Quantum potential induced UV-IR coupling in analogue Hawking radiation: From Bose-Einstein condensates to canonical acoustic black holes

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Arising out of a nonlocal nonrelativistic Bose-Einstein condensates (BEC), we present an analogue gravity model up to $\mathcal{O}(\xi^2)$ accuracy (ξ being the healing length of the condensate) in the presence of the quantum potential term for a canonical acoustic black hole in $(3+1)$D spacetime, where the series solution of the free minimally coupled KG equation for the large-length-scale massive scalar modes is derived. We systematically address the issues of the presence of the quantum potential term being the root cause of a UV-IR coupling between short-wavelength *primary* modes which are supposedly Hawking-radiated through the sonic horizon and the large-wavelength *secondary* modes. In the quantum gravity experiments of analogue Hawking radiation within the scope of the laboratory set up, this UV-IR coupling is inevitable, and one cannot get rid of these large-wavelength excitations which would grow over space by gaining energy from the short-wavelength Hawking-radiated modes. We identify the characteristic feature in the growth rate(s) that would distinguish these primary and secondary modes.

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I. INTRODUCTION

It is well known that from an experimentally realizable condensed matter model, through some rigorous mathematical framework, gravity comes out as an *emergent phenomenon* as seen by the sonic excitations. Unruh's seminal work [1] practically opened up this field of research, which has been extensively pursued over the last couple of decades and has a host of theoretical proposals around it [2,3].

Among many other "analogue models," from the standpoint of classical physics, the passing of sound waves as acoustic disturbances through a moving Newtonian fluid is the simplest and cleanest example of the condensed-matter analogue for the light waves in a curved spacetime [4–6]. The idea is that if the fluid flow ever becomes supersonic, then in that trapped region, the sound waves would never be able to fight their way back upstream, and this surface of no return in the fluid medium clearly bears the analogy of a gravitational event horizon. Although this terminology of an *acoustic analogue of the event horizon* (i.e., a sonic horizon) does not qualify by the stricter definition of *event horizon* in general relativity; nevertheless, this is actually a *Killing horizon* from which an analogue Hawking radiation can be expected. This implies the very existence of a "dumb hole," or, in other words, an *acoustic black hole* [7,8].

One can probe various aspects of curved spacetime quantum field theory (QFT) via these analogue models due to the amenability of accurate experimental control and observational verification—a quantum system characterized by very cold temperature (~100 nK), a low speed of sound, and a high degree of quantum coherence offers the best test field [9,10]. Bose-Einstein condensate (BEC), which is a superfluid quantum phase of matter, happens to be one of the most prominent candidates of all such ultracold systems [6,11] to examine and investigate some crucial features of *emergent gravity*. It does create analogue gravitational scenarios at nK temperatures within the laboratory setup and readily provides it with an experimental window to capture some key aspects of Hawking radiation,[1] which is one of the cornerstone results [12] of curved spacetime QFT.

This fact basically led to the increasing interest in using BEC[2] as a platform to observe *analogue* Hawking radiation [19–21] as a thermal bath of phonons with the temperature proportional to the *surface gravity* [3,6]. In this context, Parentani and coworkers have already proposed some novel ideas based on density correlations, studying the hydrodynamics over several length scales and even surface-gravity-independent temperature, etc. [22–25]. In order to experimentally detect analogue Hawking radiation, a stack

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[1]In order for a gravitational black hole (with mass, say, M_{BH}) to be observed to emit Hawking radiation, it must have a temperature (say, T_H) greater than that of the present-day cosmic microwave background (CMB) radiation (say, T_{CMB}). $T_{\mathrm{CMB}} = 2.725$ K, while $T_H = \frac{\hbar c^3}{8\pi G M_{\mathrm{BH}} k_{\mathrm{B}}} \approx 6.169 \times 10^{-8}\ \mathrm{K} \times \frac{M_{\mathrm{sun}}}{M_{\mathrm{BH}}}$ in SI units; and hence the direct detection of the gravitational Hawking radiation for a Schwarzschild black hole with a mass equivalent to at least the solar mass falls far below the limit of current observational techniques. So, it is extremely difficult to verify Hawking radiation in nature, the reason being that T_H is 7 orders of magnitude smaller than T_{CMB}.

[2]Apart from BEC, some other condensed matter systems, such as superfluid helium [13,14], superconductors [15,16], polariton superfluid [17], and degenerate Fermi gas [18] have also been used as tools to probe emergent gravity.

of recent works are obviously worth mentioning here [26–34].

In a BEC, the small-amplitude collective excitations [35] of the uniform density moving phase (to be precise, the first-order phase fluctuation field) obeys the quantum hydrodynamics which, ignoring the "quantum potential" term [refer to Sec. 4.2.1 of Ref. [3]], can be cast into the d'Alembertian equation of motion of a massless, minimally coupled, free scalar field on a $(3+1)$D Lorentzian manifold with an effective metric to be regarded as the acoustic metric of the curved background.[3]

Some worthwhile efforts have been taken in order to regularize the dynamics taking the quantum potential term into account [36]. In 2005, Visser *et al.* showed the emergence of a massive Klein-Gordon (KG) equation considering a two-component BEC, where a laser-induced transition between the two components was exploited [37]. Liberati *et al.* proposed a weak $U(1)$ symmetry breaking of the analogue BEC model by the introduction of an extra quadratic term in the Hamiltonian to make the scalar field massive [38]. Considering the flow in a Laval nozzle, Cuyubamba has shown the emergence of a massive scalar field in the context of analogue gravity arguing for the possibility of the observation of quasinormal ringing of the massive scalar field within the laboratory setup [39]. In the context of trans-Planckian backreaction issues on low-energy predictions in analogue gravity, Fischer *et al.* have recently done a remarkable work [40].

It is well known that, in order to obtain an analogue gravity model from a nonrelativistic BEC, one needs to get rid of the terms coming from the linearization of the quantum potential to find the effective acoustic metric on the basis of "hydrodynamical" approximation, where the contributions coming from the small-length scale regime (wavelengths shorter than the healing length ξ of the system) are neglected. However, if one goes beyond this hydrodynamic regime to access the high-frequency modes, the contribution of the quantum potential cannot be neglected, and the acoustic description can still be achieved through *eikonal approximation*—the Lorentz breaking in BEC models [3]. The main advantage of this approximation is that the "operator" $\hat{D}_2$ [refer to Eq. (13) later] can effectively be replaced by just the "function," and consequently, the entries of the acoustic metric become explicitly momentum-dependent numbers, but not operators. In this case, it gives rise to a modified dispersion relation for the quasiparticles [refer to Eq. (271) of Ref. [3]]. In this regard of the significance of the eikonal approximation, the points 4 and 5 as mentioned by Barceló *et al.* on p. 64 of Ref. [3] are of great importance.

In our previous paper [41], we looked into the effect of the Lorentz-breaking quantum potential term in a different way. This is a term of immense importance in the context of analogue gravity, because this gives rise to the dispersion relation, which is used to present an alternative scenario of the analogue Hawking radiation, bypassing the trans-Planckian problem [22–25,42,43]. But the presence of this quantum potential term in the dynamics is somewhat analogous to that of a diffusion term which should spread the small-scale modes into the large-scale ones. In our previous work, we guessed the existence of this coupling between the small- and large-scale dynamics and captured this picture through the massive large-wavelength excitations as the amplitude modes over the usual small-scale excitations. This whole work was presented on flat spacetime for the sake of simplicity and in order to introduce the idea primarily.

In the present paper, we analyze in detail the effect and consequence of the presence of the quantum potential term in the context of the spreading out of the small-scale excitations into the large-scale ones on the curved spacetime of a canonical acoustic black hole (a model first proposed by Visser [6] in 1998). This is an important analysis in its own right, because the ultraviolet (UV) to infrared (IR) coupling is inevitable in these types of systems. This, in turn, results in the presence of instabilities to the short-wavelength (UV-correspondence) modes, which are predominant in the Hawking spectrum, as seen by a free-falling observer in a local Minkowski spacetime. This is because of the fact that, at very large curvatures, the local Minkowski flat space can only account for the short-wavelength (UV-correspondence) modes. The large-wavelength (IR-correspondence) modes which would be subsequently generated out of the UV modes are characterized by a mass term solely dependent on the small scales. This fact does manifest the energetic dependence of these IR modes on the UV ones and, at some sufficiently large time, there would be a transfer of energy from the UV modes to the IR ones, and this can completely mask the Hawking signal of the Hawking-radiated modes. A detailed analysis of the UV-IR coupled dynamics is quite essential in that respect. Very recently, Vieira *et al.* have discussed the analogue Hawking radiation of the massless scalar particles and the features of the Hawking spectrum associated in the spacetime of rotating and canonical acoustic black holes [44].

Here we have obtained the series solution to the free minimally coupled "massive" KG equation on a $(3+1)$D canonical curved background.[4] And we have followed the method shown by Elizalde [45] in 1988. Although the acoustic situation is similar to the Schwarzschild geometry, the metric (i.e., the acoustic metric) itself is quite different from the standard Schwarzschild metric. As a result, our present study here is quite different in some details from the one done by Elizalde on the Scwarzschild background.

[3]Refer to Eq. (254) of Ref. [3].

[4]Refer to the line element given by Eq. (55) of Ref. [6].

Using the series solution truncated to the desired accuracy, we show that the IR modes, having grown from the UV modes (supposedly Hawking-radiated), would have a dominant *power-law growth over space*, which is characterized by the quartic power of the UV frequency of the original Hawking-radiated modes. This has the clear indication that there remains the information of the relative abundance of the Hawking-radiated quanta within the growth of the IR modes generated by the UV Hawking-radiated modes.

In view of the inevitability of the appearance of the UV-IR coupling through the presence of the quantum potential term in these analogue gravity models, the present exercise of ours does provide a better look at the prevailing situation in such systems.

This paper is organized in the following manner: In Sec. II, we start from the stepping stone which is the nonlocal Gross-Pitaevskii (GP) model for a BEC with interaction. By adopting the Madelung ansatz and adding the first-order fluctuations to the density and phase of the single-particle BEC state, we develop a setting from the condensed matter standpoint.

In Sec. III, we give the structure of the full model as proposed by us, derive the general acoustic metric of a $(3+1)$D curved background, and form the covariant "massive" KG equation for a free scalar field.

In Sec. IV, we consider the geometry of a canonical acoustic black hole where the fluid flow is assumed to be incompressible and spherically symmetric, and we derive the exact solution to the massive KG equation on this background, taking into account the radial part specifically.

In Sec. V, we conclude the paper with a detailed discussion of our results.

II. THE SETTING

To study interacting nonuniform Bose gases at very low temperatures, one uses the Bogoliubov prescription for the field operator $\hat{\psi}(t,\mathbf{r})$ that obeys the well-known commutation relations $[\hat{\psi}(t,\mathbf{r}),\hat{\psi}^{\dagger}(t,\mathbf{r}')]=\delta(\mathbf{r}-\mathbf{r}')$ and $[\hat{\psi}(t,\mathbf{r}),\hat{\psi}(t,\mathbf{r}')]=0$. In the Heisenberg picture, the dynamics of this Bose field operator $\hat{\psi}(t,\mathbf{r})$ is given by the exact equation[5]

$$i\hbar\partial_t\hat{\psi}(t,\mathbf{r})=[\hat{\psi}(t,\mathbf{r}),\hat{\mathcal{H}}]=\left(-\frac{\hbar^2}{2m}\mathbf{\nabla}^2+V_{\mathrm{ext}}(t,\mathbf{r})+\int\hat{\psi}^{\dagger}(t,\mathbf{r}')V(\mathbf{r}'-\mathbf{r})\hat{\psi}(t,\mathbf{r}')d\mathbf{r}'\right)\hat{\psi}(t,\mathbf{r}), \tag{1}$$

where the many-body Hamiltonian $\hat{\mathcal{H}}$ of the full interacting Bose system, as inserted above, is given by

$$\hat{\mathcal{H}}=\int\left(\frac{\hbar^2}{2m}\mathbf{\nabla}\hat{\psi}^{\dagger}(t,\mathbf{r})\mathbf{\nabla}\hat{\psi}(t,\mathbf{r})\right)d\mathbf{r}+\int\hat{\psi}^{\dagger}(t,\mathbf{r})V_{\mathrm{ext}}(t,\mathbf{r})\hat{\psi}(t,\mathbf{r})d\mathbf{r}+\frac{1}{2}\int\hat{\psi}^{\dagger}(t,\mathbf{r})\hat{\psi}^{\dagger}(t,\mathbf{r}')V(\mathbf{r}'-\mathbf{r})\hat{\psi}(t,\mathbf{r})\hat{\psi}(t,\mathbf{r}')d\mathbf{r}'d\mathbf{r}, \tag{2}$$

with $\hbar=h/2\pi$, h being the Planck constant, m being the mass of a single boson. Obviously, $V_{\mathrm{ext}}(t,\mathbf{r})$ is an external (trapping) potential, and $V(\mathbf{r}'-\mathbf{r})$ is the interaction potential.

Now, to the lowest-order Born approximation and at very low temperatures, one gets the license to replace the quantum field operator $\hat{\psi}(t,\mathbf{r})$ with the classical wave function $\psi(t,\mathbf{r})$ of the condensate due to the macroscopic occupation of a large number of atoms in a single quantum state (the BEC ground state). The mean-field approximation is

$$\hat{\psi}(t,\mathbf{r})\to\langle\hat{\psi}\rangle=\psi(t,\mathbf{r}), \tag{3}$$

by which one sort of neglects the noncommutativity of the field operators $\hat{\psi}(t,\mathbf{r})$ as defined above. This mean-field approximation has its implication from a physical point of view as well—since $\hat{\psi}(t,\mathbf{r})$ or $\hat{\psi}^{\dagger}(t,\mathbf{r})$ act as "annihilation" or "creation" operator(s), respectively, to annihilate or create a particle at $(t,\mathbf{r})$, now if a particle is subtracted from or added to the condensate, it does not really change the physical properties of the whole system, which is actually governed by the order parameter $\psi(t,\mathbf{r})$. Moreover, this switching from $\hat{\psi}(t,\mathbf{r})$ to the classical mean field $\psi(t,\mathbf{r})$ is accurate enough when one does not consider the realistic potential[6] but replaces $V(\mathbf{r}'-\mathbf{r})$ with some effective soft potential $V_{\mathrm{eff}}(\mathbf{r}'-\mathbf{r})$.

The minimal GP model for the *nonlocal* [46] s-wave scattering in a nonuniform BEC is characterized by the following equation:

$$\begin{aligned}i\hbar\partial_t\psi(t,\mathbf{r})&\\&=\left(-\frac{\hbar^2}{2m}\mathbf{\nabla}^2+V_{\mathrm{ext}}(t,\mathbf{r})+\mathsf{g}|\psi(t,\mathbf{r})|^2\right)\psi(t,\mathbf{r})\\&\quad+\kappa a^2\mathsf{g}\psi(t,\mathbf{r})\mathbf{\nabla}^2|\psi(t,\mathbf{r})|^2,\end{aligned} \tag{4}$$

where $\psi(t,\mathbf{r})$ is the condensate wave function which plays the role of the order parameter of the system, and thus

[5] See Eq. (5.1) of Ref. [35].

