Renormalizability, van Dam-Veltman-Zakharov discontinuity, and Newtonian singularity in higher-derivative gravity

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It was proposed that if a higher-derivative gravity is renormalizable it implies necessarily a finite Newtonian potential at the origin, but the reverse of this statement is not true. Here, we show that the reverse is true when taking into account the van Dam-Veltman-Zakharov discontinuity, which states that the theory obtained from the massive one by taking a zero mass limit is not equivalent to the theory obtained in the zero mass case. The surviving degree of freedom in the zero mass limit is an extra scalar that does not affect the light bending angle but affects the Newtonian potential. This asserts that in order to make the singularity cancellation the number of massive ghost and healthy tensors matches with that of massive ghost and healthy scalars.

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I. INTRODUCTION

It was shown that the renormalizability in higher-derivative gravity might be related to the behavior of the classical potential of the model. Explicitly, there is a conjecture that renormalizable higher-derivative gravity has a finite Newtonian potential at the origin [1,2]. This relation was first notified in Stelle's seminal work [3], which showed that the fourth-derivative gravity is renormalizable and nonunitary and has a finite potential at the origin. In this case, a massive ghost tensor and a massive healthy scalar contribute in such a manner that they cancel out the Newtonian singularity of a massless tensor. Recently, it was conjectured that the reverse of the above statement is not true, which indicates that the finiteness of the Newtonian potential at the origin is a necessary, but not sufficient condition, for the renormalizability of the model [4,5]. The model used in Refs. [5,6] includes a massive ghost tensor, massive ghost and heathy scalars, and a healthy tensor. Even though the potential is finite at $r = 0$, it is nonrenormalizable by power counting. Actually, two massive (ghost and healthy) scalars make a contribution $1/3$ to the massless and massive ghost tensors $(1 - 4/3)$. This is not the case in which the number of massive ghost (healthy) tensors matches with the number of massive healthy (ghost) scalars.

On the other hand, it is known that Fierz and Pauli (FP) in 1939 obtained five propagating degrees of freedom (d.o.f.) of a massive tensor by adding a mass term of $m^2(h_{\mu\nu}h^{\mu\nu} - h^2)/2$ to the bilinearized Einstein-Hibert action [7]. It is well known that a massless tensor has 2 d.o.f. An inevitable mismatch between massive and massless cases was first realized in 1970 by van Dam and Veltman [8] and independently by Zakharov [9] (vDVZ). Then, it is known as the vDVZ discontinuity, which states that the theory obtained from the massive one by taking the zero mass limit ($m \to 0$) is not equivalent to the theory obtained in the zero mass case ($m = 0$). Especially, the former has 3 d.o.f., while the latter has 2 d.o.f., which shows a difference of 1 d.o.f. When using the Stueckelberg formalism [10], one may find the origin of the vDVZ discontinuity [11]. After applying this formalism to the FP massive gravity action, a scalar field that was introduced by Stueckelberg to maintain the gauge symmetry was coupled to the external source. The coupling between the source and the Stueckelberg field was identified as the origin of the vDVZ discontinuity. This Stueckelberg scalar behaves as an attractive force in the theory and affects the Newtonian potential but not the light bending angle. We note that the Newtonian potential of $V_{m=0}(r) = -GM/r$ differs from $V_{m\to 0}(r) = -\frac{4}{3}GM/r$ in the zero mass limit of $e^{-mr} \approx 1$. Clearly, the scalar causes the mismatch $(-1/3)$ in the Newtonian potential between massless and massive gravities.

Hence, we propose that the vDVZ discontinuity is related closely to the singularity cancellation of the Newtonian potential at the origin.

In this work, we explore why the matching of the number of ghost and healthy modes between the spin-2 and spin-0 massive sectors is necessary to make the singularity cancellation at the origin. This will be explained by introducing the vDVZ discontinuity appearing in the zero mass limit of the massive gravity.

