Rotating and twisting locally rotationally symmetric spacetimes: A general solution

Sayuri Singh,[1,*] George F. R. Ellis,[2,†] Rituparno Goswami,[1,‡] and Sunil D. Maharaj[1,§]

[1]*Astrophysics and Cosmology Research Unit, School of Mathematics, Statistics and Computer Science, University of KwaZulu–Natal, Private Bag X54001, Durban 4000, South Africa*

[2]*Department of Mathematics and Applied Mathematics and ACGC, University of Cape Town, Cape Town 7701, South Africa*

(Received 21 July 2017; published 27 September 2017)

We derive a general solution for the most general rotating and twisting locally rotationally symmetric spacetimes. This is achieved in three steps. First, we decompose the manifold via a $1 + 1 + 2$ semitetrad formalism that yields a set of geometrical and thermodynamic scalars for the spacetime. We then recast the Einstein field equations in terms of evolution and propagation of these scalars. It is then shown that this class of spacetimes must possess self-similarity and we use this property to solve for these scalars, thus obtaining a general solution. This solution has a number of very interesting cosmological or astrophysical consequences which we discuss in detail.

DOI: 10.1103/PhysRevD.96.064049

I. INTRODUCTION

Locally rotationally symmetric (LRS) spacetimes are those that possess a continuous isotropy group at each point, which generally implies the existence of a multiply transitive isometry group acting on the spacetime manifold [1,2]. It is well known that isotropies around a point in any spacetime with a fluid can be a three-dimensional or one-dimensional subgroup of the full group of isometries; they necessarily leave the normalized four-velocity of the matter flow invariant. The 3D case implies isotropy at every point, yielding the Friedmann-Lemaître-Robertson-Walker (FLRW) models, while the 1D case corresponds to anisotropic and, in general, spatially inhomogeneous models [3–5]. These have a preferred spacelike direction e^a orthogonal to the fluid flow four-vector u^a: all spatial directions orthogonal to e^a and u^a are geometrically identical.

In the case of a perfect fluid, these spacetimes are split into three classes as described in Sec. IV, depending on whether the vorticity component Ω along the direction e^a of the fluid and the two-dimensional twist ξ of the vector field e^a are zero or not (they cannot both be nonzero in this case). However, for an imperfect fluid—for example, if there is an entropy flux—both can be nonzero. In a previous paper [6], we obtained a set of field equations and integrability conditions for the imperfect fluid case. We also proved that the LRS spacetimes with nonzero rotation and spatial twist must be self-similar. In this paper, we extend that work by obtaining a general solution to the field equations for this situation. This is achieved by using the property of the self-similarity. Also, we show that we may specify an equation of state for the isotropic pressure at an initial Cauchy surface for particular applications.

The physical and mathematical importance of the general solution obtained in this paper are as follows:

(1) First of all, this completes the general solution of *all* possible classes of LRS spacetimes that arise from the classification in terms of rotation and spatial twist.

(2) This solution is extremely important in astrophysical scenarios, as most realistic stars have nonzero rotation and nonzero entropic flux in the interior. For example, the radiative heat flux can be prominently seen for neutron stars, as the core temperature rapidly drops from 10^{11}–10^{12} K to 10^6 K within a few years [7–9]. Even for main sequence stars, the radiative heat transfer from core to the convection zone is always present [10]. Hence, any solution to the Einstein field equations that incorporates rotation, spatial twist, and heat flux simultaneously is definitely a better candidate for relativistic description of a rotating stellar interior.

(3) Similarly, this solution can be used to model a galactic dynamics [11] where rotation and entropy flux from the active galactic nucleus plays a very important role.

(4) This solution dramatically changes the usual notion of dynamic black hole formation scenarios by the gravitational collapse of massive stars. As we will see in this paper, the nature of the final spacetime singularity crucially depends on the initial data provided on any initial hypersurface, and the singularity can be timelike, spacelike, or null. In case of nonspacelike singularities, a Cauchy horizon develops which violates the cosmic censorship conjecture.

*sayurisingh22@gmail.com
†george.ellis@uct.ac.za
‡Goswami@ukzn.ac.za
§Maharaj@ukzn.ac.za

(5) For the cosmological scenario, on the scales where rotation, inhomogeneity, and cosmic background radiation play important roles in the cosmological dynamics, this solution is a better approximation than the usual perturbed Friedmann-Lemaître-Robertson-Walker models. Moreover, the scale invariance of this solution creates an interesting mapping between the geometrical scalars of the spacetime and massless scalar fields in a Minkowski background. This mapping may have far-reaching importance in a complete theory of quantum cosmology.

The paper is organized as follows: In Secs. II and III, the semitetrad formalism used is introduced in a general form and the field equations are written in terms of the geometrical scalars. In Sec. IV, a reduced set of field equations is obtained with self-similar variables. In Secs. V and VI, the general solution to the field equations is obtained for the case of LRS fluids with nonzero rotation and spatial twist. In Sec. VII, their properties are discussed for both cosmological and astrophysical scenarios.

