)}\ldots h(x_N)$$
$$\times\pi(z)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\rm connected}\Big\} = 0 \tag{A12}$$

where the hatted fields $\widehat{h(x_n)}$ and $\widehat{\pi(y_m)}$ are to be removed. We have also thrown away a sum of M terms, proportional to $\langle H\rangle$, that corresponds entirely to disconnected graphs.

(6) ϕ-sector connected amplitudes: Connected momentum-space amplitudes, with N external BEHs and M external πs, are defined in terms of ϕ-sector connected time-ordered products

$$iG_{N,M}(p_1\ldots p_N;q_1\ldots q_M)(2\pi)^4\delta^4\left(\sum_{n=1}^N p_n+\sum_{m=1}^M q_m\right)$$
$$=\prod_{n=1}^N\int d^4x_n e^{ip_nx_n}\prod_{m=1}^M\int d^4y_m e^{iq_my_m}$$
$$\times\langle 0|T[h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}}. \quad \text{(A13)}$$

The master equation (A12) can then be rewritten as

$$\lim_{k_\lambda\to 0}\{i\langle H\rangle k^2 G_{N,M+1}(p_1\ldots p_N;kq_1\ldots q_M)$$
$$-\sum_{n=1}^N G_{N-1,M+1}(p_1\ldots\widehat{p_n}\ldots p_N;(k+p_n)q_1\ldots q_M)$$
$$+\sum_{m=1}^M G_{N+1,M-1}((k+q_m)p_1\ldots p_N;q_1\ldots\widehat{q_m}\ldots q_M)\}$$
$$=0 \quad \text{(A14)}$$

with the hatted momenta $(\widehat{p_n},\widehat{q_m})$ removed in (A14), and an overall momentum conservation factor of $(2\pi)^4\delta^4(k+\sum_{n=1}^N p_n+\sum_{m=1}^M q_m)$.

(7) ϕ-propagators: Special cases of (A13) are the BEH and π propagators

$$iG_{2,0}(p_1,-p_1;)=i\int\frac{d^4p_2}{(2\pi)^4}G_{2,0}(p_1,p_2;)$$
$$=\int d^4x_1 e^{ip_1x_1}\langle 0|T[h(x_1)h(0)]|0\rangle$$
$$\equiv i\Delta_{\text{BEH}}(p_1^2)$$
$$iG_{0,2}(;q_1,-q_1)=i\int\frac{d^4q_2}{(2\pi)^4}G_{0,2}(;q_1,q_2)$$
$$=\int d^4y_1 e^{iq_1y_1}\langle 0|T[\pi(y_1)\pi(0)]|0\rangle$$
$$\equiv i\Delta_\pi(q_1^2). \quad \text{(A15)}$$

(8) ϕ-sector connected amputated 1-(h,π)-reducible (1-ϕ-R) transition matrix (T matrix): With an overall momentum conservation factor $(2\pi)^4\delta^4\times(\sum_{n=1}^N p_n+\sum_{m=1}^M q_m)$, the ϕ-sector connected amplitudes are related to ϕ-sector connected amputated T-matrix elements

$$G_{N,M}(p_1\ldots p_N;q_1\ldots q_M)$$
$$\equiv\prod_{n=1}^N[i\Delta_{\text{BEH}}(p_n^2)]\prod_{m=1}^M[i\Delta_\pi(q_m^2)]$$
$$\times T_{N,M}(p_1\ldots p_N;q_1\ldots q_M) \quad \text{(A16)}$$

so that the master equation (A12) can be written

$$\lim_{k_\lambda\to 0}\Big\{i\langle H\rangle k^2[i\Delta_\pi(k^2)]T_{N,M+1}(p_1\ldots p_N;kq_1\ldots q_M)$$
$$-\sum_{n=1}^N T_{N-1,M+1}(p_1\ldots\widehat{p_n}\ldots p_N;(k+p_n)q_1\ldots q_M)$$
$$\times[i\Delta_\pi((k+p_n)^2)][i\Delta_{\text{BEH}}(p_n^2)]^{-1}$$
$$+\sum_{m=1}^M T_{N+1,M-1}((k+q_m)p_1\ldots p_N;q_1\ldots\widehat{q_m}\ldots q_M)$$
$$\times[i\Delta_{\text{BEH}}((k+q_m)^2)][i\Delta_\pi(q_m^2)]^{-1}\Big\}=0 \quad \text{(A17)}$$

with the hatted momenta $(\widehat{p_n},\widehat{q_m})$ removed in (A17), and an overall momentum conservation factor of $(2\pi)^4\delta^4(k+\sum_{n=1}^N p_n+\sum_{m=1}^M q_m)$.

