Soft gravitons and the memory effect for plane gravitational waves

P.-M. Zhang,[1,*] C. Duval,[2,†] G. W. Gibbons,[3,4,5,‡] and P. A. Horvathy[1,4,§]

[1]*Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China*
[2]*Aix Marseille Univ, Université de Toulon, CNRS, CPT, 13288 Marseille, France*
[3]*Department of Applied Mathematics and Theoretical Physics, Cambridge University, Cambridge CB3 0WA, United Kingdom*
[4]*Laboratoire de Mathématiques et de Physique Théorique, Université de Tours, Parc de Grandmont, 37200 Tours, France*
[5]*Le Studium, Loire Valley Institute for Advanced Studies, 45000 Orleans, France*
(Received 5 June 2017; published 11 September 2017)

The "gravitational memory effect" due to an exact plane wave provides us with an elementary description of the diffeomorphisms associated with the analogue of "soft gravitons for this nonasymptotically flat system. We explain how the presence of the latter may be detected by observing the motion of freely falling particles or other forms of gravitational wave detection. Numerical calculations confirm the relevance of the first, second and third time integrals of the Riemann tensor pointed out earlier. Solutions for various profiles are constructed. It is also shown how to extend our treatment to Einstein-Maxwell plane waves and a midisuperspace quantization is given.

DOI: 10.1103/PhysRevD.96.064013

I. INTRODUCTION

The gravitational memory effect means, intuitively, that a short-burst gravitational wave changes the separation of freely falling particles (viewed here as "detectors") after the wave has passed [1,2]. The effect is potentially observable using LISA [3]; recently, Lasky informed us that aLIGO might also be able to detect memory associated with binary black hole mergers in the not-too-distant future [4]. The effect would be observed indirectly if the B-mode is detected in the cosmic microwave background (CMB) [5]. It is also relevant to recent work by Hawking *et al.* [6,7] on soft graviton theorems in their attempt to resolve the information paradox of black hole physics.

Gravitational waves had long been thought to arise from periodic sources such as binary star systems and were therefore expected to be detected through resonance. The novel idea of observing a burstlike gravitational wave through the displacement of freely falling bodies after the wave has passed was put forward in 1974 by Zel'dovich and Polnarev [1], who suggested

> *… another, nonresonance, type of detector is possible, consisting of two noninteracting bodies (such as satellites). … the distance between a pair of free bodies should change, and in principle this effect might possibly serve as a nonresonance detector. … One should note that although the distance between the free bodies will change, their relative velocity will actually become vanishingly small as the flyby event concludes.*

The idea of Zel'dovich and Polnarev was elaborated by Braginsky and Grishchuk [2], who introduced term "memory effect." Both the title ("Kinematic resonance and the memory effect in free mass gravitational antennas") and the abstract of the latter paper give a clear idea of what is involved:

> *Consideration is given to two effects in the motion of free masses subjected to gravitational waves, kinematic resonance and the memory effect. In kinematic resonance, a systematic variation in the distance between the free masses occurs, provided the masses are free in a suitable phase of the gravitational wave. In the memory effect, the distance between a pair of bodies is different from the initial distance in the presence of a gravitational radiation pulse. Some possible applications … to detect gravitational radiation …*

Braginsky and Grishchuk were clearly concerned with the motion of test masses (that is, no backreaction) moving in a weak gravitational wave. Their analysis was at the *linear* level.

Two years later, Braginsky and Thorne [8] published a short paper making a distinction between two types of bursts, namely, one *without memory* and one *with memory.* The same distinction had been made earlier in [9], but without the explicit introduction of the memory concept.

In the 1990s a *nonlinear* form of memory was discovered, independently, by Christodoulou [10,11] and by Blanchet and Damour [12]. It arises from the contribution of the emitted gravitational waves to the changing quadrupole and higher mass moments (cf. [9]). These papers obtained a permanent displacement.

Since the mid-1990s, there have been many studies of plane gravitational waves. As far as we are aware, few have

*zhpm@impcas.ac.cn
†duval@cpt.univ-mrs.fr
‡G.W.Gibbons@damtp.cam.ac.uk
§horvathy@lmpt.univ-tours.fr

2470-0010/2017/96(6)/064013(20)

dealt with the memory effect, and none with the concept of soft gravitons. However, our attention was recently drawn to two relevant papers of Harte, [13,14]. Although mainly concerned with optics, attention was drawn in [14] to a link with the memory effect that we shall elaborate on later in this paper.

In what follows we consider the effect of a fully nonlinear plane gravitational wave on a detector whose backreaction is negligible. This is done by considering geodesics in the exact plane wave background. We argue that our approach, initiated in [15] following earlier work by Souriau [16], is significantly simpler than those in [10–12], since it requires no knowledge of the source or a sophisticated understanding of nonlinear partial differential equations—just simple calculation. It is based on the idea that far from the source we may approximate the gravitational wave in the neighborhood of a detector by an exact plane wave.

We shall work in $3+1$ spacetime dimensions although the discussion of this paper readily generalizes to higher spatial dimensions. The assumption of three spatial dimensions is an obvious requirement for any discussion of physically realizable detectors such as LIGO and LISA and moreover is also made in [6,7]. However, it has been pointed out that the boundary conditions for asymptotically flat higher dimensional spacetimes differ considerably from those in four spacetime dimensions [17], which probably means that the obvious generalization of the Bondi-Metzner-Sachs (BMS) group to higher dimensions [18] is not applicable.

A detailed analysis of weak sources at the linearized level analogous to that in three dimensions [9] indicates that there is no memory effect in higher dimensions [19]. Since the analysis of the present paper reveals the importance of considering nonlinear focusing effects it may either be the case that these must be taken into account, or our assumption that, at a large distance, plane waves are a good approximation to outgoing gravitational waves, fails in higher dimensions.

The plan of the paper is as follows. In Sec. II we describe the basic geometry of plane gravitational waves and the two most useful coordinate systems used to describe them as well as the relation between them. One referred to as Brinkmann (B) coordinates [20] is global and allows the general vacuum solution to be specified in terms of two arbitrary functions of a single retarded time variable. The second, called Baldwin-Jeffery-Rosen (BJR) coordinates [21,22], depends upon the same single retarded time variable. These coordinates are adapted to a three-dimensional mutually commuting subset of the five independent Killing vectors of plane wave spacetimes. This fact renders local calculations simpler than in Brinkmann coordinates for which only a single Killing vector is manifest. The price to pay for this simplification is that the metric is now specified by a 2×2 symmetric matrix giving the metric on the transverse space, which requires solving a coupled system of Sturm-Liouville differential equations with no nontrivial global solution. This holds even in the locally flat case, as we show explicitly.

Section III is concerned with how gravitational waves are detected. This is at the heart of the gravitational memory effect and the detectability of soft gravitons. We consider how a sandwich wave (i.e., one whose curvature vanishes outside a finite interval of retarded time) affects freely falling particles initially at rest with respect to one another after the wave has passed.

In Secs. III A and III B we recall how, in linear theory, this behavior is encoded in integrals of the Riemann curvature with respect to retarded time and how these integrals serve as a diagnostic for the nature of the source.

In Sec. IV D we show how the memory effect may be illustrated by means of "Tissot" diagrams illustrating the effect of gravitational pulses on a ring of freely falling particles.

Section IV is concerned with the detailed exact behavior of these geodesics in the exact plane wave backgrounds. In Secs. IV A and IV B we do this both in Brinkmann and in BJR coordinates. In (B)-coordinates our study is numerical; however, in the latter case we can proceed analytically: by virtue of Noether's theorem, the spatial positions are independent of retarded time. This has the consequence that for pulses, the memory effect is encoded into a diffeomorphism (i.e., a coordinate transformation) taking a part of flat spacetime in standard inertial coordinates into a patch of flat space in noninertial BJR coordinates. In field theory approaches to general relativity, such as those used in [6,7], diffeomorphisms or coordinate transformations are thought of as "gravitational gauge transformations" and some gravitational gauge transformations of asymptotically flat spacetimes are associated with soft gravitons. In Sec. IV C. we argue that in our context, flat plane waves in BJR coordinates correspond to soft gravitons in the asymptotically flat spacetimes.

In Sec. V we relate our work to the light cone structure of plane gravitational waves and a well-known analysis of Penrose.

In Sec. VI we indicate how much of our work may be extended to exact solutions of Einstein-Maxwell theory. In particular, we point out that the coupled system has the Carroll symmetry identified recently [15] for pure gravitational waves.

Up to this point, our work has been purely classical. In Sec. VII we turn to possible implications for the quantum theory by considering a midisuperspace (Sec. VII A) made up of plane gravitational waves and the associated space of quantum states.

In the analogous case of electromagnetic waves there is an elaborate theory of polarization and the photon states specified by Stokes parameters correspond to points on what is called the Poincaré sphere, which carries a Pancharatnam connection. In Sec. VII B we show how this formalism may be smoothly carried over to the case of gravitons.

The subject of plane gravitational waves has a long history and many contributions and reviews distributed over many different journals in many different languages.

In reviewing the material necessary for an exact and comprehensible understanding of the memory effect and its relation to the concept of soft gravitons we have felt it necessary on the one hand to incorporate sufficient material perhaps well known to experts to make our account self-contained for nonexperts while on the other hand giving sufficient credit to the pioneers of the field without overwhelming the reader with an unmanageable list of all every contribution.

Some of the results presented here appear in summary in [23].

II. PLANE GRAVITATIONAL WAVES

We begin by reviewing some facts about plane waves [16,20–43].

A. Brinkmann and Baldwin-Jeffery-Rosen coordinates

There are two commonly used coordinate systems for plane gravitational waves:

(i) Brinkmann coordinates (B) [20,25] for which the metric is[1];

$$g=\delta_{ij}dX^i dX^j+2dUdV+K_{ij}(U)X^iX^jdU^2, \quad (2.1)$$

where the symmetric and traceless 2×2 matrix with components $K_{ij}(U)$ characterizes the profile of the wave. The only nonvanishing components of the Riemann tensor are, up to symmetry,[2]

$$R_{iUjU}(U)=-K_{ij}(U). \quad (2.2)$$

For suitable K_{ij}, the Brinkmann coordinates $(\mathbf{X},U,V)$, which are harmonic, are global [26,27]. The general form of their profile is then

$$K_{ij}(U)X^iX^j=\frac{1}{2}\mathcal{A}_+(U)((X^1)^2-(X^2)^2)+\mathcal{A}_\times(U)X^1X^2, \quad (2.3)$$

where $\mathcal{A}_+$ and $\mathcal{A}_\times$ are the amplitude of the $+$ and $\times$ polarization state.

Aside from their astrophysical applications to gravitational radiation, plane waves, in arbitrary spacetime dimensions, provide a general framework in which any "natural" nonrelativistic dynamical system with a configuration space of dimension n may be "Eisenhart" lifted to a system of null geodesics in an $(n+2)$-dimensional Lorentzian spacetime endowed with a covariantly constant null Killing vector field $\xi=\partial_V$ [44–47]. Conversely, a null reduction along the orbits of such "Bargmann" spacetimes gives rise to a possibly time-dependent dynamical system on an n-dimensional configuration space. From the "Bargmann" point of view, the metric (2.1) describes a nonrelativistic particle subjected to an (attractive or repulsive) harmonic (and generally time-dependent and anisotropic) oscillator potential.

(ii) Baldwin-Jeffery-Rosen coordinates (BJR) [21,22,28], for which

$$g=a_{ij}(u)dx^idx^j+2dudv, \quad (2.4)$$

where the 2×2 matrix $a(u)=(a_{ij}(u))$ is strictly positive. The BJR coordinates $(\mathbf{x},u,v)$ are not harmonic and are typically not global, but exhibit coordinate singularities [16,22,26,27,30,33], a fact which gave rise to much confusion in the early days of the subject. Our investigations below provide further clarification of this issue.

The relation between the two coordinate systems is given by [15,38]

$$\mathbf{X}=P(u)\mathbf{x}, \quad U=u, \quad V=v-\frac{1}{4}\mathbf{x}\cdot\dot{a}(u)\mathbf{x}, \quad (2.5)$$

with[3]

$$a(u)=P(u)^TP(u), \quad (2.6)$$

where P satisfies

$$\ddot{P}=KP. \quad (2.7)$$

For a given matrix K, this is a second-order ODE of the Sturm-Liouville type for P, which implies that $P^T\dot{P}-\dot{P}^TP=\text{const}$ Then the initial values of $\dot{P}$ and of P may be chosen so that the constant vanishes,

$$P^T\dot{P}-\dot{P}^TP=0. \quad (2.8)$$

The mapping (2.5) transforms the quadratic "potential" $K_{ij}(U)X^iX^j$ in (2.1) into a time-dependent transverse metric (2.6) and vice versa. The relation is

$$K=\frac{1}{2}P\left(\dot{b}+\frac{1}{2}b^2\right)P^{-1}, \qquad b=a^{-1}\dot{a}. \quad (2.9)$$

B. Plane waves in BJR coordinates

Up to symmetry, the only nonzero components of the Riemann tensor are

$$R_{uiuj}=-\frac{1}{2}\left(\ddot{a}-\frac{1}{2}\dot{a}a^{-1}\dot{a}\right)_{ij}, \quad (2.10)$$

yielding the Ricci tensor, whose only nonzero component is (2.9),

[1] Equation (2.1) gives the most general form of a pp-wave only in $D=4$ total dimensions; further components arise also if $D\geq 5$ [20]. In this paper, we limit ourselves to $D=4$.

[2] We use the convention $R^\mu_{\nu\rho\sigma}=2\partial_{[\rho}\Gamma^\mu_{\sigma]\nu}+\cdots$; indices are lowered according to $R_{\mu\nu\rho\sigma}=g_{\mu\lambda}R^\lambda_{\nu\rho\sigma}$.