II. LRS SPACETIMES IN SEMITETRAD FORMALISM

Because of the symmetries of LRS spacetimes, a $1+1+2$ semitetrad covariant formalism [13–15] (which is a natural extension of local $1+3$ decomposition [12]) is well suited to describing the geometry, as in this formalism the field equations become a set of coupled differential equations in covariantly defined scalar variables. In the $1+3$ decomposition, with respect to a timelike congruence, the spacetime can be locally decomposed into time and space parts. Such a timelike congruence can be defined by the matter flow lines, with the *four-velocity* defined as

$$u^a = \frac{dx^a}{d\tau}, \quad \text{with} \quad u^a u_a = -1, \tag{1}$$

where τ is the proper time along the flow lines. Given the four-velocity u^a, we have unique parallel and orthogonal *projection tensors*,

$$U^a{}_b = -u^a u_b \quad \text{and} \quad h^a{}_b = g^a{}_b + u^a u_b, \tag{2}$$

where $h^a{}_b$ is the projection tensor that projects any 4D vector or tensor onto the local 3-space orthogonal to u^a which has the volume element $\epsilon_{abc} \coloneqq \eta_{abcd} u^d$.

From this, it follows that we have two well-defined directional derivatives. The vector u^a is used to define the *covariant time derivative* along the flow lines (denoted by a dot) for any tensor $S^{a..b}{}_{c..d}$, given by

$$\dot{S}^{a..b}{}_{c..d} = u^e \nabla_e S^{a..b}{}_{c..d}. \tag{3}$$

The tensor h_{ab} is used to define the fully orthogonally *projected covariant derivative D* for any tensor $S^{a..b}{}_{c..d}$:

$$D_e S^{a..b}{}_{c..d} = h^a{}_f h^p{}_c \ldots h^b{}_g h^q{}_d h^r{}_e \nabla_r S^{f..g}{}_{p..q}, \tag{4}$$

with total projection on all free indices. In this way, the covariant derivative of u^a can be decomposed as

$$\nabla_a u_b = -u_a A_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \epsilon_{abc}\omega^c. \tag{5}$$

Here, $A_b = \dot{u}_b$ is the acceleration, $\Theta = D_a u^a$ represents the expansion of u_a, $\sigma_{ab} = (h^c{}_{(a} h^d{}_{b)} - \frac{1}{3} h_{ab} h^{cd}) D_c u_d$ is the shear tensor that denotes the rate of distortion, and ω^c is the vorticity vector denoting the rotation.

The Weyl tensor is split relative to u^a into the *electric* and *magnetic Weyl curvature* parts as

$$E_{ab} = C_{acbd} u^c u^d \quad \text{and} \quad H_{ab} = \frac{1}{2}\epsilon_{ade} C^{de}{}_{bc} u^c. \tag{6}$$

The energy-momentum tensor of matter can be decomposed similarly as

$$T_{ab} = \mu u_a u_b + q_a u_b + q_b u_a + p h_{ab} + \pi_{ab}, \tag{7}$$

where $p = (1/3) h^{ab} T_{ab}$ is the isotropic pressure, $\mu = T_{ab} u^a u^b$ is the energy density, $q_a = q_{\langle a\rangle} = -h^c{}_a T_{cd} u^d$ is the three-vector that defines the heat flux, and $\pi_{ab} = \pi_{\langle ab\rangle}$ is the anisotropic stress.

Now, in the $1+1+2$ decomposition, we choose a spacelike vector field e^a such that

$$u^a e_a = 0 \quad \text{and} \quad e^a e_a = 1. \tag{8}$$

The new projection tensor is given by

$$N_a{}^b \equiv h_a{}^b - e_a e^b = g_a{}^b + u_a u^b - e_a e^b. \tag{9}$$

This spacelike vector now naturally introduces two new derivatives, which, for any tensor $\psi_{a\ldots b}{}^{c\ldots d}$, are

$$\hat{\psi}_{a..b}{}^{c..d} \equiv e^f D_f \psi_{a..b}{}^{c..d}, \tag{10}$$

$$\delta_f \psi_{a..b}{}^{c..d} \equiv N_a{}^p \ldots N_b{}^g N_i{}^c .. N_j{}^d N_f{}^r D_r \psi_{p..g}{}^{i..j}. \tag{11}$$

The derivative (10) along the e^a vector field in the surfaces orthogonal to u^a is called the hat derivative, while the derivative (11) projected onto the sheet is called the δ derivative. This projection is orthogonal to u^a and e^a on every free index.

