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Cosmological perturbation theory in generalized Einstein-Aether models

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We investigate the evolution of cosmological perturbations in models of dark energy described by a timelike unit normalized vector field specified by a general function $\mathcal{F}(\mathcal{K})$, so-called generalized Einstein-Aether models. First we study the background dynamics of such models via a designer approach in an attempt to model this theory as dark energy. We find that only one specific form of this designer approach matches ΛCDM at background order, and we also obtain a differential equation which $\mathcal{F}(\mathcal{K})$ must satisfy for general wCDM cosmologies, where CDM refers to cold dark matter. We also present the equations of state for perturbations in generalized Einstein-Aether models, which completely parametrize these models at the level of linear perturbations. A generic feature of modified gravity models is that they introduce new degrees of freedom. By fully eliminating these we are able to express the gauge invariant entropy perturbation and the scalar, vector, and tensor anisotropic stresses in terms of the perturbed fluid variables and metric perturbations only. These can then be used to study the evolution of perturbations in the scalar, vector, and tensor sectors, and we use these to evolve the Newtonian gravitational potentials.

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I. INTRODUCTION

The nature of dark energy remains one of the biggest unsolved problems in cosmology. Numerous models of dark energy and modified gravity theories have been constructed [1] in an attempt to describe cosmological observations [2–4], with varying degrees of success. Perhaps the simplest and most successful of these is the cosmological constant which is remarkably consistent with recent observations [5,6]. However, other models must be studied in case they provide a more suitable description or otherwise to rule them out all together, both theoretically and observationally. With the advent of surveys such as DES[1] [7], Euclid[2] [8–10], LSST[3] [11,12], and SKA[4] [13–16], observational constraints on these models will undoubtedly become tighter.

An obvious way to modify gravity is to introduce a new field other than the metric and make dark energy a dynamical component. These models typically introduce scalar fields and many of these are encompassed by Horndeski [17,18], the most general scalar-tensor theory that gives rise to second-order equations of motion. This class of models include Quintessence [19–21], k-essence [22,23], Kinetic Gravity Braiding (KGB) [24], $f(R)$ gravity [25–27], and many more. Indeed, it has already been shown that it is possible to achieve a dark energy fluid with $w = -1$ exactly in, for example, Quintessence and k-essence [28], and for so-called "designer $f(R)$" [29]. However, there is no reason not to consider the new field to be a vector, and indeed such vector-tensor theories have been shown to be able to give rise to a period of accelerated expansion even without potential terms [30–38], and therefore provide an interesting avenue of research. In this paper we study so-called Einstein-Aether theories at background and perturbative order, where the vector field is constrained to be of timelike unit norm. First studied in [34], it was shown that the model would in fact slow the expansion of the Universe [39]. However, more recently, modifications to this theory have been shown to allow it to describe dark energy and still be compatible with observations [36–38]. This is done by introducing noncanonical kinetic terms parametrized by a free function $\mathcal{F}(\mathcal{K})$, where $\mathcal{K}$ determines the kinetic terms for the vector field. In principle this could take on any functional form, and in previous work in this area specific forms were chosen to work with. However, as with designer $f(R)$, we will choose a background evolution of the Universe and allow that to dictate the form of $\mathcal{F}(\mathcal{K})$ in a "designer $\mathcal{F}(\mathcal{K})$" model.

At background order, despite the many complex models of dark energy all of these can be parametrized by specifying a single function of time, the equation of state parameter, $w_{\rm de} = P_{\rm de}/\rho_{\rm de}$. Exactly how $w_{\rm de}$ behaves as a function of time will of course depend on the theory, but at this order there is nothing else to measure which will tell us about the nature of dark energy, provided FRW spacetime symmetries are respected. At the level of linear perturbations various approaches have been developed in order to try to parametrize different theories [40–50]. In this paper, we work with the equation of state for perturbations (EoS) approach [47–49]. A generic feature of modified gravity

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[3]https://www.lsst.org/
[4]https://www.skatelescope.org/

TABLE I. Expressions for Γ and Π^S in terms of the dark energy perturbed fluid variables and metric perturbations for some dark energy models and modified gravity theories, in the synchronous gauge.

Theory	Scalar anisotropic stress, $w\Pi^S$	Entropy perturbation, $w\Gamma$
Minimally coupled scalar fields	0	$(c_s^2 - \frac{dP}{d\rho})[\delta + 3H(1+w)\theta^S]$
KGB	0	$\mathcal{A}_1\delta + \mathcal{A}_2\theta^S + \mathcal{A}_3 h' + \mathcal{A}_4 h''$
EDE	$\frac{3}{2}(w - c_s^2)[\delta - 3(1+w)\eta]$	0

models is that new degrees of freedom arise at the level of perturbations. The EoS approach packages the parametrization into the gauge invariant entropy perturbation, Γ, and anisotropic stress, Π^S, by eliminating these degrees of freedom in favor of the perturbed fluid variables and metric perturbations. The perturbed conservation equation, $\delta(\nabla_\mu T^\mu{}_\nu) = 0$, gives two evolution equations for the density perturbation, $\delta\rho$, and divergence of the velocity field, θ^S. For example, in the synchronous gauge they are given by

$$\left(\frac{\delta}{1+w}\right)' = -k^2\theta^S - \frac{1}{2}h' - \frac{3\mathcal{H}}{1+w}w\Gamma, \tag{1}$$

$$(1+w)\theta^{S\prime} = \mathcal{H}(1+w)\left(3\frac{dP}{d\rho} - 1\right)\theta^S + \frac{dP}{d\rho}\delta + w\Gamma + \frac{2}{3}w\Pi^S, \tag{2}$$

where primes denote conformal time differentiation and $\mathcal{H}$ is the conformal Hubble parameter. The metric perturbations, h and η, are evolved via Einstein's equation. However, the forms of Π^S and Γ are not known, and hence (1) and (2) are not closed. If we can somehow specify Γ and Π^S as linear functions of the perturbed fluid variables, metric perturbations, and their derivatives only, these equations close, i.e. we wish to write $\Gamma = \Gamma(\delta, \theta^S, h', \eta, \ldots)$ and $\Pi^S = \Pi^S(\delta, \theta^S, h', \eta, \ldots)$, or equivalently in terms of the dark energy (de) and matter (m) fluid variables, $\Gamma = \Gamma(\delta_{\rm de}, \theta^S_{\rm de}, \delta_{\rm m}, \theta^S_{\rm m})$ and $\Pi^S = \Pi^S(\delta_{\rm de}, \theta^S_{\rm de}, \delta_{\rm m}, \theta^S_{\rm m})$. Our approach is to eliminate the internal degrees of freedom describing the dynamics of the modified gravity theory, via expressions for δ and θ^S, supplemented by the equation of motion for the vector field. In principle, the equations of motion and hence the perturbed fluid variables have already been derived in [36,37], for example, although the equations of state have not been computed. However, in most of the previous work the so-called "acceleration" term has not been included, corresponding to the c_4 term in [35]. This term is often either completely ignored or argued that a transformation of the coefficients can remove it. However, we discuss later why this is not true in general and so keep the c_4 term in our subsequent analysis. In particular, we extend on previous work done by including the c_4 term for $\mathcal{F}(\mathcal{K})$ theories in so-called generalized Einstein-Aether, as well as using the EoS formalism.

Although in this paper we use a specific Lagrangian to work with, one of the advantages of the EoS approach is that it allows the computation of cosmological perturbations in a model independent way. In [49] this approach was applied to generic scalar-tensor theories by specifying only the field content of the Lagrangian and nothing specific about its functional form. This approach also provides a set of modifications that are, in principle, easy to insert into numerical codes. Equations of state have already calculated for various different classes of theories, for example, the elastic dark energy (EDE) [51], which was shown to be equivalent to Lorentz violating, massive gravity theories [52]. They have also been calculated for general scalar-tensor theories [49] and in particular Quintessence, k-essence, KGB, and Horndeski theories [18]. In these cases, the degree of freedom to be eliminated is related to the perturbed scalar field, $\delta\varphi$, and its derivatives. This was also shown to be the case for $f(R)$ gravity and was studied in [53]. In this paper we apply the EoS approach to generalized Einstein-Aether theories. The expressions for Γ and Π^S are shown in Table I for some of these theories, in the synchronous gauge, where $\{\mathcal{A}_i\}$ are functions of background quantities and $c_s^2 = \delta P/\delta\rho$ is the squared sound speed of scalar perturbations. We do not provide the expressions in $f(R)$ gravity here as they are quite complicated; however they are presented in [53].

This paper is organized as follows. In Sec. II we present the model for generalized Einstein-Aether and derive the equations of motion. We also briefly mention subcases to this model that have been studied previously. We then study the theory at linear perturbative order (Sec. III) in the scalar, vector, and tensor sector and present expressions for the perturbed fluid variables in both the conformal Newtonian and synchronous gauges. We then proceed to derive the gauge invariant equations of state for perturbations (Sec. IV) by eliminating all the internal degrees of freedom that arise from introducing the vector field. From these we also study the evolution of the Newtonian gravitational potentials. We then conclude in Sec. V and discuss future steps.

Natural units are used throughout with $c = \hbar = 1$ and the metric signature is $(-,+,+,+)$.

II. GENERALIZED EINSTEIN-AETHER FIELD EQUATIONS

A. Field equations

The Lagrangian for generalized Einstein-Aether is [36]

$$16\pi G\mathcal{L}_A = M^2\mathcal{F}(\mathcal{K}) + \lambda(g_{\mu\nu}A^\mu A^\nu + 1), \tag{3}$$

where we introduce the vector field A^μ, which is known as the Aether field. The scalar $\mathcal{K}$ is defined by

$$\mathcal{K} = \frac{1}{M^2}K^{\alpha\beta}{}_{\mu\nu}\nabla_\alpha A^\mu \nabla_\beta A^\nu \tag{4}$$

and the rank-4 tensor is defined by

$$K^{\alpha\beta}{}_{\mu\nu} = c_1 g^{\alpha\beta}g_{\mu\nu} + c_2\delta^\alpha_\mu\delta^\beta_\nu + c_3\delta^\alpha_\nu\delta^\beta_\mu + c_4 A^\alpha A^\beta g_{\mu\nu}. \tag{5}$$

Here, $\{c_i\}$ are dimensionless constants, and M has dimensions of mass. The "kinetic tensor", $K^{\alpha\beta}{}_{\mu\nu}$, determines the derivative squared terms of the Aether field. Similar to generalization of Quintessence to k-essence, the kinetic terms have been modified to an arbitrary, dimensionless function $\mathcal{F}(\mathcal{K})$. An important feature of Einstein-Aether models is the presence of the Lagrange multiplier λ. This will constrain the Aether field to have a timelike unit norm. As we will see, this will also have an effect on the propagating degrees of freedom at the perturbative level.

The full action that we will study is then

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}R + \mathcal{L}_A\right) + S_{\rm m}, \tag{6}$$

where the action for the matter fields, $S_{\rm m}$, does not couple directly to the Aether field. The equations of motion can now be obtained by varying (6) with respect to each degree of freedom i.e. λ, A^μ, and $g^{\mu\nu}$. Variation with respect to λ yields the constraint $g_{\mu\nu}A^\mu A^\nu = -1$. The equation of motion for the Aether field, A^μ, is

$$\nabla_\alpha(\mathcal{F}_\mathcal{K}J^\alpha{}_\mu) - c_4\mathcal{F}_\mathcal{K}A^\alpha\nabla_\alpha A^\nu\nabla_\mu A_\nu = \lambda A_\mu, \tag{7}$$

where we define $J^\alpha{}_\mu = K^{\alpha\beta}{}_{\mu\nu}\nabla_\beta A^\nu$ and $\mathcal{F}_\mathcal{K} = \frac{d\mathcal{F}}{d\mathcal{K}}$, and variation with respect to the metric gives Einstein's equation in the form

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + U_{\mu\nu}, \tag{8}$$

where $T_{\mu\nu}$ is the energy-momentum tensor for the matter fields only. All contributions from the Aether field are included in $U_{\mu\nu}$ which takes the form

$$\begin{aligned} U_{\alpha\beta} = {} & \nabla_\mu[\mathcal{F}_\mathcal{K}(J_{(\alpha}{}^\mu A_{\beta)} - J^\mu{}_{(\alpha}A_{\beta)} - J_{(\alpha\beta)}A^\mu)] + \lambda A_\alpha A_\beta \\ & + \frac{1}{2}M^2\mathcal{F}g_{\alpha\beta} + c_1\mathcal{F}_\mathcal{K}(\nabla_\mu A_\alpha\nabla^\mu A_\beta - \nabla_\alpha A_\mu\nabla_\beta A^\mu) \\ & + c_4\mathcal{F}_\mathcal{K}A^\mu A^\nu\nabla_\mu A_\alpha\nabla_\nu A_\beta, \end{aligned} \tag{9}$$

where brackets around indices denote symmetrization, i.e. $J_{(\alpha\beta)} = \frac{1}{2}(J_{\alpha\beta} + J_{\beta\alpha})$.

Using (7) to eliminate λ, we find that

$$\begin{aligned} U_{\alpha\beta} = {} & \nabla_\mu(\mathcal{F}_\mathcal{K}[J_{(\alpha}{}^\mu A_{\beta)} - J^\mu{}_{(\alpha}A_{\beta)} - J_{(\alpha\beta)}A^\mu]) \\ & + c_1\mathcal{F}_\mathcal{K}(\nabla_\mu A_\alpha\nabla^\mu A_\beta - \nabla_\alpha A_\mu\nabla_\beta A^\mu) \\ & + c_4\mathcal{F}_\mathcal{K}A^\mu A^\nu\nabla_\mu A_\alpha\nabla_\nu A_\beta + [c_4\mathcal{F}_\mathcal{K}A^\mu A^\nu\nabla_\mu A^\tau\nabla_\nu A_\tau \\ & - A^\nu\nabla_\mu(\mathcal{F}_\mathcal{K}J^\mu{}_\nu)]A_\alpha A_\beta + \frac{1}{2}M^2\mathcal{F}g_{\alpha\beta}. \end{aligned} \tag{10}$$

The first line arises due to the metric variation in the Christoffel symbols [39,54], the second line comes from the variation in the c_1 and c_4 terms of (5), and the third line is due to the variation of the Lagrange multiplier and $\sqrt{-g}$ terms.

B. Background dynamics

We will assume a background cosmology described by the FRW metric,

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \tag{11}$$

and that $A^\mu = (1, 0, 0, 0)$. The reason for this choice of A^μ is to satisfy the unit norm constraint and to be compatible with the symmetries of FRW. Taking $U_{\mu\nu}$ to be the energy momentum tensor of a perfect fluid, then from U_{00} and U_{ij} we find that the background energy density and pressure are

$$\rho_A = 3\alpha H^2\left(\mathcal{F}_\mathcal{K} - \frac{\mathcal{F}}{2\mathcal{K}}\right), \tag{12}$$

$$P_A = \alpha\left[3H^2\left(\frac{\mathcal{F}}{2\mathcal{K}} - \mathcal{F}_\mathcal{K}\right) - \dot{\mathcal{F}}_\mathcal{K}H - \mathcal{F}_\mathcal{K}\dot{H}\right], \tag{13}$$

where $\alpha = c_1 + 3c_2 + c_3$, overdots denote differentiation with respect to cosmic time, t, and

$$\mathcal{K} = \frac{3\alpha H^2}{M^2}. \tag{14}$$

Note that we have absorbed a factor of $8\pi G$ into $U_{\mu\nu}$. We can also check that P_A and ρ_A satisfy the energy conservation equation

$$\dot{\rho}_A = -3H(\rho_A + P_A), \tag{15}$$

as they should by construction of (8). Note that the c_4 term plays no role in the background dynamics.