The organization of our work is as follows. In Sec. II, we study the fourth-derivative gravity as a toy model of higher-derivative gravities. We will explain the singularity cancellation by introducing the zero mass limit where the vDVZ discontinuity occurs. This model shows that the theory without any kind of nonlocality could be free from the Newtonian singularity. Section III is devoted to

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explaining the finiteness of the Newtonian potential obtained from a full sixth-derivative gravity by taking the small mass limit. We present in Sec. IV that the half sixth-derivative gravity is not a suitable model, which explains the connection between the finiteness of the potential and renormalizability. This is because this model lacks the matching of the number of healthy and ghost modes between spin-2 and spin-0 massive sectors needed to implement a singularity cancellation. In Sec. V, we introduce a polynomial form of infinite-derivative gravity to explain the singularity cancellation by making use of the vDVZ discontinuity. Finally, we discuss our results in Sec. VI.

II. FOURTH-DERIVATIVE THEORY OF GRAVITY

The fourth-derivative gravity in four dimensions is defined by

$$S_{4\text{th}} = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + \frac{\alpha}{2}R^2 + \frac{\beta}{2}R^2_{\mu\nu}\right] \tag{1}$$

with $\kappa^2 = 4\kappa_4(\kappa_4 = 8\pi G)$. This action was first employed to prove the renormalizability of fourth-derivative gravity [3], and hence it is considered a prototype of higher-derivative gravities. A key point is that the last two terms are necessary to achieve the renormalizability, but the last term induces the ghost state. We have seen that the fourth-derivative gravity is not a healthy theory because of a massive ghost tensor that violates the unitarity at tree level. Recently, it was confirmed that renormalizable higher-derivative gravities are nonunitary [6]. Hence, the ghost problem is not a relevant issue in this work. We are interested in exploring the connection between renormalizability of the theory and finiteness of the Newtonian potential at the origin.

To find the Newtonian potential, one has to first compute the propagator. For this purpose, we expand the metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ around the Minkowski metric $\eta_{\mu\nu}=\text{diag}(+,-,-,-)$. Bilinearizing the Lagrangian in Eq. (1) together with imposing the de Donder gauge of $\mathcal{L}_{\text{gf}} = -(\partial^\mu h_{\mu\nu} - \partial_\nu h/2)^2/2\lambda$, one obtains $\mathcal{L}^{\text{bil}}_{4\text{th}} = h_{\mu\nu}\mathcal{O}^{\mu\nu,\alpha\beta}h_{\alpha\beta}$ [6]. Inverting $\mathcal{O}$, one obtains the propagator for the fourth-derivative gravity,

$$\mathcal{D}^{4\text{th}}_{\mu\nu,\alpha\beta}(k) = \left[\frac{1}{k^2} - \frac{1}{k^2 - m_2^2}\right]P^{(2)} - \frac{1}{2}\left[\frac{1}{k^2} - \frac{1}{k^2 - m_0^2}\right]P^{(0-s)} + (\cdots), \tag{2}$$

where $P^{(2)}$ and $P^{(0-s)}$ represent the Barnes-Rivers operators,

$$P^{(2)}_{\mu\nu,\alpha\beta} = \frac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta}, \tag{3}$$

$$P^{(0-s)}_{\mu\nu,\alpha\beta} = \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta}, \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \tag{4}$$

while $(\cdots)$ denotes the set of terms that are irrelevant to the spectrum of the theory. Here, the spin-2 and spin-0 masses squared are given by, respectively,

$$m_2^2 = -\frac{4}{\beta\kappa^2}, \qquad m_0^2 = \frac{2}{\kappa^2(3\alpha+\beta)}. \tag{5}$$

We require $\beta < 0$ and $3\alpha + \beta > 0$ for having nontachyonic masses. The propagator (2) carries 8 d.o.f.: massless tensor $(2\ \text{d.o.f.})$ + massive tensor$(5\ \text{d.o.f.})$ + massless scalar$(1\ \text{d.o.f.})$. We would like to note that Eq. (2) without $(\cdots)$ represents a gauge-invariant part of the propagator. At this stage, it is worth noting that for large momentum we have $\theta_{\mu\nu} \sim 1$, which implies that the power counting argument is valid here because the massive tensor propagator takes the form of $\frac{P^{(2)}(\theta)}{k^2 - m_2^2} \sim \frac{1}{k^2}$. However, we have $\frac{P^{(2)}(\tilde\theta)}{k^2 - m^2} \sim \frac{k^2}{m^2}$ for the FP massive gravity with $\tilde\theta_{\mu\nu} = \eta_{\mu\nu} - k_\mu k_\nu/m^2$ [11], which implies that the power counting argument does not work here and thus one cannot deduce the renormalizability of the FP massive gravity [12].

