Causal properties of nonlinear gravitational waves in modified gravity

Arthur George Suvorov[*]

School of Physics, University of Melbourne, Parkville, Victoria 3010, Australia

Andrew Melatos[†]

School of Physics, University of Melbourne, Parkville, Victoria 3010, Australia
and Australian Research Council Centre of Excellence for Gravitational Wave Discovery (OzGrav)

Some exact, nonlinear, vacuum gravitational wave solutions are derived for certain polynomial $f(R)$ gravities. We show that the boundaries of the gravitational domain of dependence, associated with events in polynomial $f(R)$ gravity, are not null as they are in general relativity. The implication is that electromagnetic and gravitational causality separate into distinct notions in modified gravity, which may have observable astrophysical consequences. The linear theory predicts that tachyonic instabilities occur, when the quadratic coefficient a_2 of the Taylor expansion of $f(R)$ is *negative*, while the exact, nonlinear, cylindrical wave solutions presented here can be superluminal for all values of a_2. Anisotropic solutions are found, whose wave fronts trace out time- or spacelike hypersurfaces with complicated geometric properties. We show that the solutions exist in $f(R)$ theories that are consistent with Solar System and pulsar timing experiments.

I. INTRODUCTION

When a field develops a localized perturbation, information about the disturbance is communicated at some finite speed to the surrounding universe. Gravitational waves (GWs) act as the energy and information transport mechanism for time-varying gravitational fields [1–3]. Within the theory of general relativity (GR), GWs propagate in the linearized regime outside the near zone in even the most compact relativistic sources (e.g. [4,5]). Linearization schemes are convenient because tools such as multipole expansions exist for calculating amplitudes and polarizations straightforwardly given a model of the source [6–8]. At least within GR, it is well known that the phase speed of GWs is precisely the speed of light[1] in both the linear and nonlinear theories [9,10]. The waves propagate along null hypersurfaces in vacuum and thus the notions of (Maxwellian) electromagnetic and general relativistic causality coincide.

Fundamental inconsistencies between quantum field theories and GR suggest that a quantum theory of gravity will modify the geometry-matter relations of GR [11]. Within bosonic string theories, for example, the quantization of the Polyakov action introduces scalar potentials (graviton-dilaton couplings) into the Einstein action which modifies the gravitational dynamics [12,13]. Transforming into the Jordan frame shows that these dilaton-tensor theories behave like higher-order curvature theories [such as the $f(R)$ theories

considered in this paper; see below], and that GR correction terms are large when the curvatures are large [14,15]. Classically speaking, therefore, GWs in string-inspired or other gravity theories may propagate differently to their GR counterparts in the vicinity of strong sources or elsewhere [16,17]. In particular, the wave fronts may trace out hypersurfaces, which are not null, indicating that notions of causality may differ between electromagnetic and gravitational events in modified theories of gravity. Theories with massive gravitons, for example, predict that the wave fronts are frequency dependent, propagate slower than light, and trace out timelike hypersurfaces [18]. A modification of the phase speed represents the simplest kind of topological adjustment that can occur in the causal structure [19]. Other, exotic kinds of topological structures can also occur in wave fronts in $f(R)$ gravity. For example, there exist choices of f such that the gravitational past and future of some event can have a nonempty intersection, thereby violating chronology protection [20,21].

The linearized $f(R)$ theory predicts an exact dispersion relation for GWs [22]. However, dispersion relations in linear and nonlinear theories can have very different physical characters. Consider a scalar field theory whose equation of motion reads

$$0 = \phi_{,tt} - \mathbf{\nabla}^2\phi + V'(\phi), \tag{1}$$

with scalar field ϕ and potential function V. Linearization of Eq. (1) returns either the Klein-Gordon or massless wave equation depending on the coefficient of the linear term in $V'(\phi)$. Both the Klein-Gordon and massless wave equations admit propagating solutions with fixed propagation speeds

---

[*]suvorova@student.unimelb.edu.au
[†]amelatos@unimelb.edu.au
[1]We adopt natural units throughout with $G = c = 1$, although the constant c is occasionally written explicitly for emphasis.

064032-1

(e.g. [23]). However, depending on the form of V, the nonlinear dispersion relation can be modified by self-interaction [24]. For example, there exist choices of V such that Eq. (1) admits solitonlike solutions with arbitrary phase speeds (such as $V \propto \phi^3$), while other choices of V preserve the Klein-Gordon character of the dispersion relation (such as $V \propto \cos \phi$) [25–28]. Given the well-studied equivalence between $f(R)$ and scalar-tensor theories of gravity, it is reasonable to expect a similar phenomenon to occur in $f(R)$ gravity depending on the particulars of the function f [14]. Hence, one must be careful when drawing conclusions about nonlinear GWs from analysis of the corresponding linearized field equations [29,30]. This phenomenon is related to the Vainshtein mechanism [31].