The time-time component of Einstein's equation gives the modified Friedmann equation as

$$(1-\alpha\mathcal{F}_{\mathcal{K}})H^2+\frac{1}{6}\mathcal{F}M^2=\frac{8\pi G}{3}\rho_m. \tag{16}$$

If we were to demand that the theory is indistinguishable from a cosmological constant at background order, then from (16) we obtain the differential equation

$$\mathcal{K}\frac{d\mathcal{F}}{d\mathcal{K}}-\frac{1}{2}\mathcal{F}=\frac{\Lambda}{M^2}, \tag{17}$$

where we have substituted H^2 for $\mathcal{K}$ via (14). The solution to this equation is

$$\mathcal{F}=B(\pm\mathcal{K})^{1/2}-\frac{2\Lambda}{M^2}, \tag{18}$$

depending on the sign of $\mathcal{K}$ and where B is an arbitrary integration constant. The case of a general power law has been studied in [36–38] as well as more exotic forms, for example see [54,55]. Indeed, the functional form of $\mathcal{F}(\mathcal{K})$ must be specified at some point to make observational predictions. However, since $\mathcal{F}(\mathcal{K})$ could in principle be anything, it would be ideal if the form of $\mathcal{F}(\mathcal{K})$ could be found by specifying more standard parameters describing the background dynamics e.g. $w_{\rm de}$, $\Omega_{\rm de,0}$, etc. Since any new dark energy model will at least have to be compatible with ΛCDM "globally", it makes sense to demand that generalized Einstein-Aether must yield a ΛCDM cosmology and in turn, this will restrict the form of $\mathcal{F}(\mathcal{K})$. Since the background evolution of this model will be identical to ΛCDM, the effects of perturbations will become very important as it is only the dynamics at the perturbative level which will be able to distinguish this model from ΛCDM, where CDM refers to cold dark matter.

Let us now demand that the Aether field energy density and pressure obey a more general equation of state i.e. $P_A=w_{\rm de}\rho_A$, where $w_{\rm de}$ is constant. Since current observations do not yet sufficiently constrain anything other than constant $w_{\rm de}$ this is a reasonable assumption to make; however this may change in the near future. We can rewrite (13) as

$$P_A=-\rho_A-\alpha(2\mathcal{K}\mathcal{F}_{\mathcal{K}\mathcal{K}}+\mathcal{F}_{\mathcal{K}})\dot{H} \tag{19}$$

and so,

$$(1+w_{\rm de})M^2\left(\mathcal{K}\mathcal{F}_{\mathcal{K}}-\frac{1}{2}\mathcal{F}\right)=-\alpha(2\mathcal{K}\mathcal{F}_{\mathcal{K}\mathcal{K}}+\mathcal{F}_{\mathcal{K}})\dot{H}, \tag{20}$$

where we have written H^2 in terms of $\mathcal{K}$. If we can write $\dot{H}=\dot{H}(\mathcal{K})$, then (20) will give us a differential equation to solve for $\mathcal{F}(\mathcal{K})$ satisfying a certain value of $w_{\rm de}$.

We write the Friedmann equation as

$$\left(\frac{H}{H_0}\right)^2=\frac{\Omega_{\rm m,0}}{a^3}+\frac{\Omega_{\rm de,0}}{a^{3(1+w_{\rm de})}}, \tag{21}$$

where we have defined $8\pi G\rho_{\rm de}=\rho_A$, $\Omega_i=\frac{8\pi G}{3H^2}\rho_i$, and for this section only the subscript "m" refers to matter with $P_{\rm m}=0$. Differentiating this and combining with (21) to eliminate $\Omega_{\rm de,0}$ gives

$$\frac{1}{a^3}=\frac{1}{w_{\rm de}\Omega_{\rm m,0}}\left[(1+w_{\rm de})\left(\frac{H}{H_0}\right)^2+\frac{2\dot{H}}{3{H_0}^2}\right]. \tag{22}$$

We can also use the Raychaudhuri equation, given by

$$\dot{H}+H^2=-\frac{4\pi G}{3}[\rho_{\rm m}+(1+3w_{\rm de})\rho_{\rm de}]. \tag{23}$$

Inserting (12) we have that

$$\frac{\dot{H}}{{H_0}^2}+\left(\frac{H}{H_0}\right)^2=-\frac{\Omega_{\rm m,0}}{2a^3}-\frac{M^2}{6{H_0}^2}(1+3w_{\rm de})\left(\mathcal{K}\mathcal{F}_{\mathcal{K}}-\frac{1}{2}\mathcal{F}\right), \tag{24}$$

and so using (22) we find that

$$\dot{H}(\mathcal{K})=-\frac{M^2}{2}\left[\frac{\mathcal{K}}{\alpha}+w_{\rm de}\left(\mathcal{K}\mathcal{F}_{\mathcal{K}}-\frac{1}{2}\mathcal{F}\right)\right]. \tag{25}$$

Therefore, the differential equation we must solve is then

$$(1+w_{\rm de})(2\mathcal{K}\mathcal{F}_{\mathcal{K}}-\mathcal{F})=(2\mathcal{K}\mathcal{F}_{\mathcal{K}\mathcal{K}}+\mathcal{F}_{\mathcal{K}})\left[\mathcal{K}+\frac{1}{2}\alpha w_{\rm de}(2\mathcal{K}\mathcal{F}_{\mathcal{K}}-\mathcal{F})\right]. \tag{26}$$

For $w_{\rm de}=-1$, then this reduces to

$$(2\mathcal{K}\mathcal{F}_{\mathcal{K}\mathcal{K}}+\mathcal{F}_{\mathcal{K}})\left[\mathcal{K}-\frac{1}{2}\alpha(2\mathcal{K}\mathcal{F}_{\mathcal{K}}-\mathcal{F})\right]=0, \tag{27}$$

for which there are two branches of solutions,

$$\mathcal{F}=\frac{2}{\alpha}\mathcal{K}+D(\pm\mathcal{K})^{1/2}, \tag{28}$$

$$\mathcal{F}=B(\pm\mathcal{K})^{1/2}+C, \tag{29}$$

again depending on the sign of $\mathcal{K}$ and where B, C and D are integration constants. If we insert (28) into (16) we find that the Friedmann equation becomes $\rho_{\rm m}=0$; and therefore we

ignore this branch of the solution. For the other branch, we see that (29) is what we obtained before from demanding a cosmological constant, which sets $C = -2\Lambda/M^2 = -6H_0^2\Omega_{\Lambda,0}/M^2$. Therefore, the only functional form for $\mathcal{F}$ which gives rise to an exact ΛCDM cosmology, at background order, is (18). More generally, we see that the initial conditions are related via (12), such that if we specify that today $\mathcal{F}(\mathcal{K}_0) = \mathcal{F}_0$, then it must be that

$$\mathcal{F}_{\mathcal{K},0} = \frac{\Omega_{\mathrm{de},0}}{\alpha} + \frac{\mathcal{F}_0}{2\mathcal{K}_0}, \tag{30}$$

where $\mathcal{F}_{\mathcal{K},0} = \mathcal{F}_{\mathcal{K}}(\mathcal{K}_0)$ and $\mathcal{K}_0 = \mathcal{K}(a=1)$. Applying these initial conditions to (29) we find that

$$\mathcal{F} = \left(\mathcal{F}_0 + \frac{6H_0^2\Omega_{\mathrm{de},0}}{M^2}\right)\left(\frac{\mathcal{K}}{\mathcal{K}_0}\right)^{1/2} - \frac{6H_0^2\Omega_{\mathrm{de},0}}{M^2}. \tag{31}$$

At background order, we appear to have 5 parameters $\{w_{\mathrm{de}}, \Omega_{\mathrm{de},0}, \mathcal{F}_0, M, \alpha\}$ which we must specify in order to compute $\mathcal{F}$. Varying α will vary the domain over which $\mathcal{F}$ varies as a function of $\mathcal{K}$. It may also seem that α will affect the functional form of $\mathcal{F}$, as it appears explicitly in (26). However, note that this is somewhat misleading because $\mathcal{K} \propto \alpha$ and the explicit dependence of α in (26) is removed under a rescaling $\mathcal{K} \to \mathcal{K}/\alpha$. This can also be seen from (12) and (13), where the factor of α is removed under the same rescaling. Therefore, α can take on any value for the purposes of the background evolution, and so we will fix $\alpha = 1$ for the rest of this section.

The evolution of $\mathcal{F}$ for different $\{w_{\mathrm{de}}, \mathcal{F}_0, M\}$ is shown in Fig. 1. We will fix $\Omega_{\mathrm{de},0} = 0.691$ and $H_0 = 2.132 \times 10^{-42}$ hGeV, where $h = 0.678$ [5]. To study the effect of varying $\mathcal{F}_0$ we will look to the analytical solution for $w_{\mathrm{de}} = -1$ in (31), with $M = H_0$. The evolution of $\mathcal{F}$ will be such that it will be driven to $\mathcal{F}_0$ at $a = 1$, as shown in Fig. 1. The parameter $\mathcal{F}_0$ is similar to designer $f(R)$ theories where the analogous parameter in [53] was called B_0. We see that in the past $\mathcal{F}$ is approximated well by a pure power law, corresponding to the behavior of the first term in (31), since this terms dominates in the past. For $\mathcal{F}_0 \gg 6H_0^2\Omega_{\mathrm{de},0}/M^2$, this power law behavior persists into the dark energy dominated era as $\mathcal{F} \to \mathcal{F}_0$. If $\mathcal{F}_0 \lesssim 6H_0^2\Omega_{\mathrm{de},0}/M^2$ then for $(\mathcal{K}/\mathcal{K}_0)^{1/2} \gg 1$ the first term still

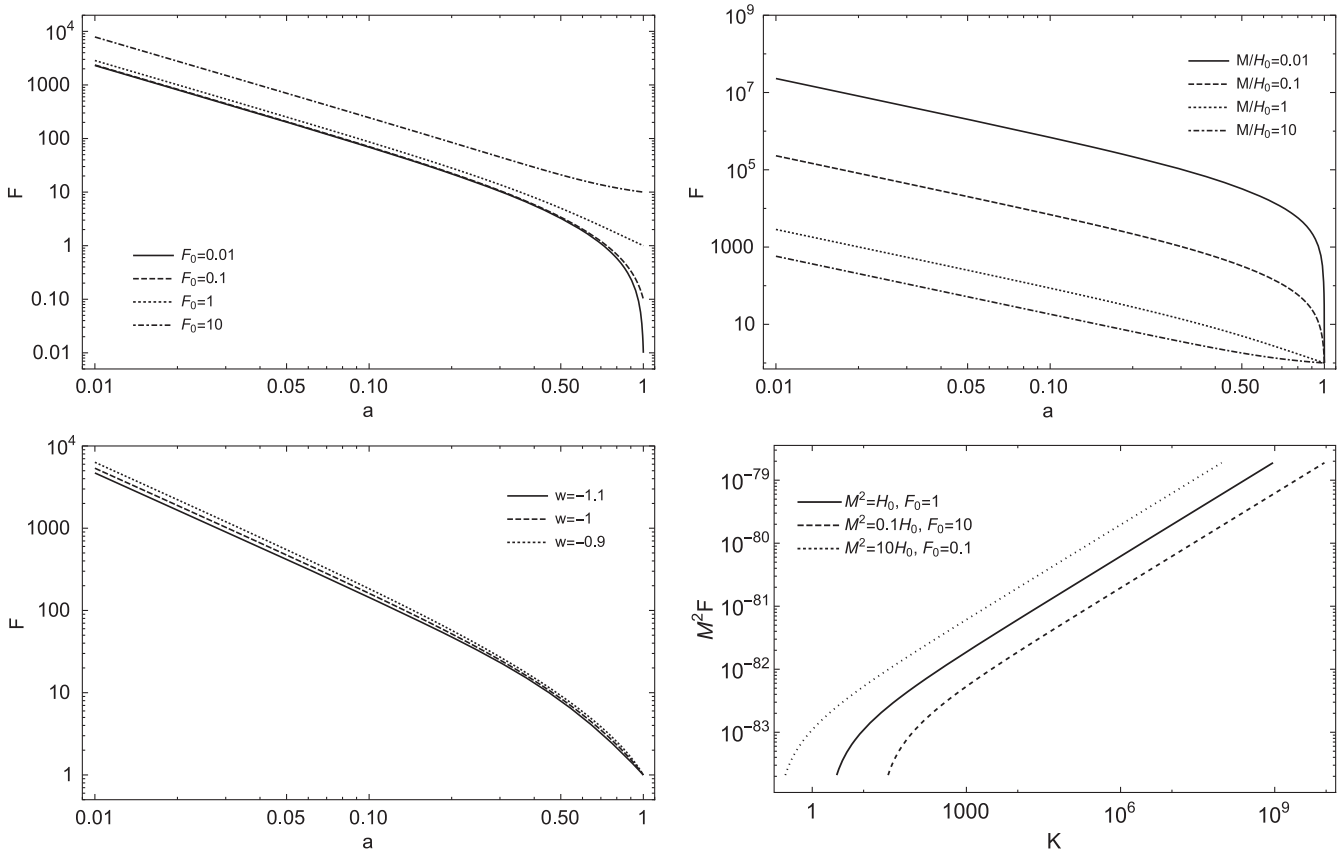

FIG. 1. *Top left panel*: Comparison of the evolution of $\mathcal{F}$ due to varying $\mathcal{F}_0$. In these models $M = H_0$ and $w_{\mathrm{de}} = -1$ are fixed. *Top right panel*: Comparison of the evolution of $\mathcal{F}$ due to the variation of M, as a multiple of H_0. In these models $\mathcal{F}_0 = 1$ and $w_{\mathrm{de}} = -1$ are fixed. *Bottom left panel*: Comparison of the evolution of $\mathcal{F}$ for varying w_{de} close to -1. In these models $\mathcal{F}_0 = 1$ and $M = H_0$ are fixed. *Bottom right panel*: Comparison of the evolution of $M^2\mathcal{F}$ for varying M^2 and $\mathcal{F}_0$, with $M^2\mathcal{F}_0/H_0^2 = 1$ and $w_{\mathrm{de}} = -1$ fixed.

dominates in (31), and we still observe the power law behavior. However, as $(\mathcal{K}/\mathcal{K}_0)^{1/2} \to 1$ the second term in (31) becomes comparable to the first and so the power law behavior is broken as $\mathcal{F} \to \mathcal{F}_0$, as seen in Fig. 1.