The spatial part of the gauge-invariant propagator (2) takes the form

$$\begin{aligned}\mathcal{P}^{4\text{th}}_{\mu\nu,\alpha\beta}(\mathbf{k}) = &-\frac{1}{\mathbf{k}^2}\left[\frac{1}{2}(\eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\kappa}) - \frac{1}{2}\eta_{\mu\nu}\eta_{\kappa\lambda}\right]\\ &+ \frac{1}{\mathbf{k}^2 + m_2^2}\left[\frac{1}{2}(\eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\kappa}) - \frac{1}{3}\eta_{\mu\nu}\eta_{\kappa\lambda}\right]\\ &- \frac{1}{\mathbf{k}^2 + m_0^2}\frac{\eta_{\mu\nu}\eta_{\kappa\lambda}}{6},\end{aligned} \tag{6}$$

where the second coefficient $1/2(= 1/3 + 1/6)$ in the first line differs from $1/3$ in the second line. We note the relation between the Newtonian potential sourced by a static mass M and propagator:

$$V(r) = \frac{\kappa_4 M}{(2\pi)^3}\int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}\mathcal{P}_{00,00}(\mathbf{k}). \tag{7}$$

Fourier transforming

$$\mathcal{P}^{4\text{th}}_{00,00}(\mathbf{k}) = \frac{1}{2}\left[-\frac{1}{\mathbf{k}^2} + \frac{4}{3}\frac{1}{\mathbf{k}^2 + m_2^2} - \frac{1}{3}\frac{1}{\mathbf{k}^2 + m_0^2}\right] \tag{8}$$

leads to the Newtonian potential as

$$V^{4\text{th}}(r) = \frac{GM}{r}\left[-1 + \frac{4}{3}e^{-m_2 r} - \frac{1}{3}e^{-m_0 r}\right], \tag{9}$$

which was already deduced by Stelle's work [3].

Here, we point out that in the limit of $r \to 0$ a massive ghost tensor contributes $4/3(=1+1/3)$ to the Newtonian potential and a massive healthy scalar contributes $-1/3$ to the potential. The singularity cancellation occurs in the fourth-derivative gravity. This model shows that the theory without any kind of nonlocality could be free from the Newtonian singularity.

We need to explain the singularity cancellation by introducing a different mechanism instead of taking the $r = 0$ limit. First of all, we wish to explain the appearance of "4/3" explicitly. In the zero mass limit of $m_2 \to 0(e^{-m_2 r} \approx 1)$, one has not 1 but 4/3 zeroth-order term, which indicates that the vDVZ discontinuity occurs in the linearized fourth-derivative gravity. We could not distinguish the zero mass limit from the $r \to 0$ limit because two cases provide the same zeroth-order term of $e^{-m_2 r} \approx 1$ in the Newtonian potential. However, one has to focus on the zero mass limit to introduce the other mechanism of the vDVZ discontinuity. The vDVZ discontinuity dictates that the gravity theory obtained from massive one with 5 d.o.f. by taking the zero mass limit is not equivalent to the gravity theory obtained in the zero mass case. Especially, the former has 3 d.o.f., while the latter has 2 d.o.f. which describes a massless tensor. A physical explanation of this phenomenon is that a massive tensor with mass m carries five polarizations, while a massless tensor carries only two [13]. In the zero mass limit of $m \to 0$, a massive tensor decomposes into massless fields of spin-2, 1, and 0. The spin-0 field couples to the trace of the stress-energy tensor. Therefore, in the zero mass limit, one does not recover the Einstein gravity but rather a scalar-tensor theory.

The discontinuity can be easily found by noting the difference in coefficients between the massless tensor propagator (the first line) and massive ghost tensor one (the second line) in Eq. (6). In the former, we have $1/2$, whereas we have $1/3$ in the latter. Thus, the zero mass limit of the massive propagator does not coincide with the massless propagator.

It is worth noting that surviving d.o.f. in the zero mass limit is 3 d.o.f. (1 d.o.f. is represented by an extra ghost scalar and the other 2 d.o.f. are given by a ghost tensor). This massive ghost scalar with $1/3$ could be identified with the Stueckelberg scalar [11], and it cancels against a massive healthy scalar with $-1/3$. On the other hand, a ghost tensor with 1 cancels out a healthy tensor (Newton term with -1). The vDVZ discontinuity explains why $\mathcal{O}(1)/r$ disappears as well. Hence, we may avoid the singularity at the origin.