[6] See the argument on p. 39 of Ref. [35].

$|\psi(t,\mathbf{r})|^2 = n(t,\mathbf{r})$ is the density of the condensate, $\mathsf{g} = 4\pi\hbar^2 a/m$ parametrizes the strength of the s-wave scattering (between different bosons in the gas) considered at the lowest-order Born approximation with a being the s-wave scattering length, and κ is some numerical prefactor of the nonlocal correction term corresponding to the specific coordinate system under consideration (for instance, $\kappa = 1/2$ for a 3D Cartesian system [41]).

The interaction term $\mathsf{g}|\psi(t,\mathbf{r})|^2\psi(t,\mathbf{r})$ in the local[7] GP equation comes by considering a δ-function approximation to the interaction potential $V(\mathbf{r}'-\mathbf{r})$ in the s-wave interaction picture $\int \psi^*(t,\mathbf{r}')V(\mathbf{r}'-\mathbf{r})\psi(t,\mathbf{r}')d\mathbf{r}'$ in a nonuniform BEC; i.e., $V(\mathbf{r}'-\mathbf{r}) \equiv V_{\text{eff}}(\mathbf{r}'-\mathbf{r}) = \mathsf{g}\delta(\mathbf{r}'-\mathbf{r})$ is substituted. This approximation is valid under the consideration $|a| \ll n^{-\frac{1}{3}}$, which is the condition of diluteness; i.e., the s-wave scattering length (that characterizes all the effects of boson-boson interaction on the physical properties of the gas) is much smaller than the average interparticle separation. Now, in a BEC, one can tune the s-wave scattering length by Feshbach resonance [see the arguments later where ϵ would be introduced in Eq. (15)], and this actually opens up the possibility of going away from the diluteness limit of $|a| \ll n^{-\frac{1}{3}}$. Keeping this in mind, a correction term [i.e., the last term in Eq. (4)] can be derived to bring in the effects of the nonlocality of the interactions. A Taylor expansion of $\psi(t,\mathbf{r}')$ about $\mathbf{r}'=\mathbf{r}$ in the interaction term $\int \psi^*(t,\mathbf{r}')V(\mathbf{r}'-\mathbf{r})\psi(t,\mathbf{r}')d\mathbf{r}'$ will give rise to the minimal correction at the second order, since the first-order correction vanishes due to spherical symmetry.

We have given a detailed description of the derivation of the above mentioned Gross-Pitaevskii equation with a nonlocal correction term in our previous paper [41]. In Eq. (4), the last term on the rhs represents the minimal nonlocal "correction" to the standard GP equation with contact interactions. If the width of the interaction potential is of the order of a microscopic length scale, say ς, then the minimal correction term due to the nonlocality turns out to be $\propto \varsigma^2$. For the sake of simplicity, here we have taken the interaction width to be of the order of the s-wave scattering length, i.e. $\varsigma \simeq a$, which will not affect any of our results qualitatively.

On considering a general single-particle state of the BEC as

$$\psi(t,\mathbf{r}) = \sqrt{n(t,\mathbf{r})}\,e^{i\vartheta(t,\mathbf{r})/\hbar} \tag{5}$$

by adopting the Madelung ansatz, Eq. (4) gives rise to a set of coupled equations[8]:

$$\text{Continuity equation:} \qquad \partial_t n(t,\mathbf{r}) + \frac{1}{m}\boldsymbol{\nabla}.\Big(n(t,\mathbf{r})\boldsymbol{\nabla}\vartheta(t,\mathbf{r})\Big) = 0. \tag{6}$$

$$\text{Euler equation:} \qquad \partial_t \vartheta(t,\mathbf{r}) + \left(\frac{[\boldsymbol{\nabla}\vartheta(t,\mathbf{r})]^2}{2m} + V_{\text{ext}}(t,\mathbf{r}) + \mathsf{g}n(t,\mathbf{r}) - \frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{n(t,\mathbf{r})}}{\sqrt{n(t,\mathbf{r})}} + \kappa a^2\mathsf{g}\boldsymbol{\nabla}^2 n(t,\mathbf{r})\right) = 0. \tag{7}$$

It is quite evident that due to the presence of the nonlocal correction term in Eq. (4), the modified form of the quantum potential as obtained from the above Eq. (7) is given by

$$\mathsf{V}_{\text{quantum}} = -\frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{n(t,\mathbf{r})}}{\sqrt{n(t,\mathbf{r})}} + \kappa a^2\mathsf{g}\boldsymbol{\nabla}^2 n(t,\mathbf{r}). \tag{8}$$

At this stage, we purposely define two independent scales in spherical polar coordinates as

$$x \equiv x^\mu = (t,r,\theta,\phi) \quad \text{and} \quad X \equiv X^{\bar{\mu}} = (\mathcal{T},R,\Theta,\Phi), \tag{9}$$

viz. the small scale and the large scale, respectively. Here $\mu,\nu\ldots$, etc. along with $\bar{\mu},\bar{\nu}\ldots$, etc. are the two different sets of free/dummy indices which separately run over the small and large scales, respectively. Later in this section, we will introduce the corresponding spacetime derivatives as well.

Now, we consider the fluctuations[9] to the density [i.e., $n(t,\mathbf{r})$] and phase [i.e., $\vartheta(t,\mathbf{r})$] of the BEC state in the following manner:

$$n \to n_0 + n_1(X,x), \qquad \vartheta \to \vartheta_0(x) + \{\vartheta_2(X)\vartheta_1(x)\}; \tag{10}$$

i.e., n_0 and $\vartheta_0(x)$ are basically the classical mean-field density and phase, respectively, such that $\langle n\rangle = n_0$ and $\langle\vartheta\rangle = \vartheta_0(x)$. These are obviously the macroscopic descriptions of the condensate within the classical regime. On the other hand, $n_1(X,x)$ is the first-order density fluctuation,

[7]See Eq. (5.2) of Ref. [35].

[8]Here, the Euler equation [Eq. (7)] is written in Hamilton-Jacobi form for convenience. One may refer to Eq. (5.15) of Ref. [35].

[9]These linearized fluctuations in the dynamical quantities are more generally referred to as acoustic disturbances. To be precise and according to convention, the low-frequency, large-wavelength disturbances are called *wind gusts*, while the high-frequency, short-wavelength disturbances are described as acoustic disturbances. Refer to p. 1770 of Ref. [6].

and $\{\vartheta_2(X)\vartheta_1(x)\}$ is the first-order phase fluctuation (accompanying amplitude modulations). In other words, these fluctuations are the quantum fields in nature and can be described as "quantum acoustic representation".[10]

Inserting Eq. (10) back into Eqs. (6) and (7), one comes up with the linearized dynamics. It is to be carefully noted here that, at this stage, the usual partial derivatives involve independent multiple scales which are mixed and clubbed together as of now. Since we are not operating with these derivatives on the density and/or phase fields at the moment, we may opt to refrain ourselves from the necessary modification of the derivative operator(s) for notational convenience. But to keep everything notationally pellucid, these usual partial derivative operators are just being "renamed" by the same notation with a tilde over each one for now. However, the modification of the derivative operator(s) is explicitly performed later in Eq. (21).

Considering $V_{\rm ext}(t,\mathbf{r})=0$ and n_0 as just a constant for simplicity, the linearized dynamics is given by Eq. (11) (obtained from the continuity equation) coupled with Eq. (12) (obtained from the Euler equation) as the following:

$$\tilde{\partial}_t n_1(X,x) + \frac{1}{m}\tilde{\boldsymbol{\nabla}}.\left(n_1(X,x)\tilde{\boldsymbol{\nabla}}\vartheta_0(x)+n_0\tilde{\boldsymbol{\nabla}}\{\vartheta_2(X)\vartheta_1(x)\}\right)=0 \quad (11)$$

and

$$\tilde{\partial}_t\{\vartheta_2(X)\vartheta_1(x)\}+\frac{1}{m}\tilde{\boldsymbol{\nabla}}\vartheta_0(x).\tilde{\boldsymbol{\nabla}}\{\vartheta_2(X)\vartheta_1(x)\}+\mathsf{g}n_1(X,x) - \frac{\hbar^2}{2m}\hat{D}_2 n_1(X,x)=0, \quad (12)$$

where

$$\hat{D}_2 \equiv \frac{2m\mathsf{g}}{\hbar^2}\left(\frac{\hbar^2}{4m\mathsf{g}n_0}-\kappa a^2\right)\tilde{\boldsymbol{\nabla}}^2 = \frac{2m\mathsf{g}}{\hbar^2}\xi^2\tilde{\boldsymbol{\nabla}}^2 \quad (13)$$

represents a second-order differential operator obtained by linearizing $\mathsf{V}_{\rm quantum}$ from Eq. (8) and ξ is the "modified" healing length corresponding to our proposed *nonlocal* model, given by

$$\xi=\left(\frac{\hbar^2}{4m\mathsf{g}n_0}-\kappa a^2\right)^{1/2}=\xi_0\epsilon; \qquad \xi_0=\frac{\hbar}{\sqrt{2m\mathsf{g}n_0}}. \quad (14)$$

This ξ_0 is nothing but the healing length[11] corresponding to the usual *local* GP model, and ϵ is a parameter set to be a very small quantity,

$$\epsilon=\left(\frac{1}{2}-8\pi\kappa a^3 n_0\right)^{1/2}. \quad (15)$$

The s-wave scattering length a can practically be increased from $-\infty$ to ∞ near a *Feshbach resonance*, as experimentally verified by Cornish *et al.* in 2000 [47]. Evidently, the tuning of a will keep increasing or decreasing the value of the parameter ϵ as per requirement [46,48]. In fact, by increasing a through Feshbach resonance, the value of $8\pi\kappa a^3 n_0$ can be made as close to $\frac{1}{2}$ as possible; and hence, naturally, ϵ can be experimentally set to be a very small quantity via Eq. (15).

From Eq. (12), $n_1(X,x)$ can be obtained as the following:

$$n_1(X,x) = -\hat{\mathcal{A}}\left(\tilde{\partial}_t\{\vartheta_2(X)\vartheta_1(x)\}+\frac{1}{m}\tilde{\boldsymbol{\nabla}}\vartheta_0(x).\tilde{\boldsymbol{\nabla}}\{\vartheta_2(X)\vartheta_1(x)\}\right),$$
$$\text{where}\quad \hat{\mathcal{A}}=\left(\mathsf{g}-\frac{\hbar^2}{2m}\hat{D}_2\right)^{-1}\simeq \mathsf{g}^{-1}(1+\xi^2\tilde{\boldsymbol{\nabla}}^2). \quad (16)$$

Here, ϵ being a very small quantity [as given by Eq. (15)] is exactly what would allow us to take a binomial approximation above. This expression of $n_1(X,x)$ is again substituted back into Eq. (11) to get a second-order partial differential equation in terms of the phase fluctuations, i.e.

$$\tilde{\partial}_t\left[-\hat{\mathcal{A}}\left(\tilde{\partial}_t\{\vartheta_2(X)\vartheta_1(x)\}+\frac{1}{m}\tilde{\boldsymbol{\nabla}}\vartheta_0(x).\tilde{\boldsymbol{\nabla}}\{\vartheta_2(X)\vartheta_1(x)\}\right)\right] + \frac{1}{m}\tilde{\boldsymbol{\nabla}}.\left(\left[-\hat{\mathcal{A}}\left(\tilde{\partial}_t\{\vartheta_2(X)\vartheta_1(x)\}+\frac{1}{m}\tilde{\boldsymbol{\nabla}}\vartheta_0(x).\tilde{\boldsymbol{\nabla}}\{\vartheta_2(X)\vartheta_1(x)\}\right)\right]\tilde{\boldsymbol{\nabla}}\vartheta_0(x)+n_0\tilde{\boldsymbol{\nabla}}\{\vartheta_2(X)\vartheta_1(x)\}\right)=0. \quad (17)$$

In the present context of passing a sonic disturbance through a barotropic inviscid fluid,[12] the background fluid flow $\mathbf{v}(x)$ is considered to be vorticity free; or in other words, locally irrotational, i.e.

[10]Refer to Eq. (242) of Ref. [3].
[11]Refer to Eq. (5.20) of Ref. [35].
[12]For a detailed justification, refer to p. 9 of Ref. [3].

$$\mathbf{v}(x) = \frac{1}{m}\tilde{\nabla}\vartheta_0(x) \equiv \frac{1}{m}\nabla\vartheta_0(x). \quad (18)$$

See Eq. (21) for the justification of $\tilde{\nabla} \to \nabla$ on $\vartheta_0(x)$ above. Clearly, the classical mean-field phase $\vartheta_0(x)$ of the BEC state [see Eq. (10)] now gets to act as the velocity potential[13] here in Eq. (18).

This background velocity being irrotational plays a very crucial role in determining the metric (which eventually turns out to be *Lorentzian*, as seen by the phonons inside the fluid medium) of this particular $(3+1)$D curved spacetime. In the present context, the fluid motion is assumed to be completely nonrelativistic; i.e., $|\mathbf{v}| \ll c$, where $c \approx 2.997 \times 10^8\ \mathrm{ms}^{-1}$ is the speed of light in vacuum.

It is worth mentioning here that the local speed of sound[14] inside the fluid medium is given by

$$c_s = \sqrt{n_0 \mathsf{g}/m}, \quad (19)$$

so obviously $0 < c_s \ll c$ in magnitude. If one starts by assuming the density n_0 to be position independent (which is pretty much the argument for considering a canonical acoustic black hole out of an incompressible and spherically symmetric fluid flow), due to the barotropic assumption, the pressure also becomes position independent. Thus, for a barotropic fluid, c_s becomes a position-independent constant; refer to Eq. (25) of Ref. [3]. In our analysis, the above Eq. (19) effectively keeps c_s as a constant throughout. This is an approximation on our part in the present context. We make this approximation here because we do not need to consider the actual structure of the sonic horizon in that detail. Obviously, when c_s becomes position dependent, the sonic horizon would have a spread over space. But what we are basically concerned with here is analyzing "secondary" waves (low-frequency sonic modes) being generated outside the acoustic horizon by gaining energy from the "primary" waves (high-frequency Hawking-radiated sonic modes) due to quantum-potential-induced UV-IR coupling in a $(3+1)$D curved spacetime. Hence, for that reason, we can safely regard c_s to be a position-independent constant, and this approximation is legit.

In terms of the velocity components, Eq. (17) can now be rewritten as

$$\tilde{\partial}_t\Big[-\hat{\mathcal{A}}\Big(\tilde{\partial}_t\{\vartheta_2(X)\vartheta_1(x)\} + \mathbf{v}(x).\tilde{\nabla}\{\vartheta_2(X)\vartheta_1(x)\}\Big)\Big] + \tilde{\nabla}.\Big(\Big[-\hat{\mathcal{A}}\Big(\tilde{\partial}_t\{\vartheta_2(X)\vartheta_1(x)\} + \mathbf{v}(x).\tilde{\nabla}\{\vartheta_2(X)\vartheta_1(x)\}\Big)\Big]\mathbf{v}(x) + \frac{n_0}{m}\tilde{\nabla}\{\vartheta_2(X)\vartheta_1(x)\}\Big) = 0, \quad (20)$$

where $\hat{\mathcal{A}} \equiv \mathsf{g}^{-1}(1 + \xi^2\tilde{\nabla}^2)$; see Eq. (16).