(9) Pion-pole dominance and 1-soft-π theorems for the T matrix: Consider the 1-soft-pion limit

$$\lim_{k_\lambda\to 0}k^2\Delta_\pi(k^2)=1 \quad \text{(A18)}$$

where the π is hypothesized to be all-loop-orders massless, and written in the Källén-Lehmann representation [26] with spectral density ρ^π_{AHM},

$$\Delta_\pi(k^2)=\frac{1}{k^2+i\epsilon}+\int dm^2\frac{\rho^\pi_{\text{AHM}}(m^2)}{k^2-m^2+i\epsilon}. \quad \text{(A19)}$$

The master equation (A12) then becomes

$$-\langle H\rangle T_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M)$$
$$=\sum_{n=1}^N T_{N-1,M+1}(p_1\ldots\widehat{p_n}\ldots p_N;p_nq_1\ldots q_M)$$
$$\times[i\Delta_\pi(p_n^2)][i\Delta_{\text{BEH}}(p_n^2)]^{-1}$$
$$-\sum_{m=1}^M T_{N+1,M-1}(q_mp_1\ldots p_N;q_1\ldots\widehat{q_m}\ldots q_M)$$
$$\times[i\Delta_{\text{BEH}}(q_m^2)][i\Delta_\pi(q_m^2)]^{-1} \quad \text{(A20)}$$

in the 1-soft-pion limit. As usual the hatted momenta $(\widehat{p_n},\widehat{q_m})$ and associated fields are removed in (A20), and an overall momentum conservation factor $(2\pi)^4\delta^4(\sum_{n=1}^N p_n+\sum_{m=1}^M q_m)$ applied.

The set of 1-soft-pion theorems (A20) has the form

$$\langle H\rangle T_{N,M+1} \sim T_{N-1,M+1} - T_{N+1,M-1}, \quad \text{(A21)}$$

relating, by the addition of a zero-momentum pion, an $N+M+1$-point function to $N+M$-point functions.

(10) The Adler self-consistency relations [but now for a gauge theory rather than global $SU(2)_L \times SU(2)_R$ [37,38]] are obtained by putting the remainder of the (A20) particles on mass shell

$$\langle H\rangle T_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) \times (2\pi)^4\delta^4\left(\sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m\right)\Bigg|^{p_1^2=p_2^2\ldots=p_N^2=m_{\mathrm{BEH}}^2}_{q_1^2=q_2^2\ldots=q_M^2=0} = 0, \quad \text{(A22)}$$

which guarantees the IR finiteness of the ϕ-sector on-shell T matrix in the SSB AHM gauge theory in Lorenz gauge, with massless π in the 1-soft-pion limit. These 1-soft-pion theorems [37,38] force the T matrix to vanish as one of the pion momenta goes to 0, provided all other physical scalar particles are on mass shell. Equation (A22) asserts the absence of infrared divergences in the physical-scalar sector in Goldstone mode. "Although individual Feynman diagrams may be IR divergent, those IR divergent parts cancel exactly in each order of perturbation theory. Furthermore, the Goldstone mode amplitude must vanish in the soft-pion limit [12]."

(11) 1-(h,π) reducibility (1-ϕ-R) and 1-(h,π) irreducibility (1-ϕ-I): With some exceptions, a ϕ-sector connected amputated transition-matrix element $T_{N,M}$ can be cut apart by cutting an internal h or π line, and is designated 1-ϕ-R. In contrast, a ϕ-sector connected amputated Green's function $\Gamma_{N,M}$ is defined to be 1-ϕ-I: i.e. it cannot be cut apart by cutting an internal h or π line.

$$T_{N,M} = \Gamma_{N,M} + (1\text{-}\phi\text{-}R). \quad \text{(A23)}$$

Both $T_{N,M}$ and $\Gamma_{N,M}$ are 1-(A_μ)-reducible (1-A^μ-R): i.e. they can be cut apart by cutting an internal transverse-vector A_μ gauge-particle line.

(12) ϕ-sector two-point functions, propagators and a three-point vertex: The special two-point functions $T_{0,2}(;q,-q)$ and $T_{2,0}(p,-p;)$, and the three-point vertex $T_{1,2}(q;0,-q)$, are 1-ϕ-I (i.e. they are not 1-ϕ-R), and are therefore equal to the corresponding 1-ϕ-I connected amputated Green's functions. The two-point functions

$$T_{2,0}(p,-p;) = \Gamma_{2,0}(p,-p;) = [\Delta_{\mathrm{BEH}}(p^2)]^{-1}$$
$$T_{0,2}(;q,-q) = \Gamma_{0,2}(;q,-q) = [\Delta_\pi(q^2)]^{-1} \quad \text{(A24)}$$

are related to the $(1h,2\pi)$ three-point $h\pi^2$ vertex

$$T_{1,2}(p;q,-p-q) = \Gamma_{1,2}(p;q,-p-q) \quad \text{(A25)}$$

by a 1-soft-pion theorem (A20)