[3] The dot stands everywhere for the derivative with respect to u.

$$R_{uu} = -\frac{1}{2}\mathrm{Tr}\left(\dot{b} + \frac{1}{2}b^2\right) \quad \text{with} \quad b = a^{-1}\dot{a}. \tag{2.11}$$

The most general flat metric obtained by solving the equation $R_{uiuj} = 0$. With initial conditions

$$a_0 = a(u_0) \quad \text{and} \quad \dot{a}_0 = \dot{a}(u_0) \tag{2.12}$$

we find

$$a(u) = \left(a_0 + \frac{1}{2}(u - u_0)\dot{a}_0\right)a_0^{-1}\left(a_0 + \frac{1}{2}(u - u_0)\dot{a}_0\right), \tag{2.13}$$

from which we infer that

$$a(u) = a_0^{\frac{1}{2}}(\mathbf{1} + (u - u_0)c_0)^2 a_0^{\frac{1}{2}}, \quad \text{where } c_0 = \frac{1}{2}a_0^{-\frac{1}{2}}\dot{a}_0 a_0^{-\frac{1}{2}} \tag{2.14}$$

and where $a_0^{\frac{1}{2}}$ is a (symmetric) square root of the positive matrix a_0.

If, in particular, the initial conditions in (2.12) are $a_0 = \mathbf{1}$ and $\dot{a}_0 = 0$, then we obtain flat spacetime in inertial coordinates, for which $a(u) = \mathbf{1}$ for all u. More generally, (2.14) allows us to recast, in any flat region, the metric (2.4) into standard Minkowskian form by a change of coordinates, $(\mathbf{x}, u, v) \mapsto (\hat{\mathbf{x}}, \hat{u}, \hat{v})$. For

$$\hat{\mathbf{x}} = (\mathbf{1} + (u - u_0)c_0)a_0^{\frac{1}{2}}\mathbf{x}, \tag{2.15a}$$

$$\hat{u} = u, \tag{2.15b}$$

$$\hat{v} = v - \frac{1}{2}\mathbf{x} \cdot (a_0^{\frac{1}{2}}c_0(\mathbf{1} + (u - u_0)c_0)a_0^{\frac{1}{2}}\mathbf{x}), \tag{2.15c}$$

whose inverse is

$$\mathbf{x} = a_0^{-\frac{1}{2}}(\mathbf{1} + (u - u_0)c_0)^{-1}\hat{\mathbf{x}}, \tag{2.16a}$$

$$u = \hat{u}, \tag{2.16b}$$

$$v = \hat{v} + \frac{1}{2}\hat{\mathbf{x}} \cdot (c_0(\mathbf{1} + (u - u_0)c_0)^{-1}\hat{\mathbf{x}}), \tag{2.16c}$$

one readily finds indeed that

$$\mathrm{g} = d\mathbf{x} \cdot a(u)d\mathbf{x} + 2dudv = d\hat{\mathbf{x}} \cdot d\hat{\mathbf{x}} + 2d\hat{u}d\hat{v}. \tag{2.17}$$

We will call $(\hat{\mathbf{x}}, \hat{u}, \hat{v})$ a "manifestly flat BJR coordinate chart."

Two metrics related by a coordinate transformation, i.e., by a diffeomorphism, are usually regarded as equivalent. However, as it stands, this statement is not very precise. One needs to specify how the diffeomorphism f acts on the spacetime $\{\mathcal{M}, g\}$ under consideration. If it is the identity outside a compact set within the spacetime manifold $\mathcal{M}$ (which we assume to be noncompact), then one typically assumes then that the two spacetimes $\{\mathcal{M}, g\}$ and $\{\mathcal{M}, f^\star g\}$, where $f^\star$ denotes pullback, are physically equivalent, i.e., "mere coordinate transformations of one another."

However if the diffeomorphism f does not vanish outside a compact set and does not tend in some appropriate sense to the identity at "infinity," more care is required. For example, in (suitably defined) asymptotically flat spacetimes, there is a class of distinguished coordinate systems related by the subset of diffeomorphisms which do not tend to the identity at infinity, but which nevertheless take asymptotically flat spacetimes to asymptotically flat spacetimes.[4] The set of such diffeomorphisms is referred to as the "asymptotic symmetry group." These include the translations and boosts.

Solutions of the Einstein equations which differ by such diffeomorphisms are not usually thought of as physically identical since they could, for example, describe two black holes moving towards one another.

In $3 + 1$ spacetime dimensions the asymptotic symmetry is well known to be the infinite-dimensional BMS group. Spacetimes which differ by the action of elements of the BMS group are typically regarded as physically distinct. This is especially so in scattering theory, both at the classical level and in attempts to construct a perturbative quantum version in which the S-matrix plays an important role, the classical theory being described by tree diagrams.[5]

Since in this approach gravitational waves carrying arbitrarily small energy have to be considered, the quantum theory has to address certain difficulties, specifically infinite quantities which arise even in electromagnetic theory in Minkowski spacetime where they are ascribed to the presence of so-called soft (i.e., zero energy) photons. At the quantum level these soft quanta are frequently assigned states in the quantum Hilbert space. At the classical level these soft photons carry vanishing electromagnetic fields and so differ only by electromagnetic gauge transformations which, however, do not tend to the identity at infinity, "at infinity" being, in this case, a neighborhood of the conformal boundary of Minkowski spacetime.[6] The work of [6,7] is an attempt to make use of much earlier work by their authors and others (referred to in detail in their papers), which extends the ideas and results obtained for photons to gravitons.

[4] One should be aware that confusion can arise if the class of permissible diffeomorphisms is not viewed *actively*. In discussing the elementary geometry of Euclidean space, the metric tensors in Cartesian coordinates and in spherical coordinates differ substantially, as do the coordinate functions themselves at large radius. However, viewed *passively*, a "change of coordinates" merely relabels the points which are left fixed.

[5] One might object that strictly from a rigorous point of view, no S-matrix exists in quantum field theories based on standard Fock space constructions of their Hilbert spaces [48] and even classically the soundness of the so-called Lorentz covariant approaches has often been questioned on causality grounds (see, e.g., [49]), but in this paper we shall set aside such doubts.

[6] We are grateful to Piotr Bizon [50] for informing us of what appears to the first mention of a memory effect in the electromagnetic case [51]. We subsequently learned from Malcolm Perry [52] that an even earlier though not very explicit mention may well come from Mott in a paper in which he computed the number of photons produced in Rutherford scattering [53].

The starting point of [6,7] (which has not been without its critics [54–56]) was to consider asymptotically flat spacetimes and the BMS group. The idea of the present paper is to consider a much simpler situation: plane wave spacetimes. In the present case the diffeomorphism defined by (2.15)–(2.16) does not tend to the identity as $|\mathbf{x}|$ or $|u|$ tend to infinity. Moreover, since every metric tensor given by (2.14) is locally flat, it is tempting to regard them, in the language of quantum field theory, as ground states or vacua.

As we recalled above, in theories with no massless excitations one usually regards all such "gauge equivalent" vacua as equivalent. But in theories with massless excitations it is customary to regard such vacua related by gauge transformations which do not tend to identity at infinity as *nonequivalent*, differing by the presence of "soft" (i.e., zero energy) quanta. Such claims are often supported by a canonical or Hamiltonian treatment in which the soft states are associated with charges or moment maps which may be expressed as surfaces integrals "at infinity" which, for asymptotically flat spacetimes, are two-surface integrals evaluated on the conformal boundary. This has been done in the asymptotically flat case in [6,7,57].

In the present case the massless excitations correspond, at the quantum level, to gravitons and so one may regard the metrics given by (2.14) as "dressed by soft gravitons," (i.e., carrying vanishing energy), the dressing being affected by the pulse of gravitational radiation itself made up of "hard" (i.e., carrying nonvanishing energy) gravitons. This interpretation is consistent with that given in [6,7] in the asymptotically flat case.

To confirm this suggestion in full mathematical detail would require a detailed treatment of what one means by "at infinity" for plane gravitational waves, their conformal boundary (cf. [58]), a canonical or Hamiltonian treatment, and the identification of possible moment maps defined as two-surface integrals "at infinity." This is an interesting and demanding challenge for the future. For the present we shall content ourselves with fleshing out some aspects of plane gravitational waves which (we feel) make our suggestion plausible at the physical level. As partial compensation we note that our results, being based on exact solutions of the Einstein equations, evade the strictures of [49] alluded to earlier.

We will consider sandwich waves, i.e., gravitational waves which are flat *outside* the sandwich but not *inside*, i.e., for $u \in [u_i, u_f]$. Our point here is that flat spacetimes both in the "before zone" $u < u_i$ and in the "after zone" $u > u_f$ [33] are *nonequivalent*.

Inside a sandwich wave we only have Ricci flatness [cf. (2.11)],

$$\mathrm{Tr}\left(\dot{b} + \frac{1}{2}b^2\right) = 0. \tag{2.18}$$

By (2.9) this is precisely the tracelessness of K.

BJR coordinates are convenient for comparing the standard linear theory in the transverse traceless gauge with the fully nonlinear theory. For plane waves in linear theory one has a metric of the form (2.4), with

$$a_{ij} = \delta_{ij} + h_{ij}(u) + \ldots \tag{2.19}$$

Thus

$$P_{ij}(u) = \delta_{ij} + \frac{1}{2}h_{ij}(u) + \ldots,$$
$$K_{ij}(u) = \frac{1}{2}\ddot{h}_{ij}(u) + \ldots \tag{2.20}$$

Thus after the wave has passed, i.e., if $K_{ij} = 0$, we have $h_{ij}(u) = h^0_{ij} + uh^1_{ij}$, where h^0_{ij} and h^1_{ij} are independent of u. If $h^1_{ij} = 0$ we have the metric (5.19) of Favata [59] in his discussion of the possibilities of detecting the memory effect with interferometers and his (5.20) transforming to manifestly flat coordinates. These agree with (2.13) and (2.15). Note that generically h_{ij} is *linear* in u.

To see that the BJR coordinates are indeed necessarily singular as stated, let us define

$$\chi = (\det a)^{\frac{1}{4}} > 0 \quad \text{and} \quad \gamma = \chi^{-2}a, \tag{2.21}$$

so that $b = \gamma^{-1}\dot{\gamma} + 2\chi^{-1}\dot{\chi}\mathbf{1}$. Since $\det\gamma = 1$, we readily obtain $\mathrm{Tr}(\gamma^{-1}\dot{\gamma}) = 0$; this allows us to show that (2.18) is equivalent to the Sturm-Liouville equation

$$\ddot{\chi} + \omega^2(u)\chi = 0, \qquad \omega^2(u) = \frac{1}{8}\mathrm{Tr}((\gamma^{-1}\dot{\gamma})^2), \tag{2.22}$$

which thus guarantees that the vacuum Einstein equations are satisfied for an otherwise arbitrary choice of the unimodular symmetric 2×2 matrix,

$$\gamma(u) = \begin{pmatrix} \alpha(u) & \beta(u) \\ \beta(u) & (1 + \beta(u)^2)/\alpha(u) \end{pmatrix}. \tag{2.23}$$

Thus the matrix $a(u)$ depends on two arbitrary functions $\alpha(u)$ and $\beta(u)$; see Eq. (2.23) and [15].

The positivity of the matrix $(\gamma^{-1}\dot{\gamma})^2$ implies that ω^2 in (2.22) is positive; the equation describes therefore an attractive oscillator with a time-dependent frequency. It follows that $\chi(u)$ is a concave function, $\ddot{\chi} < 0$, which in turn implies the vanishing of χ for some $u_{\mathrm{sing}} > u_i$,

$$\chi(u_{\mathrm{sing}}) = 0, \tag{2.24}$$

signaling a singularity of the metric (2.4). Choosing $a(u) = \mathrm{diag}(a_{11}, a_{22})$, for example, we find

$$\omega^2(u) = \frac{1}{16}\left(\frac{\dot{a}_{11}}{a_{11}} - \frac{\dot{a}_{22}}{a_{22}}\right)^2 \tag{2.25}$$

and the Sturm-Liouville equation (2.22) becomes

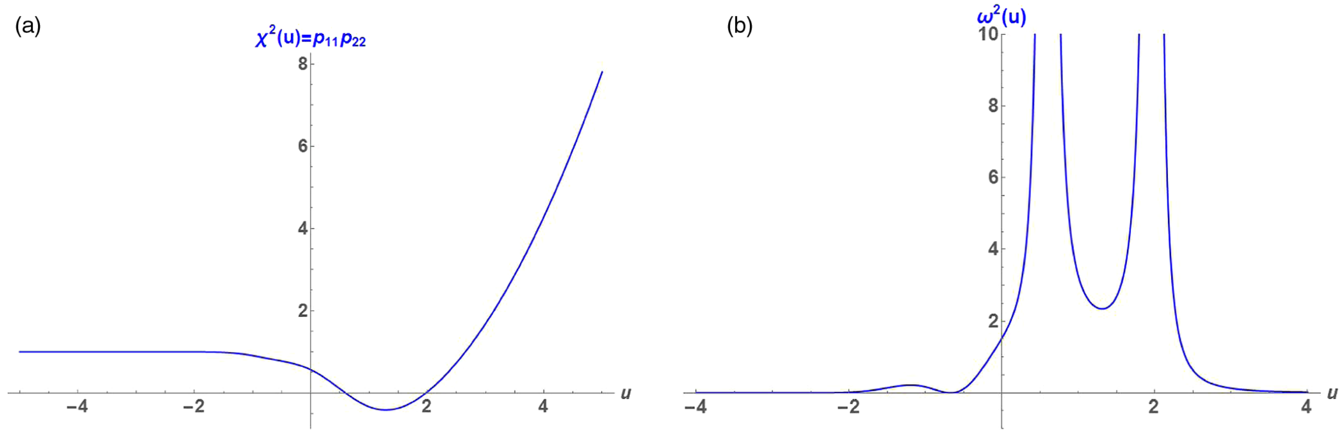

FIG. 1. (a) $\chi = (\det a)^{1/4}$ and (b) ω^2 in (2.22), respectively, calculated numerically for $\mathcal{A}_+ = \mathcal{A}_+^3$ confirm that, in BJR coordinates, the metric becomes singular at $u_{\rm sing}$.

$$\frac{\ddot{a}_{11}}{a_{11}} + \frac{\ddot{a}_{22}}{a_{22}} - \frac{1}{2}\left(\frac{\dot{a}_{11}^2}{a_{11}^2} + \frac{\dot{a}_{22}^2}{a_{22}^2}\right) = 0. \tag{2.26}$$

Expressed in terms of the matrix P, this is simply

$$\frac{\ddot{P}_{11}}{P_{11}} + \frac{\ddot{P}_{22}}{P_{22}} = 0, \tag{2.27}$$

which is indeed $\mathrm{Tr}K = 0$ since $\ddot{P}P^{-1} = K$.