Let us note the following features of the above Eq. (20):

(1) Essentially, our choice of coordinate system is the spherical polar coordinates throughout this paper.

(2) Each operation with $\hat{\mathcal{A}}$ naturally contains a Laplacian which is considered to be at small scales [i.e., $\tilde{\nabla}^2 \to \nabla^2 \equiv \partial^{\mathrm{j}}\partial_{\mathrm{j}}$, where $\mathrm{j} \equiv (r, \theta, \phi)$ is the dummy index] that comes with a prefactor of ξ^2. But $\xi^2 = \xi_0^2\epsilon^2$; see Eq. (14). This fact is being stressed now, since we will be constructing a model here up to $\mathcal{O}(\epsilon^2)$ accuracy. Hence, in the following prescription, we have to consider $\hat{\mathcal{A}}$ to involve only the small-scale derivatives and keep it unperturbed, because even the small-scale Laplacian in $\hat{\mathcal{A}}$ already bears a ϵ^2 prefactor.

Now, we explicitly mention the decomposition of the spacetime derivatives over independent *multiple scales* in order to separate out the dynamics up to $\mathcal{O}(\epsilon^2)$. The multiple-scale perturbation as considered here is defined by

$$\tilde{\partial}_\mu \to \partial_\mu + \epsilon\partial_{\bar{\mu}}, \quad (21)$$

where μ and $\bar{\mu}$ on the rhs are merely the free indices subject to the restriction[15] as mentioned previously under Eq. (9).

From now on, throughout the paper,

(1) By "small-scale spacetime derivative," we refer to this $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ on the rhs of Eq. (21).

(2) By "large-scale spacetime derivative," we refer to the above mentioned $\partial_{\bar{\mu}} \equiv \frac{\partial}{\partial X^\mu}$.

III. THE MODEL: ANALOGUE GRAVITY PERSPECTIVE

In the following prescription, we will be deriving our proposed model in detail, where we show that the underlying nonrelativistic BEC system under consideration is very much capable of simulating the massive KG equation

[13] Though it is quite obvious from Eq. (18), we still write the velocity components as $\mathbf{v}(x) = v_r(\mathbf{r})\hat{r} + v_\theta(\mathbf{r})\hat{\theta} + v_\phi(\mathbf{r})\hat{\phi}$ for the sake of clarity, where $v_r(\mathbf{r}) = \frac{1}{m}\partial_r\vartheta_0(x)$, $v_\theta(\mathbf{r}) = \frac{1}{m}\frac{\partial_\theta\vartheta_0(x)}{r}$, and $v_\phi(\mathbf{r}) = \frac{1}{m}\frac{\partial_\phi\vartheta_0(x)}{r\sin\theta}$.

[14] For reference, in the context of fluid dynamics, see Eq. (4.15) of Ref. [35].

[15] It is quite redundant from Eq. (21) that $\tilde{\partial}_t \to \partial_t + \epsilon\partial_{\mathcal{T}}$, $\tilde{\partial}_r \to \partial_r + \epsilon\partial_R$, $\tilde{\partial}_\theta \to \partial_\theta + \epsilon\partial_\Theta$, $\tilde{\partial}_\phi \to \partial_\phi + \epsilon\partial_\Phi$.

for some scalar field in a curved background. Since many elementary particles in nature do have nonzero mass, this happens to be a very significant and essential step, as already pointed out by Visser *et al.* in 2005 [37], towards building some realistic analogue models rather than just providing some fictitious mathematical methodology.

In due course, we will deduce the explicit structure of the general acoustic metric in $(3+1)$D, and here we will stick to the signature[16] of the metric tensor as $(-,+,+,+)$.

A. Dynamics at different orders of ϵ

Now, Eq. (21) is applied to Eq. (20) and gives rise to a set of equations at different orders of ϵ. The full model, considered till $\mathcal{O}(\epsilon^2)$, is given by

$$\mathcal{O}(1) \Rightarrow \partial_\mu f^{\mu\nu}\partial_\nu \vartheta_1(x) = 0, \tag{22}$$

$$\mathcal{O}(\epsilon) \Rightarrow \{\partial_\mu f_1^{\mu\bar{\mu}}\partial_{\bar{\mu}} + \partial_{\bar{\nu}} f_1^{\bar{\nu}\nu}\partial_\nu\}(\vartheta_2(X)\vartheta_1(x)) = 0, \tag{23}$$

and

$$\mathcal{O}(\epsilon^2) \Rightarrow \partial_{\bar{\mu}} f_2^{\bar{\mu}\bar{\nu}}\partial_{\bar{\nu}}\vartheta_2(X) - \mathfrak{m}^2\vartheta_2(X) = 0; \tag{24}$$

where $[f^{\mu\nu}(x)]$, $[f_1^{\mu\bar{\mu}}(x,X)]$ or $[f_1^{\bar{\nu}\nu}(X,x)]$ and $[f_2^{\bar{\mu}\bar{\nu}}(X)]$ are all constructed as symmetric 4×4 matrices explicitly written later. Here $\mathfrak{m}$ is the mass of the large-length-scale phonon modes, while it is strikingly found to be a finite function of the $\vartheta_1(x)$ field, and thus Eq. (24) is the *massive* free KG equation for the field amplitude $\vartheta_2(X)$.

It is important to acknowledge the fact that, in the standard literature, one usually identifies this Eq. (22) as the massless minimally coupled KG equation for a scalar field $\vartheta_1(x)$; see Eqs. (248) and (254) of Ref. [3]. But in our present framework, on top of this usual massless picture at $\mathcal{O}(1)$, we come up with a massive KG equation in larger-length scales at $\mathcal{O}(\epsilon^2)$ subject to some constraint, given by Eq. (23), obtained at the intermediate $\mathcal{O}(\epsilon)$.

In the beginning, the full expression of the mass term in Eq. (24) contains a factor[17] of $R^2\sin\Theta$. But through the process of scale reversion, as we are about to see in the next section, this factor of $R^2\sin\Theta$ would obviously become $r^2\sin\theta$, giving rise to a rescaled mass $\mathfrak{m}$, which gets inserted into the scale-reversed massive KG equation [i.e., Eq. (31)] later. From now on, throughout the paper, whenever we speak about $\mathfrak{m}$, we would only refer to this rescaled $\mathfrak{m}$ appearing in Eq. (31). The general expression of $\mathfrak{m}$, which was obtained using the Mathematica 9.0 package, is extremely lengthy, and hence it is not shown explicitly in this paper. However, after some physical approximations, a relatively tidier version is presented later by Eq. (A1) in Appendix A.

Later, in Sec. IV D, while deriving the full mass term, we will talk about and clarify these steps one by one in detail. Now we are going to show the structures of the f-matrices to present our model. At $\mathcal{O}(1)$ in Eq. (22), the $[f^{\mu\nu}(x)]$ matrix is of the following form:

$$[f^{\mu\nu}] = \frac{r^2\sin\theta}{\mathsf{g}}\begin{pmatrix} -1 & -v_r & -\frac{v_\theta}{r} & -\frac{v_\phi}{r\sin\theta} \\ -v_r & (c_s^2 - v_r^2) & -\frac{v_r v_\theta}{r} & -\frac{v_r v_\phi}{r\sin\theta} \\ -\frac{v_\theta}{r} & -\frac{v_\theta v_r}{r} & \frac{(c_s^2-v_\theta^2)}{r^2} & -\frac{v_\theta v_\phi}{r^2\sin\theta} \\ -\frac{v_\phi}{r\sin\theta} & -\frac{v_\phi v_r}{r\sin\theta} & -\frac{v_\phi v_\theta}{r^2\sin\theta} & \frac{(c_s^2-v_\phi^2)}{r^2\sin^2\theta} \end{pmatrix}. \tag{25}$$

In Eq. (23), $[f_1^{\mu\bar{\mu}}(x,X)]$ and $[f_1^{\bar{\nu}\nu}(X,x)]$ are the two matrices with each and every corresponding entry being exactly the same, i.e.,

$$\begin{aligned}[f_1^{\mu\bar{\mu}}] &\equiv [f_1^{\bar{\nu}\nu}] \\ &= \frac{R^2\sin\Theta r^2\sin\theta}{\mathsf{g}} \\ &\quad\times\begin{pmatrix} -1 & -v_r & -\frac{v_\theta}{R} & -\frac{v_\phi}{R\sin\Theta} \\ -v_r & (c_s^2 - v_r^2) & -\frac{v_r v_\theta}{R} & -\frac{v_r v_\phi}{R\sin\Theta} \\ -\frac{v_\theta}{r} & -\frac{v_\theta v_r}{r} & \frac{(c_s^2-v_\theta^2)}{Rr} & -\frac{v_\theta v_\phi}{Rr\sin\Theta} \\ -\frac{v_\phi}{r\sin\theta} & -\frac{v_\phi v_r}{r\sin\theta} & -\frac{v_\phi v_\theta}{Rr\sin\theta} & \frac{(c_s^2-v_\phi^2)}{R\sin\Theta\, r\sin\theta} \end{pmatrix}.\end{aligned} \tag{26}$$

And finally, the $[f_2^{\bar{\mu}\bar{\nu}}(X)]$ matrix, appearing in Eq. (24), is given by

$$[f_2^{\bar{\mu}\bar{\nu}}] = \frac{R^2\sin\Theta}{\mathsf{g}}\begin{pmatrix} -1 & -v_r & -\frac{v_\theta}{R} & -\frac{v_\phi}{R\sin\Theta} \\ -v_r & (c_s^2 - v_r^2) & -\frac{v_r v_\theta}{R} & -\frac{v_r v_\phi}{R\sin\Theta} \\ -\frac{v_\theta}{R} & -\frac{v_\theta v_r}{R} & \frac{(c_s^2-v_\theta^2)}{R^2} & -\frac{v_\theta v_\phi}{R^2\sin\Theta} \\ -\frac{v_\phi}{R\sin\Theta} & -\frac{v_\phi v_r}{R\sin\Theta} & -\frac{v_\phi v_\theta}{R^2\sin\Theta} & \frac{(c_s^2-v_\phi^2)}{R^2\sin^2\Theta} \end{pmatrix}. \tag{27}$$

Up to this point, we have pretty much sketched the basic introduction of our proposed model, describing the dynamics of the phonon modes at different length scales and at different orders of the parameter ϵ. Our main

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[16] In fact, this convention of $(-,+,+,+)$ is clearly the reason that leads to generating a "−" sign in front of $\mathfrak{m}^2$ in Eq. (24).

[17] This can be regarded as a coordinate artifact due to the spherical polar coordinates. In Cartesian coordinates, the formation of $\partial_{\bar{\mu}} f_2^{\bar{\mu}\bar{\nu}}\partial_{\bar{\nu}}\vartheta_2(X)$ while constructing Eq. (24) would not have required a multiplication by any factor from the left, and consequently, the large-scale signature would have been completely absent in the expression of $\mathfrak{m}$. In our previous paper [see Eq. (23) of Ref. [41]], we were able to construct a compact form to present the mass term because our formalism was in $(3+1)$D Cartesian coordinates.

motive is to try to investigate the massive KG equation found at $\mathcal{O}(\epsilon^2)$ in detail through a simple mathematical framework.

B. Scale reversion of the dynamics ($\partial_{\bar{\mu}} \to \frac{1}{\epsilon}\partial_\mu$)

In our previous paper [41], we analyzed the situation in a flat background. But, for the large-scale dynamics, to see the curvature of spacetime, we must revert back systematically to the small-length scales, and this is exactly where our present analysis stands out to be very different from our previous analysis on flat spacetime.

On reversion back to the small-scale dynamics from the large scales, each large-scale spacetime derivative generates a factor of $1/\epsilon$ (i.e., $\partial_{\bar{\mu}} \to \frac{1}{\epsilon}\partial_\mu$) while undergoing the switching of scales.

Till now, we have yet to talk about the constraint equation we found at $\mathcal{O}(\epsilon)$; see Eq. (23). First of all, we would start by identifying the construction of two matrices, viz. $[f^{\mu\bar{\mu}}_{1,\mathbb{I}}(x,X)]$ and $[f^{\bar{\nu}\nu}_{1,\mathbb{II}}](X,x)$, given by

$$(R^2\sin\Theta \times [f^{\mu\bar{\mu}}_{1,\mathbb{I}}]) = [f^{\mu\bar{\mu}}_1] \equiv [f^{\bar{\nu}\nu}_1] = (r^2\sin\theta \times [f^{\bar{\nu}\nu}_{1,\mathbb{II}}]), \tag{28}$$

with reference to Eq. (26). This readily gives the permit to rewrite Eq. (23) in the following manner:

$$((R^2\sin\Theta)\times\partial_\mu f^{\mu\bar{\mu}}_{1,\mathbb{I}}\partial_{\bar{\mu}} + (r^2\sin\theta)\times\partial_{\bar{\nu}} f^{\bar{\nu}\nu}_{1,\mathbb{II}}\partial_\nu)\{\vartheta_2(X)\vartheta_1(x)\} = 0. \tag{29}$$

Through scale reversion, $R^2\sin\Theta$ written above naturally becomes $r^2\sin\theta \neq 0$, $\partial_{\bar{\mu}} \to \frac{1}{\epsilon}\partial_\mu$ as just mentioned above, and we find that both $[f^{\mu\bar{\mu}}_{1,\mathbb{I}}(x,X)]$ and $[f^{\bar{\nu}\nu}_{1,\mathbb{II}}(X,x)]$ strikingly take the form of $[f^{\mu\nu}(x)]$ as given by Eq. (25), and the field amplitude $\vartheta_2(X)$ gets scale-transformed to give rise to a new complex scalar function, hence we write $\{\vartheta_2(X)\vartheta_1(x)\} \to \Xi(x)$. Thus, Eq. (29) is rewritten in the scale-reversed form as

$$\partial_\mu f^{\mu\nu}\partial_\nu \Xi(x) = 0. \tag{30}$$

Hence, we come up with exactly the same dynamics for $\Xi(x)$ at $\mathcal{O}(\epsilon)$ as we had obtained for only $\vartheta_1(x)$ at $\mathcal{O}(1)$; see Eq. (22). This clearly indicates that we are not going to get anything new at this stage, because $\mathcal{O}(\epsilon)$ dynamics practically captures the same field over small-length scales. This happens particularly because there is no source term at $\mathcal{O}(\epsilon)$.

So we have to move on to analyzing the next order, i.e. $\mathcal{O}(\epsilon^2)$ dynamics, to see the new structure in the field after having it reverted back to the small scales. By inspection, it is quite evident that $[f^{\bar{\mu}\bar{\nu}}_2(X)]$ in Eq. (24) would again take the form exactly as $[f^{\mu\nu}(x)]$ on switching back to the small scales and $\vartheta_2(X) \to \varphi(x)$ as mentioned already. Therefore, Eq. (24) gets scale-transformed as the following:

$$\partial_\mu f^{\mu\nu}\partial_\nu\varphi(x) - \epsilon^2\mathfrak{m}^2\varphi(x) = 0, \tag{31}$$

where $[f^{\mu\nu}]$ is already given by Eq. (25).