We note that the variation of the mass scale, M, also has a similar effect to varying $\mathcal{F}_0$, as the behavior of $\mathcal{F}$ will depend on the relative size of $\mathcal{F}_0$ and $6H_0^2\Omega_{\mathrm{de},0}/M^2$ from (31). Similar to α, varying M will also change the domain of $\mathcal{F}$. It may seem that M should not influence the evolution of $\mathcal{F}$ as it does not appear explicitly in (26). However, similar to α, the M dependence is hidden via $\mathcal{K} \propto M^{-2}$. Under a rescaling $\mathcal{K} \to M^2\mathcal{K}$, we see that there is in fact a M dependence in (26). However, if we instead work with the combination $M^2\mathcal{F}$, then under the rescaling we find that (26) becomes independent of M. Indeed, note that $\mathcal{F}$ appears as $M^2\mathcal{F}$ in the Lagrangian (3) and from (31) we can write this as

$$M^2\mathcal{F} = (M^2\mathcal{F}_0 + 6H_0^2\Omega_{\mathrm{de},0})\left(\frac{\mathcal{K}}{\mathcal{K}_0}\right)^{1/2} - 6H_0^2\Omega_{\mathrm{de},0}. \quad (32)$$

Hence, we see that any change in M can be offset with a change in $\mathcal{F}_0$, i.e. M and $\mathcal{F}_0$ are degenerate, as seen in Fig. 1. As expected, the choice of M does not affect the functional form of $M^2\mathcal{F}$. We will therefore fix $M = H_0$, corresponding to the approximate mass scale dark energy begins to dominate, and keep $\mathcal{F}_0$ as a free parameter.

For solutions close to $w_{\mathrm{de}} = -1$, we do not expect to see large deviations from the analytical solution. Indeed, the previous discussion about the power law behavior still holds for solutions with w_{de} sufficiently close to -1, as seen in Fig. 1. Although unfavored by current observations, dark energy models with $w_{\mathrm{de}} \neq -1$ have not been completely ruled out and so we will allow for this in the subsequent analysis.

To summarize, for the background evolution we have 3 free parameters $\{w_{\mathrm{de}}, \Omega_{\mathrm{de},0}, \mathcal{F}_0\}$ to specify, not 5, since α has no effect on the background evolution, other than a rescaling of the domain as a function of $\mathcal{K}$, and M^2 is degenerate with $\mathcal{F}_0$. While the background evolution only requires us to specify $\{w_{\mathrm{de}}, \Omega_{\mathrm{de},0}, \mathcal{F}_0, M\}$, as we will see in Sec. IV, at the level of linear perturbations the value of α and the other $\{c_i\}$ coefficients will be important.

C. Subclasses to generalized Einstein-Aether

There are a number of interesting subclasses of the generalized Einstein-Aether model that have been studied previously which we will mention here.

1. Linear Einstein-Aether

The first is perhaps the most obvious simplification to this model, other than the absence of the Aether field, and that is to set $\mathcal{F}(\mathcal{K}) = \mathcal{K}$, and indeed this is the form of Einstein-Aether that was originally proposed in [34].

In this case, the equations of motion become

$$\nabla_\tau(J^\tau{}_\mu) - c_4 A^\alpha \nabla_\alpha A^\nu \nabla_\mu A_\nu = \lambda A_\mu \quad (33)$$

and

$$\begin{aligned} U_{\alpha\beta} = {} & \nabla_\mu(J_{(\alpha}{}^\mu A_{\beta)} - J^\mu{}_{(\alpha}A_{\beta)} - J_{(\alpha\beta)}A^\mu) + c_1(\nabla_\mu A_\alpha \nabla^\mu A_\beta \\ & - \nabla_\alpha A_\mu \nabla_\beta A^\mu) + c_4 A^\mu A^\nu \nabla_\mu A_\alpha \nabla_\nu A_\beta \\ & + (c_4 A^\mu A^\nu \nabla_\mu A^\tau \nabla_\nu A_\tau - A^\nu \nabla_\mu J^\mu{}_\nu) A_\alpha A_\beta + \frac{1}{2}\mathcal{K} g_{\alpha\beta}. \end{aligned} \quad (34)$$

The energy density and pressure are then

$$\rho_A = \frac{3}{2}\alpha H^2, \qquad P_A = -\frac{3}{2}\alpha H^2 - \alpha\dot{H}. \quad (35)$$

For a universe dominated by a fluid species with equation of state $P = w_i\rho$ the scale factor is $a \propto t^{2/3(1+w_i)}$. We therefore have that

$$\frac{P_A}{\rho_A} = w_{\mathrm{de}} = -1 - \frac{2\dot{H}}{3H^2} = w_i, \quad (36)$$

i.e. the equation of state parameter for Aether field in linear Einstein-Aether matches that of other fluids present in the Universe [39]. This behavior prevents linear Einstein-Aether, $\mathcal{F}(\mathcal{K}) = \mathcal{K}$, from being a dark energy candidate and is one of the motivations for its generalization.

2. Generalized Einstein-Aether with $c_4 = 0$

As already mentioned, many previous studies of Einstein-Aether models set $c_4 = 0$. It is often argued that this can be done via a redefinition of the coefficients. However, we will see in the next section that this can only be achieved after a specific choice of A^μ which has further consequences at the level of linear perturbations. In this case, the equations of motion become

$$\nabla_\nu(\mathcal{F}_\mathcal{K} J^\nu{}_\mu) = \lambda A_\mu, \quad (37)$$

and

$$\begin{aligned} U_{\alpha\beta} = {} & \nabla_\mu(\mathcal{F}_\mathcal{K}[J_{(\alpha}{}^\mu A_{\beta)} - J^\mu{}_{(\alpha}A_{\beta)} - J_{(\alpha\beta)}A^\mu)] \\ & + c_1\mathcal{F}_\mathcal{K}(\nabla_\mu A_\alpha \nabla^\mu A_\beta - \nabla_\alpha A_\mu \nabla_\beta A^\mu) \\ & - A_\alpha A_\beta A^\nu \nabla_\mu(\mathcal{F}_\mathcal{K} J^\mu{}_\nu) + \frac{1}{2}M^2\mathcal{F} g_{\alpha\beta}. \end{aligned} \quad (38)$$

3. The khronometric model

The khronometric model [56,57] is a version of Einstein-Aether where the Aether field is constrained via a

scalar field, φ, called the khronon. In this case, the field is defined as

$$A_\mu = -\frac{\partial_\mu \varphi}{\sqrt{-g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi}}, \tag{39}$$

and so the timelike unit norm constraint is satisfied automatically. In doing so, the Aether is restricted to be orthogonal to a set of spacelike surfaces defined by φ. At background order we assume $\varphi = \varphi(t)$ and so from (39) we have that $A^\mu = (1, 0, 0, 0)$, which is the same as before. Therefore, the choice of the khronon definition has no effect on background dynamics.

The khronometric model was first proposed in [56], where φ sets a preferred global time coordinate. It was discussed how this model describes the low energy limit of the consistent extension of Horava gravity, a quantum theory of gravity. At low energies, this reduces to a Lorentz-violating scalar-tensor gravity theory. For more details see [56–59].

For this choice of the Aether field, the c_1, c_3 and c_4 terms are no longer independent. The twist vector is defined as [60]

$$\omega_\alpha = \varepsilon_{\alpha\beta\mu\nu}A^\beta\nabla^\mu A^\nu, \tag{40}$$

where $\varepsilon_{\alpha\beta\mu\nu}$ is the four-dimensional Levi-Civita symbol, and $\omega_\alpha = 0$ if A^μ is hypersurface orthogonal. If $\omega_\alpha = 0$ then

$$\begin{aligned} w^\alpha w_\alpha = 0 &= \varepsilon_{\alpha\beta\mu\nu}\varepsilon^{\alpha\gamma\rho\sigma}A^\beta A_\gamma \nabla^\mu A^\nu \nabla_\rho A_\sigma \\ &= -\delta^{\gamma\rho\sigma}_{\beta\mu\nu}A^\beta A_\gamma \nabla^\mu A^\nu \nabla_\rho A_\sigma, \end{aligned} \tag{41}$$

where $\delta^{\gamma\rho\sigma}_{\beta\mu\nu}$ is the generalized Kronecker delta. Therefore,

$$\begin{aligned} &- A^\gamma A_\gamma \nabla_\rho A_\sigma \nabla^\rho A^\sigma - A^\sigma A_\gamma \nabla_\rho A_\sigma \nabla^\gamma A^\rho - A^\rho A_\gamma \nabla_\rho A_\sigma \nabla^\sigma A^\gamma \\ &\quad + A^\gamma A_\gamma \nabla_\rho A_\sigma \nabla^\sigma A^\rho + A^\rho A_\gamma \nabla_\rho A_\sigma \nabla^\gamma A^\sigma \\ &\quad + A^\sigma A_\gamma \nabla_\rho A_\sigma \nabla^\rho A^\gamma = 0. \end{aligned} \tag{42}$$

From $A_\gamma \nabla^\rho A^\gamma = \nabla^\rho(A_\gamma A^\gamma) - A^\gamma \nabla^\rho A_\gamma$, applying the unit norm constraint gives $A_\gamma \nabla^\rho A^\gamma = 0$, and so

$$A^\rho A_\gamma \nabla_\rho A_\sigma \nabla^\gamma A^\sigma = \nabla_\rho A_\sigma \nabla^\sigma A^\rho - \nabla_\rho A_\sigma \nabla^\rho A^\sigma. \tag{43}$$

Note that the left-hand side of (43) is the c_4 term in (5). Since the terms on the right-hand side of (43) are related to the c_1 and c_3 terms, we are able to absorb c_4 into the other coefficients effectively setting $c_4 = 0$ i.e. $c_1 \to c_1' = c_1 - c_4$ and $c_3 \to c_3' = c_3 + c_4$ giving

$$K^{\alpha\beta}{}_{\mu\nu} = c_1' g^{\alpha\beta} g_{\mu\nu} + c_2 \delta^\alpha_\mu \delta^\beta_\nu + c_3' \delta^\alpha_\nu \delta^\beta_\mu. \tag{44}$$

We therefore see that it is possible to set $c_4 = 0$, but only if the choice is made that A^μ is also hypersurface orthogonal. While this has no effect at background order, we will see later that differences arise at the level of linear perturbations for the vector sector. Furthermore, this is not the only choice we can make as (43) also allows a redefinition which could remove c_1 or c_3 instead.

III. LINEAR PERTURBATIONS

We will present results for perturbations in the scalar sector in both the synchronous and conformal Newtonian gauge. We perturb the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}), \tag{45}$$

such that

$$ds^2 = a^2(\tau)[-(1 + 2\Psi)d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j], \tag{46}$$

where we now work in conformal time, τ. In the synchronous gauge we set $\Psi = 0$ and decompose h_{ij} into [61,62]

$$\begin{aligned} h_{ij} = \hat{k}_i\hat{k}_j h + \left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)6\eta + 2\hat{k}_{(i}(h^{V1}\hat{l}_{j)} + h^{V2}\hat{m}_{j)}) \\ + h^+(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j) + h^\times(\hat{l}_i\hat{m}_j - \hat{l}_j\hat{m}_i), \end{aligned}$$

where the unit vectors $\{\hat{k}, \hat{l}, \hat{m}\}$ form an orthonormal basis in k-space. Here, h and η are the scalar perturbations, h^{V1} and h^{V2} are the vector perturbations, and h^+ and $h^\times$ are the tensor perturbations. In the conformal Newtonian gauge we set $h_{ij}^{\text{scalar}} = -2\Phi\delta_{ij}$, while the vector and tensor perturbations are as before in the synchronous gauge.

We also perturb the Aether field as [38]

$$A^\mu = \bar{A}^\mu + \delta A^\mu = \frac{1}{a}(1 + X, \partial^i V + \mathrm{i}B^i), \tag{47}$$

where V is the longitudinal scalar mode and B^i is the transverse vector mode such that $k_i B^i = 0$. The unit norm constraint demands that $X = -\Psi$, and so

$$\delta A^\mu = \frac{1}{a}(-\Psi, \partial^i V + \mathrm{i}B^i). \tag{48}$$

Hence, we see that the timelike unit norm constraint means that there is only one scalar degree of freedom, V, along with a transverse vector mode, B^i. In what follows, we will suppress overbars to denote background order quantities.

The perturbed energy momentum tensor is given by

$$\delta U_{\alpha\beta} = \delta(\nabla_\mu[\mathcal{F}_\mathcal{K}(J_{(\alpha}{}^\mu A_{\beta)} - J^\mu{}_{(\alpha}A_{\beta)} - J_{(\alpha\beta)}A^\mu)]) + c_1\mathcal{F}_{\mathcal{K}\mathcal{K}}\delta\mathcal{K}(\nabla_\mu A_\alpha\nabla^\mu A_\beta - \nabla_\alpha A_\mu\nabla_\beta A^\mu) + c_1\mathcal{F}_\mathcal{K}\delta(\nabla_\mu A_\alpha\nabla^\mu A_\beta - \nabla_\alpha A_\mu\nabla_\beta A^\mu) + c_4\mathcal{F}_{\mathcal{K}\mathcal{K}}\delta\mathcal{K}A^\mu A^\nu\nabla_\mu A_\alpha\nabla_\nu A_\beta + c_4\mathcal{F}_\mathcal{K}\delta(A^\mu A^\nu\nabla_\mu A_\alpha\nabla_\nu A_\beta) + \delta([c_4\mathcal{F}_\mathcal{K}A^\mu A^\nu\nabla_\mu A^\tau\nabla_\nu A_\tau - A^\nu\nabla_\mu(\mathcal{F}_\mathcal{K}J^\mu_\nu)]A_\alpha A_\beta) + \frac{1}{2}M^2(\mathcal{F}\delta g_{\alpha\beta} + g_{\alpha\beta}\mathcal{F}_\mathcal{K}\delta\mathcal{K}). \tag{49}$$

For a general energy-momentum tensor, $E_{\mu\nu}$, we can decompose its perturbations as [61]

$$\delta E^\mu{}_\nu = (\delta\rho + \delta P)u^\mu u_\nu + \delta P\delta^\mu{}_\nu + (\rho + P)(\delta u^\mu u_\nu + \delta u_\nu u^\mu) + P\Pi^\mu{}_\nu, \tag{50}$$

where $u^\mu = \frac{1}{a}(1,0,0,0)$, $\delta u^\mu = \frac{1}{a}(0,v^i)$ and Π^μ_ν is the anisotropic stress, with the properties $u^\nu\Pi^\mu{}_\nu = 0$, $\Pi^\mu{}_\nu = \Pi_\nu{}^\mu$, and $\Pi^\mu{}_\mu = 0$. Projecting out the perturbed fluid variables, we find that

$$\delta E^0{}_0 = -\delta\rho, \tag{51}$$

$$\delta E^0{}_i = (\rho + P)v_i, \tag{52}$$

$$\delta E^i{}_0 = -(\rho + P)v_i, \tag{53}$$

$$\delta E^i{}_j = P\Pi^i_j + \delta P\delta^i{}_j. \tag{54}$$

Similar to h_{ij}, we can decompose v_i and Π_{ij} into scalar, vector, and tensor parts. They are given by [63]

$$v_i = V^S\hat{k}_i + V^{V1}\hat{l}_i + V^{V2}\hat{m}_i, \tag{55}$$

$$\Pi_{ij} = \left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\Pi^S + 2\hat{k}_{(i}(\Pi^{V1}\hat{l}_{j)} + \Pi^{V2}\hat{m}_{j)}) + \Pi^+(\hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j) + \Pi^\times(\hat{l}_i\hat{m}_j - \hat{l}_j\hat{m}_i), \tag{56}$$

whereas the transverse vector, B_i, only has vector modes i.e.