The purpose of this short paper is to demonstrate, by explicit construction, some topological properties of non-linear GWs in $f(R)$ theories of gravity. We show that the predictions offered by the linear and nonlinear theories may differ significantly. In Sec. II we define some general notions of causality that are used throughout the paper. In Sec. III we present the $f(R)$ field equations and recall some results concerning phase speeds of GWs in the linear theory. In Sec. IV we show, by constructing two exact solutions, that these relations may fail to describe the propagation speed of nonlinear GWs, that exotic topological properties can occur in the GW front defining the causal structure, and that the theories considered are consistent with Solar System and pulsar timing constraints. However, the analytic solutions exhibit certain artificial properties, which are likely to be avoided in more general, numerical solutions, a topic for future work. Some brief, additional discussion regarding $f(R)$ theories and causality is presented in Sec. V.

II. CAUSALITY IN MODIFIED GRAVITY

In any physical theory where information propagates at a finite speed, a notion of causality emerges. Given an event E_1, a second event E_2 is causally connected through electromagnetic signals to E_1 provided that it lies within the null cone originating at E_1. The same two events are causally connected gravitationally, if there exists a curve joining E_1 and E_2 that is contained within the *domain of dependence*, defined by the hypersurface traced out by the GW fronts emanating from E_1 (see e.g. Hawking and Ellis [10] for formal definitions). In vacuum GR, the domain of dependence coincides exactly with the null cone for any event, and an unambiguous notion of causality emerges. The domain of dependence, however, depends on the structure of the field equations (since it depends on the properties of GWs) and need not coincide with the null cones in modified gravity. Throughout this work we use the phrase "causal" to refer to gravitational causality unless otherwise stated.

Consider a universe where GW fronts propagate isotropi-cally with phase speed v in vacuum, and suppose some perturbation event occurs at P. Figure 1 illustrates three kinds of causal connection that can occur in such a
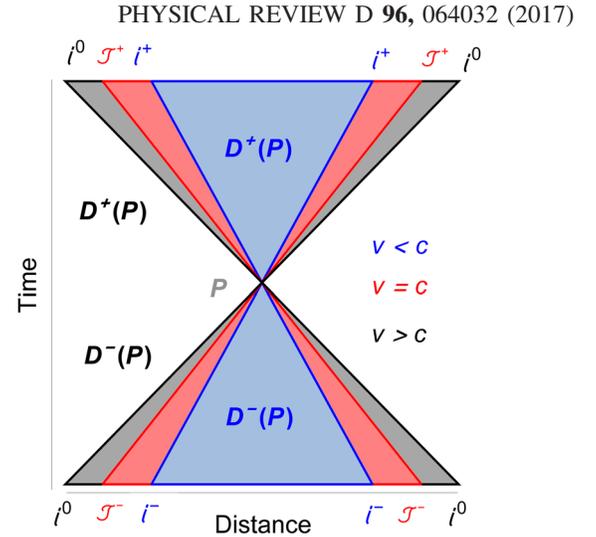

FIG. 1. Cross section of the causal domains for an event P for a universe where GWs propagate at speed v. The shaded blue region represents the causal domain of P, when GWs are subluminal ($v < c$). For $v < c$, the domain $D(P) = D^+(P) \cup D^-(P)$ is strictly contained within the null cone ($v = c$) represented by the union of the red region, which extends to future (past) null infinity $\mathcal{J}^+$ ($\mathcal{J}^-$), and the blue region, which extends to future (past) timelike infinity i^+ (i^-). The causal domain for superluminal GWs ($v > c$) is represented by the union of the gray, red, and blue regions and extends to spacelike infinity i^0. Note that, although drawn as conical structures here for simplicity, the global shape of $D(P)$ will be warped by the metric coefficients in general (see Sec. 12.6 of [33]).

universe. The domain of events which could be influenced by (influence) P is known as the future (past) domain of dependence and is denoted by $D^+(P)$ [$D^-(P)$]. The set $D(P) = D^+(P) \cup D^-(P)$ represents the causal domain of the event P. For $v < c$, there exist observers in Lorentz-boosted frames who see the wave travel at nonzero speeds less than c. The domain $D^+(P)$ [$D^-(P)$] extends to future (past) timelike infinity i^+ (i^-). In such a universe, events exist that are electromagnetically but not gravitationally connected, i.e. events which lie within the null cone originating at P but not in $D(P)$. For $v = c$, gravitational events are seen at the same time as electromagnetic ones by all observers, and the domain $D^+(P)$ [$D^-(P)$] extends to future (past) null infinity $\mathcal{J}^+$ ($\mathcal{J}^-$). This is the case in vacuum GR. If GWs are superluminal with speeds $v > c$, boosted-frame observers exist who see the waves travel at arbitrarily high speeds, the domains $D^\pm(P)$ extend to spacelike infinity i^0, and events exist that are gravitationally but not electromagnetically connected. See e.g. Refs. [25,32] for a discussion on physical consequences.