Consequently, the cancellation of singularity occurs because there is contribution (-1, $4/3$, $-1/3$) from 6 d.o.f. [a massless tensor with 2^+, a massless limit of massive ghost tensor with $3^-(=2^-+1^-)$, and a massive scalar with 1^+, where the superscripts $+(-)$ represent healthy (ghost) d.o.f.]. Considering 8 d.o.f. of the theory initially, it is clear that the vDVZ discontinuity occurs in the zero mass limit of a massive ghost tensor.

III. FULL SIXTH-DERIVATIVE THEORY OF GRAVITY

An action for full sixth-derivative gravity takes the form

$$S_{6\text{th}} = \int d^4x\sqrt{-g}\frac{1}{\kappa^2}\left[2R + \frac{\alpha_0}{2}R^2 + \frac{\beta_0}{2}R^2_{\mu\nu} + \frac{\alpha_1}{2}R\Box R + \frac{\beta_1}{2}R_{\mu\nu}\Box R^{\mu\nu}\right]. \tag{10}$$

It was proven that the action (10) becomes super-renormalizable because the superficial divergence $\delta[=4+E-\sum_{n=3}^{\infty}(n-2)V_n]$ decreases as the number of vertices V_n increases [4,6]. An important point to remember is that the same order of the last two six-derivative terms in (10) is a key to guaranteeing the renormalizability. The propagator is found to be

$$\mathcal{D}^{6\text{th}}(k) = \left[\frac{1}{k^2} + \frac{1}{m^2_{2_+} - m^2_{2_-}}\left(\frac{m^2_{2_-}}{k^2 - m^2_{2_+}} - \frac{m^2_{2_+}}{k^2 - m^2_{2_-}}\right)\right]P^{(2)} - \frac{1}{2}\left[\frac{1}{k^2} + \frac{1}{m^2_{0_+} - m^2_{0_-}}\left(\frac{m^2_{0_-}}{k^2 - m^2_{0_+}} - \frac{m^2_{0_+}}{k^2 - m^2_{0_-}}\right)\right]P^{(0-s)} + (\cdots), \tag{11}$$

where masses squared $m^2_{2\pm}$ and $m^2_{0\pm}$ are defined by

$$m^2_{2\pm} = \frac{\beta_0}{2\beta_1}\left(1 \pm \sqrt{1 + \frac{16\beta_1}{\beta_0^2}}\right),$$

$$m^2_{0\pm} = \frac{3\alpha_0 + \beta_0}{2(3\alpha_1 + \beta_1)}\left(1 \pm \sqrt{1 - \frac{8(3\alpha_1 + \beta_1)}{(3\alpha_0 + \beta_0)^2}}\right).$$

Here, one requires $\beta_0 < 0$ and $\beta_1 < 1$ to have nontachyonic masses. In this case, $\mathcal{D}^{6\text{th}}(k)$ describes propagations of 14 $(=2+5+5+1+1)$ d.o.f. Its spatial part of the gauge-invariant propagator is given by

$$\mathcal{P}^{6\text{th}}_{00,00}(\mathbf{k}) = \frac{1}{2}\left[-\frac{1}{\mathbf{k}^2} + \frac{4}{3}\frac{1}{m^2_{2_+} - m^2_{2_-}}\left(\frac{m^2_{2_+}}{\mathbf{k}^2 + m^2_{2_-}} - \frac{m^2_{2_-}}{\mathbf{k}^2 + m^2_{2_+}}\right) - \frac{1}{3}\frac{1}{m^2_{0_+} - m^2_{0_-}}\left(\frac{m^2_{0_+}}{\mathbf{k}^2 + m^2_{0_-}} - \frac{m^2_{0_-}}{\mathbf{k}^2 + m^2_{0_+}}\right)\right]. \tag{12}$$

The particle content of the model is made up of three healthy particles (a massless tensor, a massive tensor with mass m_{2+}, and a massive scalar with m_{0-}) and two ghosts (a massive tensor with m_{2-} and a massive scalar with m_{0+}).