$$B_i = B^{V1}\hat{l}_i + B^{V2}\hat{m}_i. \tag{57}$$

In a general gauge, the perturbed fluid variables from (48) in k-space are then

$$a^2\delta\rho = \alpha\left[3\mathcal{F}_{\mathcal{K}\mathcal{K}}\delta\mathcal{K}\mathcal{H}^2 + \mathcal{F}_\mathcal{K}\mathcal{H}\left(\frac{1}{2}h' - k^2V - 3\mathcal{H}\Psi\right)\right] + c_{14}\mathcal{F}_\mathcal{K}k^2(V' + \mathcal{H}V + \Psi), \tag{58}$$

$$a^2\delta P = \alpha\mathcal{F}_\mathcal{K}\left[\mathcal{H}\Psi' + (2\mathcal{H}' + \mathcal{H}^2)\Psi - \frac{1}{6}(h'' + 2\mathcal{H}h') + \frac{1}{3}k^2(V' + 2\mathcal{H}V)\right] - \alpha\mathcal{F}_{\mathcal{K}\mathcal{K}}\left[\left(\mathcal{H}' + 2\mathcal{H}^2 + \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}\mathcal{K}}}\mathcal{K}'\mathcal{H}\right)\delta K + \delta\mathcal{K}'\mathcal{H} - \frac{1}{6}\mathcal{K}'(12\mathcal{H}\Psi + 2k^2V - h')\right], \tag{59}$$

$$a^2(\rho + P)v_i = \mathrm{i}\alpha\left[(\mathcal{F}_\mathcal{K}(\mathcal{H}^2 - \mathcal{H}') - \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\mathcal{H})\xi_i - \frac{1}{2}k^2B_i\right] + \mathrm{i}\left(\frac{3}{2}c_2 + c_1\right)\mathcal{F}_\mathcal{K}k^2B_i + \mathrm{i}c_{14}[\mathcal{F}_\mathcal{K}(\xi_i'' + 2\mathcal{H}\xi_i' + (\mathcal{H}' + \mathcal{H}^2)\xi_i + k_i\Psi' + \mathcal{H}k_i\Psi) + \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'(\xi_i' + \mathcal{H}\xi_i + k_i\Psi)], \tag{60}$$

$$a^2P\Pi^i{}_j = c_{13}\left[\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\left(k^ik_jV - \frac{1}{2}h^i{}_{j'}\right) + \mathcal{F}_\mathcal{K}k^ik_j(V' + 2\mathcal{H}V) - \mathcal{F}_\mathcal{K}\left(\frac{1}{2}h^i{}_{j''} + \mathcal{H}h^i{}_{j'}\right) + \frac{1}{6}(\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'(h' - 2k^2V) + \mathcal{F}_\mathcal{K}(h'' + 2\mathcal{H}h') - 2\mathcal{F}_\mathcal{K}k^2(V' + 2\mathcal{H}V))\delta^i{}_j + \left(\mathcal{F}_\mathcal{K}\mathcal{H} + \frac{1}{2}\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\right)(k^iB_j + k_jB^i) + \frac{1}{2}\mathcal{F}_\mathcal{K}(k^iB_{j'} + k_jB^{i'})\right], \tag{61}$$

where primes denote conformal time differentiation, $c_{13} = c_1 + c_3$, $c_{14} = c_1 - c_4$, $c_{123} = c_1 + c_3 + c_3$, and $\xi_i = k_iV + B_i$.

A. Scalar sector

The scalar components of v_i and $\Pi^i{}_j$ are obtained via $V^S = \hat{k}^i v_i$ and $\Pi^S = \frac{3}{2}(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j_i)\Pi^i_j$. If we further define $\theta^S = \mathrm{i}V^S/k = \mathrm{i}k^i v_i/k^2$, then we have that

$$a^2(\rho+P)\theta^S = \alpha[\mathcal{F}_{\mathcal{K}}(\mathcal{H}' - \mathcal{H}^2) + \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\mathcal{H}]V - c_{14}[\mathcal{F}_{\mathcal{K}}(V'' + 2\mathcal{H}V' + (\mathcal{H}' + \mathcal{H}^2)V + \Psi' + \mathcal{H}\Psi) + \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'(V' + \mathcal{H}V + \Psi)], \tag{62}$$

$$\frac{2}{3}a^2P\Pi^S = c_{13}\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j_i\right)\left[\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\left(k^ik_jV - \frac{1}{2}h'^i{}_j\right) + \mathcal{F}_{\mathcal{K}}k^ik_j(V' + 2\mathcal{H}V) - \mathcal{F}_{\mathcal{K}}\left(\frac{1}{2}h''^i{}_j + \mathcal{H}h'^i{}_j\right)\right]. \tag{63}$$

Note that the expression for Π^S will simplify further once we specify the gauge. We further define the entropy perturbation, Γ, as

$$w\Gamma = \left(\frac{\delta P}{\delta\rho} - \frac{dP}{d\rho}\right)\delta. \tag{64}$$

It should be noted that whatever gauge we choose to work in, both Π^S and Γ are gauge invariant. The perturbed Aether field equation of motion is obtained from perturbing (7). Taking the i-component, the $\hat{k}^i$ direction will yield the equation of motion governing the perturbation V, given by

$$\begin{aligned} &c_1\left[V'' + 2\mathcal{H}V' + (2\mathcal{H}^2 + k^2)V + \Psi' + 2\mathcal{H}\Psi - \frac{1}{2}\hat{k}^i\hat{k}_jh^j{}_{i'}\right] + c_2\left[(k^2 + 3\mathcal{H}^2 - 3\mathcal{H}')V + 3\mathcal{H}\Psi - \frac{1}{2}h'\right] \\ &+ c_3\left[(k^2 + \mathcal{H}^2 - \mathcal{H}')V + \mathcal{H}\Psi - \frac{1}{2}\hat{k}^i\hat{k}_jh^j{}_{i'}\right] - c_4[V'' + 2\mathcal{H}V' + (\mathcal{H}' + \mathcal{H}^2)V + \Psi' + \mathcal{H}\Psi] \\ &- \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}}}(\alpha\delta\mathcal{K}\mathcal{H} + \mathcal{K}'[\alpha\mathcal{H}V - c_{14}(V' + \mathcal{H}V + \Psi)]) = 0, \end{aligned} \tag{65}$$

where we have substituted in for λ.

1. Conformal Newtonian gauge

In the conformal Newtonian gauge, where the metric perturbations are parametrized via Ψ and Φ, we have that

$$a^2\delta\rho = [3\mathcal{F}_{\mathcal{K}\mathcal{K}}\delta\mathcal{K}\mathcal{H}^2 - \mathcal{F}_{\mathcal{K}}\mathcal{H}(k^2V + 3\mathcal{H}\Psi + 3\Phi')] + c_{14}\mathcal{F}_{\mathcal{K}}k^2(V' + \mathcal{H}V + \Psi), \tag{66}$$

$$\begin{aligned} a^2\delta P = {}& \alpha\mathcal{F}_{\mathcal{K}}\left[\mathcal{H}\Psi' + (2\mathcal{H}' + \mathcal{H}^2)\Psi + \Phi'' + 2\mathcal{H}\Phi' + \frac{1}{3}k^2(V' + 2\mathcal{H}V)\right] \\ &- \alpha\mathcal{F}_{\mathcal{K}\mathcal{K}}\left[\left(\mathcal{H}' + 2\mathcal{H}^2 + \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}\mathcal{K}}}\mathcal{K}'\mathcal{H}\right)\delta K + \delta\mathcal{K}'\mathcal{H} - \frac{1}{3}\mathcal{K}'(6\mathcal{H}\Psi + 3\Phi' + k^2V)\right], \end{aligned} \tag{67}$$

$$a^2(\rho+P)\theta^S = \alpha[\mathcal{F}_{\mathcal{K}}(\mathcal{H}' - \mathcal{H}^2) + \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\mathcal{H}]V - c_{14}[\mathcal{F}_{\mathcal{K}}(V'' + 2\mathcal{H}V' + (\mathcal{H}' + \mathcal{H}^2)V + \Psi' + \mathcal{H}\Psi) + \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'(V' + \mathcal{H}V + \Psi)], \tag{68}$$

$$a^2P\Pi^S = c_{13}[\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'k^2V + \mathcal{F}_{\mathcal{K}}k^2(V' + 2\mathcal{H}V)]. \tag{69}$$

The perturbed Aether field equation of motion reads

$$\begin{aligned} &\alpha\left[(\mathcal{H}^2 - \mathcal{H}' + k^2)V + \mathcal{H}\Psi + \Phi' - \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}}}(\delta\mathcal{K}\mathcal{H} + \mathcal{K}'\mathcal{H}V)\right] \\ &+ c_{14}\left[V'' + 2\mathcal{H}V' + (\mathcal{H}^2 + \mathcal{H}')V + \Psi' + \mathcal{H}\Psi + \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}}}\mathcal{K}'(V' + \mathcal{H}V + \Psi)\right] - 2c_2k^2V = 0. \end{aligned} \tag{70}$$

2. Synchronous gauge

In the synchronous gauge, where h_{ij} is decomposed into h and η as in (47), we find that

$$a^2\delta\rho = \alpha\left[3\mathcal{F}_{\mathcal{K}\mathcal{K}}\delta\mathcal{K}\mathcal{H}^2 + \mathcal{F}_{\mathcal{K}}\mathcal{H}\left(\frac{1}{2}h' - k^2V\right)\right] + c_{14}\mathcal{F}_{\mathcal{K}}k^2(V' + \mathcal{H}V) \tag{71}$$

$$a^2\delta P = \frac{1}{3}\alpha\mathcal{F}_{\mathcal{K}}\left[k^2(V' + 2\mathcal{H}V) - \frac{1}{2}h'' - \mathcal{H}h'\right] - \alpha\mathcal{F}_{\mathcal{K}\mathcal{K}}\left[\left(\mathcal{H}' + 2\mathcal{H}^2 + \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}\mathcal{K}}}\mathcal{K}'\mathcal{H}\right)\delta K + \delta\mathcal{K}'\mathcal{H} - \frac{1}{6}\mathcal{K}'(h' + 2k^2V)\right], \tag{72}$$

$$a^2(\rho + P)\theta^S = \alpha[\mathcal{F}_{\mathcal{K}}(\mathcal{H}' - \mathcal{H}^2) + \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\mathcal{H}]V - c_{14}[\mathcal{F}_{\mathcal{K}}(V'' + 2\mathcal{H}V' + (\mathcal{H}' + \mathcal{H}^2)V) + \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'(V' + \mathcal{H}V)], \tag{73}$$

$$a^2P\Pi^S = c_{13}\left[\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\left(k^2V - \frac{1}{2}(h + 6\eta)\right) + \mathcal{F}_{\mathcal{K}}k^2(V' + 2\mathcal{H}V) - \mathcal{F}_{\mathcal{K}}\left(\frac{1}{2}(h'' + 6\eta'') + \mathcal{H}(h' + 6\eta')\right)\right]. \tag{74}$$

The perturbed equation of motion for the Aether field reads

$$\begin{aligned}&\alpha\left[(\mathcal{H}^2 - \mathcal{H}' + k^2)V - \frac{1}{2}(h' + 4\eta') - \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}}}(\delta\mathcal{K}\mathcal{H} + \mathcal{K}'\mathcal{H}V)\right]\\ &\quad + c_{14}\left[V'' + 2\mathcal{H}V' + (\mathcal{H}^2 + \mathcal{H}') + \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}}}\mathcal{K}'(V' + \mathcal{H}V)\right] + c_2(h' + 6\eta' - 2k^2V) = 0.\end{aligned} \tag{75}$$

B. Vector and tensor sectors

In the vector and tensor sectors, the vector and tensor modes of v_i and $\Pi^i{}_j$ can be computed via $V^{V1} = \hat{l}^i v_i$, $\Pi^{V1} = \hat{k}_i\hat{l}^j\Pi^i_j$, and $\Pi^+ = \frac{1}{2}(\hat{l}_i\hat{l}^j - \hat{m}_i\hat{m}^j)\Pi^i_j$. Equivalent expressions also exist for the $V2$ modes and $\Pi^\times$. Also, analogous to θ^S, we can define $\theta^{V1} = iV^{V1}/k = i\hat{l}^i v_i/k$, and so we have that

$$\begin{aligned}a^2(\rho + P)k\theta^{V1} &= \alpha[\mathcal{F}_{\mathcal{K}}(\mathcal{H}' - \mathcal{H}^2) + \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\mathcal{H}]B^{V1}\\ &\quad + \frac{1}{2}(c_3 - c_1)\mathcal{F}_{\mathcal{K}}k^2B^{V1} - c_{14}[\mathcal{F}_{\mathcal{K}}(B^{V1''}\\ &\quad + 2\mathcal{H}B^{V1'} + (\mathcal{H}' + \mathcal{H}^2)B^{V1})\\ &\quad + \mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'(B^{V1'} + \mathcal{H}B^{V1})],\end{aligned} \tag{76}$$

$$\begin{aligned}a^2P\Pi^{V1} = c_{13}\Bigg[&\frac{1}{2}\mathcal{F}_{\mathcal{K}}(kB^{V1'} - h^{V1''})\\ &+ \left(\mathcal{F}_{\mathcal{K}}\mathcal{H} + \frac{1}{2}\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\right)(kB^{V1} - h^{V1'})\Bigg],\end{aligned} \tag{77}$$

$$a^2P\Pi^+ = -c_{13}\left[\frac{1}{2}\mathcal{F}_{\mathcal{K}}h^{+''} + \left(\mathcal{F}_{\mathcal{K}}\mathcal{H} + \frac{1}{2}\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\right)h^{+'}\right]. \tag{78}$$

The time-time and traced ij-components are zero in the vector and tensor sectors since $\delta\rho$ and δP only have scalar modes.

The equation of motion for the Aether field in the $\hat{l}^i$ direction is given by

$$\begin{aligned}&\alpha\left[(\mathcal{H}^2 - \mathcal{H}')B^{V1} - \frac{1}{2}kh^{V1'} - \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}}}\mathcal{K}'\mathcal{H}B^{V1}\right]\\ &\quad + c_1k^2B^{V1} + \frac{3}{2}c_2kh^{V1'} + c_{14}\Bigg[B^{V1''} + 2\mathcal{H}B^{V1'}\\ &\quad + (\mathcal{H}' + \mathcal{H}^2)B^{V1} + \frac{\mathcal{F}_{\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}}}\mathcal{K}'(B^{V1'} + \mathcal{H}B^{V1})\Bigg] = 0.\end{aligned} \tag{79}$$

Note that the two vector and tensor modes are interchangeable. From here on we will not discriminate between them and denote them simply as θ^V, Π^V and Π^T.

C. Vector modes in the khronon

If we restrict ourselves to the case where the Aether field is defined by the khronon in (39), then we find that

$$\delta A_\mu = \frac{a}{\varphi'}\left[-\partial_\mu\delta\varphi + \partial_\mu\varphi\left(\Psi + \frac{\delta\varphi'}{\varphi'}\right)\right], \tag{80}$$

where $\delta\varphi$ is the perturbed khronon field. The time component is then $\delta A_0 = a\Psi$, which is a consequence of the timelike unit norm constraint, as in (48). However, if we calculate the spatial component we find that

$$\delta A_i = -\frac{a}{\varphi'}\partial_i\delta\varphi \Rightarrow B_i = 0, \tag{81}$$

i.e. there is no propagating transverse vector mode. Therefore, if we redefine $\frac{1}{\varphi}\partial_i\delta\varphi = \partial_i V$ then we obtain the results from Sec. III. 1. Therefore, the scalar sector for generalized Einstein-Aether and the khronon are completely equivalent [57], up to a redefinition of the coefficients discussed previously.