While we cannot solve the nonlinear equation (2.27) in general, we may proceed differently: starting with some physically relevant profile in Brinkmann coordinates and solving (2.7) numerically allows us to calculate the matrix a and to plot $\chi(u)$ and $\omega^2(u)$ in (2.21) and (2.22). This confirms the existence of a $u_{\rm sing} > u_i$ such that the metric becomes singular, $\chi(u_{\rm sing}) = 0$. For the choice $\mathcal{A}_\times = 0$, $\mathcal{A}_+ = \mathcal{A}_+^3$ in (4.9) (justified in the next section), for example, χ and ω^2 are plotted in Fig. 1.

III. DETECTION OF THE MEMORY EFFECT

A. Detection theory

We turn now to the question of the detectability of soft gravitons. As pointed out in the pioneering papers of Pirani [29,30] knowledge of the *relative* motion of freely falling particles in time-dependent gravitational fields is essential for our understanding of gravitational radiation and its detection. In practical devices, the "particles," such as mirrors in interferometers or the individual atoms in old-fashioned bar detectors, are never truly freely falling since they are subject to various forces holding them in place. Nevertheless it is the relative motions induced by external time-dependent gravitational influences that are what is actually detected.

Let us consider two infinitesimally close geodesics, X_1^μ and $X_2^\mu = X_1^\mu + \eta^\mu$, whose unit tangent vector is $\frac{dx^\alpha}{d\tau}$, where τ is their common proper time. The quantity η^μ is referred to as the connecting vector. Theories of detectors start with the equation of geodesic deviation (or Jacobi equation),[7]

$$\frac{D^2\eta^\mu}{d\tau^2} + R^\mu_{\alpha\nu\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\eta^\nu = 0, \tag{3.1}$$

and then modify it with elastic and damping terms [see, e.g., Eq. (4) of [9]]. The connecting vector satisfies

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\eta^\nu = 0. \tag{3.2}$$

The geodesic deviation has been studied by Griffiths and Podolsky [36]. For the central geodesic given by $U = \tau$, $V = -\frac{1}{2}\tau$, $X^1 = X^2 = 0$, their equations may be cast in the form

$$\frac{d^2\eta^1}{dU^2} = \frac{1}{2}\mathcal{A}_+(U)\eta^1 + \frac{1}{2}\mathcal{A}_\times(U)\eta^2, \tag{3.3a}$$

$$\frac{d^2\eta^2}{dU^2} = -\frac{1}{2}\mathcal{A}_+(U)\eta^2 + \frac{1}{2}\mathcal{A}_\times(U)\eta^1, \tag{3.3b}$$

$$\frac{d^2\eta^3}{dU^2} = 0. \tag{3.3c}$$

Given $\mathcal{A}_+(U)$ and $\mathcal{A}_\times(U)$, this is a system of second-order linear differential equations for η^i as a function of U and hence τ. Within a tubular neighborhood of any geodesic, one may introduce a Fermi coordinate system (x^0, x^i) in which the metric is locally flat and $t = x^0$ coincides with proper time τ along the geodesic. In such a local coordinate system at rest with respect to a freely falling detector, the acceleration of the separation η^i in such a local coordinate system is subject to a forcing term

[7]For any tensorial quantity $T_{\because q}$, we put $\frac{DT_{\cdots}}{d\tau} = T_{\cdots;\nu}\frac{dx^\nu}{d\tau}$.

$$-R^i_{0j0}\eta^j, \tag{3.4}$$

where 0 labels the time direction and i, j the spatial directions. For a more detailed discussion of Fermi coordinates in plane gravitational wave spacetimes, see [60].

In fact, since, if l^i is the time averaged separation, both the change $x^i = \eta^i - l^i$ in separation and the curvatures are typically small, one may approximate the geodesic deviation equation by

$$\frac{d^2x^i}{dt^2} = -R^i_{0j0}l^j. \tag{3.5}$$

Thus supposing $\dot{x}^i$ is initially zero, one has an induced velocity

$$v^i(t) = \frac{dx^i}{dt} = -\int_{t_i}^{t} dt' R^i_{0j0}(t')l^j. \tag{3.6}$$

Now in linear theory

$$R_{i0j0} = \frac{G}{3r}\frac{d^4D_{ij}}{dt^4}(t-r), \tag{3.7}$$

where D_{ij} is the quadrupole of the source, r its distance, and $u = t - r$ is retarded time. Note that in linear theory and to the approximation we are using, there is no distinction between upper and lower spatial indices. Thus for many plausible sources such as

(i) gravitational collapse of a previously time-independent object to form a black hole
(ii) or a gravitational flyby

the forcing term would be confined to a finite interval $t_i \le t \le t_f$ of time: it is *pulselike* referred to as a sandwich wave [26,27,35]. It follows that while the separation η^i may have been constant before the arrival of the pulse, it will nevertheless, in general, be *time dependent* after the pulse. In fact it was pointed out in [9] that, at the linear level, the three time integrals of the signal,

$$I^{(3)} = (I^{(3)}_{ij}) = \int_{t_i}^{t_f} dt \int_{t_i}^{t} dt' \int_{t_i}^{t'} dt'' R_{0i0j}(t'') \tag{3.8a}$$

$$I^{(2)} = (I^{(2)}_{ij}) = \int_{t_i}^{t_f} dt \int_{t_i}^{t} dt' R_{0i0j}(t') \tag{3.8b}$$

$$I^{(1)} = (I^{(1)}_{ij}) = \int_{t_i}^{t_f} dt R_{0i0j}(t), \tag{3.8c}$$

should vanish in the *collapse* case, since $\frac{dD_{ij}}{dt}$ would vanish initially and finally. By contrast, in the *flyby* case only the last integral needs to vanish, since initially and finally D_{ij} could be expected to be quadratic in time and hence only $\frac{dD^3_{ij}}{dt^3}$ would vanish initially and finally.

The analyses of Zel'dovich and Polnarev and of Braginsky and Grishchuk are entirely at the linear level and as far as the source is concerned, they simply use the analogue of (3.7) for the metric perturbation. Using the transverse traceless or radiation gauge,[8] one has

$$h^{TT}_{ij} \propto \frac{1}{r}\frac{d^2}{dt^2}D_{ij}(t-r). \tag{3.9}$$

Now from (3.9) we have

$$R_{0i0j} \propto \frac{d^2h^{TT}_{ij}}{dt^2}. \tag{3.10}$$

Thus

$$\frac{d^2x^i}{dt^2} \propto -\frac{d^2h^{TT}_{ij}}{dt^2}l^j, \tag{3.11}$$

which is consistent with

$$x^i \propto h^{TT}_{ij}l^j. \tag{3.12}$$

Braginsky and Grishchuk also suggest [their Eq. (7)] that flybys should have D_{ij} quadratic in time. Braginsky and Thorne [8] makes a distinction between two types of bursts, one without memory and one with memory, expressed in terms of a linearized description of the gravitational perturbation in the transverse traceless gauge h^{TT}_{ij} rather than the gauge-invariant Riemann tensor components R_{0i0j}. Thus,

(i) for a gravitational-wave burst without memory h^{TT}_{ij} is nonzero only in a finite interval $t_i < t < t_f$
(ii) while for a gravitational-wave burst with memory, $h^{TT}_{ij} =$ constant for $t > t_f$.

From (3.10) it follows that for bursts *without memory* the two integrals I^1 and I^2 in (3.8) should vanish, while for signals *with memory*, only I^1 needs to vanish.

To test these ideas we shall consider pulses constructed from Gaussians, and their integrals and derivatives. While not strictly sandwich waves, their curvatures vanish rapidly outside the width of the Gaussian.

(1) For a flyby the D_{ij} could be the third integral of a Gaussian and hence K_{ij} would be the *derivative* of a Gaussian; see (4.5) below.
(ii) The system considered by Thorne and Braginsky could be the *second derivative* of a Gaussian, (4.7).
(iii) For a collapse one could take $D_{ij}(u) \propto -\mathrm{erfc}(u)$, minus the complementary error function. Thus the Riemann tensor or equivalently K_{ij} would be the *third derivative* of a Gaussian, (4.9).

[8] In fact, their Eq. (1) is a plane wave in BJR coordinates with $\mathrm{Tr}(a) = 2$, which they regard as a small perturbation of flat space, i.e., when $a_{ij} = \delta_{ij}$. They write $a_{ij} = \delta_{ij} + h_{ij}$.

B. Memory via Hamilton-Jacobi theory in BJR coordinates

Since it is central to an understanding of the physical reality of the memory effect, we shall begin by giving a self-contained account of the motion of freely falling particles using the Hamilton-Jacobi method in BJR coordinates. The results agree with the derivation in [12,15] but are included for the sake of making the paper self-contained. We thus need to solve

$$g^{\mu\nu}\partial_\mu S\partial_\nu S = 2c, \qquad \frac{dx^\mu}{d\tau} = g^{\mu\nu}\partial_\nu S. \tag{3.13}$$

The coordinates v, x^i are ignorable and we may separate variables

$$2\partial_u S\partial_v S + a^{ij}\partial_i S\partial_i S = 2c, \qquad S = G(u) + vp_v + x^i p_i,$$

where p_v, p_i are constants and the motion is reduced to quadratures,

$$\frac{dx^i}{d\tau} = a^{ij}p_j, \qquad \frac{dv}{d\tau} = \dot{G}, \qquad \frac{du}{d\tau} = p_v, \tag{3.14}$$

where τ is proper time. It follows that particles which have initially constant coordinates $\mathbf{x}$ have $\mathbf{x}$ constant for all times for which the coordinates are well defined. This is key to our approach to the memory effect.

In BJR coordinates we may obtain flat spacetime for $u \le u_i$ by setting $a_{ij} = \delta_{ij}$. Thus before the pulse arrives we may make this choice. It is consistent with the Einstein vacuum equations which state that the trace of the right-hand side of (2.9) should vanish. However clearly from (2.9), we will not have $a_{ij} = \delta_{ij}$ after the pulse has passed.

At the linear level

$$\ddot{a}_{ij} \approx 2K_{ij}, \tag{3.15}$$

and in Brinkmann coordinates as an exact statement [15] $K_{ij} = -R_{iUjU}$, so at the linear level [see (2.10)] we have

$$\ddot{a}_{ij} \approx -2R_{uiuj}, \tag{3.16}$$

which is the analogue of (3.10) in the BJR gauge. We have $\frac{1}{2}b_{ij}(u_f) \approx \int_{u_i}^{u_f} du K_{ij}(u)$. Since in linear theory $b_{ij} \approx \dot{a}_{ij}$,

$$a_{ij}(u) \approx \delta_{ij} + 2\int_{u_i}^{u} du' \int_{u_i}^{u'} du'' K_{ij}(u''). \tag{3.17}$$

The particles are at rest in this coordinate system; however, their distances apart will be different after the pulse has passed, i.e., for $u > u_f$ if the metric is different and hence if the double integral in (3.17) is nonzero.

This, then, is the linear memory effect in BJR coordinates. A persistent change in the metric means a persistent change in separation. A similar conclusion in the optical context was reached in [14].

Now it is clear how this works at the nonlinear level. Equation (2.9) provides a nonlinear second-order differential equation for a_{ij} and with initial conditions that $a_{ij} = \delta_{ij}$ before the arrival of the pulse. This means that in general $a_{ij} \neq \delta_{ij}$ after the pulse has passed and so the distance between nearby freely falling particles has changed. At the linear level we can express the shift in terms of integrals of the Riemann tensor introduced in [9]. In the full nonlinear case (2.9) has no obvious explicit solution but in a perturbation expansion, it seems clear that many more such iterated integrals will crop up. In fact, after the first version of this paper was circulated, we were informed that this is indeed the case; see Sec. IV. 1. 1 of [14]. In later sections we shall explore the relevant solutions both numerically and analytically.

IV. GEODESICS

A. Geodesics in Brinkmann coordinates

Brinkmann coordinates, (2.1), are convenient for a numerical study. For simplicity, we only consider the $+$ polarization, for which

$$K_{ij}(U)X^iX^j = \frac{1}{2}\mathcal{A}_+(U)((X^1)^2 - (X^2)^2). \tag{4.1}$$

The geodesics are the solution of the uncoupled system

$$\frac{d^2X^1}{dU^2} - \frac{1}{2}\mathcal{A}_+X^1 = 0, \tag{4.2a}$$

$$\frac{d^2X^2}{dU^2} + \frac{1}{2}\mathcal{A}_+X^2 = 0, \tag{4.2b}$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}_+}{dU}((X^1)^2 - (X^2)^2) + \mathcal{A}_+\left(X^1\frac{dX^1}{dU} - X^2\frac{dX^2}{dU}\right) = 0. \tag{4.2c}$$

Fixing the initial conditions $\mathbf{X}(U_0) = \mathbf{X}_0$ and $\dot{\mathbf{X}}(U_0) = \dot{\mathbf{X}}_0$, the projection of the $4D$ worldline to the transverse plane is therefore independent of the choice of $V(U_0) = V_0$, i.e., independent of whether the motion is timelike, lightlike, or spacelike.