$$\begin{aligned}\langle H\rangle T_{1,2}(q;0,-q) &- T_{2,0}(q,-q;) + T_{0,2}(;q,-q)\\ &= \langle H\rangle T_{1,2}(q;0,-q) - [\Delta_{\mathrm{BEH}}(q^2)]^{-1} + [\Delta_\pi(q^2)]^{-1}\\ &= \langle H\rangle \Gamma_{1,2}(q;0,-q) - \Gamma_{2,0}(q,-q;) + \Gamma_{0,2}(;q,-q)\\ &= \langle H\rangle \Gamma_{1,2}(q;0,-q) - [\Delta_{\mathrm{BEH}}(q^2)]^{-1} + [\Delta_\pi(q^2)]^{-1}\\ &= 0. \end{aligned} \quad \text{(A26)}$$

(13) The LSS theorem, in the spontaneously broken AHM in Lorenz gauge, is a special case of that SSB gauge theory's Adler self-consistency relations (A22)

$$\begin{aligned}\langle H\rangle T_{0,2}(;00) &= 0,\\ \langle H\rangle \Gamma_{0,2}(;00) &= 0,\\ \langle H\rangle [\Delta_\pi(0)]^{-1} &= 0,\end{aligned} \quad \text{(A27)}$$

proving that π is massless $m_\pi^2 = 0$ (i.e. not just the much weaker theorem that the Nambu-Goldstone boson $\tilde{\pi}$ is massless). That all-loop-orders renormalized masslessness is protected/guaranteed by the global $U(1)_Y$ symmetry of the *physical states* of the gauge theory after spontaneous symmetry breaking.

(14) $T^{\mathrm{External}}_{N,M+1}$ ϕ-sector T matrix with one soft $\pi(q_\mu = 0)$ attached to an external leg: Figure 1 shows that

$$\begin{aligned}\langle H\rangle T^{\mathrm{External}}_{N,M+1}&(p_1\ldots p_N;0q_1\ldots q_M)\\ &= \sum_{n=1}^{N}[i\langle H\rangle\Gamma_{1,2}(p_n;0,-p_n)][i\Delta_\pi(p_n^2)]\\ &\quad\times T_{N-1,M+1}(p_1\ldots\widehat{p_n}\ldots p_N;p_nq_1\ldots q_M)\\ &\quad+\sum_{m=1}^{M}[i\langle H\rangle\Gamma_{1,2}(q_m;0,-q_m)][i\Delta_{\mathrm{BEH}}(q_m^2)]\\ &\quad\times T_{N+1,M-1}(q_mp_1\ldots p_N;q_1\ldots\widehat{q_m}\ldots q_M)\\ &= \sum_{n=1}^{N}(1-[i\Delta_\pi(p_n^2)][i\Delta_{\mathrm{BEH}}(p_n^2)]^{-1})\\ &\quad\times T_{N-1,M+1}(p_1\ldots\widehat{p_n}\ldots p_N;p_nq_1\ldots q_M)\\ &\quad-\sum_{m=1}^{M}(1-[i\Delta_{\mathrm{BEH}}(q_m^2)][i\Delta_\pi(q_m^2)]^{-1})\\ &\quad\times T_{N+1,M-1}(q_mp_1\ldots p_N;q_1\ldots\widehat{q_m}\ldots q_M)\end{aligned} \quad \text{(A28)}$$

where we used (A26). Now separate

$$T_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) = T^{\text{External}}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) + T^{\text{Internal}}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) \quad \text{(A29)}$$

so that

$$\langle H\rangle T^{\text{Internal}}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) = \sum_{m=1}^{M} T_{N+1,M-1}(q_m p_1\ldots p_N;q_1\ldots \widehat{q_m}\ldots q_M) - \sum_{n=1}^{N} T_{N-1,M+1}(p_1\ldots \widehat{p_n}\ldots p_N;p_n q_1\ldots q_M). \quad \text{(A30)}$$

(15) Recursive $U(1)_Y$ WTI for 1-ϕ-I connected amputated Green's functions $\Gamma_{N,M}$: Removing the 1-ϕ-R graphs from both sides of (A30) yields the recursive identity

$$\langle H\rangle \Gamma_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) = \sum_{m=1}^{M} \Gamma_{N+1,M-1}(q_m p_1\ldots p_N;q_1\ldots \widehat{q_m}\ldots q_M) - \sum_{n=1}^{N} \Gamma_{N-1,M+1}(p_1\ldots \widehat{p_n}\ldots p_N;p_n q_1\ldots q_M). \quad \text{(A31)}$$

B. W. Lee [12] gave an inductive proof for the corresponding recursive $SU(2)_L \times SU(2)_R$ WTI in the global Gell-Mann Lévy model with PCAC [17]. Specifically, he proved that, given the global $SU(2)_L \times SU(2)_R$ analogy of (A30), the global $SU(2)_L \times SU(2)_R$ analogy of (A31) follows. This he did by examination of the explicit reducibility/irreducibility of the various Feynman graphs involved.