IV. EQUATIONS OF STATE FOR PERTURBATIONS

A. Scalar sector

We now derive the equations of state, Γ and $\Pi^{S,V,T}$, in terms of the other perturbation variables by fully eliminating the internal degrees of freedom introduced by the theory i.e. V, B^i, and their derivatives. In the scalar sector we do this via the expressions for $\delta\rho$ and θ^S. Let us first work in the conformal Newtonian gauge. Initially it may not seem possible to eliminate the degrees of freedom as we have that $\theta^S \equiv \theta^S(V, V', V'')$ and $\delta\rho \equiv \delta\rho(V, V')$, i.e. we have three unknowns and only two equations. However, we can use the perturbed Aether field equation of motion (70) to reduce the dimensionality of the problem. Using this to eliminate V'' in (68) and gathering terms in V and V', we find that

$$a^2\delta\rho = c_{14}\mathcal{F}_{\mathcal{K}}k^2V' - \left[\alpha\mathcal{F}_{\mathcal{K}} - c_{14}\mathcal{F}_{\mathcal{K}} + \frac{6\alpha^2\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{H}^2}{a^2M^2}\right]\mathcal{H}k^2V + c_{14}\mathcal{F}_{\mathcal{K}}k^2\Psi - 3\alpha\mathcal{H}\left(\mathcal{F}_{\mathcal{K}} + \frac{6\alpha\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{H}^2}{a^2M^2}\right)(\mathcal{H}\Psi + \Phi'), \tag{82}$$

$$a^2\rho(1+w_{\rm de})\theta^S = \left[c_{123}\mathcal{F}_{\mathcal{K}} + \frac{2\alpha^2\mathcal{H}^2\mathcal{F}_{\mathcal{K}\mathcal{K}}}{a^2M^2}\right]k^2V + \alpha\left(\mathcal{F}_{\mathcal{K}} + \frac{6\alpha\mathcal{H}^2\mathcal{F}_{\mathcal{K}\mathcal{K}}}{a^2M^2}\right)(\mathcal{H}\Psi + \Phi'), \tag{83}$$

where we have substituted in for $\mathcal{K}$ from (14) and

$$\delta\mathcal{K} = -\frac{2\alpha\mathcal{H}}{a^2M^2}(k^2V + 3\mathcal{H}\Psi + 3\Phi'). \tag{84}$$

So we see that in fact $\theta^S \equiv \theta^S(V)$. Note that we can already see the emergence of the gauge invariant combination, $\mathcal{H}\Psi + \Phi'$, in the $0i$-component of Einstein's equation that was used in [43–45].

We can then write this system of equations as

$$a^2\begin{pmatrix}\delta\rho\\ \rho(1+w_{\rm de})\theta^S\end{pmatrix} = k^2\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}\begin{pmatrix}V'\\ V\end{pmatrix} + \begin{pmatrix}D\\ E\end{pmatrix}, \tag{85}$$

with

$$A = c_{14}\mathcal{F}_{\mathcal{K}}, \tag{86}$$

$$B = \left[c_{14}\mathcal{F}_{\mathcal{K}} - \alpha\mathcal{F}_{\mathcal{K}} - \frac{6\alpha^2\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{H}^2}{a^2M^2}\right]\mathcal{H}, \tag{87}$$

$$C = \left[c_{123}\mathcal{F}_{\mathcal{K}} + \frac{2\alpha^2\mathcal{H}^2\mathcal{F}_{\mathcal{K}\mathcal{K}}}{a^2M^2}\right], \tag{88}$$

$$D = c_{14}\mathcal{F}_{\mathcal{K}}k^2\Psi - 3\alpha\mathcal{H}\left(\mathcal{F}_{\mathcal{K}} + \frac{6\alpha\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{H}^2}{a^2M^2}\right)(\mathcal{H}\Psi + \Phi'), \tag{89}$$

$$E = \alpha\left(\mathcal{F}_{\mathcal{K}} + \frac{6\alpha\mathcal{H}^2\mathcal{F}_{\mathcal{K}\mathcal{K}}}{a^2M^2}\right)(\mathcal{H}\Psi + \Phi'). \tag{90}$$

In [49] the ABC matrix in (85) was dubbed the activation matrix, as it determines which degrees of freedom are present, or activated, in the perturbed fluid variables. Inverting this then yields expressions for V and V' in terms of $\delta\rho$, θ^S, the metric perturbations, Ψ and Φ, and their derivatives. Eliminating for these in Π^S (69), we find that we can write

$$w_{\rm de}\Pi^S = A_1\delta + A_2(1+w)\theta^S + A_3k^2\Psi + A_4(\mathcal{H}\Psi + \Phi'), \tag{91}$$

where

$$A_1 = \frac{c_{13}}{c_{14}}, \tag{92}$$

$$A_2 = \frac{3c_{13}\mathcal{H}}{3c_{123} + 2\alpha\gamma_2}\left[1 + \frac{2(\mathcal{H}' - \mathcal{H}^2)}{\mathcal{H}^2}\gamma_2 + \frac{\alpha(1+2\gamma_2)}{c_{14}}\right], \tag{93}$$

$$A_3 = \frac{2c_{13}\gamma_1}{3\alpha\mathcal{H}^2(2\gamma_1 - 1)}, \tag{94}$$

$$A_4 = \frac{2c_{13}\gamma_1(1+2\gamma_2)}{\mathcal{H}(2\gamma_1 - 1)(3c_{123} + 2\alpha\gamma_2)} \times \left[2\left(\frac{c_{13}}{c_{14}} - \frac{(\mathcal{H}' - \mathcal{H}^2)}{\mathcal{H}^2}\gamma_2\right) - 1\right], \tag{95}$$

and we define the dimensionless functions

$$\gamma_1 = \frac{\mathcal{K}\mathcal{F}_{\mathcal{K}}}{\mathcal{F}}, \qquad \gamma_2 = \frac{\mathcal{K}\mathcal{F}_{\mathcal{K}\mathcal{K}}}{\mathcal{F}_{\mathcal{K}}}, \qquad \gamma_3 = \frac{\mathcal{K}\mathcal{F}_{\mathcal{K}}\mathcal{K}\mathcal{K}}{\mathcal{F}_{\mathcal{K}\mathcal{K}}}. \tag{96}$$

In the parlance of [53], we write (91) in terms of a set of dimensionless variables given in Table II, where $h_{\parallel} = h + 6\eta$, $K = k/\mathcal{H}$, and $\epsilon_H = 1 - \mathcal{H}'/\mathcal{H}^2$. Note that these

TABLE II. Combinations of the metric perturbations and perturbed fluid variables are now written in terms of the dimensionless variables given in this table, in both the conformal Newtonian and synchronous gauges.

Variable	Conformal newtonian	Synchronous
T	$\frac{h_\parallel'}{2\mathcal{H}K^2}$	0
W	$\frac{1}{\mathcal{H}}X' - \epsilon_H(X+Y)$	$\frac{1}{\mathcal{H}}X' - \epsilon_H(X+Y)$
X	$\frac{1}{\mathcal{H}}Z' + Y$	$\frac{1}{\mathcal{H}}Z' + Y$
Y	Ψ	$\frac{1}{\mathcal{H}}T' + \epsilon_H T$
Z	Φ	$\eta - T$
Δ	$\delta + 3\mathcal{H}(1+w)\theta^S$	$\delta + 3\mathcal{H}(1+w)\theta^S$
$\hat{\Theta}$	$3\mathcal{H}(1+w)\theta^S$	$3\mathcal{H}(1+w)\theta^S + 3(1+w)T$
$\delta\hat{P}$	δP	$\delta P + P'T$

new variables are gauge invariant except T, which we be important in the synchronous gauge. From this we can write (91) as

$$w_{\rm de}\Pi^S = c_{\Pi\Delta}\Delta + c_{\Pi\Theta}\hat{\Theta} + c_{\Pi X}X + c_{\Pi Y}K^2Y, \quad (97)$$

where

$$c_{\Pi\Delta} = \frac{c_{13}}{c_{14}}, \quad (98)$$

$$c_{\Pi\Theta} = \frac{c_{13}}{3c_{123}+2\alpha\gamma_2}\left[1 - 2\left(\epsilon_H\gamma_2 + \frac{c_{13}}{c_{14}}\right)\right], \quad (99)$$

$$c_{\Pi X} = \frac{2c_{13}\gamma_1(1+2\gamma_2)}{(2\gamma_1-1)(3c_{123}+2\alpha\gamma_2)}\left[2\left(\frac{c_{13}}{c_{14}} + \epsilon_H\gamma_2\right) - 1\right], \quad (100)$$

$$c_{\Pi Y} = \frac{2c_{13}\gamma_1}{3\alpha(1-2\gamma_1)}. \quad (101)$$

In a similar fashion, we can eliminate V and V' in δP and hence write the entropy perturbation as

$$w_{\rm de}\Gamma = c_{\Gamma\Delta}\Delta + c_{\Gamma\Theta}\hat{\Theta} + c_{\Gamma W}W + c_{\Gamma X}X + c_{\Gamma Y}K^2Y, \quad (102)$$

where

$$c_{\Gamma\Delta} = \frac{\alpha(1+2\gamma_2)}{3c_{14}} - \frac{dP}{d\rho}, \quad (103)$$

$$c_{\Gamma\Theta} = \frac{\alpha}{3(3c_{123}+2\alpha\gamma_2)}\left[\left(1 - \frac{2c_{13}}{c_{14}}\right)(1+2\gamma_2) - 6\epsilon_H\gamma_2\left(1 + \frac{2}{3}\gamma_3\right)\right] + \frac{dP}{d\rho}, \quad (104)$$

$$c_{\Gamma W} = \frac{2\gamma_1(1+2\gamma_2)}{3(2\gamma_1-1)}, \quad (105)$$

$$c_{\Gamma X} = \frac{4\alpha\gamma_1}{3(2\gamma_1-1)(3c_{123}+2\alpha\gamma_2)}\left[\left(1 + \frac{c_{13}}{c_{14}}\right)(1+2\gamma_2)^2 + \frac{3c_{13}}{\alpha}\left(1 + 2\gamma_2\left[1 - \epsilon_H\left(1 + \frac{2}{3}\gamma_3\right)\right]\right)\right], \quad (106)$$

$$c_{\Gamma Y} = \frac{2\gamma_1(1+2\gamma_2)}{9(1-2\gamma_1)}. \quad (107)$$

Note that in (97) and (102) the perturbed fluid variables are those for the dark energy fluid.

In order to ensure these results are truly gauge invariant, we must do the same calculation in the synchronous gauge. However, as mentioned previously, we now have an extra variable, T, to deal with. Therefore, let us suppose that in the synchronous gauge we find that

$$w_{\rm de}\Pi^S = c_{\Pi\Delta}\Delta + c_{\Pi\Theta}\hat{\Theta} + c_{\Pi X}X + c_{\Pi Y}K^2Y + c_{\Pi T}T, \quad (108)$$

$$w_{\rm de}\Gamma = c_{\Gamma\Delta}\Delta + c_{\Gamma\Theta}\hat{\Theta} + c_{\Gamma W}W + c_{\Gamma X}X + c_{\Gamma Y}K^2Y + c_{\Gamma T}T, \quad (109)$$

with $c_{\Pi T}, c_{\Gamma T} \neq 0$. If this was the case, Π^S and Γ would not be gauge invariant due to the presence of T, and so it must be that $c_{\Pi T} = c_{\Gamma T} = 0$. Note that this was not necessary in the conformal Newtonian gauge as $T = 0$ from Table II. We also require that in both gauges the coefficients are identical i.e. $c_{\Pi,\Gamma}^{\rm CN} = c_{\Pi,\Gamma}^{\rm Sync}$, because $\Delta, \hat{\Theta}, W, X$, and Y are gauge invariant. Indeed, doing this calculation in the synchronous gauge we find that this is the case, and hence (97) and (102) constitute the gauge invariant equations of state for the perturbations and are both presented simultaneously in the conformal Newtonian and synchronous gauges via Table II. For details of this calculation in the synchronous gauge see Appendix A.

To ensure that no coefficient diverges we require that α, c_{14}, γ_1, $2\gamma_1 - 1$, and $3c_{123} + 2\alpha\gamma_2$ do not equal zero. If $\alpha = 0$ then $\mathcal{K} = 0$, removing the dynamics of Einstein-Aether completely, and so this must be excluded. As we will see later, to prevent a diverging sound speed for perturbations we must have that $c_{14} \neq 0$ from (119). The solution for $\gamma_1 = 0$ is constant $\mathcal{F}$, which is just the case of a cosmological constant with no Einstein-Aether and therefore has no perturbations, while setting $2\gamma_1 - 1 = 0$ yields $\rho_{\rm m} = 0$ from the Friedmann equation. The case for disallowing $3c_{123} + 2\alpha\gamma_2 = 0$ is more subtle. If this was true it would set the coefficient of k^2V in (83) to zero, and hence the activation matrix would be singular, i.e. we would be unable to eliminate the degrees of freedom V and V' from our equations using θ^S. However, we note that this is not a

strict condition and could in principle be true for some models as there is nothing that physically prevents this. For the designer $\mathcal{F}(\mathcal{K})$ in (31) this is nonzero and so all the $c_{\Pi,\Gamma}$ coefficients are well behaved.