The geodesic deviation equations of Griths and Podolsky, their Eq. (III.3) in [36], can be rederived from ours here. For $\eta^i = X_2^i - X_1^i$, $i = 1, 2$, this follows from the linearity of the first two equations in (4.2). As to the third one, Eq. (4.2c) entails that

$$\begin{aligned}\frac{d^2(V_2 - V_1)}{dU^2} &= -\frac{1}{4}\frac{d\mathcal{A}_+}{dU}((X_2^1)^2 - (X_1^1)^2 - (X_2^2)^2 + (X_1^2)^2) \\ &\quad + \mathcal{A}_+\left(X_2^1\frac{dX_2^1}{dU} - X_1^1\frac{dX_1^1}{dU} - X_2^2\frac{dX_2^2}{dU} - X_1^2\frac{dX_1^2}{dU}\right) \\ &= -\frac{1}{4}\frac{d\mathcal{A}_+}{dU}((\eta^1)^2 - (\eta^2)^2) + \mathcal{A}_+(\eta^1\dot{\eta}^1 - \eta^2\dot{\eta}^2)\end{aligned}$$

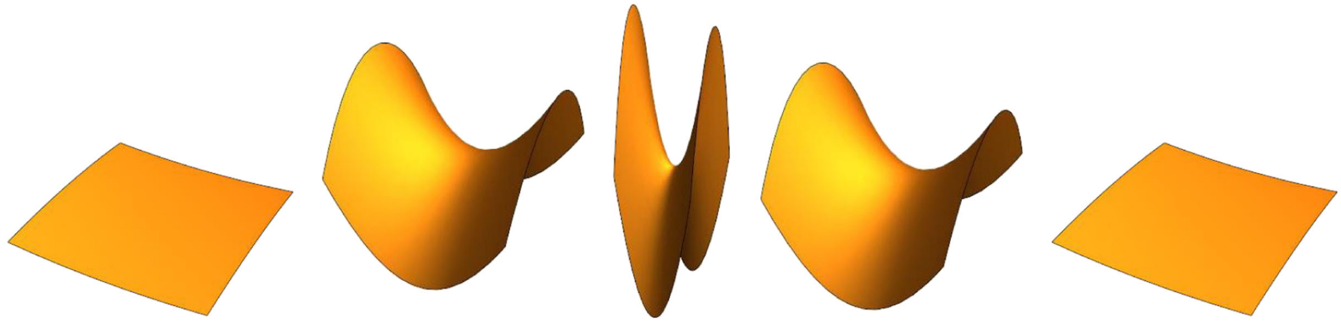

FIG. 2. Evolution of the wave profile of a Gaussian burst.

if one assumes that $X_1^i = 0$, i.e., $\eta^i = X_2^i$. The Jacobi deviation equation being linear in η^μ, we can conclude that $d^2\eta^3/dU^2 = d^2(V_2 - V_1)/dU^2 = 0$.

The system (4.2) can be solved once $\mathcal{A}_+(U)$ is given. Analytic solutions can be obtained in particular cases only, though; therefore we study our equations numerically. An insight into what happens is gained by considering Gaussians and their integrals and derivatives. The colors refer, in Figs. 3,5,6, and 7, to identical initial conditions $X_0^1 = X_0^2 = V_0 = .5, 1, 1.5$ at $U_0 \ll 0$.

(i) We start with a toy example, assuming that the gravitational burst is a simple Gaussian,

$$\mathcal{A}_+(U) = \mathcal{A}_+^0(U) \equiv \frac{1}{2} e^{-U^2}. \tag{4.3}$$

Then the integrals (3.8) are

$$I^1 = \frac{\sqrt{\pi}}{2} \mathrm{diag}(1, -1),$$
$$I^2 = I^3 = \infty \ \mathrm{diag}(1, -1). \tag{4.4}$$

The evolution of the profile and the geodesics are shown Figs. 2 and 3, respectively.

The variation of the relative (Euclidean) distance $\Delta_X(\mathbf{X}, \mathbf{Y}) = |\mathbf{X} - \mathbf{Y}|$ and of the relative velocity $\Delta_{\dot{X}} = |\dot{\mathbf{X}} - \dot{\mathbf{Y}}|$ are depicted in Fig. 4. The latter

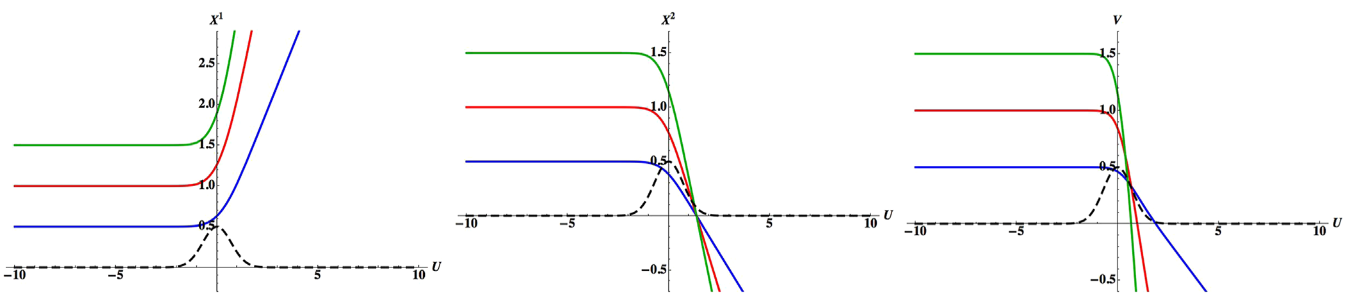

FIG. 3. Evolution of geodesics for a Gaussian burst.

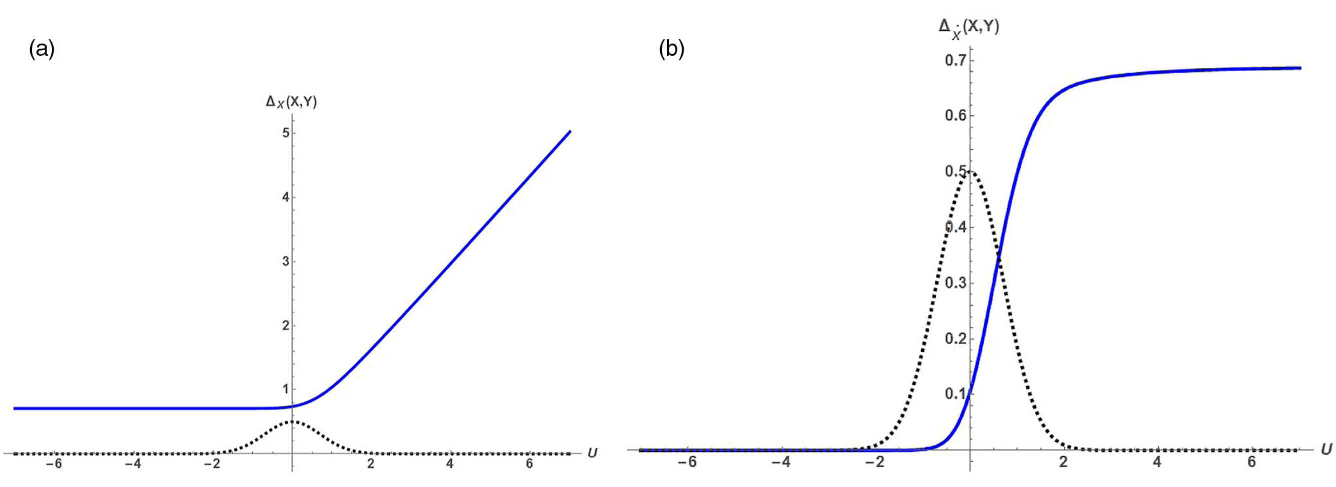

FIG. 4. In the Gaussian case, (a) two particles initially at rest recede from each other after the wave has passed. Their distance, Δ_X, increases roughly linearly in the after zone. (b) The relative velocity, $\Delta_{\dot{X}}$, jumps to an approximately constant but nonzero value.

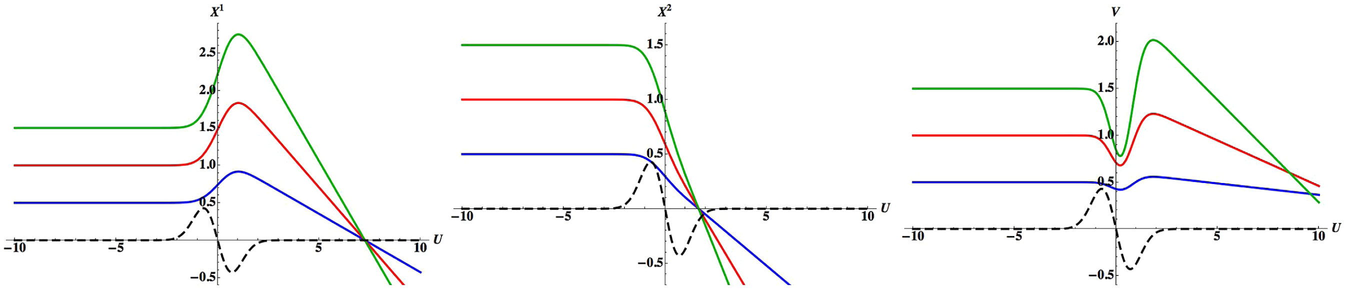

FIG. 5. Evolution of geodesics for the first derivative of a Gaussian, Eq. (4.5), appropriate for flyby.

could in principle be observed through the Doppler effect [2].

(ii) For a *flyby* the quadrupole of the source, D_{ij} in (3.7), would be the third integral of a Gaussian and hence $\mathcal{A}_+(U)$ would be proportional to the *first derivative* of a Gaussian,

$$\mathcal{A}_+(U) = \mathcal{A}_+^1(U) \equiv \frac{1}{2}\frac{d(e^{-U^2})}{dU}. \qquad (4.5)$$

The integrals (3.8) are now

$$I^1 = 0,$$
$$I^2 = \frac{\sqrt{\pi}}{2}\,\mathrm{diag}(1,-1),$$
$$I^3 = \infty\,\mathrm{diag}(1,-1), \qquad (4.6)$$

consistently with the interpretation as flyby; cf. Sec. III. The geodesics are depicted in Fig. 5.

(iii) The system considered by Braginsky and Thorne [8] would seem to correspond to the *second derivative* of a Gaussian,

$$\mathcal{A}_+(U) = \mathcal{A}_+^2(U) \equiv \frac{1}{2}\frac{d^2(e^{-U^2})}{dU^2}. \qquad (4.7)$$

The integrals (3.8) are now

$$I^1 = I^2 = 0, \qquad I^3 = \frac{\sqrt{\pi}}{2}\,\mathrm{diag}(1,-1). \qquad (4.8)$$

The geodesics are shown in Fig. 6.

(iv) In the early 1970s when it was claimed that gravitational-wave bursts had been discovered [61], it was suggested that for gravitational collapse the quadrupole momentum could be modeled by the fourth derivative of the error function −erfc [9], yielding

$$\mathcal{A}_+(U) = \mathcal{A}_+^3(U) \equiv \frac{1}{2}\frac{d^3(e^{-U^2})}{dU^3}. \qquad (4.9)$$

All integrals in (3.8) vanish now,

$$I^1 = I^2 = I^3 = 0, \qquad (4.10)$$

as expected for gravitational collapse; cf. Sec. III. The evolution is presented in Fig. 7.

B. Geodesics in BJR coordinates

Further insight can be gained by working in the BJR coordinates $(\mathbf{x}, u, v)$ used in (2.4).

Plane gravitational waves (2.1) or (2.4) have a five-dimensional isometry group [27], which has been identified recently as the "Carroll group with broken rotations" [15], implemented on spacetime as

$$\mathbf{x} \to \mathbf{x} + H(u)\mathbf{b} + \mathbf{c},$$
$$u \to u,$$
$$v \to v - \mathbf{b}\cdot\mathbf{x} - \frac{1}{2}\mathbf{b}\cdot H(u)\mathbf{b} + f, \qquad (4.11)$$

with $\mathbf{b}, \boldsymbol{r} \in \mathbb{R}^2$, and $f \in \mathbb{R}$, where $H(u)$ is the symmetric 2×2 matrix,

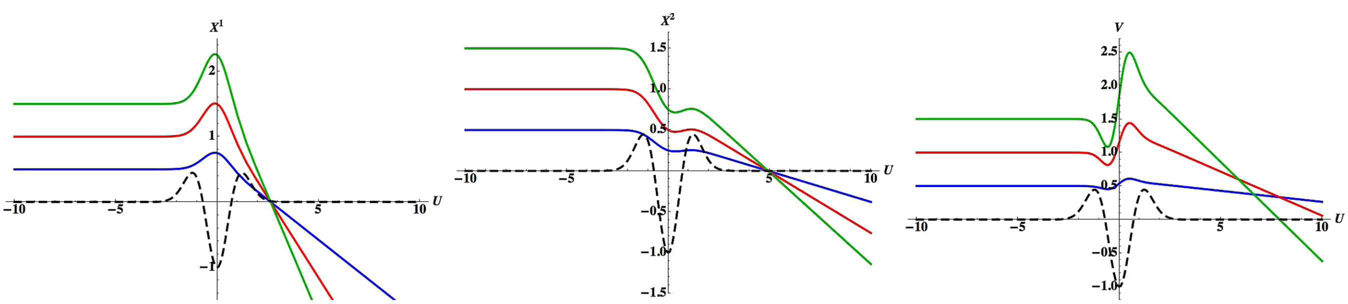

FIG. 6. The system considered by Thorne and Braginsky corresponds to the second derivative of a Gaussian, (4.7).