That proof also works for the $U(1)_Y$ SSB AHM, thus establishing our tower of 1-ϕ-I connected amputated Green's functions' recursive $U(1)_Y$ WTI (A31) for a local/gauge theory.

Rather than including the lengthy proof here, we paraphrase [12] as follows: (A26) shows that (A31) is true for $(N = 1, M = 1)$. Assume it is true for all (n, m) such that $n + m < N + M$. Consider (A30) for $n = N$, $m = M$. The two classes of graphs contributing to $T^{\text{Internal}}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M)$ are displayed in Fig. 2.

The top graphs in Fig. 2 are 1-ϕ-R. For $(n, m; n + m < N + M)$ we may use (A31), for those 1-ϕ-I Green's functions $\Gamma_{n,m}$ that contribute to (A30), to show that the contribution of 1-ϕ-R graphs to both sides of (A30) is identical.

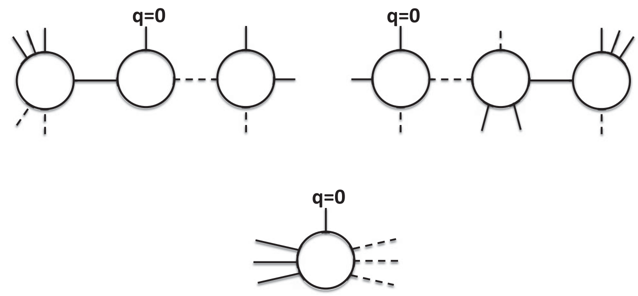

FIG. 2. Circles are 1-ϕ-I $\Gamma^{\text{E-AHM}}_{n,m}$, solid lines π, and dashed lines h, with $n + m < N + M$. One (zero-momentum) soft pion emerges in all possible ways from the connected amputated Green's functions. $\Gamma^{\text{E-AHM}}_{n,m}$ is 1-A^μ-R by cutting an A^μ line, and also 1-Φ-R by cutting a Φ line. Figure 2 is the E-AHM analogy of B. W. Lee's Fig. 11 [12]. The same graph topologies, but without internal beyond-AHM Φ, ψ heavy matter, are used in the proof of (A31) for the (unextended) AHM.

The bottom graphs in Fig. 2 are 1-ϕ-I and so already obey (A31).

(16) The LSS theorem makes tadpoles vanish,

$$\langle 0|h(x=0)|0\rangle_{\text{connected}} = i[i\Delta_{\text{BEH}}(0)]\Gamma_{1,0}(0;), \quad \text{(A32)}$$

but the $N = 0$, $M = 1$ case of (A31) reads

$$\Gamma_{1,0}(0;) = \langle H\rangle \Gamma_{0,2}(;00) = 0, \quad \text{(A33)}$$

where we used (A27), so that tadpoles all vanish automatically, and separate tadpole renormalization is unnecessary. Since we can choose the origin of coordinates anywhere we like

$$\langle 0|h(x)|0\rangle_{\text{connected}} = 0. \quad \text{(A34)}$$

(17) Renormalized gauge-independent observable $\langle H\rangle$.

$$\langle 0|H(x)|0\rangle_{\text{connected}} = \langle 0|h(x)|0\rangle_{\text{connected}} + \langle H\rangle = \langle H\rangle$$
$$\partial_\mu \langle H\rangle = 0. \quad \text{(A35)}$$

(18) Benjamin W. Lee's 1970 Cargese summer school lectures' [12] proof of ϕ-sector WTI focuses on the global $SU(2)_L \times SU(2)_R$ Gell-Mann Lévy theory and PCAC, but gives a detailed pedagogical account of the appearance of the Goldstone theorem and its true massless Nambu-Goldstone bosons, especially of the emergence of the LSS theorem, in *global* theories, and is recommended reading. We include a translation guide in Table I.

TABLE I. Derivation of Ward-Takahashi identities.