In the $1+1+2$ splitting, the 4-acceleration, vorticity, and shear split in this way as

$$\dot{u}^a = \mathcal{A} e^a + \mathcal{A}^a, \tag{12}$$

$$\omega^a = \Omega e^a + \Omega^a, \tag{13}$$

$$\sigma_{ab} = \Sigma\left(e_a e_b - \frac{1}{2} N_{ab}\right) + 2\Sigma_{(a} e_{b)} + \Sigma_{ab}. \tag{14}$$

For the electric and magnetic Weyl tensors, we get

$$E_{ab} = \mathcal{E}\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\mathcal{E}_{(a}e_{b)} + \mathcal{E}_{ab}, \quad (15)$$

$$H_{ab} = \mathcal{H}\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\mathcal{H}_{(a}e_{b)} + \mathcal{H}_{ab}. \quad (16)$$

Similarly, the fluid variables q^a and π_{ab} are split as follows:

$$q^a = Qe^a + Q^a, \quad (17)$$

$$\pi_{ab} = \Pi\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\Pi_{(a}e_{b)} + \Pi_{ab}. \quad (18)$$

By decomposing the covariant derivative of e^a in the directions orthogonal to u^a into its irreducible parts, we get

$$D_a e_b = e_a a_b + \frac{1}{2}\phi N_{ab} + \xi\epsilon_{ab} + \zeta_{ab}. \quad (19)$$

Here, $\epsilon_{ab} = \epsilon_{[ab]}$ is the volume element on the sheet, ϕ represents the *spatial expansion* of the sheet, ζ_{ab} is the *spatial shear*, i.e., the distortion of the sheet, a^a is its *spatial acceleration* (i.e., deviation from a geodesic), and ξ is its spatial *vorticity*, i.e., the "twisting" or rotation of the sheet.

III. LRS SPACETIMES AND FIELD EQUATIONS

The basic property of fluid filled LRS spacetimes is that there exists a unique, preferred spatial direction at every point, covariantly defined, which creates a local axis of symmetry. Hence, the $1+1+2$ decomposition described in the previous section is ideally suited for the study of these spacetimes, as we can immediately see that, if we choose the spacelike unit vector e^a along the preferred spatial direction, then by symmetry all of the sheet vectors and tensors vanish identically:

$$\mathcal{A}^a = \Omega^a = \Sigma_a = \mathcal{E}_a = \mathcal{H}_a = Q^a = \Pi^a = a_a = 0, \quad (20)$$

$$\Sigma_{ab} = \mathcal{E}_{ab} = \mathcal{H}_{ab} = \Pi_{ab} = \zeta_{ab} = 0. \quad (21)$$

Thus, the remaining variables are

$$\mathcal{D}_1 \coloneqq \{\mathcal{A}, \Theta, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \mu, p, Q, \Pi, \phi, \xi\} \quad (22)$$

$$= \mathcal{D}_{\text{matter}} + \mathcal{D}_{\text{geometry}}, \quad (23)$$

where

$$\mathcal{D}_{\text{matter}} \coloneqq \{\mu, p, Q, \Pi\} \quad (24)$$

are the matter variables that completely specify the energy-momentum tensor of the matter. On the other hand,

$$\mathcal{D}_{\text{geometry}} \coloneqq \{\mathcal{A}, \Theta, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \phi, \xi\} \quad (25)$$

are the geometrical variables. By decomposing the Ricci identities for u^a and e^a and the doubly contracted Bianchi identities, we then get the following field equations for LRS spacetimes: Evolution:

$$\dot{\phi} = \left(\frac{2}{3}\Theta - \Sigma\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) + 2\xi\Omega + Q, \quad (26)$$

$$\dot{\xi} = \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)\xi + \left(\mathcal{A} - \frac{1}{2}\phi\right)\Omega + \frac{1}{2}\mathcal{H}, \quad (27)$$

$$\dot{\Omega} = \mathcal{A}\xi + \Omega\left(\Sigma - \frac{2}{3}\Theta\right), \quad (28)$$

$$\dot{\mathcal{H}} = -3\xi\mathcal{E} + \left(\frac{3}{2}\Sigma - \Theta\right)\mathcal{H} + \Omega Q + \frac{3}{2}\xi\Pi, \quad (29)$$

Propagation:

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) + 2\xi^2 - \frac{2}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi, \quad (30)$$

$$\hat{\xi} = -\phi\xi + \left(\frac{1}{3}\Theta + \Sigma\right)\Omega, \quad (31)$$

$$\hat{\Sigma} - \frac{2}{3}\hat{\Theta} = -\frac{3}{2}\phi\Sigma - 2\xi\Omega - Q, \quad (32)$$

$$\hat{\Omega} = (\mathcal{A} - \phi)\Omega, \quad (33)$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + 3\Omega\mathcal{H} + \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)Q, \quad (34)$$

$$\hat{\mathcal{H}} = -\left(3\mathcal{E} + \mu + p - \frac{1}{2}\Pi\right)\Omega - \frac{3}{2}\phi\mathcal{H} - Q\xi, \quad (35)$$