Additionally, we can eliminate the metric perturbations in favor of the perturbed fluid variables for matter and dark energy as done in [18] for the Horndeski theory. This will allow us to write (97) and (102) as

$$w_{\rm de}\Pi^S_{\rm de} = c_{\Pi\Delta_{\rm de}}\Delta_{\rm de} + c_{\Pi\Theta_{\rm de}}\hat{\Theta}_{\rm de} + c_{\Pi\Delta_{\rm m}}\Delta_{\rm m} + c_{\Pi\Theta_{\rm m}}\hat{\Theta}_{\rm m} + c_{\Pi\Pi_{\rm m}}\Pi^S_{\rm m}, \tag{110}$$

$$w_{\rm de}\Gamma_{\rm de} = c_{\Gamma\Delta_{\rm de}}\Delta_{\rm de} + c_{\Gamma\Theta_{\rm de}}\hat{\Theta}_{\rm de} + c_{\Gamma\Delta_{\rm m}}\Delta_{\rm m} + c_{\Gamma\Theta_{\rm m}}\hat{\Theta}_{\rm m} + c_{\Gamma\Gamma_{\rm m}}\Gamma_{\rm m}, \tag{111}$$

where we now make explicit distinction between the perturbed fluid variables for matter and dark energy. In the notation of Table II, the perturbed Einstein equations take the form [53]

$$2W = \Omega_{\rm m}\left(\frac{3\delta\hat{P}_{\rm m}}{\rho_{\rm m}} + 2w_{\rm m}\Pi^S_{\rm m} - 3\hat{\Theta}_{\rm m}\right) + \Omega_{\rm de}\left(\frac{3\delta\hat{P}_{\rm de}}{\rho_{\rm de}} + 2w_{\rm de}\Pi^S_{\rm de} - 3\hat{\Theta}_{\rm de}\right), \tag{112}$$

$$2X = \Omega_{\rm m}\hat{\Theta}_{\rm m} + \Omega_{\rm de}\hat{\Theta}_{\rm de}, \tag{113}$$

$$-\frac{2}{3}K^2Y = \Omega_{\rm m}(\Delta_{\rm m} - 2w_{\rm m}\Pi^S_{\rm m}) + \Omega_{\rm de}(\Delta_{\rm de} - 2w_{\rm de}\Pi^S_{\rm de}), \tag{114}$$

$$-\frac{2}{3}K^2Z = \Omega_{\rm m}\Delta_{\rm m} + \Omega_{\rm de}\Delta_{\rm de}. \tag{115}$$

Substituting for these in (97) yields

$$(1 - 3c_{\Pi Y}\Omega_{\rm de})w_{\rm de}\Pi^S_{\rm de} = \left(c_{\Pi\Delta} - \frac{3}{2}c_{\Pi Y}\Omega_{\rm de}\right)\Delta_{\rm de} + \left(c_{\Pi\Theta} + \frac{1}{2}c_{\Pi X}\Omega_{\rm de}\right)\hat{\Theta}_{\rm de} - \frac{3}{2}c_{\Pi Y}\Omega_{\rm m}\Delta_{\rm m} + \frac{1}{2}c_{\Pi X}\Omega_{\rm m}\hat{\Theta}_{\rm m} + 3c_{\Pi Y}\Omega_{\rm m}w_{\rm m}\Pi^S_{\rm m}. \tag{116}$$

Similarly, the entropy perturbation becomes

$$\left(1 - \frac{3}{2}c_{\Gamma W}\Omega_{\rm de}\right)w_{\rm de}\Gamma_{\rm de} = \left(c_{\Gamma\Delta} + \frac{3}{2}c_{\Gamma W}\Omega_{\rm de}\frac{dP}{d\rho}\bigg|_{\rm de} - \frac{3}{2}c_{\Gamma Y}\Omega_{\rm de}\right)\Delta_{\rm de} + \frac{3}{2}\Omega_{\rm m}\left(c_{\Gamma W}\frac{dP}{d\rho}\bigg|_{\rm m} - c_{\Gamma Y}\right)\Delta_{\rm m} + \left[c_{\Gamma\Theta} - \frac{3}{2}c_{\Gamma W}\Omega_{\rm de}\left(1 + \frac{dP}{d\rho}\bigg|_{\rm de}\right) + \frac{1}{2}c_{\Gamma X}\Omega_{\rm de}\right]\hat{\Theta}_{\rm de} + \frac{1}{2}\left[c_{\Gamma X} - 3c_{\Gamma W}\left(1 + \frac{dP}{d\rho}\bigg|_{\rm m}\right)\right]\Omega_{\rm m}\hat{\Theta}_{\rm m} + \frac{3}{2}c_{\Gamma W}\Omega_{\rm m}w_{\rm m}\Gamma_{\rm m}. \tag{117}$$

Note that (116) and (117) are completely general and not specific to generalized Einstein-Aether. If for any theory $w_{\rm de}\Pi^S$ and $w_{\rm de}\Gamma$ can be written as (97) and (102), then (116) and (117) will also be true automatically.

From these expressions we can derive the sound speed for scalar perturbations. Starting from the perturbed conservation equations, (1) and (2), we can deduce that

$$\delta'' + \cdots + k^2c_s^2\delta = F(h, \eta, \ldots). \tag{118}$$

Therefore, extracting the coefficient of $k^2\delta$ we find that

$$c_{\rm s}^2 = \frac{1}{c_{14}}\left(c_{123} + \frac{2}{3}\alpha\gamma_2\right). \tag{119}$$

In general, the sound speed of scalar perturbations varies with time due to $\mathcal{F}$. To ensure subluminal propagation and stable growth of perturbations, we require that $0 \le \frac{1}{c_{14}}(c_{123} + \frac{2}{3}\alpha\gamma_2) \le 1$.

From here, we could attempt to obtain constraints on the $\{c_i\}$ coefficients by appealing to the behavior of perturbations in the limit of Minkowski space, as in [64]. However, as we have directly coupled the evolution of $\mathcal{F}$ to $a(t)$ via a designer approach, we argue that no sensible Minkowski limit exists for this theory once this connection has been made. For a brief discussion of this see Appendix B. In the context of the equation of state approach, in the limit of $H \to 0$ we see that $\rho, P \to 0$ from (12) and (13). Therefore, the expressions for $w_{\rm de}\Pi^S$ and $w_{\rm de}\Gamma$ cannot be computed since $w_{\rm de}\Pi^S$ appears as $P\Pi^S$ from the perturbed energy momentum tensor (54) and $w_{\rm de}\Gamma$ can be written as $w_{\rm de}\rho\Gamma = (\frac{\delta P}{\delta\rho} - \frac{dP}{d\rho})\delta\rho$.

B. Special cases

1. $w_{\rm de} = -1$

Consider the case where we have exactly $w_{\rm de} = -1$, equivalent to ΛCDM. From Sec. II. 2 we have an analytical solution given by (31) and in this case the c_Π and c_Γ coefficients reduce to

$$c_{\Pi\Delta} = \frac{c_{13}}{c_{14}}, \qquad c_{\Pi\Theta} = \frac{1}{2}(1+\epsilon_H) - \frac{c_{13}}{c_{14}},$$
$$c_{\Pi X} = 0, \qquad c_{\Pi Y} = -\frac{c_{13}}{3\alpha}\left(1 + \frac{M^2\mathcal{F}_0}{6\Omega_{\mathrm{de},0}H_0^2}\right)\left(\frac{H}{H_0}\right), \tag{120}$$

and also

$$c_{\Gamma\Delta} = -c_{\Gamma\Theta} = -\frac{dP}{d\rho} = 1, \qquad c_{\Gamma W} = c_{\Gamma X} = c_{\Gamma Y} = 0, \tag{121}$$

and hence $\Gamma = \delta$. Here we see that from $c_{\Pi Y}$, as with the background evolution, M and $\mathcal{F}_0$ are degenerate.

This case is indistinguishable from ΛCDM at background order, but at the level of linear perturbations they are not the same. Therefore, geometrical cosmological tests such as SNe and BAOs would not be able to observe a difference between ΛCDM and generalized Einstein-Aether with $w_{\mathrm{de}} = -1$, whereas probes which are sensitive to perturbations, such as weak lensing, will be different and can in principle distinguish between them.

From (31) we note that the ΛCDM limit is when $\mathcal{F}_0 = -6H_0^2\Omega_{\mathrm{de},0}/M^2$ and so $\mathcal{F} = -6H_0^2\Omega_{\mathrm{de},0}/M^2$. This case corresponds to the cosmological constant in the Friedmann equation. Indeed, this also is reflected at the level of linear perturbations since $\mathcal{F}_{\mathcal{K}} = 0$, and so all the perturbed fluid variables and the equation of motion for V in Sec. III are zero, as in ΛCDM. However, it seems that there is a discontinuity in taking the limit of $\mathcal{F}_0 \to -6H_0^2\Omega_{\mathrm{de},0}/M^2$, since in this limit the $c_{\Pi,\Gamma}$ coefficients become

$$c_{\Pi\Delta} = \frac{c_{13}}{c_{14}}, \qquad c_{\Pi\Theta} = \frac{1}{2}(1+\epsilon_H) - \frac{c_{13}}{c_{14}},$$
$$c_{\Pi X} = c_{\Pi Y} = 0 \tag{122}$$

and

$$c_{\Gamma\Delta} = -c_{\Gamma\Theta} = 1, \qquad c_{\Gamma W} = c_{\Gamma X} = c_{\Gamma Y} = 0, \tag{123}$$

i.e. Π^S and Γ are nonzero in this limit, but are zero if $\mathcal{F}_0 = -6H_0^2\Omega_{\mathrm{de},0}/M^2$ exactly. This is a property shared by $f(R)$ models in the limit of $B_0 \to 0$.

2. Power law

For a general power law with $\mathcal{F} \propto (\pm\mathcal{K})^n$ as studied in [36–38], the coefficients become

$$c_{\Pi\Delta} = \frac{c_{13}}{c_{14}}, \tag{124}$$

$$c_{\Pi\Theta} = \frac{c_{13}}{(2n+1)\alpha - 6c_2}\left[1 - 2\left(\epsilon_H(n-1) - \frac{c_{13}}{c_{14}}\right)\right], \tag{125}$$

$$c_{\Pi X} = \frac{2nc_{13}}{(2n+1)\alpha - 6c_2}\left[\frac{2c_{13}}{c_{14}} - 1 + 2\epsilon_H(n-1)\right], \tag{126}$$

$$c_{\Pi Y} = \frac{2nc_{13}}{3\alpha(1-2n)}, \tag{127}$$

and

$$c_{\Gamma\Delta} = \frac{(2n-1)\alpha}{3c_{14}} - \frac{dP}{d\rho}, \tag{128}$$

$$c_{\Gamma\Theta} = \frac{(2n-1)\alpha}{3[(2n+1)\alpha - 6c_2]}\left[1 - 2\epsilon_H(n-1) - \frac{c_{13}}{c_{14}}\right] + \frac{dP}{d\rho}, \tag{129}$$

$$c_{\Gamma W} = \frac{2}{3}n, \tag{130}$$

$$c_{\Gamma X} = \frac{4n}{3[(2n+1)\alpha - 6c_2]}\left[\frac{\alpha(2n-1)(c_{13}+c_{14})}{c_{14}} + 3c_{13}\left(1 - \frac{2}{3}\epsilon_H(n-1)\right)\right], \tag{131}$$

$$c_{\Gamma Y} = -\frac{2}{9}n. \tag{132}$$

Note that $c_{\Pi Y}$ is singular for the case of $n = \frac{1}{2}$. Although $\mathcal{F} \propto (\pm\mathcal{K})^{1/2}$ is also a solution to (26), inserting this into the Friedmann equation (16) shows that this case corresponds to an absence of dark energy at the level of background cosmology.

C. Dynamics of linear perturbations in the scalar sector

The dynamics of scalar perturbations can be computed via the perturbed fluid equations in (1) and (2). We will use the designer $\mathcal{F}(\mathcal{K})$ model via (26). Following the notation of Table II we rewrite these equations as

$$\dot{\Delta} - 3w\Delta + g_{\mathrm{K}}\epsilon_H\hat{\Theta} - 2w\Pi^S = 3(1+w)X, \tag{133}$$

$$\dot{\hat{\Theta}} + 3\left(\frac{dP}{d\rho} - w + \frac{1}{3}\epsilon_H\right)\hat{\Theta} - 3\frac{dP}{d\rho}\Delta - 2w\Pi^S - 3w\Gamma = 3(1+w)Y, \tag{134}$$

where $g_{\mathrm{K}} = 1 + \frac{K^2}{3\epsilon_H}$ and, for this section only, overdots denote differentiation with respect to the logarithmic scale factor, $\log a$. For a cold, pressureless matter fluid with $w_{\mathrm{m}} = \Pi^S_{\mathrm{m}} = \Gamma_{\mathrm{m}} = 0$ and assuming w_{de} constant, (133) and (134) yield 4 differential equations for the dark energy and matter perturbed fluid variables, given by

$$\dot{\Delta}_{\mathrm{m}} + g_{\mathrm{K}}\epsilon_H\hat{\Theta}_{\mathrm{m}} = 3X, \tag{135}$$

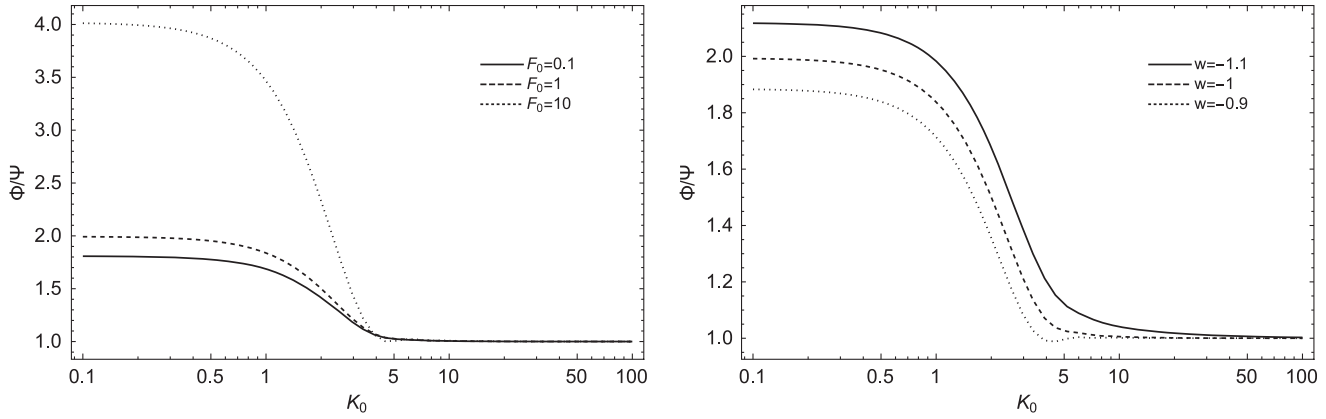

FIG. 2. *Left panel*: The spectrum of Φ/Ψ, or Z/Y, at $a=1$ as a function of scale for varying $\mathcal{F}_0$ and $w_{\rm de}=-1$. *Right panel*: The spectrum of Φ/Ψ at $a=1$ as a function of scale for a general Einstein-Aether fluid with $w_{\rm de}$ varying around -1 and $\mathcal{F}_0=1$.

$$\dot{\hat{\Theta}}_{\rm m}+\epsilon_H\hat{\Theta}_{\rm m}=3Y, \tag{136}$$

$$\dot{\Delta}_{\rm de}-3w_{\rm de}\Delta_{\rm de}+g_{\rm K}\epsilon_H\hat{\Theta}_{\rm de}-2w_{\rm de}\Pi^S_{\rm de}=3(1+w_{\rm de})X, \tag{137}$$

$$\dot{\hat{\Theta}}_{\rm de}+\epsilon_H\hat{\Theta}_{\rm de}-3w_{\rm de}\Delta_{\rm de}-2w_{\rm de}\Pi^S_{\rm de}-3w_{\rm de}\Gamma_{\rm de}=3(1+w_{\rm de})Y. \tag{138}$$

With these, the dynamics of the Newtonian gravitational potential, $Z=\Phi$, can be computed directly from the perturbed Einstein equation in (114) or via $\dot{Z}=X-Y$ from the definition of Z in Table II. Note that in Table II the variables and derivatives are in conformal time, not the scale factor. To solve these equations we will opt to specify $\Pi^S_{\rm de}$ and $\Gamma_{\rm de}$ terms of the perturbed fluid variables for dark energy and matter, given in (116) and (117). The variables X and Y are also specified in terms of the perturbed fluid variables via the perturbed Einstein equations in (112) and (113). We note that this is not the only way to proceed. For example, instead of the perturbed fluid variables we could have opted to work with the metric perturbation variables W, X, Y, and Z. For more details see [53]. We set the initial conditions as described in [53]. They are set at $z=100$ such that $\Delta_{\rm de}=\hat{\Theta}_{\rm de}=0$, $\Omega_{\rm m}\Delta_{\rm m}=-\frac{2}{3}K^2Z$, $\Omega_{\rm m}\hat{\Theta}_{\rm m}=2X$, and $X=Y=Z$. Since the behavior of the perturbations will also depend of the specific choice of $\{c_i\}$ and not just α, we will fix $c_1=1$, $c_2=1$, $c_3=1$, and $c_4=-3$. This choice is somewhat arbitrary, other than ensuring the subluminal propagation of the perturbations (119).