Property	This paper	B. W. Lee [12]
Lagrangian invariant	BRST	Global group
Structure group	$U(1)_Y$	$SU(2)_L \times SU(2)_R$
Local/gauge group	$U(1)_Y$	
Rigid/global group	$U(1)_Y$	$SU(2)_L \times SU(2)_R$
Global currents	J^μ_{AHM}	$\vec{V}^\mu; \vec{A}^\mu$
PCAC	no	Yes
Current divergence	$H m_A \partial_\beta A^\beta$	$0; f_\pi m_\pi^2 \vec{\pi}$
$L_{\text{Gauge Fixing}}$	Lorenz	
Gauge	Lorenz	
Ghosts $\bar{\eta}, \omega$	Decouple	
Conserved current	Physical states	Lagrangian
Physical states	$A_\mu, h, \pi, \Phi, \psi$	$s, \vec{\pi}$
Interaction	Weak	Strong
Fields	$A_\mu, H, \pi, \bar{\eta}, \omega, \Phi, \psi$	$\sigma, \vec{\pi}$
BEH scalar	$h = H - \langle H \rangle$	$s = \sigma - \langle \sigma \rangle$
VEV	$\langle H \rangle$	$\langle \sigma \rangle = v = f_\pi$
Particles in loops	Physicalandghosts	$s, \vec{\pi}$
Renormalization	All loop orders	All loop orders
Amplitudes		G
Connected amplitudes	$G_{N,M}$	H
No pion-pole singularity		$\bar{H}$
1-ϕ-I or R	$h,\ \pi$	$s, \vec{\pi}$
Connected $\Gamma_{N,M}$	Amputated	Amputated
Connected $T_{N,M}$	Amputated	Amputated
NGB after SSB	$\tilde{\pi}$	$\tilde{\vec{\pi}}$
LSS theorem	$\langle H \rangle \Gamma^{t_1 t_2}_{0,2}(;00) = 0$	$f_\pi \Gamma^{t_1 t_2}_{0,2}(;00)$ $= \epsilon \delta^{t_1 t_2} = 0$
Explicit breaking		$\epsilon = f_\pi m_\pi^2$
ϕ-sector $T_{N,M}$	1-ϕ-R 1-A_μ-R, 1-Φ-R	1-ϕ-R
ϕ-sector $\Gamma_{N,M}$	1-ϕ-I 1-A_μ-R, 1-Φ-R	1-ϕ-I
T-matrix	$T_{N,M}$	T
ϕ-sector Green's F's	$\Gamma_{N,M}$	$\Gamma_{N,M}$
External $\pi(q_\mu = 0)$	$T^{\text{External}}_{N,M+1}$	T_1
Internal $\pi(q_\mu = 0)$	$T^{\text{Internal}}_{N,M+1}$	T_2
BEH propagator	Δ_{BEH}	Δ_σ
Transverse propagator	$\Delta^{\mu\nu}_A$	
Pion propagator	Δ_π	$\delta^{t_i t_j} \Delta_\pi$
SSB	Goldstone mode	Goldstone mode
Goldstone theorem	Physical states	Goldstone mode
LSS theorem	One-dimensional line	One-dimensional boundary of Two-dimensional quarter-plane

APPENDIX B: $U(1)_Y$ ϕ-SECTOR WTI WHICH INCLUDE THE ALL-LOOP-ORDERS CONTRIBUTIONS OF CERTAIN ADDITIONAL VIRTUAL $U(1)_Y$ *CP*-CONSERVING MATTER REPRESENTATIONS Φ, ψ IN THE E-AHM

We focus on the rigid/global extended-AHM current

$$J^\mu_{\text{E-AHM}} = J^\mu_{\text{AHM}}(A^\mu, \phi) + J^\mu_{\text{beyondAHM}}(\Phi, \Psi) \quad \text{(B1)}$$

of the extended Abelian Higgs model, the spontaneously broken gauge theory of a complex spin $S = 0$ scalar $\phi = \frac{1}{\sqrt{2}}(H + i\pi)$, a massive $U(1)_Y$ $S = 1$ transverse gauge field A_μ, and certain $S = 0$ scalars Φ and anomaly-canceling $S = \frac{1}{2}$ fermions ψ originating in beyond-AHM models.

In order to force renormalized connected amplitudes with an odd number of π s to vanish, the new particles Φ, ψ are taken in this paper to conserve *CP*.

The classical equations of motion reveal that, due to gauge-fixing terms in the BRST-invariant Lagrangian, the classical current (B1) is not conserved. In Lorenz gauge

$$\begin{aligned} \partial_\mu J^\mu_{\text{E-AHM}} &= H m_A F_A, \\ m_A &= eY_\phi \langle H\rangle, \\ F_A &= \partial_\beta A^\beta, \end{aligned} \tag{B2}$$

with F_A being the gauge-fixing function.

The purpose of this appendix is to derive a tower of $U(1)_Y$ extended WTI that exhausts the information content of (B2), and severely constrains the dynamics (i.e. the connected time-ordered products) of the physical states of the SSB *extended* AHM. We make use here of all of the results in Appendix A concerning J^μ_{AHM}.