Propagation/evolution:

$$\hat{\mathcal{A}} - \dot{\Theta} = -(\mathcal{A} + \phi)\mathcal{A} + \frac{1}{3}\Theta^2 + \frac{3}{2}\Sigma^2 - 2\Omega^2 + \frac{1}{2}(\mu + 3p - 2\Lambda), \quad (36)$$

$$\dot{\mu} + \hat{Q} = -\Theta(\mu + p) - (\phi + 2\mathcal{A})Q - \frac{3}{2}\Sigma\Pi, \quad (37)$$

$$\dot{Q} + \hat{p} + \hat{\Pi} = -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\frac{4}{3}\Theta + \Sigma\right)Q - (\mu + p)\mathcal{A}, \quad (38)$$

$$\dot{\Sigma} - \frac{2}{3}\hat{\mathcal{A}} = \frac{1}{3}(2\mathcal{A} - \phi)\mathcal{A} - \left(\frac{2}{3}\Theta + \frac{1}{2}\Sigma\right)\Sigma - \frac{2}{3}\Omega^2 - \mathcal{E} + \frac{1}{2}\Pi, \quad (39)$$

$$\dot{\mathcal{E}} + \frac{1}{2}\dot{\Pi} + \frac{1}{3}\hat{Q} = +\left(\frac{3}{2}\Sigma - \Theta\right)\mathcal{E} - \frac{1}{2}(\mu + p)\Sigma - \frac{1}{2}\left(\frac{1}{3}\Theta + \frac{1}{2}\Sigma\right)\Pi + 3\xi\mathcal{H} + \frac{1}{3}\left(\frac{1}{2}\phi - 2\mathcal{A}\right)Q, \quad (40)$$

Constraint:

$$\mathcal{H} = 3\xi\Sigma - (2\mathcal{A} - \phi)\Omega. \quad (41)$$

IV. MOST GENERAL CLASS OF LRS SPACETIMES

As described in [3], if we consider a perfect fluid form of matter with $Q = \Pi = 0$, then the propagation equations evolve consistently in time if and only if

$$\Omega\xi = 0. \quad (42)$$

The above relation then naturally divides perfect fluid LRS spacetimes in three distinct subclasses [2,3]:

(1) LRS class I ($\Omega \neq 0, \xi = 0$). These are stationary inhomogeneous rotating solutions.

(2) LRS class II ($\xi = 0 = \Omega$). These are inhomogeneous orthogonal family of solutions that can be either static or dynamic. Spherically symmetric solutions are a subclass of this class.

(3) LRS class III ($\xi \neq 0, \Omega = 0$). These are homogeneous orthogonal models with a spatial twist.

In a recent paper [6], we established the existence of and found the necessary and sufficient conditions for the general class of solutions of locally rotationally symmetric spacetimes that have nonvanishing rotation and spatial twist simultaneously: that is, for this class of spacetimes, we have by definition

$$\Omega\xi \neq 0. \quad (43)$$

By the above, these solutions must be imperfect fluid models. We also provided a brief algorithm indicating how to solve the system of field equations with the given Cauchy data on an initial spacelike Cauchy surface. The important features of this class of spacetimes are as follows:

(1) The necessary condition for a LRS spacetime to have nonzero rotation and spatial twist simultaneously is the presence of nonzero heat flux Q, which is bounded from both sides.

(2) In these spacetimes, *all* scalars Ψ obey the following consistency relation:

$$\forall\Psi, \quad \dot{\Psi}\Omega = \hat{\Psi}\xi. \quad (44)$$

This equation can be easily derived by noting that, for any scalar Ψ in a general LRS spacetime, we have $\nabla_a\Psi = -\dot{\Psi}u_a + \hat{\Psi}e_a$ and $\epsilon^{ab}\nabla_a\nabla_b\Psi = 0$. Also, Eq. (44), which is required by (43), implies self-similarity, for it applies to all scalars, and it is unchanged under the transformation $\tau \to a\tau$, $\rho \to a\rho$, where τ and ρ are the curve parameters of the integral curves of u and e, respectively.

(3) The above symmetries generate further constraints, and hence the total set of constraint equations are now $\mathcal{C} \equiv \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$, where

$$\mathcal{C}_1 \coloneqq \mathcal{H} = 3\xi\Sigma - \left(2\mathcal{A} + \frac{\Omega}{\xi}\left(\Sigma - \frac{2}{3}\Theta\right)\right)\Omega, \quad (45)$$

$$\mathcal{C}_2 \coloneqq \phi = -\frac{\Omega}{\xi}\left(\Sigma - \frac{2}{3}\Theta\right), \quad (46)$$

$$\mathcal{C}_3 \coloneqq Q = -\frac{\frac{\Omega}{\xi}}{1 + (\frac{\Omega}{\xi})^2}(\mu + p + \Pi), \quad (47)$$

$$\mathcal{C}_4 \coloneqq \mathcal{E} = \frac{\Omega}{\xi}\mathcal{A}\left(\Sigma - \frac{2}{3}\Theta\right) - \Sigma^2 + \frac{1}{3}\Theta\Sigma + \frac{2}{9}\Theta^2 + 2(\xi^2 - \Omega^2) + \frac{(\frac{\Omega}{\xi})^2}{1 + (\frac{\Omega}{\xi})^2}(\mu + p + \Pi) - \frac{1}{2}\Pi - \frac{2}{3}\mu. \quad (48)$$