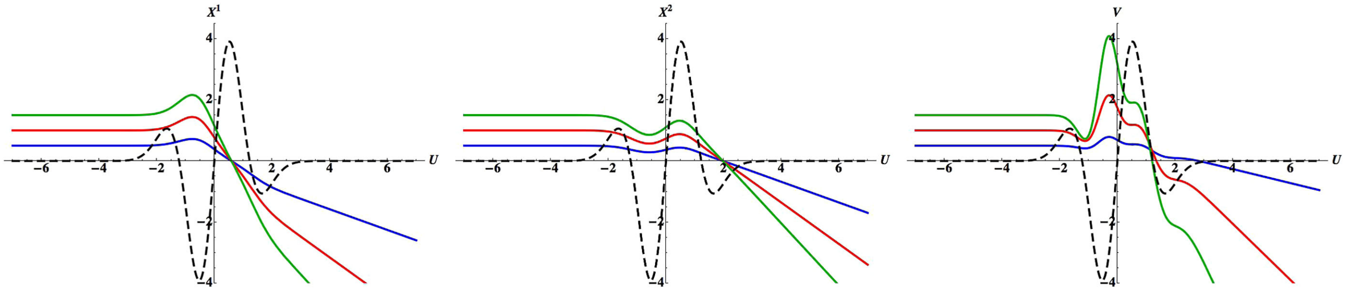

FIG. 7. Geodesics for particles initially at rest for $\mathcal{A}^3_+(U)$ in (4.9), modeling gravitational collapse.

$$H(u) = \int_{u_0}^{u} a(t)^{-1} dt. \tag{4.12}$$

Here $a(u)$ is the transverse-space metric in (2.4) [16].

Noether's theorem associates with the Carroll symmetry five conserved quantities, associated to these isometries. For the geodesic flow parametrized by some s, they are [16,23]

$$\boldsymbol{p} = a(u)\frac{d\mathbf{x}}{ds}, \qquad \mathbf{k} = \mathbf{x}(u) - H(u)\boldsymbol{p}, \qquad \mu = \frac{du}{ds}. \tag{4.13}$$

An extra constant of the motion we identify with the *kinetic energy* is

$$e = \frac{1}{2} g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{1}{2}\frac{d\mathbf{x}}{ds}\cdot a(u(s))\frac{d\mathbf{x}}{ds} + \mu\frac{dv}{ds}. \tag{4.14}$$

Geodesics are timelike/lightlike/spacelike, depending on the sign of e. Timelike means $e < 0$, implying that $\mu \neq 0$ since $a(u) > 0$; the same condition holds also for null geodesics, $e = 0$. Therefore from now on we put $\mu = 1$, which amounts to choosing u as a parameter. Then the quantities listed in (4.13) are interpreted as conserved linear momentum, boost momentum, and "μ".[9]

The geodesics may be expressed using the Noetherian quantities above [15,16], via

$$\mathbf{x}(u) = H(u)\boldsymbol{p} + \mathbf{k},$$
$$v(u) = -\frac{1}{2}\boldsymbol{p}\cdot H(u)\boldsymbol{p} + eu + d, \tag{4.15}$$

where d is a constant of integration. These equations are consistent with (3.14) with $p_v = 1$, as expected. Note that once the values of the conserved quantities are chosen, the only quantity to calculate here is the matrix-valued function $H(u)$ in (4.12). Thus the latter determines both the action of the isometries and the evolution of causal geodesics. In flat Minkowski space with the choice $u_0 = 0$, we have $H(u) = u\mathbf{1}$, yielding free motion

$$\mathbf{x}(u) = u\boldsymbol{p} + \mathbf{k},$$
$$v(u) = \left(-\frac{1}{2}|\boldsymbol{p}|^2 + e\right)u + v_0. \tag{4.16}$$

Returning to the general case, the isometries act on the constants of the motion as

$$(\boldsymbol{p}, \mathbf{k}, e, d) \to (\boldsymbol{p} + \mathbf{b}, \mathbf{k} + \mathbf{c}, e, d + f - \mathbf{b}\cdot\mathbf{k}), \tag{4.17}$$

leaving e invariant [15,16]. They can be used therefore to "straighten out" a geodesic by carrying it to one with $\boldsymbol{p} = 0$, $\mathbf{k} = \mathbf{x}_0$, and $d = 0$, yielding

$$\mathbf{x}(u) = \mathbf{x}_0 = \mathrm{const}, \qquad v = eu, \tag{4.18}$$

as shown on Fig. 8. Therefore we have, for each sign of e, just one type of "vertical" geodesic [16,33]. Conversely, any geodesic is obtained from one of form (4.18) by an isometry.

C. The geodesics in the flat before zone or after zone

We first study the geodesics in the flat spacetime zones outside a sandwich by making use of the results of Sec. II B.

Let us suppose that $a(u) = \mathbf{1}$ in the before zone [33], i.e., for $u < u_i$; then the concavity of the function $\chi(u)$ mentioned above implies that the BJR coordinate system suffers a singularity at some time u_{sing} such that $\chi(u_{\mathrm{sing}}) = 0$, as illustrated in Fig. 1. Note that u_{sing} may lie in or outside the sandwich $[u_i, u_f]$. This coordinate system, used in Eqs. (4.15), is therefore legitimate for $u < u_{\mathrm{sing}}$ only, which we will assume henceforth.

Consider a system of particles at rest (detectors or dust [16]) in the before zone. Their geodesics are given, in natural flat BJR coordinates, by

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_0 \quad \text{and} \quad \hat{v} = e(\hat{u} - \hat{u}_0) + \hat{v}_0 \tag{4.19}$$

which identifies the quantities $\hat{\mathbf{x}}_0$ and $\hat{v}_0$ as initial values.

For the flat metric (2.14) with general initial condition matrix $c_0 \neq 0$, the matrix (4.12) is

[9]When viewed as a Bargmann space of a nonrelativistic particle in one lower dimension, μ (chosen here to be unity) is indeed interpreted as the mass.

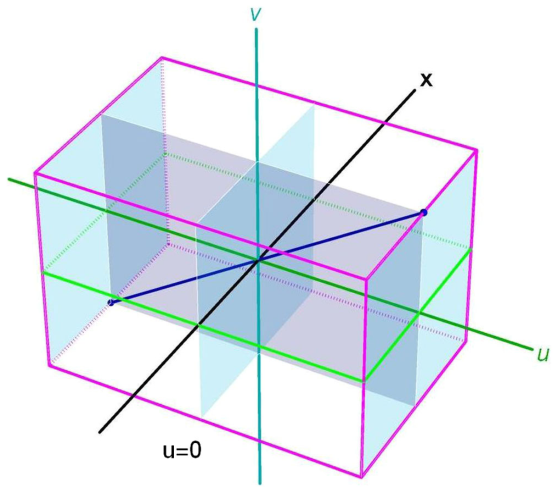

FIG. 8. Each geodesic can be "straightened out" by a suitable action of the (Carroll) isometry group.

$$H(u) = -a_0^{-\frac{1}{2}}c_0^{-1}[(\mathbf{1} + (u - u_0)c_0)^{-1} - \mathbf{1}]a_0^{-\frac{1}{2}}. \quad (4.20)$$

Then a further tedious calculation yields the first integrals $\boldsymbol{p}$, $\mathbf{k}$, and d in (4.15), namely,

$$\boldsymbol{p} = -a_0^{\frac{1}{2}}c_0\hat{\mathbf{x}}_0, \quad \mathbf{k} = a_0^{-\frac{1}{2}}\hat{\mathbf{x}}_0, \quad d = \hat{v}_0 - e\hat{u}_0 + \frac{1}{2}\hat{\mathbf{x}}_0 \cdot c_0\hat{\mathbf{x}}_0. \quad (4.21)$$

Moreover, another lengthy calculation yields, using (4.21) and (4.19), that the geodesics (4.15) are expressed, in original BJR coordinates, as

$$\mathbf{x}(u) = [-H(u)a_0^{\frac{1}{2}}c_0 + a_0^{-\frac{1}{2}}]\hat{\mathbf{x}}, \quad (4.22a)$$

$$u = \hat{u}, \quad (4.22b)$$

$$v(u) = \hat{v} + \frac{1}{2}\hat{\mathbf{x}} \cdot [c_0 - c_0 a_0^{\frac{1}{2}}H(u)a_0^{\frac{1}{2}}c_0]\hat{\mathbf{x}}. \quad (4.22c)$$

These equations may be extended into the sandwich (provided the singularity is avoided) using $H(u)$ given by (4.12). Note that Eqs. (2.21) and (2.22) hold everywhere, including the inside zone.

In the new BJR coordinate system given by (4.22) [which we will still denote by $(\hat{\mathbf{x}}, \hat{u}, \hat{v})$], the metric can be recast into the form

$$\mathrm{g} = d\mathbf{x} \cdot a(u)d\mathbf{x} + 2dudv = d\hat{\mathbf{x}} \cdot \hat{a}(u)d\hat{\mathbf{x}} + 2d\hat{u}d\hat{v}, \quad (4.23a)$$

$$\hat{a}(u) = \left(a_0^{-\frac{1}{2}} - c_0 a_0^{\frac{1}{2}}H(u)\right)a(u)\left(a_0^{-\frac{1}{2}} - H(u)a_0^{\frac{1}{2}}c_0\right); \quad (4.23b)$$

cf. (2.14).

We would like to emphasize that the descriptions in Brinkmann and respectively in BJR coordinates are consistent: numerical calculations show that pushing forward to B coordinates a solution constructed in BJR coordinates yields a trajectory which coincides with the one calculated independently in B coordinates, as long as the BJR coordinate system is regular.

D. Tissot indicatrices and gravitational waves

Textbooks providing an account of the action gravitational waves on a ring of freely falling particles are often illustrated by a series of time frames showing how the ring is squashed and stretched as the wave passes over it. See, e.g., [34]. This representation has an interesting connection with Tissot's indicatrix [62,63], which was originally introduced in *cartography* to illustrate the distortions brought about by map projections.

Suppose we have a projection $\phi: S^2 \to \mathbb{R}^2$ from the surface of the Earth to a flat sheet of paper equipped with Cartesian coordinates x, y; let $g_{xx}(x,y), g_{xy}(x,y) = g_{yx}(x,y), g_{yy}(x,y)$ be the components of the push forward to the flat sheet of paper of the curved metric on the Earth's surface. Tissot's indicatrix at the point $p \in S^2$ with coordinates (x_p, y_p) is the ellipse

$$g_{xx}(x_p, y_p)x^2 + 2g_{xy}(x_p, y_p)xy + g_{yy}(x_p, y_p)y^2 = 1 \quad (4.24)$$

and is the image under ϕ of the unit disc in the tangent space of S^2 [62,63].

If for some reason the metric of the surface of the Earth varied with time then so would Tissot's indicatrix:

$$g_{xx}(x_p, y_p, t)x^2 + 2g_{xy}(x_p, y_p, t)xy + g_{yy}(x_p, x_p, t)y^2 = 1. \quad (4.25)$$

Returning to gravitational waves, we note that the two-dimensional sections of the wave fronts at constant time in Brinkmann coordinates are given by $U = \text{const}$, $V = \text{const}$; in Cartesian coordinates X^i carry a flat, time-independent Euclidean metric. These are mapped into two-dimensional sections of the wave fronts at constant times in BJR co-ordinates $u = \text{const}$, $v = \text{const}$ by $X^i = P^i_j(u)x^j$ as in (2.5), which carry a flat time-dependent Euclidean metric $a_{ij}(u)$ in x^i coordinates. Note that these two-surfaces do not in general coincide in spacetime since while U and u are identical, V and v differ.

The family of timelike geodesics $x^i = \text{const}$ do *not* have $X^i = \text{const}$ in Brinkmann coordinates. This means that an *initially* (i.e., before the pulse) circular disc of geodesics in X^i coordinates, $\mathbf{X} \cdot \mathbf{X} \leq 1$ for $U < U_i$, projects to a time-*independent* circle in x^i coordinates, $\mathbf{x} \cdot \mathbf{x} \leq 1$ for all u, i.e., even during and after the sandwich, $u \geq u_i$. However, their inverse image in Brinkmann coordinates is a time-dependent ellipse,

$$1 = \mathbf{x} \cdot \mathbf{x} = \mathbf{X} \cdot (PP^T)^{-1}\mathbf{X}. \quad (4.26)$$

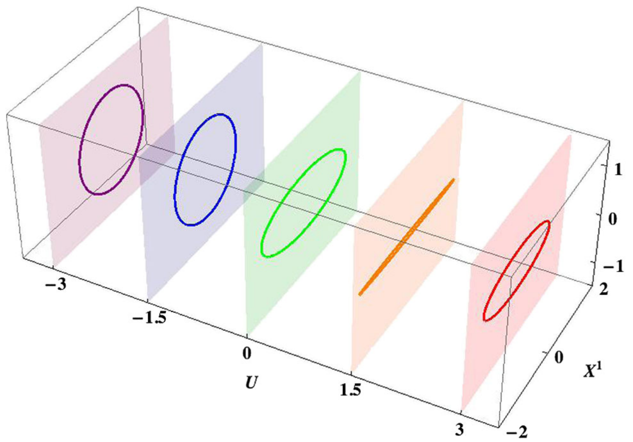

FIG. 9. Tissot spacetime diagram for the linear polarization $\mathcal{A}_+(U) = d(e^{-U^2})/dU$, $\mathcal{A}_\times(U) = 0$, for values $u = -3$ (purple), $u = -1.5$ (blue), $u = 0$ (green), $u = 1.5$ (orange), $u = 3$ (red).

Since in Brinkmann coordinates the metric is Euclidean, the coordinates represent proper distance measured within that two-surface.

The deformation of the Tissot circle is illustrated by the spacetime diagram in Fig. 9 for the linear polarization $\mathcal{A}_+(U) = d(e^{-U^2})/dU$, $\mathcal{A}_\times(U) = 0$ appropriate to model flyby, as argued above. Similar diagrams could be obtained for circular polarization, and also for nonburstlike profiles as in the case of primordial gravitational waves. The deformation starts before $u = 0$, since the burst has a finite thickness. A similar diagram is presented in [23] in the gravitation-collapse case $\mathcal{A}_+(U) = d^3(e^{-U^2})/dU^3$.