Figure 2 represents the causal domain of the event P in a universe where GW fronts no longer trace out two cones (future and past) joined at P but rather some other topological surface; i.e. GW propagation is not isotropic. Note that in this particular illustration we have that $D^+(P) \cap D^-(P) = \varnothing$. There is no reason to assume *a priori*
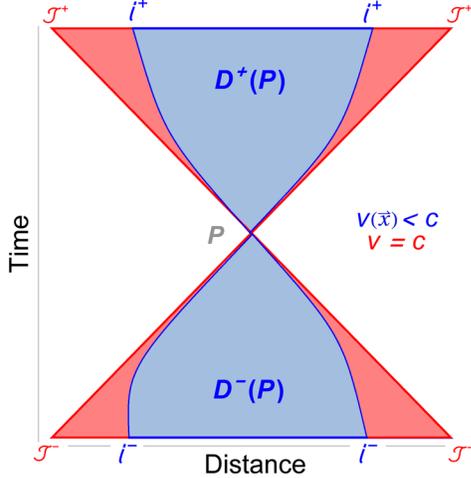
FIG. 2.   Cross section of the causal domain for an event P within a universe where GW fronts propagate anisotropically and subluminally. The domain $D(P) = D^+(P) \cup D^-(P)$ has some nontrivial topological structure; i.e $\partial D(P)$ is complicated. In this particular example, $D(P)$ is completely confined within the null cone, and as such extends to future (past) timelike infinity i^+ (i^-).

that this holds for general theories of gravity; i.e. closed timelike curves can exist in $D^+(P) \cap D^-(P)$ in general.[2]

Below we show that, in $f(R)$ gravity, both of the situations depicted in Figs. 1 and 2 may occur for nonlinear GWs.

III. LINEAR WAVES

A. Field equations

In an $f(R)$ theory of gravity, the Ricci scalar R is replaced by an arbitrary function of this quantity, $f(R)$, in the Einstein-Hilbert action. The vacuum field equations read (e.g. [14])

$$0 = f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f'(R), \quad (2)$$

where $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, and $\Box = \nabla_\mu\nabla^\mu$ symbolizes the d'Alembert operator.

B. Linear theory

Following Berry and Gair [22] we consider f to be an analytic function about $R = 0$ so that it can be expressed as a power series:

$$f(R) = a_0 + a_1 R + \frac{a_2}{2!}R^2 + \frac{a_3}{3!}R^3 + \cdots, \quad (3)$$

where the a_i are the Maclaurin coefficients [11,14]. We set $a_0 = 0$ to expand about a Minkowski background, though some of the results carry over to other backgrounds as well, e.g. (anti–)de Sitter. Perturbing the metric according to

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (4)$$

we find (2) reduces to

$$0 = \Box\bar{h}_{\mu\nu} \quad (5)$$

to linear order. We introduce the trace-reversed potential

$$\bar{h}_{\mu\nu} = a_1\left(h_{\mu\nu} - \frac{1}{2}h_{\sigma\rho}\eta^{\sigma\rho}\eta_{\mu\nu}\right) - a_2 R^{(1)}\eta_{\mu\nu}, \quad (6)$$

enforce the generalized de Donder gauge $\nabla^\mu\bar{h}_{\mu\nu} = 0$, and write the linearized Ricci scalar [to order $\mathcal{O}(h)$] as $R^{(1)}$. Equation (5) implies the existence of two tensor polarization modes for general $f(R)$ theories [22], just like in GR.

The trace of (2) shows that the linearized Ricci scalar satisfies a Klein-Gordon equation of the form

$$0 = 3a_2\Box R^{(1)} - a_1 R^{(1)}, \quad (7)$$

indicating that there is also a propagating scalar mode for $a_2 \neq 0$ (massive for $a_1 \neq 0$) in addition to the two tensor modes of GR [34]. An important feature, for our purposes, is that Eq. (7) predicts the existence of scalar modes with group velocity[3] $(a_2 \neq 0)$

$$c_g = \frac{\sqrt{\omega^2 - a_1(3a_2)^{-1}}}{\omega}, \quad (8)$$

where ω is the wave frequency [22,23]. As such, the velocity of a linear GW in $f(R)$ gravity is uniquely determined by the value of the coefficients a_2 and a_1 given (3). In particular, expression (8) demands $a_1 a_2 > 0$ to ensure $c_g < 1$, so that tachyonic instabilities are avoided [37,38].