(1) We study a certain total differential of a connected time-ordered product,

$$\begin{aligned} &\partial_\mu \langle 0|T[J^\mu_{\text{E-AHM}}(z) \\ &\quad\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}}, \end{aligned} \tag{B3}$$

written in terms of the *physical states* of the complex scalar ϕ. Here we have N external renormalized scalars $h = H - \langle H\rangle$ (coordinates x, momenta p), and M external ($CP = -1$) renormalized pseudoscalars π (coordinates y, momenta q).

(2) Conservation of the global $U(1)_Y$ current for the *physical states*: Strict quantum constraints are imposed that force the relativistically covariant theory of a massive transverse gauge boson to propagate *only* its true number of quantum spin $S = 1$ degrees of freedom. Physical states and their time-ordered products, but not the BRST-invariant Lagrangian, obey the gauge-fixing condition $F_A = \partial_\beta A^\beta = 0$ in Lorenz gauge [25],

$$\begin{aligned} &\langle 0|T[(\partial_\beta A^\beta(z)) \\ &\quad\times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}} \\ &\quad = 0, \end{aligned} \tag{B4}$$

which restores conservation of the rigid/global $U(1)_Y$ extended current for physical states

$$\begin{aligned} &\langle 0|T[(\partial_\mu J^\mu_{\text{E-AHM}}(z)) \\ &\quad\times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}} \\ &\quad = 0. \end{aligned} \tag{B5}$$

It is in this time-ordered-product sense that the rigid global extended $U(1)_Y$ current $J^\mu_{\text{E-AHM}}$ is conserved, and it is this conserved current that generates our tower of $U(1)_Y$ extended WTI. These extended WTI severely constrain the dynamics of ϕ.

(3) Vintage QFT and canonical quantization: Equal-time commutators are imposed on the exact renormalized beyond-AHM fields, yielding equal-time quantum commutators at space-time points y, z.

$$\begin{aligned} \delta(z_0 - y_0)[J^0_{\text{beyondAHM}}(z), H(y)] = 0, \\ \delta(z_0 - y_0)[J^0_{\text{beyondAHM}}(z), \pi(y)] = 0. \end{aligned} \tag{B6}$$

Only certain $U(1)_Y$ matter particles Φ, ψ obey this condition.

(a) Renormalized $\langle H\rangle$ is defined to match the (unextended) AHM. Our extended $U(1)_Y$ WTI therefore require that all of the new spin $S = 0$ fields in $J^\mu_{\text{beyondAHM}}$ have zero VEV:

$$\langle \Phi_{\text{beyondAHM}}\rangle = 0. \tag{B7}$$

Only certain $U(1)_Y$ matter particles Φ obey this condition.

(4) Certain connected surface integrals must vanish: As appropriate to our study of massless π, we again use pion-pole dominance to derive 1-soft-pion theorems, and require that the *connected* surface integral

$$\begin{aligned} &\lim_{k_\lambda\to 0}\int d^4z e^{ikz}\partial_\mu\langle 0|T[(J^\mu_{\text{beyondAHM}}(z)) \\ &\quad\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}} \\ &= \int d^4z\partial_\mu\langle 0|T[(J^\mu_{\text{beyondAHM}}(z)) \\ &\quad\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}} \\ &= \int_{3-\text{Surface}\to\infty} d^3z\hat{z}_\mu{}^{3-\text{surface}} \\ &\quad\times\langle 0|T[(J^\mu_{\text{beyondAHM}}(z)) \\ &\quad\times h(x_1)\ldots h(x_N)\pi_(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}} \\ &= 0, \end{aligned} \tag{B8}$$

where we have used Stokes' theorem, and $\hat{z}_\mu{}^{3-\text{surface}}$ is a unit vector normal to the 3-surface. The time-ordered product constrains the 3-surface to lie on or inside the light cone.

At a given point on the surface of a large enough 4-volume $\int d^4z$ (i.e. the volume of all space-time), all fields are asymptotic in states and out states, are properly quantized as free fields, with each field species orthogonal to the others, and are evaluated at equal times, making time ordering unnecessary at ($z^{3-\text{surface}}\to\infty$).

Only certain $U(1)_Y$ massive matter particles Φ, ψ obey this condition.

(5) Extended master equation: Using (B5) and (B6) in (B3) to form the right-hand side, and (B8) in (B3) to

form the left-hand side, we write the *extended* master equation, which relates *connected* time-ordered products,

$$\lim_{k_\lambda\to 0}\int d^4z e^{ikz}\times\Big\{-\langle H\rangle\partial^z_\mu\langle 0|T[(\partial^\mu\pi(z)) \times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}} - \sum_{m=1}^{M} i\delta^4(z-y_m)\langle 0|T[h(z)h(x_1)\ldots h(x_N) \times \pi(y_1)\ldots\widehat{\pi(y_m)}\ldots\pi(y_M)]|0\rangle_{\text{connected}} + \sum_{n=1}^{N} i\delta^4(z-x_n)\langle 0|T[h(x_1)\ldots\widehat{h(x_n)}\ldots h(x_N) \times \pi(z)\pi(y_1)\ldots\pi(y_M)]|0\rangle_{\text{connected}}\Big\} = 0, \tag{B9}$$

where the hatted fields $\widehat{h(x_n)}$ and $\widehat{\pi(y_m)}$ are to be removed. We have also thrown away a sum of M terms, proportional to $\langle H\rangle$, that corresponds entirely to disconnected graphs.