E. Permanent displacements?

Equation (4.15) implies that BJR solutions with $\boldsymbol{p} = 0$ are trivial *for any profile*,

$$\mathbf{x}(u) = \mathbf{x}_0, \qquad v = e(u - u_0) + v_0, \tag{4.27}$$

for all u, i.e., in the before, inside, and after zones. This happens in particular for particles which are at rest in the before zone whose conserved momentum vanishes because $\boldsymbol{p} = a\dot{\mathbf{x}}$; cf. (4.13). It is worth emphasizing that the memory effect *does* arise even in this case: nontrivial behavior in B coordinates arises entirely from the relation [15]

$$\mathbf{X}(u) = P(u - u_0)\mathbf{x}_0. \tag{4.28}$$

But such particles are *not* in general at rest in the after zone because $\dot{P} \neq 0$ in general, whereas some important papers on the memory effect [1,2,10] predict precisely that: particles at rest in the before zone could end up at *rest* but *displaced* in the after zone. Indeed, according to some authorities, this is taken as a definition of the memory effect.

A possible indication that this might not be possible comes from the particular cases studied in Sec. IV A which show constant but nonzero asymptotic velocity in the after zone (except for $\mathbf{x}_0 = 0$). Moreover, the relative velocities depend on $\mathbf{x}_0$, contradicting the expectations of Zel'dovich and Polnarev [1] cited in the Introduction.

One may ask whether one may have a smooth interpolation between $P(u) = \mathbf{1}$ in the before zone and a constant diagonal matrix $P_\infty \neq \mathbf{1}$ in the after zone. For example, is there a smooth function $f(u)$ such that

$$P(u) = (1 - f(u))\mathbf{1} + f(u)P_\infty \quad \text{with}$$
$$f(u) = \begin{cases} 0 & u \leq u_i \\ 1 & u \geq u_f \end{cases} ? \tag{4.29}$$

If we further assume that P_∞ is diagonal and $P_\infty = \mathrm{diag}(\pi_1, \pi_2)$ with $\pi_{1,2} = \mathrm{const} \neq 1$, we find that

$$K(u) = \ddot{P}.P^{-1} = \ddot{f}(u)\mathrm{diag}\left(\frac{-1 + \pi_1}{1 - f + \pi_1 f}, \frac{-1 + \pi_2}{1 - f + \pi_2 f}\right). \tag{4.30}$$

In order to satisfy the vacuum Einstein equations K must be traceless which is however readily seen to contradict the assumption that f is *smooth*: $f(u)$ should be linear with nonzero slope in the inside zone, joined by horizontal lines in the before and after zones and therefore nondifferentiable at $u = u_i, u_f$.

If this rather special example could be generalized, one might conclude that no static displacement is possible unless some sort of *impulsive waves* with nonsmooth profiles are considered [16,24,42,43].

V. NULL GEODESICS, LIGHT CONES, AND GLOBAL GEOMETRY

A. The memory effect and optics

So far we have only considered freely falling particles. However, as remarked in [14], the memory effect also influences the motion of light. One way to see this is to recall that Maxwell's equations in a curved vacuum spacetime may be interpreted as flat spacetime electrodynamics in an "impedance-matched" medium. Using the results of [15] we see that in BJR coordinates the permittivity ϵ^{ab} and permeability μ^{ab} (with $\epsilon^{ab} = \mu^{ab}$) satisfy $\epsilon^{ab} = \mu^{ab} = \delta^{ab}$ before the gravitational wave arrives but after it has passed. They are given by

$$\epsilon^{ij} = \sqrt{\det a(u)}(a(u)^{-1})^{ij}, \quad \epsilon^{33} = \sqrt{\det a(u)}, \quad \epsilon^{3i} = 0, \tag{5.1}$$

and since $\epsilon^{ij} \neq \delta^{ij}$, the wave has left a memory on the effective optical medium.

B. Light cones and causality

In an insightful account of the global geometry of plane gravitational waves Penrose [35] showed that in general they are not globally hyperbolic and as a consequence they cannot be isometrically embedded into a higher dimensional flat space with just a single time coordinate [35]. Penrose mainly worked in Brinkmann coordinates [20] although he does allude to the existence of BJR coordinates which he ascribes to Rosen [22].

Penrose obtains, for a sandwich wave, the formula

$$V = F_{ij}(U)X^iX^j + V_0 \tag{5.2}$$

for the light cone of a point $p = (\mathbf{X}_0, U_0, V_0)$, where the symmetric matrix F with components $F_{ij} \approx (U - U_0)^{-1}\delta_{ij}$ near p must satisfy

$$\dot{F} + F^2 - K = 0. \tag{5.3}$$

Penrose considers the case when p is located in the flat region before the pulse arrives. He shows that the metric F becomes singular within a finite amount of u time [16]. This allows him to obtain his nonglobal hyperbolicity result. He points out that this phenomenon is closely related to the singularity of BJR coordinates discussed by [22,26,27].

Penrose's results are readily rederived by translating our result from BJR to Brinkmann coordinates. *Null geodesics* are characterized by

$$e = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{5.4}$$

Special null geodesics, defined by the vanishing of the linear momentum, $\boldsymbol{p} = 0$, are thus simply

$$\mathbf{x}(u) = \mathbf{x}_0, \qquad v(u) = v_0. \tag{5.5}$$

Moreover, (2.5) gives us the image of the special null geodesics (5.5), namely,

$$\mathbf{X}(U) = P(U)\mathbf{x}_0, \tag{5.6a}$$

$$V(U) = v_0 - \frac{1}{2}\mathbf{X}\cdot\dot{P}P^{-1}\mathbf{X}. \tag{5.6b}$$

Then the three-dimensional light cone in $\mathbb{R}^4$ generated by null geodesics through some point is thus defined by the equation

$$V = v_0 - \frac{1}{2}\mathbf{X}\cdot F\mathbf{X}, \tag{5.7}$$

where $F = \dot{P}P^{-1}$ satisfies (5.3) in view of (2.7). Our equations above thus reproduce (7.1) and (7.2) of [35] up to a factor $\frac{1}{2}$ and a sign, due to different conventions.

Null geodesics in plane gravitational waves have recently received an extended study in [13,14].

VI. EXACT EINSTEIN-MAXWELL PLANE WAVES

Exact Einstein-Maxwell plane waves were first considered in [21] in BJR coordinates. Here we shall follow [64]. For the sake of comparison, we will temporarily adhere to their signature conventions. Their metric in Brinkmann coordinates is

$$g = -\delta_{ij}dX^idX^j + dUdV - K(\mathbf{X}, U)dU^2. \tag{6.1}$$

Their vector potential is taken to be

$$A = A_i(U)dX^i = d(A_iX^i) - X^iA_i'(U)dU. \tag{6.2}$$

In fact we shall find it useful to use the last term on the rhs of (6.2) which differs from $A_i(U)dX^i$ by a gauge transformation. The Maxwell field,

$$F = A_i'(U)dU \wedge dX^i, \tag{6.3}$$

solves $\star dF = 0$. Then the Einstein equation is equivalent to

$$\frac{\partial^2 K}{\partial X^i \partial X^i} = 4GA_i'A'_i, \tag{6.4}$$

where G is Newton's constant and we are using Heaviside units ($4\pi\epsilon_0 = 1$). We choose the solution

$$K(\mathbf{X}, U) = \mathcal{A}_+(U)((X^1)^2 - (X^2)^2) + 2\mathcal{A}_\times(U)X^1X^2 + 8G|\mathbf{A}'(U)|^2((X^1)^2 + (X^2)^2), \tag{6.5}$$

which merely differs from (2.3) in an additional quadratic (in Bargmann language, a "time-dependent oscillator" [45,46,65]) term.

The passage to BJR coordinates proceeds in a way similar to the pure gravity case, (2.5).

Note that since the gravitational wave and the electromagnetic wave are essentially independent in Brinkmann coordinates, we can specify $\mathcal{A}_+(U)$, $\mathcal{A}_\times(U)$, and $A_i(U)$ independently. There is no *graviton-photon* or *photon-graviton* conversion, even though the metric has backreacted to the presence of the electromagnetic field.

This looks very different in the BJR coordinates, though, in which no simple "superposition principle" holds. A special case is that one can superpose polarization states in Brinkmann coordinates, but not in a literal fashion in BJR coordinates [66–68].

As pointed out in [64] the coupled Einstein-Maxwell system has five Killing fields, three of which mutually commute—in fact, the generators of the isometry group found for a pure plane gravitational wave [15]—namely, *the Carroll group* with broken rotations, implemented as in (4.11).

The proof is straightforward: everything we developed here and in our previous paper [15] goes through unchanged. The metric $a = (a_{ij}(u))$ is related to the wavefront $K = (K_{ij}(u))$ in the usual manner; the only difference is that the tracelessness of K is replaced by (6.4). But this does not affect the general form of the metric, cf. (2.4), whose isometries span the Carroll group in $2+1$ dimensions with broken rotations.

VII. MIDISUPERSPACE QUANTIZATION OF PLANE GRAVITATIONAL WAVES

A. Midisuperspace of plane gravitational waves

We have seen that the midisuperspace[10] of Ricci flat plane gravitational waves is parametrized by the two real functions $\mathcal{A}_+(U)$ and $\mathcal{A}_\times(U)$. This is an infinite-dimensional vector space $\mathcal{W}$ and in what follows it will be convenient to assume that $\mathcal{A}_+(U)$ and $\mathcal{A}_\times(U)$ are in $L^2(\mathbb{R})$ so as to permit Fourier analysis. Thus we take

$$\mathcal{W} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}). \tag{7.1}$$

The rotation group $SO(2)$ acts on $\mathcal{W}$,

$$\begin{matrix} X^1 \to \cos\alpha X^1 + \sin\alpha X^2 \\ X^2 \to \cos\alpha X^2 - \sin\alpha X^1 \end{matrix} \Rightarrow \begin{matrix} \mathcal{A}_+ \to \cos 2\alpha \mathcal{A}_+ - \sin 2\alpha \mathcal{A}_\times \\ \mathcal{A}_\times \to \cos 2\alpha \mathcal{A}_\times + \sin 2\alpha \mathcal{A}_+ \end{matrix}. \tag{7.2}$$

Thus $\mathcal{W}$ carries a helicity-2 representation of SO(2), as expected. Note that two metrics related by a rotation are geometrically identical but we choose to distinguish them because the action of the rotation does not tend to the identity at infinity. In other words we are imagining some reference system "at infinity" relative to which it is meaningful to speak of the orientation $X^1 - X^2$ space.

The real vector space $\mathcal{W}$ admits a symplectic form Ω. Let us introduce the notation $\mathcal{C} = (\mathcal{A}_+, \mathcal{A}_\times)$ for a general vector in $\mathcal{W}$. Then for two vectors $\mathcal{C}_1$ and $\mathcal{C}_2$ we define

$$\Omega(\mathcal{C}_1, \mathcal{C}_2) = \int_{-\infty}^{\infty} \left(\mathcal{A}_{+1} \frac{d\mathcal{A}_{+2}}{dU} - \mathcal{A}_{+2} \frac{d\mathcal{A}_{+1}}{dU} + \mathcal{A}_{\times 1} \frac{d\mathcal{A}_{\times 2}}{dU} - \mathcal{A}_{\times 2} \frac{d\mathcal{A}_{\times 1}}{dU} \right) dU. \tag{7.3}$$

Note that if one regards V as the time coordinate, then $\mathcal{W}$ is well defined and independent of V and therefore the symplectic form Ω is independent of "time." The hypersurfaces $U = \text{const}$, while not null, act here as surrogates for Cauchy surfaces.[11]

In order to quantize this sector of quantum Einstein theory, we now pass to the complexification $W_\mathbb{C}$ of the classical real symplectic vector space $\mathcal{W}$ and to extend Ω to $\mathcal{W}_\mathbb{C}$ in a $\mathbb{C}$-linear fashion. This enables us to endow $\mathcal{W}_\mathbb{C}$ with a sesquilinear form,

$$\langle \mathcal{C} | \mathcal{C} \rangle = \frac{i}{2} \Omega(\bar{\mathcal{C}}, \mathcal{C}), \tag{7.4}$$

where $\bar{\mathcal{C}}$ denotes the complex conjugate of $\mathcal{C}$. However $\langle \mathcal{C} | \mathcal{C} \rangle$ is not positive definite. In order to render $\Omega(\mathcal{C}, \mathcal{C})$ positive definite, we must restrict $\langle \mathcal{C} | \mathcal{C} \rangle$ to a $\mathbb{C}$-linear subspace $\mathcal{H} \subset \mathcal{W}_\mathbb{C}$ on which $\langle \mathcal{C} | \mathcal{C} \rangle$ is positive definite on which $\langle \mathcal{A}_+ | \mathcal{A}_\times \rangle$ is positive definite.

This is conventionally achieved in quantum field theory by restricting to functions in $\mathcal{W}_\mathbb{C}$ which have "positive frequency" with respect to the coordinate U. If U is chosen to increase to the future, then that means that $\mathcal{A}_+$ and $\mathcal{A}_\times$ only contain Fourier components with $\omega < 0$. One then has

$$\mathcal{W}_\mathbb{C} = \mathcal{H} \oplus \bar{\mathcal{H}}. \tag{7.5}$$

The space of quantum states $\mathcal{H}$ in this sector of the entire Hilbert space of Einstein quantum gravity may be identified with the vacuum Einstein equations which are analytically continued to complex values of the Brinkmann coordinates X^1, X^2, U, V, which are holomorphic in the lower half of the U-plane.

One might then envisage an entire free "one graviton" Hilbert space by considering gravitational waves moving in all possible directions but not interacting, the continuous direct sum

$$\int_{S^2} \sin\theta d\theta d\phi \mathcal{H}_\mathbf{n}, \tag{7.6}$$

where $\mathbf{n} \in S^2$ labels the direction in the space of the plane waves. Following the conventional rules of perturbative quantum field theory one might then pass to the free Fock space based on $\mathcal{H}$. Free correlation functions would then be defined on symmetric products of the complexified plane-wave spacetime. The inclusion of interactions then however presents severe difficulties. Moreover, at the classical level spacetime singularities are encountered when plane waves collide [70,71].