Equation (31) is purposely multiplied by a real scalar constant g/c_s and gives rise to

$$\partial_\mu f^{\mu\nu}_{\text{New}}\partial_\nu\varphi(x) - \frac{\mathsf{g}}{c_s}\epsilon^2\mathfrak{m}^2\varphi(x) = 0, \tag{32}$$

where the contravariant f-matrix is now scaled up as

$$f^{\mu\nu}_{\text{New}} = \frac{\mathsf{g}}{c_s}[f^{\mu\nu}] = \frac{r^2\sin\theta}{c_s}\begin{pmatrix} -1 & -v_r & -\frac{v_\theta}{r} & -\frac{v_\phi}{r\sin\theta} \\ -v_r & (c_s^2 - v_r^2) & -\frac{v_r v_\theta}{r} & -\frac{v_r v_\phi}{r\sin\theta} \\ -\frac{v_\theta}{r} & -\frac{v_\theta v_r}{r} & \frac{(c_s^2 - v_\theta^2)}{r^2} & -\frac{v_\theta v_\phi}{r^2\sin\theta} \\ -\frac{v_\phi}{r\sin\theta} & -\frac{v_\phi v_r}{r\sin\theta} & -\frac{v_\phi v_\theta}{r^2\sin\theta} & \frac{(c_s^2 - v_\phi^2)}{r^2\sin^2\theta} \end{pmatrix}. \tag{33}$$

This process of scale reversion is actually a very important step here. Had we not switched back to the small scales, the spacetime metric arising out of $[f^{\bar{\mu}\bar{\nu}}_2(X)]$ corresponding to the large-scale dynamics would have defined the background geometry to be effectively flat, because its entries are basically the velocity field components, all of which were retained at small scales [see Eq. (27)] and thus act as just constants with respect to the large-scale spacetime derivatives.

C. The covariant massive KG equation

In order to cast the above Eq. (32) into a GCT-invariant (i.e., invariance under General Coordinate Transformation or in mathematical language - *Diffeomorphism*) form, it is required to identify the corresponding covariant structure, and hence the introduction of the effective metric (or in other words, the acoustic metric) in place of the respective f-matrix is essential. From this point onward, we talk about the *covariant* massive minimally coupled free KG equation.

Let $[g_{\mu\nu}(x)]$ be the general acoustic metric that actually defines the $(3+1)$D curved spacetime under consideration with its determinant, given by $g = \det[g_{\mu\nu}(x)]$. Considering Eq. (32), one identifies

$$f^{\mu\nu}_{\text{New}} = \sqrt{|g|}g^{\mu\nu} \tag{34}$$

$$\Rightarrow \quad \det[f^{\mu\nu}_{\rm New}] = \det[\sqrt{|g|}g^{\mu\nu}] = (\sqrt{|g|})^4 \det[g^{\mu\nu}] = (\sqrt{|g|})^4 g^{-1} = g.$$

$$\because \quad \det[f^{\mu\nu}_{\rm New}] = -c_s^2 r^4 \sin^2\theta, \qquad \therefore \quad g = -c_s^2 r^4 \sin^2\theta,$$

and obviously,
$$g^{\mu\nu} = \frac{1}{\sqrt{|g|}} f^{\mu\nu}_{\rm New} = \frac{1}{c_s r^2 \sin\theta} f^{\mu\nu}_{\rm New}. \tag{35}$$

Now, from Eq. (35), it is trivial to find the acoustic metric, which is of the following form:

$$[g_{\mu\nu}] = \begin{pmatrix} -(c_s^2 - \mathbf{v}^2) & -v_r & -v_\theta r & -v_\phi r\sin\theta \\ -v_r & 1 & 0 & 0 \\ -r v_\theta & 0 & r^2 & 0 \\ -r\sin\theta v_\phi & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{36}$$

It should be noted that, in general relativity, the spacetime metric (which does bear the feature of the background geometry) is related to the distribution of matter (i.e., the stress-energy tensor) through Einstein's field equations, whereas the acoustic metric $[g_{\mu\nu}(x)]$ here happens to be related to the background velocity field $[\mathbf{v}(\mathbf{r})]$ as well as the local speed of sound (c_s) in a much simpler algebraic fashion. Some striking features of this $[g_{\mu\nu}(x)]$ from its topological aspects and regarding "stable causality" have been discussed by Visser in pp. 1773–1774 of Ref. [6].

Finally, Eq. (32) is rewritten in the standard covariant form, given by

$$\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu)\varphi(x) - \frac{1}{\sqrt{|g|}}\frac{\mathsf{g}}{c_s}\epsilon^2\mathbf{m}^2\varphi(x) = 0,$$
$$\text{i.e.} \qquad (\nabla_\mu\nabla^\mu - \mathcal{M}^2)\varphi(x) = 0, \tag{37}$$

where ∇_μ is obviously the covariant derivative, and the final mass term $\mathcal{M}$ is determined through

$$\mathcal{M}^2 = \frac{1}{\sqrt{|g|}}\frac{\mathsf{g}}{c_s}\epsilon^2\mathbf{m}^2, \tag{38}$$

where $g = -c_s^2 r^4 \sin^2\theta$ in our present model [refer to Eq. (35)].

One thing to be understood is that if the mass term is dropped (i.e., $\mathcal{M} = 0$) from the solution of the scalar function $\varphi(x)$ as determined through Eq. (37) [or, in other words, Eq. (31)], then what we come up with as a solution is nothing but $\vartheta_1(x)$ via Eq. (22).

IV. THE CANONICAL ACOUSTIC BLACK HOLE

With a strong motive to examine how closely the acoustic metric can mimic the standard Schwarzschild geometry in gravity, one usually considers some specific symmetry in the analogue spacetime in order to move ahead. If one starts by considering an analogue gravity scenario in a spherically symmetric flow (we will consider the flow to be nonrelativistic here) of a barotropic, incompressible, inviscid fluid, one comes up with a solution called the *canonical acoustic black hole*, found by Visser [6] in 1998.

In principle, we would restrict ourselves only to stationary,[18] nonrotating, asymptotically flat canonical acoustic black holes. Thus, in our following prescription, the notions of *apparent* and *event horizons (acoustic)*[19] coincide, and the distinction becomes immaterial. In the language of standard general relativity, an event horizon is a null hypersurface that separates those spacetime points that are connected to infinity through a timelike path from those that are not [49].

Since a canonical acoustic black hole (BH), as considered here, is indeed stationary and asymptotically flat, every event horizon is a Killing horizon for some Killing vector field—say, σ^μ. Due to the time-translational and axial symmetry of the metric [refer to Eq. (43) later], obviously there are two Killing vector fields, viz. $\sigma_{(t)} \equiv \partial_t$ and $\sigma_{(\phi)} \equiv \partial_\phi$, which go from timelike to spacelike and vice versa at the event horizon. The *Killing horizon* is formally defined to be the null hypersurface on which the Killing vector field becomes null.

The acoustic (Killing) horizon is formed once the radial component of the background fluid velocity (v_r) exceeds the local speed of sound (c_s); refer to Eq. (45) of Ref. [6].

A. Massive KG equation in canonical spacetime

Since the classical mean field n_0 and $\vartheta_0(x)$ must satisfy the continuity equation [Eq. (6)], clearly

$$0 = \partial_t n_0 + \frac{1}{m}\boldsymbol{\nabla}.(n_0\boldsymbol{\nabla}\vartheta_0(x)) = n_0\boldsymbol{\nabla}.\mathbf{v}(x)$$
$$\Rightarrow |\mathbf{v}(x)| \propto \frac{1}{r^2}. \tag{39}$$

And thus, through a normalization constant finite $r_0 > 0$, the background velocity field[20] is set to be

$$|\mathbf{v}(\mathbf{r})| = v_r = c_s\frac{r_0^2}{r^2}, \qquad (\forall\ 0 < r < \infty). \tag{40}$$

Considering $v_\theta = 0 = v_\phi$, Eq. (36) gives rise to the exact acoustic metric that describes the present scenario. The line element is given by

[18]Stationary solutions are of special interest and significance because they are regarded as the "end states" of a gravitational collapse.

[19]The event horizon is a global feature; it could be difficult to actually locate such a boundary when handed with a metric in an arbitrary set of coordinates. Usually, it is defined to be the boundary of the region from which even the null geodesics cannot escape—strictly speaking, this is the *future* event horizon.

[20]Refer to Eq. (54) of Ref. [6].

$$ds^2 = -c_s^2 dt^2 + (dr \pm v_r dt)^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (41)$$

It is to be noted that when $v_r > 0$, which is also the usual convention, there would be a "−" sign in front of $v_r dt$ in the second term on the rhs of the above Eq. (41). Otherwise, there would be a "+" sign when $v_r < 0$; i.e., when the fluid flow is considered to be in the opposite direction.

Instead of the laboratory time t, one can now introduce the analogue Schwarzschild time coordinate τ via the simple coordinate transformation as

$$t \to \tau = t \mp \left(\frac{r_0}{2c_s}\tan^{-1}(r/r_0) + \frac{r_0}{4c_s}\ln\left|\frac{1-\frac{r}{r_0}}{1+\frac{r}{r_0}}\right|\right), \quad (42)$$

and, using Eqs. (40) and (41), it readily gives rise to a somewhat "Schwarzschild-like" line element[21] describing a canonical acoustic black hole, given by

$$ds^2 = -\frac{c_s^2}{r^4}\Delta(r)d\tau^2 + \frac{r^4}{\Delta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$\text{where} \qquad \Delta(r) = r^4 - r_0^4. \quad (43)$$

It is redundant to read off the acoustic metric from the above line element [Eq. (43)] as the following:

$$[g_{\mu\nu}]_{\text{Canonical BH}} \equiv \left(-\frac{c_s^2}{r^4}\Delta(r), \frac{r^4}{\Delta(r)}, r^2, r^2\sin^2\theta\right), \quad (44)$$

and we see that the spacetime of a canonical acoustic BH is asymptotically flat and naturally has a "physical singularity" at $r = 0$, which is again quite obvious from Eq. (40)—the background fluid velocity diverges at the center of the canonical acoustic BH. Evidently, r_0 is the Killing horizon (or the *sonic horizon*, to be more precise) of the canonical acoustic black hole. As far as the physical picture is concerned, beyond this point $r = r_0$; the fluid essentially becomes supersonic with respect to an observer sitting at some large $r \to \infty$; i.e., $v \geq c_s$ holds true $\forall r \leq r_0$, which is again quite obvious from Eq. (40) as well.

Through the acoustic metric $[g_{\mu\nu}(x)]_{\text{Canonical BH}}$ in Eq. (44), the covariant massive KG equation [Eq. (37)] in the spacetime of a canonical acoustic black hole boils down to the following form:

$$-\frac{r^4}{c_s^2\Delta(r)}\partial_{\tau\tau}\varphi(\tau,\mathbf{r}) + \frac{1}{r^2}\partial_r\left(\frac{\Delta(r)}{r^2}\partial_r\varphi(\tau,\mathbf{r})\right)$$
$$+\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\varphi(\tau,\mathbf{r})) + \frac{1}{r^2\sin^2\theta}\partial_{\phi\phi}\varphi(\tau,\mathbf{r})$$
$$-\mathcal{M}^2\varphi(\tau,\mathbf{r}) = 0. \quad (45)$$

The above Eq. (45) has some important features:

(1) The spacetime given by Eq. (43) is clearly *static* and has a time-translational symmetry. Thus, the temporal part of $\varphi(\tau,\mathbf{r})$ that solves the above differential equation [Eq. (45)] can easily be separated out as $e^{-i\omega\tau}$ ($\forall 0 < \omega < \infty$), where ω is the frequency (or equivalently, the *energy* in $\hbar = 1$ units) of the particles associated with the $\varphi(\tau,\mathbf{r})$ field.

(2) $[g_{\mu\nu}(x)]_{\text{Canonical BH}}$ describes a spacetime that also has a rotational invariance with respect to ϕ, and similarly the azimuthal part of the solution to Eq. (45) is obviously $e^{i\mathsf{m}\phi}$, where $\mathsf{m} = \pm 1, \pm 2, \pm 3, \ldots$ is the azimuthal quantum number.

(3) From Eq. (45), it is evident that the general angular solution can be given in terms of the standard spherical harmonics,

$$\mathcal{Y}^l_{\mathsf{m}}(\theta,\phi) = \mathcal{P}^l_{\mathsf{m}}(\cos\theta)e^{i\mathsf{m}\phi}, \quad (46)$$

where $\mathcal{P}^l_{\mathsf{m}}$'s are obviously the Legendre polynomials, with l being an integer such that $|\mathsf{m}| \leq l$.

B. The radial solution

Therefore, to solve Eq. (45), we can consider the following ansatz:

$$\varphi(\tau,\mathbf{r}) = \frac{1}{r}\mathcal{R}(r)\mathcal{Y}^l_{\mathsf{m}}(\theta,\phi)e^{-i\omega\tau}, \quad (47)$$

where $\mathcal{R}(r)$ is just the radial function to be determined.

Substituting Eq. (47) back into Eq. (45), we find that

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{\Delta(r)}{r^2}\frac{d}{dr}\left(\frac{\mathcal{R}(r)}{r}\right)\right)$$
$$+\left[\frac{\omega^2 r^4}{c_s^2\Delta(r)} - \left(\mathcal{M}^2 + \frac{l(l+1)}{r^2}\right)\right]\frac{\mathcal{R}(r)}{r} = 0. \quad (48)$$

Now, this has become a linear second-order ordinary differential equation in r for an undetermined function $\mathcal{R}(r)$. We go on reducing Eq. (48) further to check the singularity (if any) at various points, because in order to solve the radial differential equation, we are about to pick the Frobenius ansatz for $\mathcal{R}(r)$ and adopt the method of series solution.

By inspection, we find the nature of singularities (for detailed explanations, see Appendix B) of the above differential equation [Eq. (48)] as follows:

(1) $r = 0$ is a *regular* singular point. (It is to be understood clearly that $r = 0$ is indeed a point of physical singularity for the metric [Eq. (44)] itself, but not for the above ordinary differential equation [Eq. (48)]. For Eq. (48), $r = 0$ is a regular or removable singular point. Refer to Appendix B for a detailed argument).

(2) $r = r_0$ is also a *regular* singular point.

(3) The point $r \to \infty$ is an *irregular* singular point.

We introduce a new coordinate χ in order to simplify the structure of the above Eq. (48). The coordinate transformation, actually known as the *Eddington-Finkelstein*

[21] Refer to Eq. (56) of Ref. [44].

tortoise coordinates (also known as *Regge-Wheeler* coordinates), basically allows one to use the new coordinate χ even in the interior region of the acoustic black hole (i.e., when $r < r_0$).