IV. EXACT SOLUTIONS

In this section we construct three explicit examples of nonlinear wave solutions to (2). In Sec. IV A, we present a class of solutions which admit an arbitrary phase speed independent of the value of a_2, the situation depicted in Fig. 1. We also derive a class of generalized Peres waves which propagate anisotropically in Sec. IV B, the situation depicted in Fig. 2. In both cases, we work with the function

$$f(R) = R + \frac{a_2}{2!}R^2 + \frac{a_3}{3!}R^3 + \frac{a_k}{\Gamma(k+1)}R^k, \quad (9)$$

---

[2]This scenario can also occur in GR for universes filled with exotic matter, e.g. the Gödel solution [1]. In this work we consider vacuum spacetimes only.

[3]Note that there is a misplaced minus sign in Eq. (30) in Ref. [22]; Berry and Gair's no-tachyon condition should read $\Upsilon^2 < 0$ to be consistent with the usual Starobinsky $[f'(R) > 0]$ and Dolgov-Kawasaki $[f''(R) \geq 0]$ conditions [35,36].

where $\Gamma(\xi) = \int_0^\infty d\tau \tau^{\xi-1} e^{-\tau}$ is the usual gamma function, k is an integer greater than 3, and at least two of the a_2, a_3, and a_k are nonzero.

Functions of the form (9) have been considered in the literature as geometric models of dark energy (e.g. [39]). In this context, the parameters appearing in (9) have been constrained through Solar System experiments [40], supernova Ia luminosity distance data [41], and stochastic gravitational wave background limits [42]. While in dark energy models one typically sets $0 \le k < 1$, the Maclaurin expansion (3) does not exist for k in this range, since $f'(0)$ diverges and feeds into Eq. (2). Hence, we consider k to be a positive integer here. In fact the exact solutions presented below exist formally for all real k (though not for any a_k), so some models may admit tachyonic gravitational waves, even when the theory cannot be linearized about a Minkowski background. A discussion of astrophysical constraints on theories of the form (9) is presented in Sec. IV C.

A. Arbitrary phase speed

We construct an exact solution that is cylindrically symmetric. Such solutions can be described by the Jordan-Ehlers-Kompaneets line element in Weyl coordinates (t, ρ, ϕ, z) [43]:

$$ds^2 = e^{-2\psi}[e^{2\gamma}(-dt^2 + d\rho^2) + \rho^2 d\phi^2] + e^{2\psi}dz^2, \quad (10)$$

where ψ and γ are functions of t and ρ [1,29]. In GR, vacuum GW solutions, represented by (10) or otherwise, must necessarily have unit propagation speed (see Theorem 8.8 of [33]). Many exact, cylindrical GW solutions are known [44–47]. While noncylindrical GWs exist (e.g. in Ref. [48] or any multipole with nonzero azimuthal wave number; see also below), cylindrical GWs suffice to demonstrate the points considered here.

Consider the metric (10) for the choices

$$\psi = 0 \quad (11)$$

and

$$\gamma = \frac{1}{2}\ln\{A(1-v^2)\operatorname{csch}[\delta + \omega(t - v\rho)]^2\}, \quad (12)$$

where $\operatorname{csch}(\xi) = 2/(e^\xi - e^{-\xi})$ is the hyperbolic cosecant function (which is singular at $\xi = 0$), ω represents a frequency, $A \ne 0$ is an amplitude factor,[4] δ represents a phase shift, and $v \ne 1$ is a phase velocity. In the zero-frequency limit $\omega \to 0$ we recover the Minkowski spacetime. It can be verified by direct computation that the metric given through (11) and (12) is a solution to (2) for f given by (9), provided that (i) the wave satisfies an

amplitude-frequency relation, as for any nonlinear wave [49], of the form $(a_3 \ne 0)$

$$\frac{\omega^2}{A} = \frac{\alpha - 3a_2(k-2)}{4a_3(k-3)}; \quad (13)$$

and (ii) the coefficient a_k is given by

$$a_k = \frac{2^{k-2}\Gamma(k+1)[\alpha - 3a_2(k-2)]^{-k}}{a_3^{2-k}(k-3)^{3-k}}$$
$$\times [4a_3\alpha(k-3) - 3a_2^2\alpha(k-2)$$
$$+ 9a_2^3(k-2)^2 - 12a_2a_3(k-3)(2k-3)], \quad (14)$$

where

$$\alpha = [9a_2^2(k-2)^2 - 24a_3(k-3)(k-1)]^{1/2}. \quad (15)$$

The parameter v, which takes any value except unity,[5] is the phase speed of the solitonic GW described by (11)–(14). Therefore, tachyonic GWs may exist regardless of the sign or value of a_2, contrary to the prediction (8) from the linear theory outlined in Sec III B. Furthermore, the metric is discontinuous for $\delta = 0$ in Weyl coordinates along the curve $t = v\rho$. The causal domain for an event occurring at the origin, which emits GWs described by (11) and (12), is represented by Fig. 1 except that the case $v = c$ is not permitted. In particular, both sub- and superluminal nonlinear modes exist regardless of the value of a_2. To the authors' knowledge, the metric given by (11) and (12) is reported here for the first time.