[10] "Superspace" was a term coined by Wheeler to denote the configuration space of all Riemannian 3-metrics modulo diffeomorphisms. He thought of it as the natural arena for quantum gravity. Strictly speaking, when one quantizes, one passes to the reduced phase space, obtained by taking into account the Hamiltonian and diffeomorphism constraints. This amounts to considering the space of Cauchy data, or equivalently, classical histories, that is, classical solutions of the Einstein equations modulo diffeomorphism equivalence. A symmetry reduction (but still with infinite dimensions) is called a midisuperspace. A symmetry reduction to finite dimensions is called a minisuperspace. The reader may consult [69] for a review.

[11] Our choice of the sign in front of $2dUdV$ in our metric complicates this because it implies that either V or U decreases into the future. In order to ensure that $\mathrm{g} = -dT^2 + dZ^2 + \cdots$ we need to put $U = \frac{1}{\sqrt{2}}(Z+T)$, $V = \frac{1}{\sqrt{2}}(Z-T)$, or vice versa, for example. Often U and V are thought of as retarded and advanced times, i.e., $U = T - Z$ and $V = T + Z$. This does not quite work with our conventions.

B. Stokes parameters and the Poincaré sphere

The only covariant treatment of this is at the linear level and notationally rather complicated [72]. A treatment of electromagnetic waves in a pp-wave background is given in [73]. See also [74]. Hence we shall follow the obvious analogy with the electromagnetic case. We begin by stating our conventions about Fourier transforms. For a real valued function of $f(U)$ we define its Fourier transform $\tilde{f}(\omega)$ by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(U) e^{i\omega U} dU. \tag{7.7}$$

The Fourier inversion theorem states that

$$\begin{aligned} f(U) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega U} d\omega \\ &= \frac{1}{\pi}\int_{0}^{\infty} \mathfrak{Re}(|\tilde{f}(\omega)| \cos(\omega t - \psi(\omega))) d\omega. \end{aligned} \tag{7.8}$$

Now in the case of a coherent classical electromagnetic wave the transverse electric field has two real components $E_1(U)$ and $E_2(U)$ with Fourier transforms $\tilde{E}_1(\omega)$ and $\tilde{E}_2(\omega)$, and we shall take their gravitational analogues to be $\mathcal{A}_+(U)$ and $\mathcal{A}_\times(U)$ with Fourier transforms $\tilde{\mathcal{A}}_+(\omega)$ and $\tilde{\mathcal{A}}_\times(\omega)$. From now on we shall work at fixed ω and suppress it in most of the formulas which follow. We define the following four real Stokes parameters [75], which we combine in a Stokes four-vector S^μ given by

$$\begin{aligned} (S^0, S^1, S^2, S^3) = (&|\tilde{\mathcal{A}}_+|^2 + |\tilde{\mathcal{A}}_\times|^2, |\tilde{\mathcal{A}}_+|^2 \\ &- |\tilde{\mathcal{A}}_\times|^2, 2\mathfrak{Re}\tilde{\mathcal{A}}_+ \overline{\tilde{\mathcal{A}}_\times}, 2\mathfrak{Im}\tilde{\mathcal{A}}_+ \overline{\tilde{\mathcal{A}}_\times}). \end{aligned} \tag{7.9}$$

It follows that

$$-(S^0)^2 + (S^1)^2 + (S^2)^2 + (S^3)^2 = \eta_{\mu\nu} S^\mu S^\nu = 0. \tag{7.10}$$

That is, for a coherent state, the Stokes four-vector S^μ is a future directed null vector passing through the origin of an auxiliary Minkowski spacetime. For a statistical ensemble of gravitational waves, the definition of the Stokes four-vector contains a statistical average or expectation value denoted by $E[\cdot]$, and thus

$$\begin{aligned} S^\mu = E[(&|\tilde{\mathcal{A}}_+|^2 + |\tilde{\mathcal{A}}_\times|^2, |\tilde{\mathcal{A}}_+|^2 \\ &- |\tilde{\mathcal{A}}_\times|^2, 2\mathfrak{Re}\tilde{\mathcal{A}}_+ \overline{\tilde{\mathcal{A}}_\times}, 2\mathfrak{Im}\tilde{\mathcal{A}}_+ \overline{\tilde{\mathcal{A}}_\times})]. \end{aligned} \tag{7.11}$$

It then follows that S^μ is future directed timelike or null, i.e.,

$$-(S^0)^2 + (S^1)^2 + (S^2)^2 + (S^3)^2 \leq 0. \tag{7.12}$$

It is possible to encode the Stokes four-vector in a 2×2 Hermitian matrix positive semidefinite coherence matrix ρ which has some analogies to a density matrix in quantum mechanics. Indeed, if the ensemble is a quantum ensemble this analogy holds fairly closely.

We set

$$\begin{aligned} \rho &= E\left[\begin{pmatrix} |\tilde{\mathcal{A}}_+|^2, & \tilde{\mathcal{A}}_+ \overline{\tilde{\mathcal{A}}_\times} \\ \overline{\tilde{\mathcal{A}}_+}\tilde{\mathcal{A}}_\times, & |\tilde{\mathcal{A}}_\times|^2 \end{pmatrix}\right] \\ &= \frac{1}{2}\begin{pmatrix} S^0 + S^1, & S^2 + iS^3 \\ S^2 - iS^3, & S^0 - S^1 \end{pmatrix}. \end{aligned} \tag{7.13}$$

As long as the Stokes four-vector S^μ lies inside the future light cone, the Hermitian matrix ρ will be positive definite since $\mathrm{tr}\rho = S^0 > 0$ and $\det\rho = -\frac{1}{4}\eta_{\mu\nu}S^\mu S^\nu > 0$. If the Stokes four-vector lies on the light cone, then $\det\rho = 0$. If one introduces the Jones complex valued two-vector [76]

$$J = \begin{pmatrix} \tilde{\mathcal{A}}_+ \\ \tilde{\mathcal{A}}_\times \end{pmatrix}, \tag{7.14}$$

then

$$\rho = E[JJ^\dagger]. \tag{7.15}$$

In the coherent case, the Poincaré sphere [77] is obtained by normalizing the Jones two-vector

$$J^\dagger J = 1, \tag{7.16}$$

since this implies that $S^0 = 1$. The spinor geometry behind this construction has recently been described in [78].

In the coherent electromagnetic case it is customary to describe the polarization states by plotting the curve $(E_1, E_2) = (\mathfrak{Re}\tilde{E}_1 e^{-i\omega t}, \mathfrak{Re}\tilde{E}_2 e^{-i\omega t})$ in the (X^1, X^2) plane. If one normalizes the Jones two-vector

$$J = \begin{pmatrix} \tilde{E}_1 e^{-i\omega t} \\ \tilde{E}_2 e^{-i\omega t} \end{pmatrix} \tag{7.17}$$

such that

$$J^\dagger J = |\tilde{E}_1 e^{-i\omega t}|^2 + |\tilde{E}_2 e^{-i\omega t}|^2 = 1, \tag{7.18}$$

one may introduce parameters such that

$$J = \begin{pmatrix} \cos\frac{\theta}{2} e^{i(-\omega t + \delta_1)} \\ \sin\frac{\theta}{2} e^{i(-\omega t + \delta_2)} \end{pmatrix}. \tag{7.19}$$

Now (7.18) defines a unit three-sphere in four-dimensional Euclidean space. As time progresses, points on the three-sphere are moved along the orbits of the $U(1)$ action $J \to e^{-i\omega t} J$. However, the angle θ and the relative phase $\delta = \delta_2 - \delta_1$, $-\pi \leq \delta \leq \pi$ are unchanged. As this happens, the electric vector (E_1, E_2) sweeps out an ellipse lying inside a rectangle of sides $(\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$ whose major axis makes an angle $\frac{1}{2}\arctan(\tan\theta\cos\delta)$ with the E_x axis. If $\delta > 0$, the polarization is right handed; if $\delta \leq 0$, it is left handed. The orbits in S^3 are called Hopf fibers and the space of such orbits is called the Poincaré sphere. Points on the north and south poles $\theta = \pm\frac{\pi}{2}$, respectively, correspond to plane polarized states and points on the equator $\theta = 0$ to circularly polarized states. The remaining states are elliptically polarized. All but the plane polarized states have a "handedness."

The foregoing theory may readily be adapted to the complex polarization gravitational wave amplitudes $\tilde{\mathcal{A}}_+ e^{-\omega t}$ and $\tilde{\mathcal{A}}_\times e^{-\omega t}$. However, there is no direct analogue of the electric field vector other than the tensor K_{IJ} and so the image of the electric vector executing an ellipse does not seem to have a direct analogue. However, an important aspect brought out above is that gravitational waves also have a handedness. It is this handedness of primordial gravitational waves which is responsible for the generation of the so-called "B-mode" of the electromagnetic waves making up the cosmic background radiation, whose presence is predicted by theories of inflation [5].

The mechanism for the transfer of gravitational wave energy to electromagnetic wave energy is the effect of gravitational waves described in the present paper on freely falling electrically charged particles (free electrons in the primordial plasma around recombination in the case of the CMB) envisaged as an abstract possibility in [79].[12] The charged particles are necessary because, as we noted in the previous section, there is no direct conversion of gravitational waves into electromagnetic waves. One might almost claim that if the B-mode is observed then the gravitational memory effect will be, albeit indirectly, observed.

The effect of a polarized monochromatic gravitational wave may be seen by solving the equations of geodesic deviation (3.3), assuming

$$\mathcal{A}_+ = \mathcal{C}_+ \sin(\omega U), \qquad \mathcal{A}_\times = \mathcal{C}_\times \sin(\omega U + \phi), \tag{7.20}$$

where the frequency ω; amplitudes $\mathcal{C}_+, \mathcal{C}_\times$; and relative phase ϕ are constants.

We conclude by remarking on the analogy between the use of the Poincaré sphere and the way a two-state system, up to and over all phases, corresponds to the Bloch sphere [80]. However, depending upon the spin or helicity of the states, the action of a physical rotation through an angle α on the spheres will differ. For spin $\frac{1}{2}$, one has $\delta \to \delta + 2s\alpha$. For quantum systems there is a notion of Berry or Aharonov-Bohm transport [81–83]. In the case of spin 1 states, this corresponds to parallel transport on complex projective space $\mathbb{CP}^2$ [84]. However, in the case of polarized states in optics, this corresponds to Pancharatnam transport [85–87]. Pancharatnam's condition of maximum parallelism between two waves with Jones two-vectors is $J^\dagger J' \geq 0$ and in particular that $J^\dagger J' \geq 0$ is real. If J and $J + dJ$ are two neighboring states we have by virtue of the normalization condition $J^\dagger J = 1$

$$J^\dagger dJ + dJ^\dagger J = 0, \tag{7.21}$$

so we define Pancharatnam parallel transport of the phase by

$$J^\dagger dJ = 0 = \bar{J}_1 dJ_1 + \bar{J}_2 dJ_2. \tag{7.22}$$

Now we introduce the stereographic coordinate on S^2 by

$$\zeta = \frac{J_2}{J_1} = \tan\frac{\theta}{2} e^{-2i\delta}. \tag{7.23}$$

Pancharatnam's rule for parallel transport reads

$$d\ln Z^1 + \frac{\bar{\zeta} d\zeta}{1 + |\zeta|^2} = 0, \qquad \text{that is,}$$

$$i(d\tau + d\delta_1) + \frac{\bar{\zeta} d\zeta}{1 + |\zeta|^2} = 0, \tag{7.24}$$

which corresponds to the $U(1)$ connection and curvature,

$$A = -i\frac{\bar{\zeta} d\zeta}{1 + |\zeta|^2} \qquad \text{and} \qquad F = dA = \frac{-i d\bar{\zeta} \wedge d\bar{\zeta}}{(1 + |\zeta|^2)^2}, \tag{7.25}$$

respectively. Parallel transport around a simple closed curve γ enclosing a domain D produces total holonomy,

$$\int_D F = \frac{1}{2}\Omega, \tag{7.26}$$

where Ω is the solid angle subtended by the loop γ at the center of the sphere. The factor of $\frac{1}{2}$ arises because A is the spin connection of the metric on S^2 and satisfies the minimal Dirac requirement:

$$\int_{S^2} F = 2\pi. \tag{7.27}$$

The Levi-Cività connection, whose curvature $2F = K$ is the Gauss curvature, is twice as large and its curvature $2F = K$ is the Gauss curvature [thought of as an $\mathfrak{so}(2)$ valued 2-form], which satisfies the Gauss-Bonnet condition

$$\int_{S^2} 2F = \int_{S^2} K = 4\pi. \tag{7.28}$$

[12] However, it should be pointed out that these authors mention neither the CMB nor polarization effects.

VIII. CONCLUSION

In this paper we have clarified the physically important notions of "gravitational memory" and of "soft graviton" in a simple and easily calculable model that nevertheless permits a mathematically rigorous treatment and that captures all the relevant physics. We present exact solutions of Einstein's equations describing plane gravitational waves of arbitrary polarization in the two most useful coordinates. We obtained exact expressions for the geodesics in both sets of coordinates. This allowed us to exhibit the action of a finite duration pulse of gravitational radiation on freely falling particles initially at rest in an inertial coordinate system in a portion of flat Minkowski spacetime to the past of the pulse.

Integrating the geodesic equations in BJR coordinates became possibly due to their manifest Carroll symmetry, (4.11), leading to the conserved quantities (4.13).

Plane gravitational waves have long been known to have a five-parameter isometry group [16,27]. The generating Killing vectors have, in Brinkmann coordinates, the components of our P matrix (2.7) as coefficients [33,39]. However, being solutions of a Sturm-Liouville equation, these coefficients are not known in general.

In BJR coordinates the symmetry is manifest and the associated conserved quantities can be calculated by calculating the matrix H in (4.12). The price to pay is that it is now the correspondence B $\Leftrightarrow$ BJR that requires solving a Sturm-Liouville equation: the difficulty is thus transferred to the transformation between the two sets of coordinates.