It is important to verify that all of these new constraints evolve consistently in time. This is indeed the case, as these constraints are derived by taking all of the scalars $\Psi \in \mathcal{D}_1$ and using Eq. (44) (which is true for all epochs) together with the field equations. Therefore, the time derivatives of these new constraints will identically vanish using (44) and the field equations as we feed the solutions back to the same system. Therefore, solving for the set of variables

$$\mathcal{D}_2 \coloneqq \{\mathcal{A}, \Theta, \xi, \Sigma, \Omega, \mu\} \quad (49)$$

will automatically specify the rest,

$$\mathcal{D}_3 \coloneqq \{Q, \phi, \mathcal{E}, \mathcal{H}, p\}, \quad (50)$$

where we assume an equation of state for p of the form

$$p = p(\mu, \Pi, Q). \quad (51)$$

We note that the anisotropic pressure Π is not restricted by the constraints: there is no algebraic equation linking it to other thermodynamic variables. Hence, this quantity should be specified at any initial Cauchy surface separately (subject to the energy conditions), and it would then evolve in time via the field equations.

V. THE REDUCED SET OF FIELD EQUATIONS FOR SELF-SIMILAR VARIABLES

We will now use the property of self-similarity for the most general class of LRS spacetimes to further reduce the set of independent field equations. Let us consider the set of variables

$$\mathcal{D}_4 \coloneqq \{\mathcal{A}, \Theta, \xi, \Sigma, \Omega\} \subset \mathcal{D}_2. \tag{52}$$

Then, from the kinematical equations for LRS spacetimes,

$$\nabla_a u_b = -u_a e_b \mathcal{A} + e_a e_b \left(\frac{1}{3}\Theta + \Sigma\right) + \Omega \varepsilon_{ab} + N_{ab}\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right), \tag{53}$$

$$D_a e_b = \frac{1}{2}\phi N_{ab} + \xi \epsilon_{ab}, \tag{54}$$

it is clear that, for any element $f \in \mathcal{D}_4$, we must have

$$f(\tau, \rho) = a f(a\tau, a\rho), \tag{55}$$

as u^a, e^a, N^{ab}, and ϵ^{ab} are dimensionless. Hence, without any loss of generality, all of these quantities can be written as

$$f \equiv \frac{f_0(z)}{\rho}, \tag{56}$$

where

$$z = \frac{\tau}{\rho}, \tag{57}$$

and f_0 is dimensionless. Also, from the Einstein field equations $G_{ab} = T_{ab}$, we can easily see, as before, that all elements $g \in \mathcal{D}_5$, where

$$\mathcal{D}_5 \coloneqq \{\mu, \Pi\} = \mathcal{D}_2 - \mathcal{D}_4, \tag{58}$$

must satisfy

$$g(\tau, \rho) = a^2 g(a\tau, a\rho). \tag{59}$$

Therefore, these quantities can be generally written as

$$g \equiv \frac{g_0(z)}{\rho^2}. \tag{60}$$

Now the *dot* and *hat* derivatives of all these elements can be written in terms of the dimensionless variable z in the following way: for $f \in \mathcal{D}_4$,

$$\dot{f} = \frac{f_{0,z}}{\rho^2}, \tag{61}$$

$$\hat{f} = -\frac{(f_0 + z f_{0,z})}{\rho^2}, \tag{62}$$

and for $g \in \mathcal{D}_5$,

$$\dot{g} = \frac{g_{0,z}}{\rho^3}, \tag{63}$$

$$\hat{g} = -\frac{(2g_0 + z g_{0,z})}{\rho^3}. \tag{64}$$

Using the above results, the nontrivial field equations become the following ordinary differential equations:

$$\phi_{0,z} = \left[\frac{2}{3}\Theta_0 - \Sigma_0\right]\left[\mathcal{A}_0 - \frac{1}{2}\phi_0\right] + 2\xi_0\Omega_0 + Q_0, \tag{65}$$

$$\xi_{0,z} = \left[\frac{1}{2}\Sigma_0 - \frac{1}{3}\Theta_0\right]\xi_0 + \left[\mathcal{A}_0 - \frac{1}{2}\phi_0\right]\Omega_0 + \frac{1}{2}\mathcal{H}_0, \tag{66}$$

$$\Omega_{0,z} = \mathcal{A}_0\xi_0 + \Omega_0\left[\Sigma_0 - \frac{2}{3}\Theta_0\right], \tag{67}$$

$$\mathcal{H}_{0,z} = -3\xi_0\mathcal{E}_0 + \left[\frac{3}{2}\Sigma_0 - \Theta_0\right]\mathcal{H}_0 + \Omega_0 Q_0 + \frac{3}{2}\xi_0\Pi_0, \tag{68}$$