It should be noted that for an arbitrary value of k, the solution given by (11) and (12) only exists in the special case, where a_k is given by (14). There is no reason *a priori* to favor or disfavor theories that satisfy (14). The main purpose of the solution is to demonstrate that the linear criterion $a_2 > 0$ does not guarantee the absence of tachyonic GWs. Incidentally, we show in Sec. IV C that the constraint (14) is consistent with various astrophysical tests for a variety of values of a_2.

As a side remark, in GR, it is well known that the gravitational collapse of stars with mass beyond the Tolman-Oppenheimer-Volkoff limit strips away information concerning the collapsing stellar remnant due to the no-hair theorems [51,52]. Information is removed by the formation of horizons, which causally separate regions within the spacetime [53]. In $f(R)$ theories of the form (9), which permit the existence of superluminal GWs, it is possible that gravitational information can leak beyond the electromagnetic event horizons, which traditionally define black hole boundaries [54] (cf. [55]).

---

[4]Note that for $v > 1$ $(v < 1)$ we require $A < 0$ $(A > 0)$ to ensure that the metric (10) has a Lorentzian signature.

[5]The metric given by (11) and (12) is genuinely singular for $v = 1$ since the Kretschmann invariant $\mathcal{K} = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ diverges there [50].

B. Anisotropic propagation

We show that arbitrary domains of dependence exist in theories of the form (9) by considering a class of GWs that are not cylindrically symmetric. We consider a class of generalized Peres waves, which are described by the line element [56],

$$ds^2 = U(\rho)(-dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2) + 2\lambda(\rho, \phi, z + t)(dt + dz)^2 \quad (16)$$

in Weyl coordinates for some functions U and λ. In GR, the Peres waves (16) are defined with $U = 1$ and represent a subclass of the well-studied pp waves [1]. A Peres wave represents a GW whose source is electromagnetic in origin. A perturbation of the Faraday tensor in some region of spacetime defines initial conditions, which induce GWs of the form (16) [57].

In GR, the Einstein equations reduce to the requirement that λ be harmonic in ρ and ϕ, i.e.

$$0 = \lambda_{,\rho\rho} + \rho^{-1}\lambda_{,\rho} + \rho^{-2}\lambda_{,\phi\phi}. \quad (17)$$

However, for certain special choices of a_k and U, the metric (16) is an exact solution to the field equations (2) for *any* function λ. Explicitly, we find that the metric (16) is a solution to (2) for

$$U(\rho) = \frac{A}{\omega^2 \rho^2}, \quad (18)$$

$$\frac{A}{\omega^2} = \frac{\alpha + 3a_2(k-2)}{2(k-1)}, \quad (19)$$

and

$$a_k = \frac{2^{3-2k}3^{1-k}\Gamma(k+1)(\alpha - 3a_2)[\alpha + 3a_2(k-2)]^{k-2}}{(1-k)^{k-1}(k-3)}, \quad (20)$$

where A and ω are constants, the parameter α is defined through (15), a_2 and a_3 are arbitrary, and the function λ is arbitrary. To the authors' knowledge, the metric given by (16) with (18)–(20) is reported here for the first time.

The domain of dependence associated with a generalized Peres wave is arbitrary, because the function λ is arbitrary. Consider, for example, the case

$$\lambda = \begin{cases} \exp\{-A[(t+z)^2 - u(\rho)^2]^{-1}\} & \text{for } (t+z)^2 > u(\rho)^2, \\ 0 & \text{otherwise,} \end{cases} \quad (21)$$

for some function u and amplitude $A > 0$. For $z = 0$, the function λ tends to zero along the curve $t^2 = u(\rho)^2$. Hence the metric (16) continuously tends to the (conformal) Minkowski spacetime outside this domain but may have discontinuous derivatives along this boundary [58,59]. Hence the causal domain for an event occurring at the origin is defined as the region $u(\rho)^2 \leq (t + z)^2$, which is arbitrary since u is arbitrary. This situation is represented by Fig. 2, where we have $\partial D^\pm(O) = \{(t, \rho): u(\rho) = \pm|t|\}$ for $z = 0$. Choices of u exist that yield $D^+(O) \cap D^-(O) \neq \varnothing$, indicating that the notions of past and future can become conflated when generalized Peres waves with general λ are permitted.

C. Astrophysical constraints

In this section we review briefly, for completeness, astrophysical constraints on polynomial $f(R)$ theories given by (9). To this end we introduce the parameterized post-Newtonian (PPN) Eddington parameters γ^{PPN} and β^{PPN}, which may be written as [60,61]

$$\gamma^{\text{PPN}} - 1 = -\frac{f''(R)^2}{f'(R) + 2f''(R)} \quad (22)$$

and

$$\beta^{\text{PPN}} - 1 = \frac{f'(R)f''(R)}{8f'(R) + 12f''(R)^2}\frac{d\gamma^{\text{PPN}}}{dR}, \quad (23)$$

in a general $f(R)$ theory.