(a) $U(1)_Y$ Ward-Takahashi identities for the ϕ-sector of the E-AHM: The extended master equation (B9) governing the ϕ-sector of the E-AHM is idential to the master equation (A12) governing the ϕ-sector of the (unextended) AHM. This proves that, for each $U(1)_Y$ WTI that is true in the AHM, an analogous $U(1)_Y$ WTI is true for the E-AHM. Appendix A proved $U(1)_Y$ WTI relations among 1-ϕ-R ϕ-sector T-matrix elements $T_{N,M}$, as well as $U(1)_Y$ WTI relations among 1-ϕ-I ϕ-sector Green's functions $\Gamma_{N,M}$, in the spontaneously broken AHM. Analogous $U(1)_Y$ WTI relations among 1-ϕ-R ϕ-sector T-matrix elements $T^{\text{E-AHM}}_{N,M}$, as well as analogous $U(1)_Y$ WTI relations among 1-ϕ-I ϕ-sector Green's functions $\Gamma^{\text{E-AHM}}_{N,M}$, are therefore here proved true for the spontaneously broken *extended* AHM.

But there is one *huge* difference. The renormalization of our $U(1)_Y$ WTI, governing ϕ-sector $T^{\text{E-AHM}}_{N,M}$ and ϕ-sector $\Gamma^{\text{E-AHM}}_{N,M}$, now includes the all-loop-orders contributions of virtual gauge bosons, ϕ scalars, ghosts, new beyond-AHM scalars and new beyond-AHM fermions: i.e. A^μ, $h, \pi, \bar\eta, \omega, \Phi, \psi$ respectively.

(10) Adler self-consistency relations, but now for the E-AHM gauge theory,

$$\langle H\rangle T^{\text{E-AHM}}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) \times(2\pi)^4\delta^4\left(\sum_{n=1}^N p_n+\sum_{m=1}^M q_m\right)\Big|^{p_1^2=p_2^2\ldots=p_N^2=m^2_{\text{BEH}}}_{q_1^2=q_2^2\ldots=q_M^2=0} = 0. \tag{B10}$$

These prove the IR finiteness of the ϕ-sector on-shell connected T matrix in the E-AHM gauge theory, with massless π, in Lorenz gauge, in the 1-soft-pion limit.

(11) 1-(h,π) reducibility (1-ϕ-R) and 1-(h,π) irreducibility (1-ϕ-I): With some exceptions, the extended ϕ-sector connected amputated T-matrix elements $T^{\text{E-AHM}}_{N,M}$ can be cut apart by cutting an internal h or π line: they are designated 1-ϕ-R. In contrast, the extended ϕ-sector Green's functions $\Gamma^{\text{E-AHM}}_{N,M}$ are defined to be 1-ϕ-I: i.e. they cannot be cut apart by cutting an internal h or π line.

$$T^{\text{E-AHM}}_{N,M}=\Gamma^{\text{E-AHM}}_{N,M}+(1\text{-}\phi\text{-}R). \tag{B11}$$

As usual, both $T^{\text{E-AHM}}_{N,M}$ and $\Gamma^{\text{E-AHM}}_{N,M}$ are 1-(A^μ)-reducible (1-A^μ-R), i.e. they can be cut apart by cutting an internal transverse-vector A^μ gauge-particle line. They are also 1-Φ-reducible (1-Φ-R), i.e. they can be cut apart by cutting an internal Φ-scalar line.

(12) ϕ-sector two-point functions, propagators and a three-point vertex: The two-point functions

$$T^{\text{E-AHM}}_{2,0}(p,-p;)=\Gamma^{\text{E-AHM}}_{2,0}(p,-p;)=[\Delta_{\text{BEH}}(p^2)]^{-1},\quad T^{\text{E-AHM}}_{0,2}(;q,-q)=\Gamma^{\text{E-AHM}}_{0,2}(;q,-q)=[\Delta_\pi(q^2)]^{-1}, \tag{B12}$$

are related to the $(1h,2\pi)$ three-point $h\pi^2$ vertex

$$T^{\text{E-AHM}}_{1,2}(p;q,-p-q)=\Gamma^{\text{E-AHM}}_{1,2}(p;q,-p-q) \tag{B13}$$

by a 1-soft-pion theorem (B18)

$$\langle H\rangle T^{\text{E-AHM}}_{1,2}(q;0,-q)=[\Delta_{\text{BEH}}(q^2)]^{-1}-[\Delta_\pi(q^2)]^{-1}. \tag{B14}$$