$$\Sigma_{0,z} - \frac{2}{3}\Theta_{0,z} = -\phi_0\mathcal{A}_0 + \frac{2}{9}{\Theta_0}^2 + \frac{1}{2}{\Sigma_0}^2 - 2{\Omega_0}^2 + \frac{1}{3}\mu_0 + p_0 - \frac{2}{3}\Theta_0\Sigma_0 - \mathcal{E}_0 + \frac{1}{2}\Pi_0, \tag{69}$$

$$\mathcal{E}_{0,z} + -\frac{1}{3}\mu_{0,z} + \frac{1}{2}\Pi_{0,z} = +\left[\frac{3}{2}\Sigma_0 - \Theta_0\right]\mathcal{E}_0 + 3\xi_0\mathcal{H}_0 - \frac{1}{3}(\mu_0 + p_0) + \frac{1}{2}Q_0\phi_0 - \left(\frac{1}{6}\Theta_0 - \frac{1}{4}\Sigma_0\right)\Pi_0 - \frac{1}{2}[\mu_0 + p_0]\Sigma_0. \tag{70}$$

It can be shown that the rest of the field equations become redundant when the following set of dimensionless constraints $\tilde{\mathcal{C}} \equiv \{\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2, \tilde{\mathcal{C}}_3, \tilde{\mathcal{C}}_4\}$ hold, which are easily derived by using Eqs. (56) and (60) on the set of original constraints $\mathcal{C}$:

$$\tilde{\mathcal{C}}_1\colon\ \mathcal{H}_0 = 3\xi_0\Sigma_0 - \left[2\mathcal{A}_0 + \frac{\Omega_0}{\xi_0}\left(\Sigma_0 - \frac{2}{3}\Theta_0\right)\right]\Omega_0, \tag{71}$$

$$\tilde{\mathcal{C}}_2\colon\ \phi_0 = -\frac{\Omega_0}{\xi_0}\left(\Sigma_0 - \frac{2}{3}\Theta_0\right), \tag{72}$$

$$\tilde{\mathcal{C}}_3\colon\ Q_0 = -\frac{\frac{\Omega_0}{\xi_0}}{1 + (\frac{\Omega_0}{\xi_0})^2}(\mu_0 + p_0 + \Pi_0), \tag{73}$$

$$\tilde{\mathcal{C}}_4\colon \mathcal{E}_0 = \frac{\Omega_0}{\xi_0}\mathcal{A}_0\left(\Sigma_0 - \frac{2}{3}\Theta_0\right) - {\Sigma_0}^2 + \frac{1}{3}\Theta_0\Sigma_0 + \frac{2}{9}{\Theta_0}^2 + 2({\Sigma_0}^2 - {\Omega_0}^2) + \frac{(\frac{\Omega_0}{\xi_0})^2}{1+(\frac{\Omega_0}{\xi_0})^2}(\mu_0 + p_0 + \Pi_0) - \frac{1}{2}\Pi_0 - \frac{2}{3}\mu_0. \quad (74)$$

VI. GENERAL SOLUTION TO THE FIELD EQUATIONS

To find the general solution of the reduced set of the field equations, we note that these equations, along with (44), generate the constraint set $\tilde{\mathcal{C}} \equiv \{\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2, \tilde{\mathcal{C}}_3, \tilde{\mathcal{C}}_4\}$. Hence, the field equations are encoded in (44) and the set of constraints, and it suffices to solve (44) along with the constraint to obtain a complete solution to the spacetime. Hence, we use Eqs. (61), (62), (63), and (64) in (44) and obtain

$$\frac{f_{0,z}}{f_0} = \frac{-\xi_0}{\Omega_0 + z\xi_0}, \quad (75)$$

$$\frac{g_{0,z}}{g_0} = \frac{-2\xi_0}{\Omega_0 + z\xi_0}. \quad (76)$$

Now, letting $f_0 = \Omega_0$, we get

$$\frac{\Omega_{0,z}}{\Omega_0} = \frac{-\xi_0}{\Omega_0 + z\xi_0}, \quad (77)$$

and letting $f_0 = \xi_0$, we get

$$\frac{\xi_{0,z}}{\xi_0} = \frac{-\xi_0}{\Omega_0 + z\xi_0}. \quad (78)$$

The above two equations are coupled first order ordinary differential equations for Ω_0 and ξ_0, and the general solution is given by

$$\xi_0(z) = -\frac{1}{Az + B}, \quad (79)$$

$$\Omega_0(z) = -\frac{B}{A(Az + B)}, \quad (80)$$

where A and B are constants of integration. Now, using these solutions in (75) and (76), we get the following decoupled equations:

$$\frac{f_{0,z}}{f_0} = -\frac{A}{Az + B}, \quad (81)$$

$$\frac{g_{0,z}}{g_0} = -\frac{2A}{Az + B}. \quad (82)$$

The general solutions for the Eqs. (81) and (82) are given by

$$f_0 = \frac{C_f}{Az + B}, \quad (83)$$

$$g_0 = \frac{C_g}{(Az + B)^2}. \quad (84)$$

Here, C_f and C_g are integration constants related to each of the kinematic and dynamic variables, f_0 and g_0. Thus, the set C of arbitrary integration constants that we must specify to obtain the general solution for the most general LRS spacetime is given by

$$C \equiv (A, B, C_{\mathcal{A}}, C_\Theta, C_\Sigma, C_\mu, C_\Pi), \quad (85)$$

where we must have $A \neq 0$ and $B \neq 0$ for Eq. (43) to be true. The rest of the variables can then be easily obtained by using the constraint equations.