Table I presents a summary of data collected from recent Solar System and pulsar timing experiments when interpreted as constraints on the parameters γ^{PPN} and β^{PPN} [62–66]. The PPN parameters (22) and (23) are evaluated at the measured value of the background scalar curvature R_0, which is determined through the Friedmann-Lemaître-Robertson-Walker relationship $R_0 = 12c^{-2}H_0^2$, where H_0 is the Hubble constant [67]. We assume that the Hubble radius takes the value $cH_0^{-1} = 4.0 \times 10^3$ Mpc.

TABLE I. Selected Solar System and pulsar timing constraints on the PPN Eddington parameters γ^{PPN} and β^{PPN}.

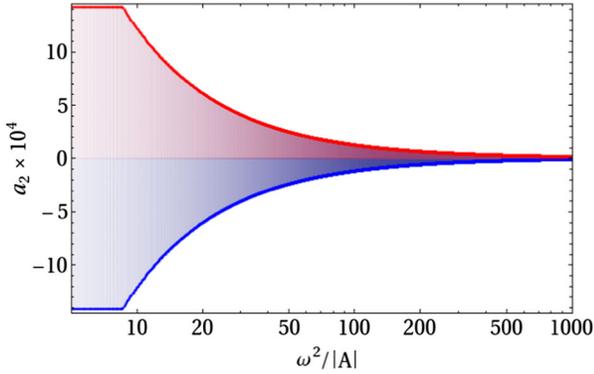| Experiment	Constraint	Reference
Precession of Mercury	$-3.0 \times 10^{-3} < 2\gamma^{\text{PPN}} - \beta^{\text{PPN}} - 1 < 3.0 \times 10^{-3}$	[62]
Lunar laser ranging (Nordtvedt effect)	$-1.7 \times 10^{-3} < 4\beta^{\text{PPN}} - \gamma^{\text{PPN}} - 3 < 0.3 \times 10^{-3}$	[63]
Very long baseline interferometry	$-4.0 \times 10^{-4} < \gamma^{\text{PPN}} - 1 < 4.0 \times 10^{-4}$	[64]
Cassini tracking	$-0.2 \times 10^{-5} < \gamma^{\text{PPN}} - 1 < 4.4 \times 10^{-5}$	[65]
Timing of PSR B1913 + 16	$\beta^{\text{PPN}} - 1 < 1.1 \times (\gamma^{\text{PPN}} - 1)$	[66]

FIG. 3. Allowed values of a_2 and $\omega^2/|A|$ (shaded region) in arbitrary units for the $f(R)$ theory given by (9) consistent with the data presented in Table I, under the assumption that a_k is given by (14) for $k = 9$. Theories for which the linear theory does (does not) predict tachyons are shaded in blue (red).

Using the experimental bounds on the parameters γ^{PPN} and β^{PPN} described in Table I, one can place constraints on the parameters a_2, a_3, and a_k appearing in the function (9) and consequently constrain the set of allowed amplitude-frequency relationships (13). We focus on the case presented in Sec. IV A, where we assume that the parameter a_k is given by (14). In Fig. 3 we present values of ω^2/A and a_2, which share the same units of length squared, consistent with the data presented in Table I, for the illustrative choice $k = 9$. Figure 3 demonstrates that tachyonic GWs with amplitude-frequency relation $\omega^2/A = \mu$ for some μ are permitted within astrophysically constrained $f(R)$ theories given by (9) provided that a_2 takes a value within the shaded region. In particular, theories for which the linear theory does (does not) predict tachyons according to (8) are shown in the blue (red) region.

V. DISCUSSION

In this paper we study the causal properties of some nonlinear GWs in vacuum $f(R)$ theories of gravity. It is found that the causal domains admit certain exotic features. The phase speeds of the waves can also be arbitrary for a wide range of functions $f(R)$, a result which does not hold in the linear regime [22]. The results suggest that the notion of causality is sensitive to the particulars of the modified theory of gravity under investigation [54]. For example, we show that the restrictions on $f(R)$ derived previously to avoid the existence of linear superluminal GW modes [22,37,38] must be augmented to avoid the existence of such modes in the nonlinear regime. We emphasize that it is unclear whether the exact, nonlinear solutions discussed here can actually be emitted by a realistic source with a time-dependent quadrupole moment (cf. the discussion in Ref. [68]). A full investigation of causality in this context relies on solving the initial boundary-value problem for a particular experiment, something falling outside the scope of this work.