We investigate how the ratio of the Newtonian potentials vary with scale. From Fig. 2, we see that at $a=1$, the large scale behavior of Φ/Ψ is highly dependent on $\mathcal{F}_0$, while this is less so for $w_{\rm de}$ near -1. We see that Φ/Ψ tends to a constant in both the large and small K_0 regimes. In all cases the small scale behavior is such that $\Phi=\Psi$, and so this indicates a vanishing $w_{\rm de}\Pi^S_{\rm de}$ for small scales. Note that $K_0=1$ corresponds to a scale of $3.35\times10^{-4}\ h{\rm Mpc}^{-1}$.

In the regime $K\gg1$ we find that the $\{\hat{\Theta}_i\}$ are negligible, and so we can write $w_{\rm de}\Pi^S_{\rm de}\approx c_{\Pi\Delta_{\rm de}}\Delta_{\rm de}+c_{\Pi\Delta_{\rm m}}\Delta_{\rm m}$. From equations (135) to (138) we compute the second order differential equations for $\{\Delta_i\}$, given by

$$\ddot{\Delta}_{\rm m}+(2-\epsilon_H)\dot{\Delta}_{\rm m}-\frac{3}{2}\Omega_{\rm m}\Delta_{\rm m}=\frac{3}{2}\Omega_{\rm de}\Delta_{\rm de}, \tag{139}$$

$$\ddot{\Delta}_{\rm de}+(5-\epsilon_H)\dot{\Delta}_{\rm de}+\frac{2}{3}c_{\Pi\Delta_{\rm de}}K^2\Delta_{\rm de}=-\frac{2}{3}c_{\Pi\Delta_{\rm m}}K^2\Delta_{\rm m}, \tag{140}$$

where we have also used the Einstein equations for X (113) and Y (114), with $w_{\rm de}=-1$. Note that in (139) the secondary source term arising from $w_{\rm de}\Pi^S_{\rm de}$ is subdominant compared to $\Omega_{\rm de}\Delta_{\rm de}$, and so we have neglected this. From (113) and (114), for small scales we have that

$$\frac{Y}{Z}=1-\frac{2\Omega_{\rm de}(c_{\Pi\Delta_{\rm de}}\Delta_{\rm de}+c_{\Pi\Delta_{\rm m}}\Delta_{\rm m})}{\Omega_{\rm de}\Delta_{\rm de}+\Omega_{\rm m}\Delta_{\rm m}}, \tag{141}$$

where the second term must be negligible from Fig. 2. In order to explain this, note that from (140) we must have that the solution tends to the particular solution

$$c_{\Pi\Delta_{\rm de}}\Delta_{\rm de}=-c_{\Pi\Delta_{\rm m}}\Delta_{\rm m}. \tag{142}$$

Hence, the second term in (141) is always negligible regardless of what the $\{c_{\Pi\Delta_i}\}$ are. Therefore, a vanishing anisotropic stress at small scales is a generic feature of these designer $\mathcal{F}(\mathcal{K})$ models.

Using (142) in (139), we find that this becomes the standard differential equation for the matter overdensity with Newton's constant replaced with an effective Newton's constant, $G_{\rm eff}$, given by

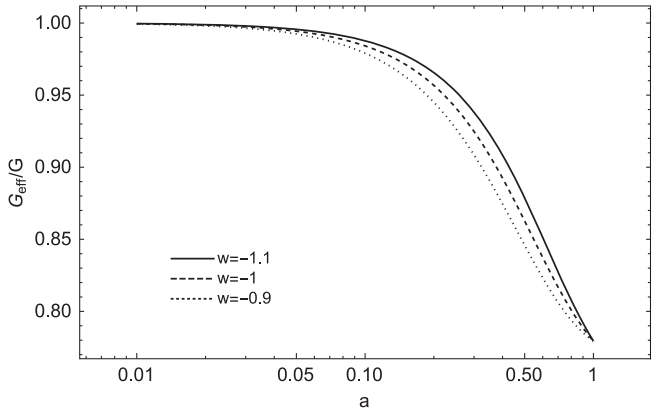

FIG. 3. The evolution of the effective Newton's constant, $G_{\rm eff}/G$, is shown for varying $w_{\rm de}$ around -1.

$$\frac{G_{\rm eff}}{G} = 1 - \frac{\Omega_{\rm de} c_{\Pi\Delta_{\rm m}}}{\Omega_{\rm m} c_{\Pi\Delta_{\rm de}}}, \tag{143}$$

and the evolution of this is shown in Fig. 3. We see that the ratio $G_{\rm eff}/G$ is always of order unity but that for our choice of $\{c_i\}$ it decreases to $G_{\rm eff} \approx 0.78G$ at $a = 1$, which should lead to a suppression of structure at late times compared to ΛCDM. We leave this as a matter for future investigation. We also observe that increasing $w_{\rm de}$ causes $G_{\rm eff}/G$ to decay faster at early times, while the opposite is true for decreasing $w_{\rm de}$. It is interesting to note that the value of $G_{\rm eff}/G$ for different $w_{\rm de}$ initially diverge and then converge again at $a = 1$. Note that what we have called $G_{\rm eff}$ is different to that in [37], for example, which is derived from the modified Poisson equation.

We also investigate the evolution for the Newtonian potential, Φ, as a function of a and this is shown in Fig. 4. We see that for a designer $\mathcal{F}(\mathcal{K})$ model which mimics a ΛCDM background the evolution is now sensitive to the scale, where $K_0 = k/H_0$, unlike the case of a cosmological constant. The amplitude of Φ grows with respect to ΛCDM for large scales, while for smaller scales the amplitude is suppressed. For scales $K_0 \lesssim 1$, we see that Φ initially grows before reaching a maximum and then decays due to the increasing contribution from dark energy. A similar feature was also observed in [37] for their power law model of $\mathcal{F}$.

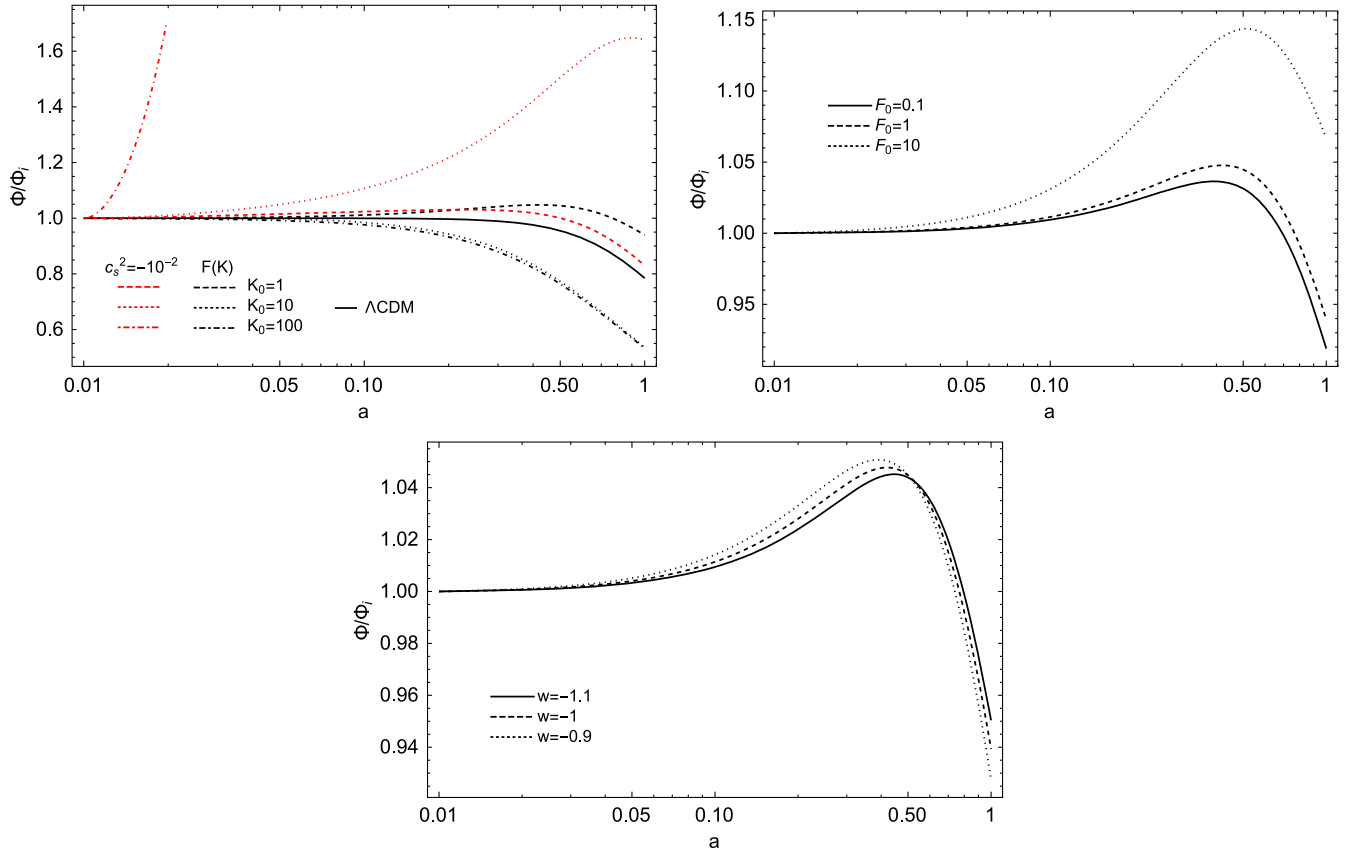

FIG. 4. *Top left panel*: The evolution of the Newtonian potential, Φ, in ΛCDM (black solid line) and for different scales in a designer $\mathcal{F}(\mathcal{K})$ model (dashed and dotted lines) for $\mathcal{F}_0 = 1$ and $w_{\rm de} = -1$. Note that the potential for the ΛCDM model is scale independent. For comparison we also show the evolution of Φ with the presence of a dark energy fluid with $w_{\rm de} = -1$ and a constant negative squared sound speed of $c_{\rm s}^2 = -10^{-2}$ (red lines), calculated using (144). *Top right panel*: The evolution of Φ in a generalized Einstein-Aether universe with varying $\mathcal{F}_0$ for $w_{\rm de} = -1$ and $K_0 = 1$ fixed. *Bottom panel*: The evolution of Φ for a general Einstein-Aether fluid with $w_{\rm de}$ varying around -1, with $\mathcal{F}_0 = 1$ and $K_0 = 1$ fixed.

We note that this is very similar to other models which introduce a new cosmological fluid with a negative squared sound speed, $c_s^2 = \delta P/\delta\rho$. We solve the differential equation governing the evolution of Φ [65,66]

$$\frac{d^2\Phi}{da^2} + \left(\frac{1}{\mathcal{H}}\frac{d\mathcal{H}}{da} + \frac{4}{a} + 3\frac{c_s^2}{a}\right)\frac{d\Phi}{da} + \left[\frac{2}{a\mathcal{H}}\frac{d\mathcal{H}}{da} + \frac{1}{a^2}(1+3c_s^2) + \frac{c_s^2k^2}{a^2\mathcal{H}^2}\right]\Phi = 0, \quad (144)$$

provided there is zero anisotropic stress and so $\Phi = \Psi$. In models where $c_s^2 < 0$ we observe the same behavior for Φ rising to a maximum before decaying, as seen in Fig. 4. In these models, the initial growth is due to an imaginary c_s^2 causing an unstable growth of perturbations. However, as dark energy begins to dominate Φ decays as in ΛCDM. This feature is enhanced for smaller scales until the effect of dark energy in unable to overcome the unstable growth of perturbations and Φ grows exponentially as seen for $K_0 = 100$ in Fig. 4. While a fluid with $c_s^2 < 0$ is unphysical, it is interesting to note that this feature appears in a designer $\mathcal{F}(\mathcal{K})$ universe without the need for $c_s^2 < 0$. Indeed, the $\{c_i\}$ coefficients were chosen to avoid this. Moreover, we see that for designer $\mathcal{F}(\mathcal{K})$ the opposite occurs compared to ΛCDM and that as we go to smaller scales this feature is suppressed rather than enhanced.

D. Vector and tensor sectors

We can also calculate the equation of state for the vector sector. In this case, the function we specify is $\Pi^V = \Pi^V(\theta^V)$. Since we only have one function, θ^V, to eliminate the vector degree of freedom, B^V, it may not seem possible as $\theta^V \equiv \theta^V(B^V, B^{V'}, B^{V''})$, as seen from (76). However, in a similar process to the scalar sector, we can use the perturbed equation of motion (79) to eliminate derivatives of B^V. In doing so, (76) becomes

$$a^2\rho(1+w_{de})\theta^V = \frac{1}{2}c_{13}\mathcal{F}_\mathcal{K}(kB^V - h^{V'}). \quad (145)$$

Inserting this into (77), we obtain the equation of state for perturbations in the vector sector as

$$w_{de}\Pi^V_{de} = [(1-3w_{de})(1+w_{de})\mathcal{H}]\theta^V_{de} + (1+w_{de})\theta^{V'}_{de}. \quad (146)$$

Note that this is exactly the same as the perturbed conservation equation and is, therefore, a tautology. To proceed we use the vector Einstein equations, given by

$$-\frac{1}{2a^2}h^{V'} = 8\pi G\rho_m(1+w_m)\theta^V_m + \rho_{de}(1+w_{de})\theta^V_{de}, \quad (147)$$

$$\frac{1}{6\mathcal{H}^2}h^{V''} + \frac{1}{3\mathcal{H}}h^{V'} = \Omega_m w_m\Pi_m + \Omega_{de}w_{de}\Pi_{de}. \quad (148)$$

Differentiating (147) and eliminating for $\theta^{V'}_{de}$ and the metric perturbations in (146), we find that

$$w_{de}\Pi^V_{de} = \mathcal{H}(1+w_{de})\theta^V_{de} + \frac{\Omega_m}{\Omega_{de}}[\mathcal{H}(1+w_m)\theta^V_m - w_m\Pi^V_m]. \quad (149)$$

For the tensor sector, since there are no new tensor degrees of freedom, Π^T can only be a function of h^T and its derivatives. Therefore, (78) immediately constitutes the equation of state for tensor perturbations and is given by

$$3\alpha\mathcal{H}^2\left(\mathcal{F}_\mathcal{K} - \frac{\mathcal{F}}{2\mathcal{K}}\right)w_{de}\Pi^T_{de} = -c_{13}\left(\mathcal{F}_\mathcal{K}\mathcal{H} + \frac{1}{2}\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\right)h^{T'} - \frac{1}{2}c_{13}\mathcal{F}_\mathcal{K}h^{T''}. \quad (150)$$

We can, therefore, derive the modification to the propagation speed of gravitational waves, due to the presence of the Aether field. Projecting out the tensor mode of the ij-component of the Einstein equation (8) yields

$$a^2(\hat{l}_i\hat{l}^j - \hat{m}_i\hat{m}^j)\delta G^i_j = h^{T''} + 2\mathcal{H}h^{T'} + k^2h^T = a^2(\hat{l}_i\hat{l}^j - \hat{m}_i\hat{m}^j)\delta U^i_j = 2a^2P\Pi^T, \quad (151)$$

assuming that the matter energy-momentum tensor contributes zero anisotropic stress. Hence, from (150) we find that

$$(1+c_{13}\mathcal{F}_\mathcal{K})h^{T''} + 2\left[\mathcal{H} + c_{13}\left(\mathcal{F}_\mathcal{K}\mathcal{H} + \frac{1}{2}\mathcal{F}_{\mathcal{K}\mathcal{K}}\mathcal{K}'\right)\right] \times h^{T'} + k^2h^T = 0, \quad (152)$$

and so gravitational waves propagate with speed

$$c^2_{grav} = \frac{1}{1+c_{13}\mathcal{F}_\mathcal{K}}. \quad (153)$$

We see that, in general, the propagation speed of gravitational waves is time dependent via $\mathcal{F}$. This is consistent with the result in [37]. It is often argued that on the grounds of causality that we should constrain $c_{grav} \leq 1$, as was said for the scalar perturbations (119). Indeed, this is the standard argument that was often made in previous work, for example see [64] and Appendix B. However, if gravitational waves were to propagate subluminally we would expect the existence of gravitational Cherenkov radiation, of which very stringent constraints have been placed [67]. See also [36] for a discussion. It was also noted in [67] that the constraint for $c_{grav} \geq 1$ were much weaker. Moreover, given that this is already a Lorentz violating theory it could be argued that

$c_{\rm grav} \geq 1$ may not be a problem; however we do not discuss this further.