For example, using the constraint $\tilde{\mathcal{C}}_1$ [Eq. (71)], we get the magnetic part of the Weyl scalar as follows:

$$\mathcal{H} = \frac{C_{\mathcal{H}}}{(Az + B)^2}, \quad (86)$$

where we have

$$C_{\mathcal{H}} = -3C_\Sigma + \left(2C_{\mathcal{A}} + \frac{B}{A}\left(C_\Sigma - \frac{2}{3}C_\Theta\right)\right)\frac{B}{A}. \quad (87)$$

Again, using the constraint $\tilde{\mathcal{C}}_2$ [Eq. (72)], we get

$$\phi_0 = \frac{C_\phi}{Az + B}; \qquad C_\phi = -\frac{B}{A}\left(C_\Sigma - \frac{2}{3}C_\Theta\right). \quad (88)$$

The variables Q_0 and $\mathcal{E}_0$ can similarly be obtained using Eqs. (73) and (74) subject to the dimensionless algebraic equation of state $p_0 = p_0(\mu_0, Q_0, \Pi_0)$, which must be provided separately along with the field equations. Once an equation of state in the form of (51) is given, it is, in principle, possible to obtain such a dimensionless equation of state, as all of the elements of $\mathcal{D}_{\mathrm{matter}}$ have the same symmetries as (59), and hence the dimensionless part can be extracted from all of them.

Thus, we obtain the solution for all of the scalar variables of the set $\mathcal{D}_1$, which completes the general solution. One can, in principle, obtain the metric elements from the definition of these covariant scalars. However, it is important to note that all physical properties of the LRS spacetime can be obtained directly from these covariant scalars, as all of them have well-defined geometrical and physical meaning. In the next section, we will discuss some of the physical properties of these solutions in both astrophysical and cosmological scenarios.

VII. COSMOLOGICAL AND ASTROPHYSICAL PROPERTIES OF THIS GENERAL SOLUTION

This class of solutions has some very interesting properties for both cosmological and stellar collapse scenarios, which we list below. We can immediately see that there is a spacetime singularity along the curve $B\rho + A\tau = 0$ which is similar to the cosmological singularity of the FLRW or Lemaître-Tolman-Bondi universes (or the corresponding black hole singularities if we take the collapsing branch of the solutions). Apart from this, there are no other singular points on the manifold.

(1) The most interesting feature of the singularity in this class of spacetime is it can be made timelike, spacelike, or null by choice of the ratio of the constants A and B. In other words, the ratio of rotation (Ω) and spatial twist (ξ) at any initial Cauchy surface completely determines the nature of the initial (or final) singularity, and this gives a range of different possibilities.

(ii) For the cosmological scenario, let us consider both A and B to be greater than zero, In that case, the initial singularity is along the line $B\rho + A\tau = 0$. This "big bang" is no longer instantaneous, and it can be spacelike, timelike, or null. Thus, the section of the manifold that depicts the universe is given by

$$\rho > 0, \qquad \tau > -(B/A)\rho. \tag{89}$$

For an expanding universe with positive energy density, we must have $\Theta > 0$ and $\mu > 0$, and hence we must choose the constants:

$$C_\Theta > 0; \qquad C_\mu > 0. \tag{90}$$

For the cosmological case, we can choose *dustlike* matter with

$$p_0 = 0, \tag{91}$$

$$C_\Pi = 0 \Rightarrow \Pi_0 = 0. \tag{92}$$

Now we can immediately see that, in this case, $\dot{\Theta} < 0, \dot{\mu} < 0$. There is no bounce in this cosmology, as the expansion goes to zero asymptotically. Furthermore, it is interesting to note that, at spacelike infinity, i_0 (where $\rho \to \infty$), timelike infinity $i+$ (where $\tau \to \infty$), and future null infinity $\mathcal{I}+$, all of the kinematical and dynamical quantities vanish, making the spacetime asymptotically Minkowski. Hence, we get a cosmology that is *future asymptotically simple*. Fig. 1.

(3) Another interesting case happens when the curves $B\rho + A\tau =$ const. are null. In this case, the initial singularity is *incoming null*. Then, for any observer on the worldline $\rho = 0$ ($\tau > 0$), observation along the past null cone will depict a universe with homogeneous density, which contrasts with the fact that, on a given time slice, the density is inhomogeneous.