Can the ideas in this paper be tested observationally? In principle, yes, although any astrophysical tests are likely to be confounded by systematic uncertainties introduced by complicated electromagnetic emission physics in the source. For example, if an event occurs which emits electromagnetic and gravitational radiation, some observers may witness the electromagnetic pulses but not the gravitational ones (if $v < c$) or vice versa (if $v > c$). Likewise, observers at rest equidistant from the source in different planes may or may not experience the gravitational radiation if the propagation is anisotropic (see Fig. 2). Moreover, in relativistic systems like a black hole surrounded by an accretion disk, electromagnetic and gravitational wave modes induce backreactions on the disk which may not be felt simultaneously if the phase speeds are different, because of how the metric and Faraday tensors enter into the various magnetohydrodynamic couplings [69–73]. Again, designing a "clean" experiment of this sort is a major challenge in an astrophysical context.

The recent detection of GWs by the Laser Interferometer Gravitational-Wave Observatory (LIGO) has opened up new avenues for experimentally determining the phase speed of GWs [74–76]. Given the small number of operational interferometers at present, which makes localization and real-time electromagnetic follow-up a difficult task, direct bounds placed on the phase speed of GWs are fairly weak at this stage [77–79] (however see Table 5 of Ref. [80]). To the authors' knowledge, no LIGO-related experimental bounds on the structure of $D^{\pm}$ exist. Detection of an inherently nonlinear property of GWs, such as the Christodoulou memory [81,82], would be useful in this direction. It is interesting to compare predictions of the amplitude of the nonlinear memory for various theories of gravity; one can have different memory amplitudes for different GW polarizations; see e.g. Eqs. (7) and (8) of [83].

ACKNOWLEDGMENTS

[1] J. B. Griffiths and J. Podolsky, *Exact Spacetimes in Einstein's General Relativity* (Cambridge University Press, Cambridge, England, 2009).

[2] K. S. Thorne, Phys. Rev. 138, B251 (1965).

[3] H. Bondi, M. G. J. Van der Burg, and A. W. K. Metzner, Proc. R. Soc. A 269, 21 (1962).

[4] L. Blanchet, Living Rev. Relativ. 9, 4 (2006).

[5] B. J. Owen, L. Lindblom, C. Cutler, B. F. Schutz, A. Vecchio, and N. Andersson, Phys. Rev. D 58, 084020 (1998).

[6] R. A. Isaacson, Phys. Rev. 166, 1263 (1968).

[7] K. S. Thorne, Rev. Mod. Phys. 52, 299 (1980).

[8] J. Näf, P. Jetzer, and M. Sereno, Phys. Rev. D 79, 024014 (2009).

[9] A. Einstein and N. Rosen, J. Franklin Inst. 223, 43 (1937).

[10] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1975).

[11] S. Capozziello and V. Faraoni, *Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics* (Springer, New York, 2010).

[12] G. T. Horowitz and A. R. Steif, Phys. Rev. Lett. 64, 260 (1990).

[13] T. Damour and A. M. Polyakov, Gen. Relativ. Gravit. 26, 1171 (1994).

[14] A. de Felice and S. Tsujikawa, Living Rev. Relativ. 13, 3 (2010).

[15] S. Capozziello, S. J. Gabriele Gionti, and D. Vernieri, J. Cosmol. Astropart. Phys. 01 (2016) 015.

[16] M. Gasperini and M. Giovannini, Phys. Rev. D 47, 1519 (1993).

[17] M. D. Roberts, Int. J. Mod. Phys. A 09, 167 (1994).

[18] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) 140, 372 (1982).

[19] K. Izumi, Phys. Rev. D 90, 044037 (2014).

[20] S. W. Hawking, Phys. Rev. D 46, 603 (1992).

[21] K. Izumi and Y. C. Ong, Classical Quantum Gravity 30, 184008 (2013).

[22] C. P. L. Berry and J. R. Gair, Phys. Rev. D 83, 104022 (2011).

[23] A. Jeffrey and T. Taniuti, *Nonlinear Wave Propagation* (Academic, New York, 1964).

[24] J. B. Keller and S. Kogelman, SIAM J. Appl. Math. 18, 748 (1970).

[25] L. Gonzalez-Mestres, arXiv:hep-ph/9610474.

[26] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum, Gen. Relativ. Gravit. 39, 1651 (2007).

[27] S. Kichenassamy, *Nonlinear Wave Equations* (Marcel Dekker, New York, 1996).

[28] F. Verhulst, *Methods and Applications of Singular Perturbations* (Springer, New York, 2005).

[29] A. G. Suvorov and A. Melatos, Phys. Rev. D 94, 044045 (2016).

[30] B. J. Carr and E. Verdaguer, Phys. Rev. D 28, 2995 (1983).

[31] A. I. Vainshtein, Phys. Lett. B 39, 393 (1972).

[32] V. A. Kostelecký, Phys. Rev. D 69, 105009 (2004).

[33] Y. Choquet-Bruhat, *General Relativity and the Einstein Equations* (Oxford University, New York, 2009).

[34] M. E. S. Alves, O. D. Miranda, and J. C. N. de Araujo, Phys. Lett. B 679, 401 (2009).

[35] A. A. Starobinsky, JETP Lett. 86, 157 (2007).