In this case, the Newtonian potential generated by static mass M is derived to be

$$V^{6\mathrm{th}}(r) = \frac{GM}{r}\left[-1 + \frac{4}{3}\frac{m_{2_+}^2 e^{-m_{2_-} r} - m_{2_-}^2 e^{-m_{2_+} r}}{m_{2_+}^2 - m_{2_-}^2} - \frac{1}{3}\frac{m_{0_+}^2 e^{-m_{0_-} r} - m_{0_-}^2 e^{-m_{0_+} r}}{m_{0_+}^2 - m_{0_-}^2}\right]. \quad (13)$$

The massive particle content is made by taking the small mass limit ($e^{-m_i r} \approx 1$) of massive ghost and healthy tensors with 6 d.o.f.,

$$\frac{4}{3}\left\{\frac{m_{2_+}^2}{m_{2_+}^2 - m_{2_-}^2}, -\frac{m_{2_-}^2}{m_{2_+}^2 - m_{2_-}^2}\right\} \Rightarrow \frac{4}{3} = 1 + \frac{1}{3}, \quad (14)$$

which provides $4/3$ in the zeroth-order amplitude. Here, we call the zero mass limit as the small mass limit because the masses are not zero but they are so small that one can take $e^{-m_i r} \approx 1$. Also, the massive healthy and ghost scalars with 2 d.o.f. provide

$$-\frac{1}{3}\left\{\frac{m_{0_+}^2}{m_{0_+}^2 - m_{0_-}^2}, -\frac{m_{0_-}^2}{m_{0_+}^2 - m_{0_-}^2}\right\} \Rightarrow -\frac{1}{3}, \quad (15)$$

which provides $-1/3$ in the zeroth-order amplitude. Considering 14 d.o.f. of the theory, we have 10 $(= 2^+ + 3^- + 3^+ + 1^+ + 1^-)$ d.o.f. in the Newtonian potential. This shows clearly that the vDVZ discontinuity occurs in the small mass limit of massive ghost and healthy tensors. We note that the singularity cancellation in $V_{6\mathrm{th}}(r)$ occurs either in the small mass limit or the $r \to 0$ limit, which gives the same zeroth-order approximation of $e^{-m_i^2 r} \approx 1$ for $i = 2_\pm, 0_\pm$.

This model gives an affirmative answer to the conjecture that the cancellation mechanism of the singularity is the matching of the ghost and healthy modes between the spin-2 and spin-0 massive sectors: $(3^-, 3^+)$ and $(1^+, 1^-)$.

IV. HALF SIXTH-DERIVATIVE THEORY OF GRAVITY

It was proposed that if a higher-derivative gravity is renormalizable it implies necessarily a finite Newtonian potential at the origin, but the reverse of this statement is not true. A relevant action is given by the half six-derivative gravity as [5,6,14]

$$S_{\mathrm{h6th}} = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + a_0 R^2 + a_1 R\Box R + b_0 R_{\mu\nu}^2\right]. \quad (16)$$

This action is not renormalizable because different derivative orders lose renormalizability [4]. A matching factor of the sixth-derivative term $R_{\mu\nu}\Box R^{\mu\nu}$ was missed in the action (16). Its propagator takes the form

$$\mathcal{D}^{\mathrm{h6th}}(k) = \left[\frac{1}{k^2} + \frac{1}{k^2 - \tilde m_2^2}\right]P^{(2)} - \frac{1}{2}\left[\frac{1}{k^2} + \frac{1}{\tilde m_{0_+}^2 - \tilde m_{0_-}^2}\left(\frac{\tilde m_{0_-}^2}{k^2 - \tilde m_{0_+}^2} - \frac{\tilde m_{0_+}^2}{k^2 - \tilde m_{0_-}^2}\right)\right]P^{(0-s)} + (\cdots), \quad (17)$$

where masses squared $\tilde m_2^2$ and $\tilde m_{0\pm}^2$ are defined by

$$\tilde m_2^2 = -\frac{4}{b_0\kappa^2},$$
$$\tilde m_{0\pm}^2 = \frac{3a_0 + b_0 \pm \sqrt{(3a_0 + b_0)^2 - 24a_1/\kappa^2}}{6a_1}.$$