V. DISCUSSION AND CONCLUSIONS

In this paper the background dynamics of generalized Einstein-Aether are studied using a designer approach. We find that only one form of $\mathcal{F}$ gives rise to a fluid species with $w_{\rm de} = -1$ exactly (18) for a "designer" $\mathcal{F}(\mathcal{K})$ model. However, we see that at the level of linear perturbations this model is not the same as ΛCDM. We obtain a differential equation for general values of constant $w_{\rm de}$ (26), which is solved numerically to see how this model behaves as we vary the parameters in the theory, shown in Fig. 1. We also find that the background evolution is independent of the choice of $\{c_i\}$. For $w_{\rm de} = -1$ there is an analytical solution for $\mathcal{F}$ given by (31).

We have also provided expressions for the perturbed fluid variables in generalized Einstein-Aether models, in the scalar, vector, and tensor sectors. These vector-tensor theories have noncanonical kinetic terms and are modified by a free function, $\mathcal{F}(\mathcal{K})$. While some work has been done on these theories, the c_4 term in (5) is often set to zero. It is often argued that this can be done via a redefinition of the coefficients, which is true only if the Aether field is hypersurface orthogonal i.e. as in the khronometric model (39). A consequence of this is that no transverse vector mode propagates at the level of linear perturbations. To keep things more general we keep the c_4 term in our analysis.

The EoS approach to cosmological perturbations provides a way of parametrizing dark energy models and modified gravity theories via the gauge invariant entropy perturbation and anisotropic stresses. This is done by fully eliminating the internal degrees of freedom introduced by this theory. In this paper, we have provided expressions for these in terms of linear functions of the perturbed variables and metric perturbations, $\Pi^S_{\rm de} = \Pi^S_{\rm de}(\Delta_{\rm de}, \hat{\Theta}_{\rm de}, X, Y)$ and $\Gamma_{\rm de} = \Gamma_{\rm de}(\Delta_{\rm de}, \hat{\Theta}_{\rm de}, W, X, Y)$, given in (97) and (102). They have been expressed in an explicitly gauge invariant form thanks to a new set of notation. Furthermore, via the Einstein equations, we are also able to specify them in terms of the perturbed fluid variables for dark energy and matter only i.e. $\Pi^S_{\rm de} = \Pi^S_{\rm de}(\Delta_{\rm de}, \Delta_{\rm m}, \hat{\Theta}_{\rm de}, \hat{\Theta}_{\rm m}, \Pi_{\rm m})$ and $\Gamma_{\rm de} = \Gamma_{\rm de}(\Delta_{\rm de}, \Delta_{\rm m}, \hat{\Theta}_{\rm de}, \hat{\Theta}_{\rm m}, \Gamma_{\rm m})$, given by (116) and (117). We note that there seems to be a discontinuity in taking the ΛCDM limit in a designer $\mathcal{F}(\mathcal{K})$ model. From these, we solve for the evolution of the Newtonian gravitational potentials via the perturbed fluid equations for varying parameters, shown in Figs. 2 and 4. In a designer $\mathcal{F}(\mathcal{K})$ we find that $w_{\rm de}\Pi^S_{\rm de} \to 0$ for $K \gg 1$, independent of the choice of $\{c_i\}$. We also provide expressions for $\Pi^{V,T}$ in the vector and tensor sectors, given by (149) and (150).

Of course, the motivation for this analysis is to obtain observables in cosmology and see how they compare to ΛCDM. We have now provided the necessary expressions in order to solve the perturbed fluid equations and obtain spectra. In principle, this should be easy to incorporate into existing numerical codes. Similar to [47–49], we would like to explore a broader class of vector-tensor models, without ever having to specify a specific model. What if we know nothing about the background Lagrangian other than its field content? Can anything be said more broadly about general vector-tensor theories of gravity and their application to dark energy? This is similar to work done in [50], but instead adopting a covariant approach as was done in [49] for scalar-tensor theories. We leave this as a matter for future work.

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APPENDIX A: EQUATIONS OF STATE FOR PERTURBATIONS IN THE SYNCHRONOUS GAUGE

In the synchronous gauge, we have that

$$a^2\delta\rho = \mathcal{F}_{\mathcal{K}}\Big[c_{14}k^2V' + (c_{14} - \alpha(1+2\gamma_2))\mathcal{H}k^2V + \frac{1}{2}\alpha\mathcal{H}(1+2\gamma_2)h'\Big], \tag{A1}$$

$$a^2\rho(1+w_{\rm de})\theta^S = \frac{1}{6}\mathcal{F}_{\mathcal{K}}[(2k^2V - h')(3c_{123} + 2\alpha\gamma_2) - 12c_{13}\eta'], \tag{A2}$$

where

$$\delta\mathcal{K} = \frac{2\alpha\mathcal{H}}{a^2M^2}\left(\frac{1}{2}h' - k^2V\right). \tag{A3}$$

We can then write this system of equations as

$$a^2\begin{pmatrix}\delta\rho\\ \rho(1+w_{\rm de})\theta^S\end{pmatrix} = k^2\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}\begin{pmatrix}V'\\ V\end{pmatrix} + \begin{pmatrix}D\\ E\end{pmatrix}, \tag{A4}$$

with

$$A = c_{14}\mathcal{F}_{\mathcal{K}}, \tag{A5}$$

$$B = [c_{14} - \alpha(1+2\gamma_2)]\mathcal{H}\mathcal{F}_{\mathcal{K}}, \tag{A6}$$

$$C = \frac{1}{3}\mathcal{F}_{\mathcal{K}}(3c_{123} + 2\alpha\gamma_2), \tag{A7}$$

$$D = \frac{1}{2}\alpha\mathcal{H}\mathcal{F}_{\mathcal{K}}(1 + 2\gamma_2)h', \tag{A8}$$

$$E = -\frac{1}{6}\mathcal{F}_{\mathcal{K}}[(3c_{123} + 2\alpha\gamma_2)h' + 12c_{13}\eta']. \tag{A9}$$

Inverting this will give us expressions for V and V' in terms of $\delta\rho$, θ^S, the metric perturbations, h and η, and their derivatives. Eliminating for these in Π^S (74), we find that we can write (91) as

$$w\Pi^S = c_{\Pi\Delta}\Delta + c_{\Pi\Theta}\hat{\Theta} + c_{\Pi X}X + c_{\Pi Y}K^2Y, \tag{A10}$$

where the c_Π coefficients are given in (98) to (101). In order to show this, we use the conservation equation (15) to find that

$$3(1 + w_{\rm de}) = \epsilon_H\frac{2\gamma_1(1 + 2\gamma_2)}{2\gamma_1 - 1} \tag{A11}$$

and replace for this in $3(1 + w)T$, arising from $\hat{\Theta}$ in Table II. From this it can be shown that the coefficient $c_{\Pi T} = 0$, as discussed previously.

Similarly, we do the same for the entropy perturbation by eliminating V and V' in δP and hence find that

$$w\Gamma = c_{\Gamma\Delta}\Delta + c_{\Gamma\Theta}\hat{\Theta} + c_{\Gamma W}W + c_{\Gamma X}X + c_{\Gamma Y}K^2Y, \tag{A12}$$

where the c_Γ coefficients are as before in (103) to (107). To show this, we note that there is a term proportional to $3(1 + w_{\rm de})\frac{dP}{d\rho}T$. As before, we use (A11) to replace $3(1 + w_{\rm de})$ and also compute that

$$\frac{dP}{d\rho} = \frac{a^2P'}{a^2\rho'} = \epsilon_H\left(\frac{2\gamma_2}{1 + 2\gamma_2}\right)\left(1 + \frac{2}{3}\gamma_3\right) + \frac{2}{3}\epsilon_H - 1 - \frac{\epsilon_H'}{3\mathcal{H}\epsilon_H}. \tag{A13}$$

After substituting in for these it can be shown that $c_{\Gamma T} = 0$. Hence, (97) and (102) constitute the gauge invariant equations of state for the perturbations.

APPENDIX B: CONSTRAINTS ON COEFFICIENTS IN MINKOWSKI SPACE

We would like to obtain constraints on the $\{c_i\}$ coefficients by studying the behavior of perturbations in Minkowski space. We largely follow the procedure defined in [64], extending their results to include $c_4 \neq 0$. The Lagrangian which governs the perturbations is obtained by perturbing the degrees of freedom in the background Lagrangian to quadratic order. This would then give rise to linear equations of motion for the perturbations. Schematically, we are computing $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} + \delta^2\mathcal{L}$, where $\delta^2\mathcal{L}$ denotes the Lagrangian quadratic in perturbations. Again suppressing overbars to denote unperturbed variables, from (3) we have that

$$\delta^2\mathcal{L} = M^2(\mathcal{F}_{\mathcal{K}\mathcal{K}}(\delta\mathcal{K})^2 + \mathcal{F}_{\mathcal{K}}\delta^2\mathcal{K}) + 2A^\mu\delta A_\mu\delta\lambda, \tag{B1}$$

since $\lambda = 0$ in Minkowski space.

Perturbing the Aether as $A^\mu \to A^\mu + \delta A^\mu = (1,0,0,0) + v^\mu$ and assuming the metric to be flat, we can compute $M^2\delta^2\mathcal{K}$ by perturbing the Aether field and expanding out to quadratic order, to give

$$M^2\delta^2\mathcal{K} = c_1\partial_\mu v^\nu\partial^\mu v_\nu + c_2(\partial_\mu v^\mu)^2 + c_3\partial_\mu v^\nu\partial_\nu v^\mu + c_4A^\mu A^\nu\partial_\mu v^\rho\partial_\nu v_\rho + 2\delta\lambda A^\mu v_\mu. \tag{B2}$$

Similarly we can calculate $M^2\delta K$ to be

$$\frac{1}{2}M^2\delta\mathcal{K} = c_1\partial_\mu A^\nu\partial^\mu v_\nu + c_2\partial_\mu A^\mu\partial_\nu v^\nu + c_3\partial_\mu A^\nu\partial_\nu v^\mu + c_4A^\mu v^\nu\partial_\mu A^\rho\partial_\nu A_\rho + c_4A^\mu A^\nu\partial_\mu A^\rho\partial_\nu v_\rho. \tag{B3}$$

From this we see that in Minkowski space $\delta\mathcal{K} = 0$ since $\partial_\mu A^\nu = 0$, which will also be true for the unperturbed value of $\mathcal{K}$. The second order Lagrangian is therefore given by

$$\delta^2\mathcal{L} = \mathcal{F}_{\mathcal{K}}[-c_{14}\dot{v}^2 + c_1\partial_i v^j\partial^i v_j + c_2(\partial_i v^i)^2 + c_3\partial_i v^j\partial_j v^i], \tag{B4}$$

where $\dot{v}^2 = \dot{v}^i\dot{v}_i$, and we have used $v^0 = 0$. By analogy to the cosmological perturbations, we decompose the perturbation into a scalar and vector part,

$$v^i = \partial^i V + iB^i = S^i + T^i, \tag{B5}$$

such that $k^iT_i = 0$. Inserting this into (B4), we find that we can write it as the sum of two uncoupled Lagrangians for the fields S^i and T^i, since any cross terms are zero by the scalar-vector decomposition of the perturbation. They are given by

$$\mathcal{L}_S = \mathcal{F}_{\mathcal{K}}[-c_{14}\dot{S}^2 + c_1\partial_i S^j\partial^i S_j + c_2(\partial_i S^i)^2 + c_3\partial_i S^j\partial_j S^i], \tag{B6}$$

$$\mathcal{L}_T = \mathcal{F}_{\mathcal{K}}[-c_{14}\dot{T}^2 + c_1\partial_i T^j\partial^i T_j]. \tag{B7}$$

Here we see the problem with the Minkowski limit for the designer model, with $\mathcal{F} = B(\mathcal{K})^{1/2} + C$. Since $\mathcal{K} \propto H^2$, in the Minkowski limit where $\mathcal{K} \to 0$ we have that $\mathcal{F}_{\mathcal{K}} \to \infty$, and hence the second order Lagrangian is not well-defined.

Constraints can still be obtained for the $\{c_i\}$ coefficients, but not for the designer model. To compare with results from [64,68] we will set $\mathcal{F}_{\mathcal{K}} = 1$.

Hence, the equations of motion from (B6) and (B7) are then given by

$$\ddot{S}_i - \frac{c_{123}}{c_{14}}\partial^j\partial_j S_i = 0, \qquad \ddot{T}_i - \frac{c_1}{c_{14}}\partial^j\partial_j T_i = 0, \tag{B8}$$

where we have used $\partial_i S_j = \partial_j S_i$ from the definition in (B5). Therefore, we see that S_i and T_i propagate with sound speeds $c_s^2 = \frac{c_{123}}{c_{14}}$ and $c_s^2 = \frac{c_1}{c_{14}}$ respectively. Imposing that the propagation speeds are less than c and to avoid them being imaginary, leading to an exponential growth in perturbations, we require

$$0 \leq \frac{c_{123}}{c_{14}} \leq 1 \quad \text{and} \quad 0 \leq \frac{c_1}{c_{14}} \leq 1. \tag{B9}$$

Also, following the process of [64], considerations of the quantum Hamiltonian gives an additional constraint of $c_{14} < 0$ to prevent ghosts. Heuristically we can see this from (B4), as $c_{14} < 0$ ensures that the kinetic term is the correct sign, however see [64] for a full treatment of the quantization of this theory.

TABLE III. Summary of the constraints on the $\{c_i\}$ coefficients, obtained from Minkowski space and gravitational waves.

Constraints	Reason
(a) $0 \leq \frac{c_{123}}{c_{14}} \leq 1$	Nontachyonic and subluminal propagation of scalar modes
(b) $0 \leq \frac{c_1}{c_{14}} \leq 1$	Nontachyonic and subluminal propagation of vector modes
(c) $c_{13} \geq 0$	Subluminal propagation of gravitational waves
(d) $c_{14} < 0$	No ghosts
(e) $c_{123} \leq 0$	(a) and (d)
(f) $c_1 \leq 0$ and $c_4 \geq 0$	(b) and (d)
(g) $c_2 \leq 0$	(c) and (e)
(h) $c_3 \geq 0$	(c) and (f)
(i) $\alpha \leq 0$	(e) and (g)

Let us summarize the constraints we have obtained. As in [64], we can also infer further constraints from those already obtained, allowing us to get more useful constraints on the individual coefficients and also combinations of them that appear frequently. They are shown in Table III and are also consistent with those obtained in [68].

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