[36] A. D. Dolgov and M. Kawasaki, Phys. Lett. B 573, 1 (2003).

[37] H. J. Schmidt, Astron. Nachr. 307, 339 (1986).

[38] K. S. Stelle, Gen. Relativ. Gravit. 9, 353 (1978).

[39] L. Amendola and S. Tsujikawa, Phys. Lett. B 660, 125 (2008).

[40] T. Faulkner, M. Tegmark, E. F. Bunn, and Y. Mao, Phys. Rev. D 76, 063505 (2007).

[41] M. Martinelli, A. Melchiorri, O. Mena, V. Salvatelli, and Z. Gironés, Phys. Rev. D 85, 024006 (2012).

[42] S. Capozziello, M. De Laurentis, S. Nojiri, and S. D. Odintsov, Gen. Relativ. Gravit. 41, 2313 (2009).

[43] P. Jordan, J. Ehlers, and W. Kundt, Akad. Wiss. Mainz. Math. Naturwiss. Kl. 2, 21 (1960).

[44] A. Ashtekar, J. Bičák, and B. G. Schmidt, Phys. Rev. D 55, 687 (1997).

[45] A. Ashtekar, J. Bičák, and B. G. Schmidt, Phys. Rev. D 55, 669 (1997).

[46] J. Weber and J. A. Wheeler, Rev. Mod. Phys. 29, 509 (1957).

[47] N. Rosen, Bull. Res. Counc. Isr. 3, 328 (1954).

[48] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2009).

[49] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations* (Wiley, New York, 1979).

[50] C. J. S. Clarke, *The Analysis of Space-Time Singularities* (Cambridge University Press, Cambridge, England, 1993).

[51] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars* (Wiley, New York, 1983).

[52] M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, Cambridge, England, 1996).

[53] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).

[54] K. Izumi, Phys. Rev. D 90, 044037 (2014).

[55] A. Ori, Phys. Rev. D 61, 024001 (1999).

[56] A. Peres, Phys. Rev. Lett. 3, 571 (1959).

[57] F. Canfora, L. Parisi, and G. Vilasi, Theor. Math. Phys. 152, 1069 (2007).

[58] A. Tomimatsu, Gen. Relativ. Gravit. 21, 613 (1989).

[59] V. A. Belinski and V. E. Zakharov, Sov. Phys. JETP 48, 985 (1978).

[60] S. Capozziello, A. Stabile, and A. Troisi, Mod. Phys. Lett. A 21, 2291 (2006).

[61] S. Capozziello and A. Troisi, Phys. Rev. D 72, 044022 (2005).

[62] I. I. Shapiro, in *General Relativity and Gravitation 12*, edited by N. Ashby et al. (Cambridge University Press, Cambridge, England, 1993).

[63] J. G. Williams, X. X. Newhall, and J. O. Dickey, Phys. Rev. D 53, 6730 (1996).

[64] S. S. Shapiro, J. L. Davis, D. E. Lebach, and J. S. Gregory, Phys. Rev. Lett. 92, 121101 (2004).

[65] B. Bertotti, L. Iess, and P. Tortora, Nature (London) 425, 374 (2003).

[66] G. Esposito-Farése, AIP Conf. Proc. 736, 35 (2004).

[67] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010).

[68] H. Bondi, F. A. E. Pirani, and I. Robinson, Proc. R. Soc. A 251, 519 (1959).

[69] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University, New York, 1998).

[70] E. Barausse and T. P. Sotiriou, Classical Quantum Gravity **30**, 244010 (2013).

[71] D. N. Page and K. S. Thorne, Astrophys. J. **191**, 499 (1974).

[72] K. S. Thorne, Astrophys. J. **191**, 507 (1974).

[73] A. G. Suvorov and A. Melatos, Phys. Rev. D **93**, 024004 (2016).

[74] B. P. Abbott *et al.*, Phys. Rev. Lett. **116**, 061102 (2016).

[75] B. P. Abbott *et al.*, Phys. Rev. Lett. **116**, 241103 (2016).

[76] V. Connaughton *et al.*, Astrophys. J. Lett. **826**, L6 (2016).

[77] J. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, Mod. Phys. Lett. A **31**, 1675001 (2016).

[78] T. E. Collett and D. Bacon, Phys. Rev. Lett. **118**, 091101 (2017).

[79] D. Blas, M. M. Ivanov, I. Sawicki, and S. Sibiryako, JETP Lett. **103**, 624 (2016).

[80] B. P. Abbott *et al.*, Phys. Rev. Lett. **118**, 221101 (2017).

[81] D. Christodoulou, Phys. Rev. Lett. **67**, 1486 (1991).

[82] M. Favata, Astrophys. J. Lett. **696**, L159 (2009).

[83] S. M. Du and A. Nishizawa, Phys. Rev. D **94**, 104063 (2016).