The propagator (17) describes $9\ (= 2 + 5 + 1 + 1)$ d.o.f. of the theory. In this case, the (00,00)-spatial part of the propagator is given by

$$\mathcal{P}^{\mathrm{h6th}}_{00,00}(\mathbf{k}) = \frac{1}{2}\left[-\frac{1}{\mathbf{k}^2} + \frac{4}{3}\frac{1}{\mathbf{k}^2 + \tilde m_2^2} - \frac{1}{3}\frac{1}{\tilde m_{0_+}^2 - \tilde m_{0_-}^2}\left(\frac{\tilde m_{0_+}^2}{\mathbf{k}^2 + \tilde m_{0_-}^2} - \frac{\tilde m_{0_-}^2}{\mathbf{k}^2 + \tilde m_{0_+}^2}\right)\right]. \quad (18)$$

The potential is given by

$$V^{\mathrm{h6th}}(r) = \frac{GM}{r}\left[-1 + \frac{4}{3}e^{-\tilde m_2 r} - \frac{1}{3}\frac{\tilde m_{0_+}^2 e^{-\tilde m_{0_-} r} - \tilde m_{0_-}^2 e^{-\tilde m_{0_+} r}}{\tilde m_{0_+}^2 - \tilde m_{0_-}^2}\right]. \quad (19)$$

The singularity is cancelled, despite the fact that there is no massive healthy tensor to balance a massive ghost scalar. It seems that this case gives a negative answer to the conjecture that the cancellation mechanism of the singularity requires the matching of the ghost and healthy modes between the spin-2 and spin-0 massive sectors. Even though the cancellation of singularity occurs in the $r \to 0$ limit of $e^{-\tilde m_i r} \approx 1$ for $i = 2, 0_\pm$, this is not the case. Considering 9 d.o.f. of the theory, we have 7 $(= 2^+ + 3^- + 1^+ + 1^-)$ d.o.f. in the Newtonian potential. Here, the vDVZ discontinuity occurs only in the small mass limit of a massive ghost tensor, leaving a mismatch for a massive ghost scalar with 1^-. The number of massive excitations in each sector should be the same. That is, there exists a massive healthy tensor to each ghost mode in scalar sector and vice versa. However, we have a particle content of $(3^-, \bullet)$ and $(1^+, 1^-)$. In this case, a massive healthy tensor with 5 d.o.f. is necessary to make a renormalizable theory like the full sixth-derivative gravity (10) and to make a balance with a massive ghost scalar. Then, $\bullet$ is given by 3^+.

Similarly, we propose that the other half six-derivative gravity,

$$\tilde{S}_{\rm h6th} = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + a_0R^2 + b_0R_{\mu\nu}^2 + b_1R_{\mu\nu}\Box R^{\mu\nu}\right]. \tag{20}$$

is not renormalizable but it has a finite Newtonian potential at the origin. This model provides a particle content of $(3^-, 3^+)$ and $(1^+, \bullet)$. Here, a massive ghost scalar $(\bullet)$ is needed to make a renormalizable theory like the full sixth-derivative gravity (10).

V. INFINITE-DERIVATIVE GRAVITY

In this section, we wish to comment on the connection between a nonsingular Newtonian potential and the vDVZ discontinuity in infinite-derivative gravity (IDG) [15,16]. It was shown that the infinite-derivative gravity has provided a finite Newtonian at the origin [16–19].

A simplest model of IDG is given by [2]

$$S_{\rm IDG} = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + G_{\mu\nu}\frac{a(\Box_\Lambda)-1}{\Box}R^{\mu\nu}\right]. \tag{21}$$

The propagator of this IDG takes the form

$$\mathcal{D}^{\rm IDG}(k) = \frac{1}{k^2a(-k^2/\Lambda^2)}\left[P^{(2)} - \frac{P^{(0-s)}}{2}\right], \tag{22}$$

which shows a ghost-free propagator of a massless tensor. If one chooses $a(\Box_\Lambda) = e^{-\Box/\Lambda^2}$ [20], there is no room to introduce masses of massive tensors. Its potential is found to be

$$V^{\rm IDG}(r) = -\frac{GM}{r}{\rm Erf}\left(\frac{\Lambda r}{2}\right), \tag{23}$$

which is a nonsingular potential at the origin because the error function takes the form of ${\rm Erf}(x) \sim 2x/\sqrt{\pi}$ as $x \to 0$. Here, the error function is defined through (7) by

$${\rm Erf}\left(\frac{\Lambda r}{2}\right) = \frac{2}{\pi}\int_0^\infty d|\mathbf{k}|\frac{e^{-|\mathbf{k}|^2/\Lambda^2}\sin[|\mathbf{k}|r]}{|\mathbf{k}|}. \tag{24}$$