(4) A similar picture can be obtained for collapsing stellar configurations with $A < 0$ and $B > 0$. In that case, the section of the manifold $\rho > 0$ and $\tau < (B/|A|)\rho$ depicts a regular collapsing region which is *past asymptotically simple*. To get a collapsing branch of the solution with positive matter density, we must have $\Theta < 0$ and $\mu > 0$. Hence, we choose

$$C_\Theta < 0; \qquad C_\mu > 0. \tag{93}$$

Also, here we should specify the equation of state linking the isotropic pressure to other thermodynamic variables and, separately, specify the constant

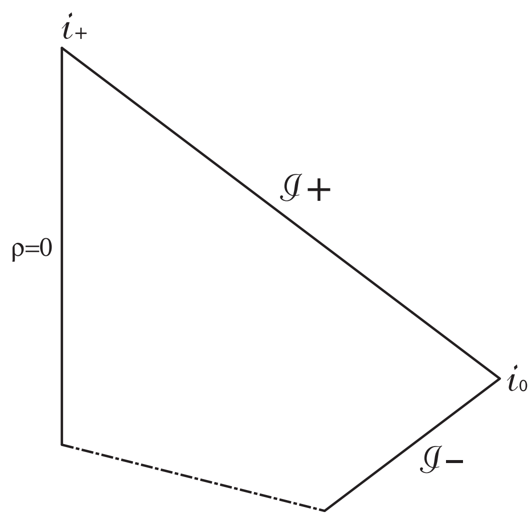

FIG. 1. Penrose diagram of expanding cosmologies which are future asymptotically simple.

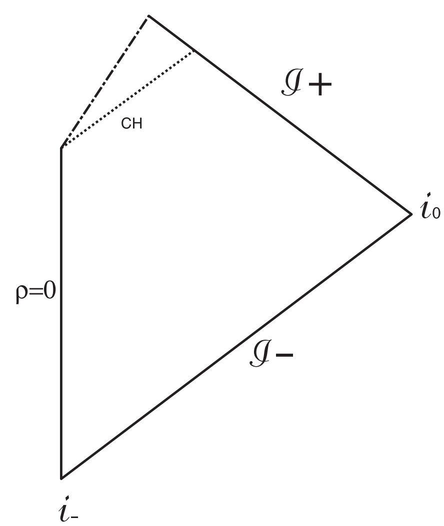

FIG. 2. Penrose diagram of a collapsing stellar structure where the future singularity is nonspacelike. This leads to the development of a Cauchy horizon (CH) in the spacetime.

C_Π at the initial Cauchy surface subject to the energy conditions. We can easily check to see that, in this case, $\dot{\Theta} < 0, \dot{\mu} > 0$. Hence, the collapse continues till $\Theta \to -\infty$ and $\mu \to \infty$. This is a final singularity at $\tau = (B/|A|)\rho$, and we can easily see that this singularity can be timelike, spacelike, or null, which will have important consequences in terms of the cosmic censorship conjecture. Fig. 2.

ACKNOWLEDGMENTS

S. S., G. F. R. E., and R. G. are supported by the National Research Foundation (NRF), South Africa. S. D. M. acknowledges that this work is based on research supported by the South African Research Chair Initiative of the Department of Science and Technology and the National Research Foundation.

[1] G. F. R. Ellis, J. Math. Phys. (N.Y.) **8**, 1171 (1967).
[2] J. M. Stewart and G. F. R. Ellis, J. Math. Phys. (N.Y.) **9**, 1072 (1968).
[3] H. V. Elst and G. F. R. Ellis, Classical Quantum Gravity **13**, 1099 (1996).
[4] G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. **12**, 108 (1969).
[5] A. R. King and G. F. R. Ellis, Commun. Math. Phys. **31**, 209 (1973).
[6] S. Singh, G. F. R. Ellis, R. Goswami, and S. D. Maharaj, Phys. Rev. D **94**, 104040 (2016).
[7] S. Shapiro and S. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars* (John Wiley & Sons, New York, 1983).
[8] See M. C. Miller's, home page, http://www.astro.umd.edu/~miller/nstar.html.
[9] A. A. Svidzinsky, Astrophys. J. **590**, 386 (2003).
[10] R. Mitalas and K. R. Sills, Astrophys. J. **401**, 759 (1992).
[11] M. Noguchi, Astrophys. J. **514**, 77 (1999).
[12] G. F. R. Ellis and H. van Elst, in *Proceedings of the NATO Advanced Study Institute on Theoretical and Observational Cosmology, Cargèse, France, 1998*, edited by M. Lachièze-Rey (Kluwer Academic, Boston, 1999), p. 1.
[13] C. A. Clarkson and R. K. Barrett, Classical Quantum Gravity **20**, 3855 (2003).
[14] G. Betschart and C. A. Clarkson, Classical Quantum Gravity **21**, 5587 (2004).
[15] C. Clarkson, Phys. Rev. D **76**, 104034 (2007).