In the present context,[22] the coordinate transformation is given by

$$r \to \chi \equiv \chi(r) = \pm \frac{r}{c_s} \mp \frac{r_0}{2c_s} \tan^{-1}\left(\frac{r}{r_0}\right) \pm \frac{r_0}{4c_s} \ln\left|\frac{1-\frac{r}{r_0}}{1+\frac{r}{r_0}}\right|. \tag{49}$$

Conventionally, $\chi = +\frac{r}{c_s} - \frac{r_0}{2c_s}\tan^{-1}(\frac{r}{r_0}) + \frac{r_0}{4c_s}\ln|\frac{1-\frac{r}{r_0}}{1+\frac{r}{r_0}}|$, and it is evident that

(1) χ approaches 0 as $r \to 0$.
(2) χ approaches $-\infty$ as $r \to r_0$ from either side of the acoustic Killing horizon.
(3) As $r \to +\infty$, χ approaches $+\infty$.

Hence, in the exterior region of the acoustic black hole, i.e. $\forall r_0 < r < +\infty$, χ is found to be continuous: $-\infty < \chi < +\infty$.

The tortoise coordinate is intended to grow infinitely at the appropriate rate such as to cancel out the singular behavior of the spacetime at $r = r_0$ [the coordinate singularity is quite vivid from Eq. (44)], which is essentially nothing but the artifact of the choice of coordinates.

Via the above transformation described in Eq. (49), one can easily reduce Eq. (48) to the following form:

$$\frac{d^2\mathcal{R}(r)}{d\chi^2} + \left[\omega^2 - \left(\mathcal{M}^2 + \frac{\bar{l}}{r^2} + \frac{4r_0^4}{r^6}\right)c_s^2\left(1 - \frac{r_0^4}{r^4}\right)\right]\mathcal{R}(r) = 0,$$
$$\text{where} \qquad \bar{l} \equiv l(l+1). \tag{50}$$

Our aim is to find a series solution of the above Eq. (50) valid in the exterior region of the spacetime; i.e., $\forall r > r_0$. We pick an ansatz, as chosen by Elizalde [45], of the following form:

$$\mathcal{R}(r) = \alpha e^{\pm i[\mathbb{k}\chi + h(\rho)]}, \tag{51}$$

where

$$\rho = 1 - \frac{r_0}{r}, \qquad h(\rho) = \beta \ln(1-\rho) + \sum_{n=1}^{\infty}(a_n \rho^n),$$
$$\text{necessarily,} \qquad a_1 \neq 0. \tag{52}$$

Here α is any arbitrary constant, while $\mathbb{k}$ in Eq. (51) and β in Eq. (52) are the constants to be determined. One should note that all the a_n's above are nothing but the Frobenius coefficients.

In order to avoid any conflict of notations, we would like to clearly mention here that this n in Eq. (52) is simply the dummy index of the infinite sum and has nothing to do with the n which was introduced previously as density in Eq. (5). From now on, we would only consider the $+$ sign in front of i on the rhs of Eq. (51), but one should notice that an ansatz with just $-i$ there would also do equally.

Clearly, in the exterior region, i.e. $\forall r \geq r_0$, we always have $0 \leq \rho \leq 1$ [see Eq. (52)], because

$$\lim_{r \to r_0} \rho = 0; \qquad \lim_{r \to \infty} \rho = 1. \tag{53}$$

This is exactly what justifies the form of $h(\rho)$ as considered in Eq. (52) to be legit in the exterior region.

Most of the tedious algebraic expressions are explicitly shown in the appendices, and we will be sketching only the important steps here. Inserting Eqs. (51) and (52) back into Eq. (50), followed by further simplifications [see Eqs. (C1) and (C2) in Appendix C for details], one can rewrite Eq. (50) in terms of the variable ρ as the following:

$$\begin{aligned}
&\left(-\frac{1}{r_0^2}\right)(1-\rho)^{10} \\
&\times\Bigg[\frac{\mathbb{k}^2 r_0^2}{(1-\rho)^{10}} + \frac{2\beta\mathbb{k}(\rho^3-4\rho^2+6\rho-4)\rho r_0 c_s}{(1-\rho)^9} + \beta c_s^2\left\{\frac{1}{(1-\rho)^4} - 1\right\}\left\{\frac{\beta - i}{(1-\rho)^4} - \beta + 5i\right\} \\
&+ \sum_{n=1}^{\infty}\Bigg\{\frac{n^2(\rho^3-4\rho^2+6\rho-4)^2 a_n^2 c_s^2 \boxed{\rho^{2n}}}{(1-\rho)^6} - \frac{1}{(1-\rho)^8}(i c_s n a_n \boxed{\rho^n}(\rho^3-4\rho^2+6\rho-4)(-c_s(1-\rho)[n(\rho^4-5\rho^3+10\rho^2-10\rho+4) \\
&+ \rho(2i\beta(\rho^3-4\rho^2+6\rho-4) + 5\rho^3 - 19\rho^2 + 26\rho - 14)] - 2i\mathbb{k}r_0))\Bigg\}\Bigg] \\
&+ \left[\omega^2 + \frac{1}{r_0^2}\{c_s^2\rho(\rho-1)^2(\rho^3-4\rho^2+6\rho-4)(\bar{l} + 4(\rho-1)^4) + \mathcal{M}^2\rho(\rho^3-4\rho^2+6\rho-4)r_0^2 c_s^2\}\right] = 0.
\end{aligned} \tag{54}$$

[22] In the case of the usual Schwarzschild metric in gravity, this tortoise coordinate becomes $\chi = r + r_s \ln|\frac{r}{r_s} - 1|$, where r_s is the Schwarzschild radius. See Eq. (5.108) of Ref. [49].

By clubbing the corresponding coefficients of the various powers of ρ from the first square bracket of the above equation, one keeps all of them on the left-hand side, while the second square bracket of the above equation is giving rise to the n-independent terms, all of which are moved to the right-hand side. Thus, the above Eq. (54) can be neatly rewritten as Eq. (C3); see Appendix C.

In order to find the recursion relation, this Eq. (C3) can now be compacted as

$$\text{lhs of Eq. (C3)} = \underbrace{\sum_{n=1}^{\infty}\sum_{k=0}^{10}\rho^{n+k}\mathfrak{f}_k^{\mathbb{I}}(n)}_{\mathcal{S}_1(\text{say})} + \underbrace{\sum_{n=1}^{\infty}\sum_{p=0}^{10}\rho^{2n+p}\mathfrak{f}_p^{\mathbb{II}}(n)}_{\mathcal{S}_2(\text{say})}$$

$$\Updownarrow$$

$$\text{rhs of Eq. (C3)} = \mathcal{F}(\rho) \equiv F_0\rho^0 + F_1\rho^1 + \cdots + F_{11}\rho^{11}, \tag{55}$$

having identified that the respective coefficient(s) of each power of ρ on both sides by some specified functions, given by $\mathfrak{f}_k^{\mathbb{I}}(n)$ $(\forall k)$ and $\mathfrak{f}_p^{\mathbb{II}}(n)$ $(\forall p)$, are obviously defined in consistence with their corresponding explicit forms written on the lhs of Eq. (C3). On the other hand, $F_0, F_1, \ldots, F_{11}$, as described above in Eq. (55), are all independent of n. These are the respective coefficients of $\rho^0, \rho^1, \ldots, \rho^{11}$ in the full source term $\mathcal{F}(\rho)$.

Our motive is to exhaust each and every term of $\mathcal{F}(\rho)$ by the corresponding term(s) picked from $\mathcal{S}_1$ and $\mathcal{S}_2$ via the power matching of ρ on both sides of Eq. (C3) and then try investigating the *recursion relation*. For the sake of clarity and lucidness, a vast part of the calculation[23] is shown in detail and step by step in Appendix C.

At this point, we need to refer to Eqs. (C6), (C9), (C11), and (C13) from Appendix C, and having these equations clubbed together, one can rewrite Eq. (55) in the following manner:

$$\begin{aligned}
&\because\quad (\mathcal{S}_1+\mathcal{S}_2) = \mathcal{F}(\rho)\\
&\Rightarrow \sum_{n=1}^{12-k-1}\sum_{k=0}^{10}\rho^{n+k}\mathfrak{f}_k^{\mathbb{I}}(n) + \sum_{n=1}^{\frac{12-p_1}{2}-1}\sum_{p_1=0,2,..}^{10}\rho^{2n+p_1}\mathfrak{f}_{p_1}^{\mathbb{II}}(n) + \sum_{n=1}^{\frac{13-p_2}{2}-1}\sum_{p_2=1,3,..}^{9}\rho^{2n+p_2}\mathfrak{f}_{p_2}^{\mathbb{II}}(n)\\
&\quad + \sum_{j=12,13,..}^{\infty}\left[\sum_{k=0}^{10}\mathfrak{f}_k^{\mathbb{I}}(j-k) + \underbrace{\sum_{p=0,1,..}^{10}\mathfrak{f}_p^{\mathbb{II}}\left(\frac{j-p}{2}\right)}_{\forall\ (j-p)=0,2,4,\ldots}\right]\rho^j\\
&= F_0\rho^0 + F_1\rho^1 + F_2\rho^2 + \cdots + F_{11}\rho^{11}\\
&\quad [\text{obviously}, F_0 = -\omega^2 r_0^2,\quad F_{11} = 0 \text{ from the rhs of Eq. (C3)}].
\end{aligned} \tag{56}$$

From the above Eq. (56), now one can evaluate the undetermined constants (viz. $\Bbbk$, β, a_n's) one by one which were introduced previously in Eqs. (51) and (52).

(1) By equating the coefficients of ρ^0 on both sides of Eq. (56), we get

$$-\Bbbk^2 r_0^2 = -\omega^2 r_0^2 \Rightarrow \Bbbk = \pm\omega. \tag{57}$$

It is to be noted that we take $\Bbbk = +\omega$ from now on in order to consider only the outgoing modes from the sonic horizon towards the external observer.

(2) By equating the coefficients of ρ^1 on both sides of Eq. (56), we get

$$\begin{aligned}
&8a_1c_s(-r_0\omega+2ic_s)\\
&\quad -4c_s(c_s(4i\beta+\bar{l}+4)+r_0^2\mathcal{M}^2c_s-2\beta r_0\omega)=0\\
&\quad \Rightarrow \beta = \frac{2a_1(4c_s^2+r_0^2\omega^2)+r_0\omega c_s(\bar{l}+\mathcal{M}^2r_0^2+4)}{8c_s^2+2r_0^2\omega^2}\\
&\qquad\qquad + i\left(\frac{c_s^2(\bar{l}+\mathcal{M}^2r_0^2+4)}{4c_s^2+r_0^2\omega^2}\right).
\end{aligned} \tag{58}$$

(3) By equating the coefficients of ρ^2 on both sides of Eq. (56), we get a_2 given by Eq. (C15); see Appendix C.

(4) And so on and so forth, by equating the coefficients of ρ^{11} on both sides of Eq. (56) in the same manner, one can determine a_{11} explicitly (in terms of a_1, which is kept nonzero arbitrary since the beginning).

[23]The calculation being extremely tedious, a part of it has been worked out via the Mathematica 9.0 package; however, the key steps are mentioned systematically.

(5) Finally, for any general j, one can find the coefficient a_j ($\forall j = 12, 13, 14, \ldots$) from the recursion relation, which is deduced later in Appendix C; see Eq. (C14).

It is to be noted that the recursion relation arises out of the square bracket on the lhs of Eq. (56) after having all the source terms fully exhausted. Though the explicit form of the mass term $\mathcal{M}$ is yet to be shown, the coefficients $\Bbbk$, β, a_n's ($\forall n \neq 1$) are all determined at this stage. Hence, through Eqs. (51) and (52), one basically gets the full structure of the radial solution $\mathcal{R}(r)$.

In order to find $\mathcal{M}$ explicitly, one requires the exact form of $\vartheta_1(x)$, which is nothing but the solution of Eq. (22).

C. Obtaining the usual massless scalar field $\vartheta_1(x)$

Like Eq. (47), one can consider an ansatz for the massless scalar field of the following form:

$$\vartheta_1(x) = \frac{1}{r}\mathcal{R}_1(r)\mathcal{Y}^l_{\mathrm{m}}(\theta,\phi)e^{-i\omega_1\tau}, \qquad \text{typically} \quad \omega_1 \gg \omega. \tag{59}$$

Inserting this into Eq. (45) with $\mathcal{M} = 0$ gives rise to a radial equation [similarly to Eq. (50)] of the following form:

$$\frac{d^2\mathcal{R}_1(r)}{d\chi^2} + \left[\omega_1^2 - \left(\frac{\bar{l}}{r^2} + \frac{4r_0^4}{r^6}\right)c_s^2\left(1 - \frac{r_0^4}{r^4}\right)\right]\mathcal{R}_1(r) = 0. \tag{60}$$

Keeping Eqs. (51) and (52) in mind, we choose $\mathcal{R}_1(r)$ to be of the following form:

$$\mathcal{R}_1(r) = \alpha_1 e^{\pm i[\Bbbk_1\chi + h_1(\rho)]}, \tag{61}$$

where

$$\rho = 1 - \frac{r_0}{r}, \qquad h_1(\rho) = \beta_1 \ln(1-\rho) + \sum_{n=1}^{\infty}(b_n\rho^n), \qquad \text{necessarily,} \quad b_1 \neq 0. \tag{62}$$

By inspection, we figure out the following:

(1) Just as with Eq. (57), we conclude that

$$\Bbbk_1 = \pm\omega_1, \tag{63}$$

and we again take the "+" sign for the outgoing modes.

(2) If we just drop the mass term in Eq. (58), we simply come up with β_1. Therefore,

$$\beta_1 \equiv \beta|_{\mathcal{M}=0} = \left(b_1 + \frac{(\bar{l}+4)r_0\omega_1 c_s}{8c_s^2 + 2r_0^2\omega_1^2}\right) + i\left(\frac{(\bar{l}+4)c_s^2}{4c_s^2 + r_0^2\omega_1^2}\right). \tag{64}$$

(3) Similarly, by having the mass term dropped from the expression of a_2 in Eq. (C15), we come up with b_2 given by Eq. (C16) in Appendix C.

(4) And so on, till b_{11} in the same manner, followed by the recursion relation for some general b_j ($\forall j = 12, 13, 14, \ldots$), helps determine the rest of the coefficients explicitly and all in terms of b_1.

1. Outside at a finite distance from the sonic horizon ($r \gtrsim r_0$)

If one considers the massless solution [see Eq. (59)] to be residing just outside the sonic horizon with respect to some external observer, then the radial coordinate of $\vartheta_1(x)$ is obviously almost of the same order of r_0—i.e., $r \gtrsim r_0$.