It is important to note that, although one does not require the small mass limit, the singularity disappears due to the nonlocality and the effect depends on a form of $a(\Box)$. Hence, it seems that the singularity cancellation has nothing to do with the vDVZ discontinuity. Interestingly, the super-renormalizable and nonlocal massive gravity has provided a massive propagator [21],

$$\mathcal{D}^{\rm SN}(k) = \frac{e^{-H(k^2/M^2)}}{k^2-m^2}\left[P^{(2)} - \frac{P^{(0-s)}}{2} + \xi\left(P^{(1)} + \frac{\tilde{P}^{(0)}}{2}\right)\right], \tag{25}$$

which is the same form as that of a massless tensor except an overall factor. If one takes the zero mass limit of $m \to 0$, the massive propagator reduces smoothly to the massless one, which shows that there is no vDVZ discontinuity.

To study the connection between a finite Newtonian potential and the vDVZ discontinuity, we consider a polynomial action of IDG defined by [2]

$$\tilde{S}^{\rm IDG} = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}[-2R + RF_1(\Box)R + R_{\mu\nu}F_2(\Box)R^{\mu\nu}], \tag{26}$$

where

$$F_1(\Box) = \alpha_0 + \alpha_1\Box + \ldots + \alpha_N\Box^N, \tag{27}$$

$$F_2(\Box) = \beta_0 + \beta_1\Box + \ldots + \beta_N\Box^N. \tag{28}$$

The $N = 0[N = 1]$ model corresponds to the fourth-derivative gravity (1) [sixth-derivative gravity (10)] with different coefficients. It requires the two polynomials of F_1 and F_2 to be of the same order. A potential generated by a static mass M can be expressed as

$$\tilde{V}^{\rm IDG}(r) = -\frac{2GM}{\pi r}\int_0^\infty\frac{dk}{k}\sin[kr]\left[\frac{4}{3}\frac{1}{P_{2N+2}(k)} - \frac{1}{3}\frac{1}{Q_{2N+2}(k)}\right], \tag{29}$$

where $P_{2N+2}(k)$ and $Q_{2N+2}(k)$ are polynomials of spin-2 and spin-0 massive sectors given by

$$P_{2N+2}(k) = 1 + \frac{1}{2}[\beta_0k^2 - \beta_1k^4 + \ldots + (-1)^N\beta_Nk^{2N+2}], \tag{30}$$

$$Q_{2N+2}(k) = 1 - (3\alpha_0 + \beta_0)k^2 + (3\alpha_1 + \beta_1)k^4 + \ldots + (-1)^N(3\alpha_N + \beta_N)k^{2N+2}. \tag{31}$$

Factorizing P_{2N+2} and Q_{2N+2}, one introduces masses of spin-0 and spin-2 massive sectors to have all simple poles as

$$0 < m^2_{(k)0} < m^2_{(k)1} < \cdots < m^2_{(k)N} \quad \text{and} \quad m^2_{(k)i} \neq m^2_{(k)j}, \quad i \neq j \tag{32}$$

for $k = 0$, 2. After contour integration, one arrives at the potential

$$\tilde{V}^{\rm IDG}(r) = -\frac{GM}{r}\left[1 - \frac{4}{3}\sum_{i=0}^{N}\prod_{j\neq i}\frac{m^2_{(2)j}}{m^2_{(2)j}-m^2_{(2)i}}e^{-m_{(2)i}r} + \frac{1}{3}\sum_{i=0}^{N}\prod_{j\neq i}\frac{m^2_{(0)j}}{m^2_{(0)j}-m^2_{(0)i}}e^{-m_{(0)i}r}\right]. \tag{33}$$

In the small mass limit of $e^{-m_{(2)i}r}\approx 1$ and $e^{-m_{(0)i}r}\approx 1$, one finds $[\cdots]$ in (33) as

$$-\frac{4}{3}\sum_{i=0}^{N}\prod_{j\neq i}\frac{m^2_{(2)j}}{m^2_{(2)j}-m^2_{(2)i}} + \frac{1}{3}\sum_{i=0}^{N}\prod_{j\neq i}\frac{m^2_{(0)j}}{m^2_{(0)j}-m^2_{(0)i}}. \tag{34}$$