(13) The LSS theorem in Lorenz-gauge E-AHM is the $N=0$, $M=1$ case of (B10),

$$\langle H\rangle T^{\text{E-AHM}}_{0,2}(;00)=0,\quad \langle H\rangle\Gamma^{\text{E-AHM}}_{0,2}(;00)=0,\quad \langle H\rangle[\Delta_\pi(0)]^{-1}=0, \tag{B15}$$

proving that π is still massless in the E-AHM, whose all-loop-orders renormalized masslessness is protected/guaranteed by the global $U(1)_Y$ symmetry of the physical states of the E-AHM gauge theory after SSB.

(14) $T^{\text{E-AHM;External}}_{N,M+1}$ are the 1-ϕ-R ϕ-sector connected amputated T-matrix elements, with one soft

$\pi(q_\mu = 0)$ attached to an external leg, as shown in Fig. 1. With the separation

$$
\begin{aligned}
&T^{\text{E-AHM}}_{N,M+1}(p_1 \ldots p_N; 0 q_1 \ldots q_M) \\
&\quad = T^{\text{E-AHM;External}}_{N,M+1}(p_1 \ldots p_N; 0 q_1 \ldots q_M) \\
&\qquad + T^{\text{E-AHM;Internal}}_{N,M+1}(p_1 \ldots p_N; 0 q_1 \ldots q_M)
\end{aligned} \tag{B16}
$$

we have the recursive $U(1)_Y$ T-matrix WTI

$$
\begin{aligned}
&\langle H \rangle T^{\text{E-AHM;Internal}}_{N,M+1}(p_1 \ldots p_N; 0 q_1 \ldots q_M) \\
&\quad = \sum_{m=1}^{M} T^{\text{E-AHM}}_{N+1,M-1}(q_m p_1 \ldots p_N; q_1 \ldots \widehat{q_m} \ldots q_M) \\
&\qquad - \sum_{n=1}^{N} T^{\text{E-AHM}}_{N-1,M+1}(p_1 \ldots \widehat{p_n} \ldots p_N; p_n q_1 \ldots q_M).
\end{aligned} \tag{B17}
$$

(15) Recursive $U(1)_Y$ WTI for 1-ϕ-I ϕ-sector connected amputated extended Green's functions $\Gamma^{\text{E-AHM}}_{N,M}$ are a solution to (B17),

$$
\begin{aligned}
&\langle H \rangle \Gamma^{\text{E-AHM}}_{N,M+1}(p_1 \ldots p_N; 0 q_1 \ldots q_M) \\
&\quad = \sum_{m=1}^{M} \Gamma^{\text{E-AHM}}_{N+1,M-1}(q_m p_1 \ldots p_N; q_1 \ldots \widehat{q_m} \ldots q_M) \\
&\qquad - \sum_{n=1}^{N} \Gamma^{\text{E-AHM}}_{N-1,M+1}(p_1 \ldots \widehat{p_n} \ldots p_N; p_n q_1 \ldots q_M).
\end{aligned} \tag{B18}
$$

(16) The LSS theorem (B15) makes tadpoles vanish,

$$
\langle 0 | h(x=0) | 0 \rangle_{\text{connected}} = i[i\Delta_{\text{BEH}}(0)] \Gamma^{\text{E-AHM}}_{1,0}(0;), \tag{B19}
$$

but the $N = 0$, $M = 1$ case of (B18) reads

$$
\Gamma^{\text{E-AHM}}_{1,0}(0;) = \langle H \rangle \Gamma^{\text{E-AHM}}_{0,2}(;00) = 0, \tag{B20}
$$

where we have used (B15), so that tadpoles all vanish automatically, and separate tadpole renormalization is unnecessary. Since we can choose the origin of coordinates anywhere we like

$$
\langle 0 | h(x) | 0 \rangle_{\text{connected}} = 0. \tag{B21}
$$

(17) Renormalized gauge-independent observable $\langle H \rangle$.

$$
\begin{aligned}
\langle 0 | H(x) | 0 \rangle_{\text{connected}} &= \langle 0 | h(x) | 0 \rangle_{\text{connected}} + \langle H \rangle \\
&= \langle H \rangle \\
\partial_\mu \langle H \rangle &= 0.
\end{aligned} \tag{B22}
$$

[1] This paper replaces part of B. W. Lynn, G. D. Starkman, and R. Stora, arXiv:1509.06471 [Phys. Rev. D (to be published)]. The strategy, mathematical methods, exposition, results, and conclusions here are unchanged. To better emphasize $U(1)_Y \otimes$ BRST symmetry, we have supplemented vintage QFT with BRST explanation.
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