In this regime, the measure of the variable $\rho = 1 - \frac{r_0}{r}$ gives a very small number, and thus one can fairly restrict oneself to the first order of ρ while neglecting its higher powers throughout the calculations. And therefore, Eq. (62) is approximated to

$$h_1(\rho) \approx \beta_1 \ln(1-\rho) + b_1\rho. \tag{65}$$

Now, inserting Eqs. (63), (64), and (65) back into Eq. (61), we get

$$\mathcal{R}_1(r) \approx \alpha_1 \exp\left[\pm i\omega_1\chi \pm ib_1\left(1 - \frac{r_0}{r}\right) \pm \left\{\frac{(\bar{l}+4)c_s^2}{4c_s^2 + r_0^2\omega_1^2} + i\left(-b_1 - \frac{r_0\omega_1 c_s(\bar{l}+4)}{8c_s^2 + 2r_0^2\omega_1^2}\right)\right\}\ln\frac{r}{r_0}\right]. \tag{66}$$

Hence, from Eq. (59),

$$\boxed{\vartheta_1(x) \approx \frac{1}{r}\alpha_1\left(\frac{r}{r_0}\right)^{\pm\frac{(\bar{l}+4)c_s^2}{4c_s^2 + r_0^2\omega_1^2}} \exp\left[\pm i\left\{\omega_1\chi + b_1\left(1 - \frac{r_0}{r}\right) + \left(-b_1 - \frac{r_0\omega_1 c_s(\bar{l}+4)}{8c_s^2 + 2r_0^2\omega_1^2}\right)\ln\frac{r}{r_0}\right\}\right]\mathcal{Y}^l_{\mathrm{m}}(\theta,\phi)e^{-i\omega_1\tau}.} \tag{67}$$

The above Eq. (67) gives the final expression of the massless scalar field approximated up to linear ρ. Between the $\pm$ signs inside the exponent above, one should consider the "$-$" sign for the ingoing modes from the sonic horizon towards the center of the acoustic black hole, and the "$+$" sign for the outgoing modes. An important thing to be noted here is that the spatial growth of these short-wavelength modes goes as $\sim r^{(\omega_1^{-2})}$. This will later be compared with the growth of the large-wavelength amplitude modes $\varphi(x)$.

Now, we move ahead to find the expression of the mass term.

D. Deriving the full mass term $\mathcal{M}$

We can identify the following steps taken towards arriving at the complete expression of $\mathfrak{m}$, for a canonical acoustic BH, given by Eq. (A1) (see Appendix A):

(1) Equation (21) was applied on Eq. (20) throughout, keeping the small-scale Laplacians and the background velocity field $\mathbf{v}(x)$ unperturbed. Thus, we ended up with an enormously large set of terms and got them all segregated according to the different orders of the parameter ϵ.

(2) Out of that huge lot, we collected a pack of terms at $\mathcal{O}(\epsilon^2)$ and equated their sum total to zero in order to form an equation in larger scales where $\vartheta_1(x)$ was treated effectively as a constant.

(3) Among the terms written on the lhs of this equation, a number of terms got compacted as $\partial_{\bar{\mu}} f_2^{\bar{\mu}\bar{\nu}} \partial_{\bar{\nu}} \vartheta_2(X)$, while the rest were being identified as something proportional to the amplitude field, i.e. $-\mathfrak{m}^2\vartheta_2(X)$. The expression of $\mathfrak{m}$, until this step, would naturally contain a factor of $R^2 \sin\Theta$ while appearing in Eq. (24).

(4) After the scale reversion (see Sec. III B), this factor of $R^2 \sin\Theta$ became simply $r^2 \sin\theta$ and gave rise to the expression of a rescaled mass $\mathfrak{m}$ inserted in Eq. (31).

(5) In order to keep things from getting too messy and unnecessarily cluttered, we consider the background velocity to be $|\mathbf{v}(\mathbf{r})| = v_r(r)$, which is exactly nothing but the case for a canonical acoustic black hole; see Eq. (40). Thus, we get a tidier expression for $\mathfrak{m}$, given by Eq. (A1).

With the $\vartheta_1(x)$ field in hand, as shown[24] in Eq. (67), one can just readily evaluate $\mathfrak{m}|_{\text{for Canonical BH}}$ through Eq. (A1). Now, finding the final mass term $\mathcal{M}$ for a canonical acoustic black hole is simply redundant and a one-step process. Using Eq. (38),

$$\mathcal{M}^2|_{\text{for Canonical BH}} = \frac{1}{c_s r^2 \sin\theta} \frac{\mathrm{g}}{c_s} \epsilon^2 \mathfrak{m}^2 \Big|_{\text{for Canonical BH}}. \tag{68}$$

Since our formalism is restricted only within the domain of a canonical acoustic black hole, from now on, we can call off the subscript for the mass term(s) and write just $\mathcal{M}$ to refer to the mass term as expressed by the above Eq. (68).

While deriving $\mathcal{M}$, we again restrict ourselves to considering only the most dominant term(s).

After some trivial and tedious algebra, the expression of the mass term is finally given by

$$\mathcal{M} = \xi\left[\left(\frac{-176c_s^2\omega_1^2 + r_0^2\omega_1^4}{256c_s^4r_0^2} + i\frac{-48c_s^2\omega_1 + 3r_0^2\omega_1^3}{32c_s^3r_0^3}\right)\rho^{-4} + \mathcal{O}(\rho^{-3}) + \cdots\right]^{1/2} \tag{69}$$

$$\approx \mathcal{M}_{\mathcal{O}(\rho^{-4})}, \tag{70}$$

where

$$\boxed{\mathcal{M}^2_{\mathcal{O}(\rho^{-4})} = \frac{\xi^2}{(1-\frac{r_0}{r})^4} \frac{1}{32c_s^4r_0^3}\left(\frac{r_0}{8}(-176c_s^2\omega_1^2 + r_0^2\omega_1^4) + i3c_s(-16c_s^2\omega_1 + r_0^2\omega_1^3)\right).} \tag{71}$$

It is interesting to note that, as far as the most dominant terms are concerned, $\mathcal{M}_{\mathcal{O}(\rho^{-4})}$ happens to be independent of the choice of a particular spherical harmonic while picking $\vartheta_1(x)$ from Eq. (67). Hence, this is the most general expression of the mass term of the phonon modes associated with the $\varphi(x)$ field in a canonical spacetime within the regime not too far from the sonic horizon.

One may be interested in a more accurate measure of $\mathcal{M}$, and hence the subleading contributions could be relevant in that scenario. See Eq. (A2) in Appendix A, where we have given the next two subleading contributions in the expression of the mass term.

As is quite evident from the above Eq. (71), $\mathcal{M}_{\mathcal{O}(\rho^{-4})}$ does depend on the position r, and this kind of coordinate dependence of the mass term appears in many contexts of

[24]Out of two possible signs, we are interested only in the outgoing modes, and hence we consider the $+i$ in the exponent of the rhs of Eq. (67) throughout the paper.

physics—e.g., Ref. [37], where Visser *et al.* encountered a position-dependent mass. But if $\mathcal{M}$ is chosen from Eq. (69), then for an observer sitting at a very large r, the mass term becomes a real constant, i.e. $\lim_{r\to\infty}\mathcal{M} = \frac{\xi\omega_1^2}{c_s^2}$ in the asymptotic limit for any arbitrary $\omega_1 \in \Re$.

To keep the notations simpler, from now on, we will refer to $\mathcal{M}_{\mathcal{O}(\rho^{-4})}$ in Eq. (71) by calling it simply $\mathcal{M}$, since the following calculations are solely based on the leading-order contributions in the mass term.

E. Obtaining the massive scalar field $\varphi(x)$

Finally, we are on the verge of deriving the expression of the massive scalar field. As with Eq. (66), one can now obtain the radial contribution to the massive field up to linear ρ in order to consider only the leading-order contributions.

With the mass term $\mathcal{M}$ in hand, one obtains β from Eq. (58). Then, using Eqs. (51) and (52), we come up with

$$\mathcal{R}(r) \approx \alpha \exp\left[\pm i\omega\chi \pm i a_1\left(1-\frac{r_0}{r}\right) \pm \left\{\left(\frac{c_s^2(\bar{l}+\mathcal{M}^2 r_0^2+4)}{4c_s^2+r_0^2\omega^2}\right) + i\left(-a_1 - \frac{r_0\omega c_s(\bar{l}+\mathcal{M}^2 r_0^2+4)}{8c_s^2+2r_0^2\omega^2}\right)\right\}\ln\frac{r}{r_0}\right]. \tag{72}$$

Thus, the massive scalar field [see Eq. (47)] is finally given by

$$\varphi(x) \approx \frac{1}{r}\alpha \exp\left[\pm i\omega\chi \pm i a_1\left(1-\frac{r_0}{r}\right) \pm \left\{\left(\frac{c_s^2(\bar{l}+\mathcal{M}^2 r_0^2+4)}{4c_s^2+r_0^2\omega^2}\right) + i\left(-a_1 - \frac{r_0\omega c_s(\bar{l}+\mathcal{M}^2 r_0^2+4)}{8c_s^2+2r_0^2\omega^2}\right)\right\}\ln\frac{r}{r_0}\right]\mathcal{Y}^l_{\mathrm{m}}(\theta,\phi)e^{-i\omega\tau}$$
$$= \boxed{\begin{aligned}&\frac{1}{r}\alpha\left(\frac{r}{r_0}\right)^{\pm\left(\frac{2c_s^2(\bar{l}+4)+2c_s^2 r_0^2\Re(\mathcal{M}^2)+r_0^3\omega\Im(\mathcal{M}^2)}{8c_s^2+2r_0^2\omega^2}\right)}\\ &\times\exp\left[\pm i\left\{\omega\chi + a_1\left(1-\frac{r_0}{r}\right) - \left(a_1 + \frac{c_s r_0(\omega(\bar{l}+4)+r_0^2\omega\Re(\mathcal{M}^2)-2c_s r_0\Im(\mathcal{M}^2))}{8c_s^2+2r_0^2\omega^2}\right)\ln\frac{r}{r_0}\right\}\right]\mathcal{Y}^l_{\mathrm{m}}(\theta,\phi)e^{-i\omega\tau},\end{aligned}} \tag{73}$$

where $\mathcal{M}$ is picked from Eq. (71), with $\Re(\mathcal{M}^2)$ and $\Im(\mathcal{M}^2)$ being its real and imaginary parts, respectively.

The above expression clearly indicates that at a fixed r, the growth rate over space is $\sim r^{\frac{\|\mathcal{M}^2\|}{\omega^2}}$. So, from Eq. (71), the growth rate of these large-wavelength modes, for a specific mode of frequency ω, turns out to be actually $\sim r^{(\omega_1^4)}$, which encodes the information of the supposedly Hawking-radiated modes [i.e., the $\vartheta_1(x)$ field]. This gives rise to the low-frequency (or larger-wavelength) band of ω—i.e., the $\varphi(x)$ field, which, in the absence of the mass term (or in other words, when $\xi^2 \approx 0$), is not distinguishable from the primary ω_1 modes. Obviously, the smaller ω modes would grow faster and effectively extract more energy from the ω_1 modes which are supposedly Hakwing radiated.

V. DISCUSSION

In the present paper, we systematically analyzed the consequences of the presence of the quantum potential term in the dynamics of a condensate on the perspectives of analogue Hawking radiation. Here we have worked out this formulation for a canonical acoustic BH configuration in $(3+1)$D spacetime. That the quantum potential term causes a UV-IR coupling which can be separated as an independent dynamics at larger length scales without disturbing the Lorentz invariance (strictly speaking, actually *Diffeomorphism*) of the basic KG equation (massless) is something that we have already shown [41], and the present work extends the same method to curved spacetime.

The presence of the UV-IR coupling resulting from the quantum potential would cause short-wavelength modes to lose energy to large-wavelength ones, which show up as massive amplitude excitations of the high-frequency Hawking-radiated modes. In the actual experimental evaluation of analogue Hawking radiation, one cannot neglect these large-wavelength modes, which will grow from primary Hawking-radiated quanta and would cause an "information loss" of the actually Hawking-radiated modes.

Our present analysis shows that the growth rate of these large-wavelength (ω) modes, in a canonical spacetime, holds the clue to keep the underlying physics consistent. In general, a massless scalar field would grow over space near the analogue acoustic Killing/sonic horizon (the region which is accessible in the experiments) as something $\sim r^{(\omega_1^{-2})}$. On the contrary, the massive secondary excitations generated by these primary modes would grow over space as $\sim r^{(\omega_1^4\omega^{-2})}$ for large ω_1, but ω can be obtained easily from the temporal profile, i.e. $e^{-i\omega\tau}$ in Eq. (73), of the large-wavelength signal as received by the external observer. So a careful observation of the ω_1 dependance of the growth rates of these secondary modes can actually reveal the relative abundance of the originally Hawking-radiated quanta in the $(3+1)$D canonical spacetime.

These massive amplitude modes arise from the quantum connection which is $\mathcal{O}(\epsilon^2)$ small. But, at the same time, one should be ensured that ω_1 is typically large, and that makes this mechanism of secondary excitation generation absolutely relevant in the quantum fluids, like BEC.

We present in this paper a detailed derivation and analysis of these excitations generated by quantum potential which, in every likelihood, would be a dominant contributor to the loss of correlations which are instrumented in probing the analogue Hawking effect in such systems. We hope to extend our present analysis in deriving the correction to the correlations of Hawking-radiated quanta to other low-dimensional experimentally relevant systems within the scope of our framework.

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APPENDIX A: THE MASS TERM IN DETAILS

In order to tackle some untidy and too cluttered expressions in a proper presentable manner, we will be introducing a few new symbols here, viz. $\mathcal{Q}_1$, $\mathcal{Q}_2$, etc., whenever required.