Considering the relation that is valid for any set of numbers a_j,

$$\sum_{i=0}^{N}\prod_{j\neq i}\frac{a_j}{a_j-a_i} = 1, \tag{35}$$

we find that the sum of zeroth-order terms is zero

$$-1 + \frac{4}{3} - \frac{1}{3} = 0. \tag{36}$$

Considering the propagator in the integrand of (29), the total d.o.f. is $2+12N(=2+10N+2N)$. However, we have $2+8N(=2^+ + (3N)^- + (3N)^+ + N^+ + N^-)$ d.o.f. in the Newtonian potential (33). This shows clearly that the vDVZ discontinuity occurs in the small mass limit of the spin-2 massive sector. Here, we decompose $(3N)^-$ into $(N+2N)^-$ in the spin-2 massive sector where N^- is represented by N ghost Stueckelberg scalars, while $(2N)^-$ is represented by N massless ghost tensors. This happens to $(3N)^+$ for massive healthy tensors, similarly. They are equivalent to writing $4/3$ in (36) to be $1/3+1$. The last two $N^+(N^-)$ can be represented by N healthy (ghost) massive scalars, providing $-1/3$ in (36). We have a particle content of $[(3N)^-, (3N)^+]$ and $[(N)^-, (N)^+]$. It explains why the matching of the number of healthy and ghost modes between the spin-2 and spin-0 massive sectors is essential to make the singularity cancellation in the Newtonian potential.

Finally, this might correspond to the condition of super-renormalizability [2,22]. At this point, it would be better to distinguish three types of renormalizable theory: (i) finite, in which no counterterms are needed at all; (ii) super-renormalizable, in which only a finite number of graphs need overall counterterms; and (iii) renormalizable, in which infinitely many graphs need overall counterterms. (But note that they only normalize a finite set of terms in the basic Lagrangian since we assumed renormalizability of the theory.)

VI. DISCUSSIONS

First of all, we have shown that the vDVZ discontinuity is related closely to the singularity cancellation of the Newtonian potential. For this purpose, we have chosen the zero (small) mass limit of $m_i \to 0$ instead of the $r \to 0$ limit.

In this work, we have explored why the matching of the number of ghost and healthy modes between the spin-2 and spin-0 massive sectors is necessary to make the singularity cancellation. This was explained by introducing the vDVZ discontinuity appearing in the zero mass limit of higher-derivative gravity. Therefore, if a higher-derivative gravity is renormalizable, it implies necessarily a finite Newtonian potential at the origin. Furthermore, the reverse of this statement seems to be true. Although a counterexample of (16) that is not renormalizable provides a finite potential at the origin, the vDVZ discontinuity occurs only in the small mass limit of a massive ghost tensor. In the model of (16), a massive healthy tensor is needed to make a renormalizable theory which amounts to happening that the vDVZ discontinuity occurs in the small mass limit of both massive ghost and healthy tensors. This leads to a balance between the attractive forces and repulsive forces in each sector as well as a specific matching of the number of tensor and scalar modes.

We note that the effect of singularity cancellation and the vDVZ discontinuity are linear effects involving the independent contribution of scalars and tensors. Hence, it might not be clear that the cancellation may hold in these theories at the nonlinear level. However, it seems that the UV divergences of quantum theory are related to the Newtonian singularity. This means that the Newtonian singularity is indeed the simplest UV divergence due to the interaction. Also, we mention that two nonlinear issues of the Vainshtein radius [23] and Boulware-Deser ghost [24] concerning the vDVZ discontinuity are not directly related to the renormalizability.

On the other hand, the other physical observable of light deflection (bending angle) does not depend on the massive spin-0 sector of R^2 and $R\Box R$ [25,26]. Thus, it suggests that the UV divergence of quantum theory is not closely related to the light bending angle.

Finally, the two tracks were found to arrive at a finite Newtonian at the origin. One is to use the IDG action (21) without ghost fields. In this case, the singularity disappeared due to the nonlocality. The other is to consider the IDG action (26) with ghost fields. The ghost scalar and tensors are needed to have a finite potential at the origin.

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