The following expression of $\mathfrak{m}$ is written in the case of a canonical acoustic black hole, where the background velocity is selected by Eq. (40):

$$\mathfrak{m}^2|_{\text{for Canonical BH}} = \frac{\xi_0^2}{\mathsf{g}}\frac{\sin\theta}{r^2}\frac{1}{\vartheta_1(x)}\mathcal{Q}_1(x)|_{\text{for Canonical BH}}, \quad \text{where} \tag{A1a}$$

$$\begin{aligned}\mathcal{Q}_1(x)|_{\text{for Canonical BH}} &\equiv \mathcal{Q}_1(x)|_{v_\theta=0=v_\phi}\\
&= \Bigg[r^4 v_r''\frac{\partial^2\vartheta_1}{\partial t\partial r} + 2r^3\frac{\partial^3\vartheta_1}{\partial t^2\partial r} + 4r^3 v_r'\frac{\partial^2\vartheta_1}{\partial t\partial r} + r^2\frac{\partial^4\vartheta_1}{\partial t^2\partial\theta^2} + \frac{r^2}{\sin^2\theta}\frac{\partial^4\vartheta_1}{\partial t^2\partial\phi^2} + r^2\cot\theta\frac{\partial^3\vartheta_1}{\partial t^2\partial\theta} + r^2 v_r'\frac{\partial^3\vartheta_1}{\partial t\partial\theta^2}\\
&\quad + \frac{r^2}{\sin^2\theta}v_r'\frac{\partial^3\vartheta_1}{\partial t\partial\phi^2} + r^2\cot\theta v_r'\frac{\partial^2\vartheta_1}{\partial t\partial\theta} + r^4\frac{\partial^4\vartheta_1}{\partial t^2\partial r^2} + 3r^4 v_r'\frac{\partial^3\vartheta_1}{\partial t\partial r^2} + v_r\bigg\{2r^2\frac{\partial^4\vartheta_1}{\partial t\partial r\partial\theta^2} + \frac{2r^2}{\sin^2\theta}\frac{\partial^4\vartheta_1}{\partial t\partial r\partial\phi^2}\\
&\quad + 2r^2\cot\theta\frac{\partial^3\vartheta_1}{\partial t\partial r\partial\theta} + 2r^2\frac{\partial^2\vartheta_1}{\partial t\partial r} + 2r^2 v_r'\frac{\partial^3\vartheta_1}{\partial r\partial\theta^2} + \frac{2r^2}{\sin^2\theta}v_r'\frac{\partial^3\vartheta_1}{\partial r\partial\phi^2} + 2r^2\cot\theta v_r'\frac{\partial^2\vartheta_1}{\partial r\partial\theta} + 2r^4\frac{\partial^4\vartheta_1}{\partial t\partial r^3}\\
&\quad + 6r^3\frac{\partial^3\vartheta_1}{\partial t\partial r^2} + r^4 v_r^{(3)}\frac{\partial\vartheta_1}{\partial r} + 4r^3 v_r''\frac{\partial\vartheta_1}{\partial r} + 2r^2 v_r'\frac{\partial\vartheta_1}{\partial r} + 4r^4 v_r'\frac{\partial^3\vartheta_1}{\partial r^3} + 3r^4 v_r''\frac{\partial^2\vartheta_1}{\partial r^2} + 10r^3 v_r'\frac{\partial^2\vartheta_1}{\partial r^2}\bigg\}\\
&\quad + r^2 v_r^2\left(\frac{\partial^4\vartheta_1}{\partial r^2\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^4\vartheta_1}{\partial r^2\partial\phi^2} + \cot\theta\frac{\partial^3\vartheta_1}{\partial r^2\partial\theta} + 4r\frac{\partial^3\vartheta_1}{\partial r^3} + 2\frac{\partial^2\vartheta_1}{\partial r^2} + r^2\frac{\partial^4\vartheta_1}{\partial r^4}\right)\\
&\quad + r^4 v_r' v_r''\frac{\partial\vartheta_1}{\partial r} + 2r^3(v_r')^2\frac{\partial\vartheta_1}{\partial r} + 2r^4(v_r')^2\frac{\partial^2\vartheta_1}{\partial r^2}\Bigg].\end{aligned} \tag{A1b}$$

In the above expression, $v_r' \equiv \frac{\partial v_r}{\partial r}$, $v_r'' \equiv \frac{\partial^2 v_r}{\partial r^2}$, and $v_r^{(3)} \equiv \frac{\partial^3 v_r}{\partial r^3}$.

Only contributions up to leading order are considered in the final expression of the mass term in Eq. (71). Here we have given a more accurate measure by taking into account the next two subleading contributions, given by

$$\mathcal{M} \approx [\mathcal{Q}_3(r) + i\mathcal{Q}_4(r)]^{1/2}, \tag{A2}$$

where

$$\mathcal{Q}_3(r) = \frac{\xi^2\omega_1^2(-176c_s^2 + r_0^2\omega_1^2)}{256c_s^4 r_0^2(1-\frac{r_0}{r})^4} + \frac{\xi^2\omega_1^2(64c_s^4(88+7\bar{l}) - 8c_s^2(-113+\bar{l})r_0^2\omega_1^2 - 6r_0^4\omega_1^4)}{256c_s^4 r_0^2(4c_s^2 + r_0^2\omega_1^2)(1-\frac{r_0}{r})^3} + \frac{\xi^2\omega_1^2\bar{l}}{16c_s^2 r_0^2(1-\frac{r_0}{r})^2}$$

and

$$\mathcal{Q}_4(r) = \frac{\xi^2\omega_1(-48c_s^2 + 3r_0^2\omega_1^2)}{32c_s^3 r_0^3(1-\frac{r_0}{r})^4} + \frac{\xi^2\omega_1(64c_s^4(20+\bar{l}) + 2c_s^2(66-7\bar{l})r_0^2\omega_1^2 - 17r_0^4\omega_1^4)}{32c_s^3 r_0^3(4c_s^2 + r_0^2\omega_1^2)(1-\frac{r_0}{r})^3} + \frac{\xi^2\omega_1\bar{l}}{4c_s r_0^3(1-\frac{r_0}{r})^2}.$$

APPENDIX B: NATURE OF SINGULARITIES OF THE RADIAL DIFFERENTIAL EQUATION FOR THE CANONICAL ACOUSTIC BH

In this section, we investigate the nature of singularities of Eq. (48) at various points. We first rewrite it in the following manner:

$$\frac{d^2}{dr^2}\mathcal{R}(r)+\underbrace{\frac{4r_0^4}{r(r^4-r_0^4)}}_{\mathsf{p}(r)\text{ (say)}}\frac{d}{dr}\mathcal{R}(r)+\underbrace{\frac{\omega^2r^{10}-c_s^2\mathcal{M}^2(r^4-r_0^4)r^6-c_s^2(r^4-r_0^4)(l(l+1)r^4+4r_0^4)}{c_s^2r^2(r^4-r_0^4)^2}}_{\mathsf{q}(r)\text{ (say)}}\mathcal{R}(r)=0. \tag{B1}$$

(1) At $r=0$: $\lim_{r\to 0}\mathsf{p}(r)\to-\infty$, $\lim_{r\to 0}\mathsf{q}(r)\to\infty$; i.e., there is a singularity at the $r=0$ point. But we find that

$$\lim_{r\to 0}[r\mathsf{p}(r)]=-4,\qquad \lim_{r\to 0}[r^2\mathsf{q}(r)]=4, \tag{B2}$$

which refer to $r=0$ as a *regular* singular point.

(2) At $r=r_0$: $\lim_{r\to r_0}\mathsf{p}(r)\to\infty$, $\lim_{r\to r_0}\mathsf{q}(r)\to\infty$; i.e., there is again a singularity at the $r=r_0$ point. But

$$\lim_{r\to r_0}[(r-r_0)\mathsf{p}(r)]=1,\qquad \lim_{r\to r_0}[(r-r_0)^2\mathsf{q}(r)]=\frac{\omega^2r_0^2}{16c_s^2} \tag{B3}$$

hold true, which guarantees that $r=r_0$ is a *regular* singular point.

(3) At $r\to\infty$: We take $r=\frac{1}{r_*}$; thus, as $r\to\infty$, $r_*\to 0$. Now, obviously $\mathsf{p}(r=\frac{1}{r_*})$ becomes some $\mathsf{p}_1(r_*)$ and $\mathsf{q}(r=\frac{1}{r_*})$ becomes some $\mathsf{q}_1(r_*)$. We have

$$\lim_{r_*\to 0}\left[\frac{2}{r_*}-\frac{\mathsf{p}_1(r_*)}{r_*^2}\right]\to\infty,\qquad \lim_{r_*\to 0}\left[\frac{\mathsf{q}_1(r_*)}{r_*^4}\right]\to\infty, \tag{B4}$$

which basically means that $r_*\to 0$ or $r\to\infty$ is a singular point, while

$$\lim_{r_*\to 0}\left[r_*\left(\frac{2}{r_*}-\frac{\mathsf{p}_1(r_*)}{r_*^2}\right)\right]=2,\qquad \underbrace{\lim_{r_*\to 0}\left[r_*^2\left(\frac{\mathsf{q}_1(r_*)}{r_*^4}\right)\right]\to\infty.}_{\text{i.e., the limit does not exist}} \tag{B5}$$

APPENDIX C: THE NITTY-GRITTY DETAILS OF THE FROBENIUS SERIES SOLUTION AND THE RECURSION RELATION FOR THE CANONICAL ACOUSTIC BH

Inserting Eqs. (51) and (52) into Eq. (50), we calculate the lhs of Eq. (50) part by part.

The first part of the lhs of Eq. (50)

$$\begin{aligned}\Rightarrow\frac{d^2\mathcal{R}(r)}{d\chi^2}&=\mathcal{Q}_2(r)\times\left(-\frac{1}{r^{10}}\right)\Big[\mathbb{k}^2r^{10}+\beta c_s^2(r^4-r_0^4)(r^4(-i+\beta)-(-5i+\beta)r_0^4)-2\mathbb{k}r^5\beta c_s(r^4-r_0^4)\\&+\sum_{n=1}^{\infty}\Big\{-ina_nc_sr_0\underbrace{\left(1-\frac{r_0}{r}\right)^n}(r^3+r_0r^2+r_0^2r+r_0^3)(c_sr_0r^3(n-1)+c_sr_0^2r^2(n-1)+c_sr_0^4(2i\beta+n+5)\\&+c_sr_0^3r(n-1)+c_s(-2-2i\beta)r^4+2i\mathbb{k}r^5)+n^2a_n^2c_s^2r_0^2\underbrace{\left(1-\frac{r_0}{r}\right)^{2n}}(r^3+r_0r^2+r_0^2r+r_0^3)^2\Big\}\Big],\end{aligned} \tag{C1}$$

where the above two terms are underbraced for a reason, because, while switching from $r\to\rho$, these two terms obviously become ρ^n and ρ^{2n}, respectively, which are boxed in Eq. (54).

And the second part of the lhs of Eq. (50)

$$\Rightarrow \left[\omega^2 - \left(\mathcal{M}^2 + \frac{\bar{l}}{r^2} + \frac{4r_0^4}{r^6}\right)c_s^2\left(1 - \frac{r_0^4}{r^4}\right)\right]\mathcal{R}(r) = \mathcal{Q}_2(r) \times \left[\omega^2 - \frac{c_s^2(r^4 - r_0^4)(\bar{l}r^4 + \mathcal{M}^2 r^6 + 4r_0^4)}{r^{10}}\right],$$
$$\text{where}\quad \mathcal{Q}_2(r) = \alpha \exp\left[i\left\{\frac{\Bbbk}{4c_s}\left(4r - 2r_0\tan^{-1}\left(\frac{r}{r_0}\right) + r_0 \ln\left|\frac{1 - \frac{r}{r_0}}{1 + \frac{r}{r_0}}\right|\right) + \sum_n \left(a_n\left(1 - \frac{r_0}{r}\right)^n\right)\right\}\right]\left(\frac{r_0}{r}\right)^{i\beta}. \quad \text{(C2)}$$

Clearly, $\mathcal{Q}_2(r) \neq 0$, because $\alpha \neq 0$. Having the above two Eqs. (C1) and (C2) attached together, followed by a trivial algebraic manipulation, we actually come up with Eq. (54). Now, we present the corresponding coefficients of $\rho^n, \rho^{n+1}, \ldots \forall n = 1(1)\infty$ in a more jotted down way by rewriting the parent equation [i.e., Eq. (50)] in the following manner:

$$\begin{aligned}
&\underbrace{-\Bbbk^2 r_0^2} + \sum_{n=1}^{\infty}\Big[8ina_nc_s(2nc_s + i\Bbbk r_0)\underbrace{\rho^n} + 4na_nc_s(7\Bbbk r_0 + 2c_s(4\beta - 14in - 7i))\underbrace{\rho^{n+1}}\\
&\quad + 4ina_nc_s(c_s(48i\beta + 89n + 89) + 10i\Bbbk r_0)\underbrace{\rho^{n+2}} + 10na_nc_s(3\Bbbk r_0 + 2c_s(26\beta - 34in - 51i))\underbrace{\rho^{n+3}}\\
&\quad + 4ina_nc_s(7c_s(30i\beta + 31n + 62) + 3i\Bbbk r_0)\underbrace{\rho^{n+4}} + 2na_nc_s(\Bbbk r_0 + c_s(448\beta - 388in - 970i))\underbrace{\rho^{n+5}}\\
&\quad + ina_nc_s^2(656i\beta + 493n + 1479)\underbrace{\rho^{n+6}} + 110na_nc_s^2(3\beta - 2in - 7i)\underbrace{\rho^{n+7}} + 22ina_nc_s^2(5i\beta + 3n + 12)\underbrace{\rho^{n+8}}\\
&\quad + 2na_nc_s^2(11\beta - 6in - 27i)\underbrace{\rho^{n+9}} + ina_nc_s^2(2i\beta + n + 5)\underbrace{\rho^{n+10}} - 16n^2a_n^2c_s^2\underbrace{\rho^{2n}} + 112n^2a_n^2c_s^2\underbrace{\rho^{2n+1}}\\
&\quad - 356n^2a_n^2c_s^2\underbrace{\rho^{2n+2}} + 680n^2a_n^2c_s^2\underbrace{\rho^{2n+3}} - 868n^2a_n^2c_s^2\underbrace{\rho^{2n+4}} + 776n^2a_n^2c_s^2\underbrace{\rho^{2n+5}}\\
&\quad - 493n^2a_n^2c_s^2\underbrace{\rho^{2n+6}} + 220n^2a_n^2c_s^2\underbrace{\rho^{2n+7}} - 66n^2a_n^2c_s^2\underbrace{\rho^{2n+8}} + 12n^2a_n^2c_s^2\underbrace{\rho^{2n+9}} - n^2a_n^2c_s^2\underbrace{\rho^{2n+10}}\Big]\\
&= \underbrace{-\omega^2 r_0^2} + 4c_s(c_s(4i\beta + \bar{l} + 4) + \mathcal{M}^2r_0^2c_s - 2\beta\Bbbk r_0)\underbrace{\rho} - 2c_s(c_s(-8\beta^2 + 68i\beta + 7\bar{l} + 60)\\
&\quad + 3\mathcal{M}^2r_0^2c_s - 10\beta\Bbbk r_0)\underbrace{\rho^2} + 4c_s(\mathcal{M}^2r_0^2c_s + 5(-\beta\Bbbk r_0 + c_s(-4\beta^2 + 24i\beta + \bar{l} + 20)))\underbrace{\rho^3}\\
&\quad - c_s(\mathcal{M}^2r_0^2c_s + 5(-2\beta\Bbbk r_0 + 3c_s(-12\beta^2 + 64i\beta + \bar{l} + 52)))\underbrace{\rho^4} + 2c_s(-\beta\Bbbk r_0 + 3c_s(-40\beta^2 + 204i\beta + \bar{l} + 164))\underbrace{\rho^5}\\
&\quad + c_s^2(208\beta^2 - 1044i\beta - \bar{l} - 836)\underbrace{\rho^6} - (\beta^2 - 5i\beta - 4)\{120c_s^2\underbrace{\rho^7} + 45c_s^2\underbrace{\rho^8} - 10c_s^2\underbrace{\rho^9} + c_s^2\underbrace{\rho^{10}}\}. \qquad \text{(C3)}
\end{aligned}$$

Each power of ρ, in the above equation, is again purposely underbraced to depict the whole picture as vividly as possible in front of the reader.

In the following calculation, we would go on reducing Eq. (55) and evaluate the summations systematically. The intermediate steps are shown here to arrive at Eq. (56) in Sec. IV B. From Eq. (55),

$$\mathcal{S}_1 = \sum_{n=1}^{\infty}\sum_{k=0}^{10}\rho^{n+k}\mathfrak{f}_k^{\mathbb{I}}(n) = \left[\sum_{n=1}^{\infty}\rho^{n+0}\mathfrak{f}_0^{\mathbb{I}}(n) + \sum_{n=1}^{\infty}\rho^{n+1}\mathfrak{f}_1^{\mathbb{I}}(n) + \sum_{n=1}^{\infty}\rho^{n+2}\mathfrak{f}_2^{\mathbb{I}}(n) + \cdots + \sum_{n=1}^{\infty}\rho^{n+10}\mathfrak{f}_{10}^{\mathbb{I}}(n)\right]. \qquad \text{(C4)}$$

For any particular $0 \leq k \leq 10$, the general term being singled out from $\mathcal{S}_1$ is

$$\sum_{n=1}^{\infty} \rho^{n+k} \mathfrak{f}'_k(n) = \sum_{n=1}^{12-k-1} \rho^{n+k} \mathfrak{f}'_k(n) + \underbrace{\sum_{n=12-k}^{\infty} \rho^{n+k} \mathfrak{f}'_k(n)}_{\boxed{n+k=\lambda+12,\ \text{say}}}$$
$$= \underbrace{\sum_{n=1}^{12-k-1} \rho^{n+k} \mathfrak{f}'_k(n)}_{\text{finite sum}\quad \forall\ k} + \underbrace{\sum_{\lambda=0,1,..}^{\infty} \rho^{\lambda+12} \mathfrak{f}'_k(\lambda+12-k)}_{\boxed{\lambda+12=j,\ \text{say}}} \tag{C5}$$

$$\therefore\quad \mathcal{S}_1 = \sum_{n=1}^{12-k-1} \sum_{k=0}^{10} \rho^{n+k} \mathfrak{f}'_k(n) + \underbrace{\sum_{j=12,13,..}^{\infty} \left(\sum_{k=0}^{10} \mathfrak{f}'_k(j-k) \right) \rho^j}_{\mathfrak{s}_1(\text{say})}. \tag{C6}$$

Similarly, like Eq. (C4),

$$\mathcal{S}_2 = \sum_{n=1}^{\infty} \sum_{p=0}^{10} \rho^{2n+p} \mathfrak{f}''_p(n)$$
$$= \Bigg[\underbrace{\sum_{n=1}^{\infty} \rho^{2n+0} \mathfrak{f}''_1(n) + \sum_{n=1}^{\infty} \rho^{2n+2} \mathfrak{f}''_2(n) + \cdots + \sum_{n=1}^{\infty} \rho^{2n+10} \mathfrak{f}''_{10}(n)}_{\text{say, } \mathcal{S}'_2 \ \text{ where } \ \forall\ p \equiv p_1 = 0,2,\ldots,10} + \underbrace{\sum_{n=1}^{\infty} \rho^{2n+1} \mathfrak{f}''_1(n) + \sum_{n=1}^{\infty} \rho^{2n+3} \mathfrak{f}''_3(n) + \cdots + \sum_{n=1}^{\infty} \rho^{2n+9} \mathfrak{f}''_9(n)}_{\text{say, } \mathcal{S}''_2 \ \text{ where } \ \forall\ p \equiv p_2 = 1,3,\ldots,9} \Bigg]. \tag{C7}$$

Now, the general term from the above $\mathcal{S}'_2$ is singled out as the following:

$$\sum_{n=1}^{\infty} \rho^{2n+p_1} \mathfrak{f}''_{p_1}(n) = \sum_{n=1}^{\frac{12-p_1}{2}-1} \rho^{2n+p_1} \mathfrak{f}''_{p_1}(n) + \underbrace{\sum_{n=\frac{12-p_1}{2}}^{\infty} \rho^{2n+p_1} \mathfrak{f}''_{p_1}(n)}_{\boxed{2n+p_1=\lambda_1+12,\ \text{say}}}$$
$$= \underbrace{\sum_{n=1}^{\frac{12-p_1}{2}-1} \rho^{2n+p_1} \mathfrak{f}''_{p_1}(n)}_{\text{finite sum}\quad \forall\ p_1} + \underbrace{\sum_{\lambda_1=0,2,..}^{\infty} \rho^{\lambda_1+12} \mathfrak{f}''_{p_1}\left(\frac{\lambda_1+12-p_1}{2} \right)}_{\boxed{\lambda_1+12=j,\ \text{say}}} \tag{C8}$$

$$\therefore\quad \mathcal{S}'_2 = \sum_{n=1}^{\frac{12-p_1}{2}-1} \sum_{p_1=0,2,..}^{10} \rho^{2n+p_1} \mathfrak{f}''_{p_1}(n) + \underbrace{\sum_{j=12,14,..}^{\infty} \left(\sum_{p_1=0,2,..}^{10} \mathfrak{f}''_{p_1}\left(\frac{j-p_1}{2} \right) \right) \rho^j}_{\mathfrak{s}_2(\text{say})}. \tag{C9}$$

Again, one can single out the general term from $\mathcal{S}''_2$ as well:

$$\begin{aligned}\sum_{n=1}^{\infty}\rho^{2n+p_2}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p_2}(n) &= \sum_{n=1}^{\frac{13-p_2}{2}-1}\rho^{2n+p_2}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p_2}(n) + \underbrace{\sum_{n=\frac{13-p_2}{2}}^{\infty}\rho^{2n+p_2}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p_2}(n)}_{\boxed{2n+p_2=\lambda_2+12,\ \text{say}}} \\ &= \underbrace{\sum_{n=1}^{\frac{13-p_2}{2}-1}\rho^{2n+p_2}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p_2}(n)}_{\text{finite sum}\quad\forall\ p_2} + \underbrace{\sum_{\lambda_2=1,3,..}^{\infty}\rho^{\lambda_2+12}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p_2}\left(\frac{\lambda_2+12-p_2}{2}\right)}_{\boxed{\lambda_2+12=j,\ \text{say}}}\end{aligned} \tag{C10}$$

$$\therefore\quad \mathcal{S}_2^{\mathbb{I}\mathbb{I}} = \sum_{n=1}^{\frac{13-p_2}{2}-1}\sum_{p_2=1,3,..}^{9}\rho^{2n+p_2}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p_2}(n) + \underbrace{\sum_{j=13,15,..}^{\infty}\left(\sum_{p_2=1,3,..}^{9}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p_2}\left(\frac{j-p_2}{2}\right)\right)\rho^{j}}_{\mathfrak{s}_3(\text{say})}. \tag{C11}$$

One can notice that

(1) For $\mathfrak{s}_2$ in Eq. (C9), the necessary condition is that $(j-p_1)$ has to always be even with $\forall p_1 = 0, 2, \ldots, 10$ and $\forall j = 12, 14, \ldots, \infty$.

(2) For $\mathfrak{s}_3$ in Eq. (C11), the necessary condition is that $(j-p_2)$ has to always be even with $\forall p_2 = 1, 3, \ldots, 9$ and $\forall j = 13, 15, \ldots, \infty$.

Clearly, the j's are different in $\mathfrak{s}_2$ and $\mathfrak{s}_3$, but since the corresponding ρ^j's in Eqs. (C9) and (C11) are all linearly independent, we can practically club these two separate infinite summations into a single one. Thus,

$$\begin{aligned}\mathcal{S}_2 = (\mathcal{S}_2^{\mathbb{I}} + \mathcal{S}_2^{\mathbb{I}\mathbb{I}}) &\xrightarrow[\text{summations}]{\text{infinite}} (\mathfrak{s}_2 + \mathfrak{s}_3) \\ &= \sum_{j=12,14,..}^{\infty}\left(\sum_{p_1=0,2,..}^{10}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p_1}\left(\frac{j-p_1}{2}\right)\right)\rho^{j} + \sum_{j=13,15,..}^{\infty}\left(\sum_{p_2=1,3,..}^{9}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p_2}\left(\frac{j-p_2}{2}\right)\right)\rho^{j} \\ &= \underbrace{\sum_{j=12,13,..}^{\infty}\left(\sum_{p=0,1,..}^{10}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p}\left(\frac{j-p}{2}\right)\right)\rho^{j}}_{\mathfrak{s}_{2,3}(\text{say})}\quad (\text{provided}\quad \forall\ (j-p) = 0, 2, 4, \ldots).\end{aligned} \tag{C12}$$

And through Eqs. (C6) and (C12),

$$\begin{aligned}(\mathcal{S}_1 + \mathcal{S}_2) \xrightarrow[\text{summations}]{\text{infinite}} (\mathfrak{s}_1 + \mathfrak{s}_{2,3}) &= \sum_{j=12,13,..}^{\infty}\left(\sum_{k=0}^{10}\mathfrak{f}^{\mathbb{I}}_{k}(j-k)\right)\rho^{j} + \sum_{j=12,13,..}^{\infty}\left(\sum_{p=0,1,..}^{10}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p}\left(\frac{j-p}{2}\right)\right)\rho^{j} \\ &= \sum_{j=12,13,\ldots}^{\infty}\left[\sum_{k=0}^{10}\mathfrak{f}^{\mathbb{I}}_{k}(j-k) + \underbrace{\sum_{p=0,1,..}^{10}\mathfrak{f}^{\mathbb{I}\mathbb{I}}_{p}\left(\frac{j-p}{2}\right)}_{\forall\ (j-p)=0,2,4,\ldots,\infty}\right]\rho^{j}.\end{aligned} \tag{C13}$$

After having these Eqs. (C6), (C9), (C11), and (C13) clubbed together, we have written the lhs of Eq. (56), where the last square bracket now generates a recursion relation quite naturally. This helps us to find out the Frobenius coefficient(s) for any arbitrary $j = 12, 13, 14, \ldots$, provided $(j-p) = 0, 2, 4, \ldots$ holds true $\forall p = 0, 1, \ldots, 10$. The recursion relation is explicitly given by

$$
\begin{aligned}
a_j = & -\frac{1}{32j(2c_s j + i\omega r_0)}\Big[a_{j-1}(-16(j-1)\{28jc_s - 14c_s + i(8\beta c_s + 7r_0\omega)\}) \\
& + a_{j-2}(16(j-2)\{89jc_s - 89c_s + i(48\beta c_s + 10r_0\omega)\}) + a_{j-3}(-40(j-3)\{68jc_s - 102c_s + i(52\beta c_s + 3r_0\omega)\}) \\
& + a_{j-4}(16(j-4)\{217jc_s - 434c_s + i(210\beta c_s + 3r_0\omega)\}) + a_{j-5}(-8(j-5)\{388jc_s - 970c_s + i(448\beta c_s + r_0\omega)\}) \\
& + a_{j-6}(4(j-6)c_s\{493j - 1479 + 656i\beta\}) + a_{j-7}(-440(j-7)c_s\{2j - 7 + 3i\beta\}) + a_{j-8}(88(j-8)c_s\{3j - 12 + 5i\beta\}) \\
& + a_{j-9}(-8(j-9)c_s\{6j - 27 + 11i\beta\}) + a_{j-10}(4(j-10)c_s\{j - 5 + 2i\beta\}) \\
& + \boxed{a^2_{\frac{j}{2}}}(16ic_s j^2) + \boxed{a^2_{\frac{j-1}{2}}}(-112ic_s(j-1)^2) + \boxed{a^2_{\frac{j-2}{2}}}(356ic_s(j-2)^2) + \boxed{a^2_{\frac{j-3}{2}}}(-680ic_s(j-3)^2) \\
& + \boxed{a^2_{\frac{j-4}{2}}}(868ic_s(j-4)^2) + \boxed{a^2_{\frac{j-5}{2}}}(-776ic_s(j-5)^2) + \boxed{a^2_{\frac{j-6}{2}}}(493ic_s(j-6)^2) + \boxed{a^2_{\frac{j-7}{2}}}(-220ic_s(j-7)^2) \\
& + \boxed{a^2_{\frac{j-8}{2}}}(66ic_s(j-8)^2) + \boxed{a^2_{\frac{j-9}{2}}}(-12ic_s(j-9)^2) + \boxed{a^2_{\frac{j-10}{2}}}(ic_s(j-10)^2)\Big]. \qquad \text{(C14)}
\end{aligned}
$$

It is quite evident that the boxed coefficients written above do not contribute anything to a_j only when $(j-p)$ is found to be an odd number $\forall p = 0, 1, \ldots, 10$. Here, in Eq. (C14), a_j is expressed in terms of the coefficients, all of which are predetermined, and thus the recursion relation is consistent.

By equating the coefficients of ρ^2 on both sides of Eq. (56), we get the coefficient a_2:

$$
\begin{aligned}
& \frac{1}{(2c_s + ir_0\omega)^2}(4c_s(2(a_1 - 2a_2)(2c_s + ir_0\omega)^2(r_0\omega - 4ic_s) + \mathcal{M}^4 r_0^4 c_s^3 + \mathcal{M}^2 r_0^2 c_s(-16ir_0\omega c_s + 2(\bar{l} - 10)c_s^2 + r_0^2\omega^2) \\
& \quad + c_s(-8i(\bar{l} - 4)r_0\omega c_s + (\bar{l} - 12)\bar{l}c_s^2 - (\bar{l} + 20)r_0^2\omega^2))) = 0, \\
& \Rightarrow a_2 = \frac{1}{4(4c_s^2 + r_0^2\omega^2)^2(16c_s^2 + r_0^2\omega^2)}(2a_1(16c_s^2 + r_0^2\omega^2)(4c_s^2 + r_0^2\omega^2)^2 + r_0\omega c_s(-\mathcal{M}^2 r_0^2(\mathcal{M}^2(r_0^4\omega^2 c_s^2 - 20r_0^2 c_s^4) \\
& \quad + 2(\bar{l} + 44)r_0^2\omega^2 c_s^2 + 8(18 - 5\bar{l})c_s^4 + r_0^4\omega^4) - (\bar{l}(\bar{l} + 72) + 144)r_0^2\omega^2 c_s^2 + 4(\bar{l}(5\bar{l} - 28) - 128)c_s^4 + (\bar{l} + 20)r_0^4\omega^4)) \\
& \quad + \frac{i}{(4c_s^2 + r_0^2\omega^2)^2(16c_s^2 + r_0^2\omega^2)}(2c_s^2(\mathcal{M}^4(2r_0^4 c_s^4 - r_0^6\omega^2 c_s^2) + \mathcal{M}^2 r_0^2(-2(\bar{l} + 9)r_0^2\omega^2 c_s^2 + 4(\bar{l} - 10)c_s^4 + r_0^4\omega^4) \\
& \quad - (\bar{l}(\bar{l} + 10) - 40)r_0^2\omega^2 c_s^2 + 2(\bar{l} - 12)\bar{l}c_s^4 + 2(\bar{l} + 8)r_0^4\omega^4)). \qquad \text{(C15)}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad b_2 \equiv a_2|_{\mathcal{M}=0} = & \left(\frac{b_1}{2} + \frac{4(\bar{l}(5\bar{l} - 28) - 128)r_0\omega_1 c_s^5 - (\bar{l}(\bar{l} + 72) + 144)r_0^3\omega_1^3 c_s^3 + (\bar{l} + 20)r_0^5\omega_1^5 c_s}{4(4c_s^2 + r_0^2\omega_1^2)^2(16c_s^2 + r_0^2\omega_1^2)}\right) \\
& + i\left(\frac{-2(\bar{l}(\bar{l} + 10) - 40)r_0^2\omega_1^2 c_s^4 + 4(\bar{l} + 8)r_0^4\omega_1^4 c_s^2 + 4(\bar{l} - 12)\bar{l}c_s^6}{(4c_s^2 + r_0^2\omega_1^2)^2(16c_s^2 + r_0^2\omega_1^2)}\right). \qquad \text{(C16)}
\end{aligned}
$$

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