Particles initially at rest have vanishing momentum $\boldsymbol{p} = 0$ and their trajectory in BJR coordinates is therefore, for all smooth wave profiles, the simply straight one in (4.18). Thus after the pulse their transverse positions remain at rest in the noninertial BJR coordinate system. The memory effect is *not lost*, however: it is encoded in the diffeomorphism which we calculate explicitly, relating the past inertial coordinates to the future noninertial coordinate system. This diffeomorphism, which is in principle constructible from observations using gravitational wave detectors, does not tend to identity at infinity.

Flat plane wave solutions of Einstein's vacuum equations Eq. (2.14) in noninertial coordinates are more general than just Minkowski and may be thought of as soft gravitons dressing the initial Minkowski vacuum state.

The extension to Einstein-Maxwell theory is straightforward and a midisuperspace quantization can be given.

ACKNOWLEDGMENTS

We are grateful to Paul Lasky for instructing us of LIGO's possible future ability to observe the memory effect. Anthony Lasenby kindly informed us of his researches on the velocity memory effect [88]. Correspondence from Abraham Harte is also acknowledged. G. W. G. would like to thank the Laboratoire de Mathématiques et de Physique Théorique de l'Université de Tours for hospitality and the Région Centre for a "Le Studium" research professorship. P. H. is grateful for hospitality at the Institute of Modern Physics of the Chinese Academy of Sciences in Lanzhou. Support by the National Natural Science Foundation of China (Grant No. 11575254) is acknowledged.

Note added.—Recently, we were informed of the related work of A. Lasenby [88], who arrived, independently, at similar conclusions. He also called our attention to another important reference on the velocity memory effect [89]. Our results here are fully consistent with those of Grishchuk and Polnarev [89], and also with those of Bondi and Pirani [33], who state that after the waves have passed detectors originally at rest will move with constant but not zero relative velocity.

[1] Ya. B. Zel'dovich and A. G. Polnarev, Radiation of gravitational waves by a cluster of superdense stars, Astron. Zh. **51**, 30 (1974) Sov. Astron. **18**, 17 (1974).
[2] V. B. Braginsky and L. P. Grishchuk, Kinematic resonance and the memory effect in free mass gravitational antennas, Zh. Eksp. Teor. Fiz. **89**, 744 (1985) Sov. Phys. JETP **62**, 427 (1985)].
[3] M. Favata, The gravitational-wave memory effect, Classical Quantum Gravity **27**, 084036 (2010).
[4] P. D. Lasky, E. Thrane, Y. Levin, J. Blackman, and Y. Chen, Detecting Gravitational-Wave Memory with LIGO: Implications of GW150914, Phys. Rev. Lett. **117**, 061102 (2016).
[5] M. Kamionkowski, A. Kosowsky, and A. Stebbins, Statistics of cosmic microwave background polarization, Phys. Rev. D **55**, 7368 (1997).
[6] S. W. Hawking, M. J. Perry, and A. Strominger, Soft Hair on Black Holes, Phys. Rev. Lett. **116**, 231301 (2016).
[7] S. W. Hawking, M. J. Perry, and A. Strominger, Superrotation charge and supertranslation hair on black holes, J. High Energy Phys. 05 (2017) 161.
[8] V. B. Braginsky and K. S. Thorne, Gravitational-wave burst with memory and experimental prospects, Nature (London) **327**, 123 (1987).
[9] G. W. Gibbons and S. W. Hawking, Theory of the detection of short bursts of gravitational radiation, Phys. Rev. D **4**, 2191 (1971).
[10] D. Christodoulou, Nonlinear Nature of Gravitation and Gravitational Wave Experiments, Phys. Rev. Lett. **67**, 1486 (1991).

[11] K. S. Thorne, Gravitational-wave bursts with memory: The Christodoulou effect, Phys. Rev. D **45**, 520 (1992).

[12] L. Blanchet and T. Damour, Hereditary effects in gravitational radiation, Phys. Rev. D **46**, 4304 (1992).

[13] A. I. Harte, Strong lensing, plane gravitational waves and transient flashes, Classical Quantum Gravity **30**, 075011 (2013).

[14] A. I. Harte, Optics in a nonlinear gravitational plane wave, Classical Quantum Gravity **32**, 175017 (2015).

[15] C. Duval, G. W. Gibbons, P. A. Horvathy, and P.-M. Zhang, Carroll symmetry of plane gravitational waves, Classical Quantum Gravity **34**, 175003(2017).

[16] J-M. Souriau, Ondes et radiations gravitationnelles, *Colloques Internationaux du CNRS No 220, Paris* (1973), pp. 243–256.

[17] S. Hollands, A. Ishibashi, and R. M. Wald, BMS supertranslations and memory in four and higher dimensions, Classical Quantum Gravity **34**, 155005 (2017).

[18] M. A. Awada, G. W. Gibbons, and W. T. Shaw, Conformal supergravity, twistors and the super BMS group, Ann. Phys. (N.Y.) **171**, 52 (1986).

[19] D. Garfinkle, S. Hollands, A. Ishibashi, A. Tolish, and R. M. Wald, The memory effect for particle scattering in even spacetime dimensions, Classical Quantum Gravity **34**, 145015 (2017).

[20] M. W. Brinkmann, On Riemann spaces conformal to Euclidean spaces, Proc. Natl. Acad. Sci. U.S.A. **9**, 1 (1923); Einstein spaces which are mapped conformally on each other, Math. Ann. **94**, 119 (1925).

[21] O. R. Baldwin and G. B. Jeffery, The relativity theory of plane waves, Proc. R. Soc. A **111**, 95 (1926).

[22] N. Rosen, Plane polarized waves in the general theory of relativity, Phys. Z. Sowjetunion, **12**, 366 (1937).

[23] P.-M. Zhang, C. Duval, G. W. Gibbons, P. A. Horvathy, The memory effect for plane gravitational waves, Phys. Lett. B **772**, 743 (2017).

[24] R. Penrose, The geometry of impulsive gravitational waves, in *General Relativity, Papers in Honour of J. L. Synge*, edited by L. O'Raifeartaigh (Clarendon Press, Oxford, 1972), pp. 101–115.

[25] A. Peres, PP Waves, Phys. Rev. Lett. **3**, 571 (1959).

[26] H. Bondi, Plane gravitational waves in general relativity, Nature (London) **179**, 1072 (1957).

[27] H. Bondi, F. A. E. Pirani, and I. Robinson, Gravitational waves in general relativity. 3. Exact plane waves, Proc. R. Soc. A **251**, 519 (1959).

[28] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, New York, 1975), 4th ed., Chap. 13, p. 345.

[29] F. A. E. Pirani, On the physical significance of the Riemann tensor, Acta Phys. Pol. **15**, 389 (1956) Republication of: On the physical significance of the Riemann tensor, Gen. Relativ. Gravit. **41**, 1215 (2009).

[30] F. A. E. Pirani, Invariant formulation of gravitational radiation theory, Phys. Rev. **105**, 1089 (1957).

[31] J. Ehlers and W. Kundt, Exact solutions of the gravitational field equations, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, London, 1962).

[32] D. Kramer, H. Stephani, M. McCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge Univ. Press, Cambridge, England, 1980).

[33] H. Bondi and F. A. E. Pirani, Gravitational waves in general relativity. 13: Caustic property of plane waves, Proc. R. Soc. A **421**, 395 (1989).

[34] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).

[35] R. Penrose, A remarkable property of plane waves in general relativity, Rev. Mod. Phys. **37**, 215 (1965).

[36] J. B. Griffiths and J. Podolsky, *Exact Space-Times in Einstein's General Relativity* (Cambridge Univ. Press, Cambridge, England, 2009).

[37] J. W. Van Holten, Gravitational waves and black holes: An introduction to general relativity, Fortschr. Phys. **45**, 439 (1997); The gravitational field of a light wave, Fortschr. Phys. **59**, 284 (2011).

[38] G. W. Gibbons, Quantized fields propagating in plane wave space-times, Commun. Math. Phys. **45**, 191 (1975).

[39] C. G. Torre, Gravitational waves: Just plane symmetry, Gen. Relativ. Gravit. **38**, 653 (2006).

[40] C. Barrabes and P. A. Hogan, *Advanced General Relativity (Gravity Waves, Spinning Particles, and Black Holes)*, International Series of Monographs on Physics (Oxford UP, New York, 2013).

[41] G. M. Shore, A new twist on the geometry of homogeneous plane gravitational waves, arXiv:1705.09533.

[42] D. Bini, V. Ferrari, and J. Ibanez, Finite energy wave packets of gravitational radiation, Nuovo Cimento Soc. Ital. Fis. **103B**, 29 (1989).

[43] R. Steinbauer, Geodesics and geodesic deviation for impulsive gravitational waves, J. Math. Phys. (N.Y.) **39**, 2201 (1998); On the geometry of impulsive gravitational waves, arXiv:gr-qc/9809054.

[44] L. P. Eisenhart, Dynamical trajectories and geodesics, Ann. Math. **30**, 591 (1928).

[45] C. Duval, G. Burdet, H. P. Künzle, and M. Perrin, Bargmann structures and Newton-Cartan theory, Phys. Rev. D **31**, 1841 (1985).

[46] C. Duval, G. W. Gibbons, and P. Horvathy, Celestial mechanics, conformal structures and gravitational waves, Phys. Rev. D **43**, 3907 (1991).

[47] M. Cariglia, C. Duval, G. W. Gibbons, and P. A. Horvathy, Eisenhart lifts and symmetries of time-dependent systems, Ann. Phys. (Amsterdam) **373**, 631 (2016).

[48] R. Haag, On quantum field theories, Kong. Dan. Vid. Sel. Mat. Fys. Med. **29N12**, 1 (1955) Philos. Mag. Ser. 5 **46**, 376 (1955).

[49] R. Penrose, On Schwarzschild causality: A problem for Lorentz covariant general relativity in *Essays in General Relativity: A Festschrift for Abraham Taub*, edited by F. J. Tipler (Academic Press, New York, 1980), pp. 1–12.

[50] Piotr Bizon (private communication).

[51] A. Staruszkiewicz, Gauge invariant surface contribution to the number of photons integral, Acta Phys. Pol. B **12**, 327 (1981).

[52] Malcolm Perry (private communication).

[53] N. F. Mott, Influence of radiative forces on the scattering of electrons, *Proc. Cambridge Philos. Soc.* 27, 255 (1931).

[54] M. Mirbabayi and M. Porrati, Dressed Hard States and Black Hole Soft Hair, Phys. Rev. Lett. **117**, 211301 (2016).

[55] R. Bousso and M. Porrati, Soft hair as a soft wig, arXiv:1706.00436.

[56] R. Bousso and M. Porrati, Observable supertranslations, arXiv:1706.09280.

[57] G. Compère and J. Long, Vacua of the gravitational field, J. High Energy Phys. 07 (2016) 137.

[58] D. Marolf and S. F. Ross, Plane waves: To infinity and beyond!, Classical Quantum Gravity **19**, 6289 (2002).

[59] M. Favata, Post-Newtonian corrections to the gravitational-wave memory for quasi-circular, inspiralling compact binaries, Phys. Rev. D **80**, 024002 (2009).

[60] M. Rakhmanov, Fermi-normal, optical, and wave-synchronous coordinates for spacetime with a plane gravitational wave, Classical Quantum Gravity **31**, 085006 (2014).

[61] J. Weber, Evidence for Discovery of Gravitational Radiation, Phys. Rev. Lett. **22**, 1320 (1969); Anisotropy and Polarization in the Gravitational-Radiation Experiments, Phys. Rev. Lett. **25**, 180 (1970).

[62] N. A. Tissot, *Mémoire sur la Représentation des Surfaces et les Projections des Cartes Géographiques* (Gauthier Villars, Paris, 1881).

[63] D. M. Goldberg and J. R. Gott III, Flexion and skewness in map projections of the Earth, Cartographica **42**, 297 (2007).

[64] A. Balakin, J. W. van Holten, and R. Kerner, Motions and worldline deviations in Einstein-Maxwell theory, Classical Quantum Gravity **17**, 5009 (2000).

[65] P. M. Zhang, P. A. Horvathy, K. Andrzejewski, J. Gonera, and P. Kosinski, Newton-Hooke type symmetry of anisotropic oscillators, Ann. Phys. (Amsterdam) **333**, 335 (2013).

[66] N. J. Poplawski, On the polarization of nonlinear gravitational waves, arXiv:1110.0051.

[67] B. Cropp and M. Visser, General polarization modes for the Rosen gravitational wave, Classical Quantum Gravity **27**, 165022 (2010).

[68] B. Cropp and M. Visser, Polarization modes for strong-field gravitational waves, J. Phys. Conf. Ser. **314**, 012073 (2011).

[69] D. Giulini, The superspace of geometrodynamics, Gen. Relativ. Gravit. **41**, 785 (2009).

[70] K. A. Khan and R. Penrose, Scattering of two impulsive gravitational plane waves, Nature (London) **229**, 185 (1971).

[71] J. B. Griffiths, *Colliding Plane Waves in General Relativity*, Oxford Mathematical Monographs (Clarendon, Oxford, UK, 1991).

[72] A. M. Anile and B. A. Breuer, Gravitational Stokes parameters, Astrophys. J. **189**, 39 (1974).

[73] S. Hacyan, Electromagnetic waves and Stokes parameters in the wake of a gravitational wave, Gen. Relativ. Gravit. **44**, 2923 (2012).

[74] C. Duval, Polarized spinoptics and symplectic physics, arXiv:1312.4486.

[75] G. G. Stokes, On the composition and resolution of streams of polarized light from different sources, *Trans. Cambridge Philos. Soc.* 9, 399 (1845); *Mathematical and Physical Papers* (Cambridge University Press, Cambridge, England, 1901), Vol. 3.

[76] R. C. Jones, A new calculus for the treatment of optical systems I. Description and discussion of the calculus, J. Opt. Soc. Am. **31**, 488 (1941).

[77] H. Poincaré, *Théorie Mathématique de la Lumière* (Georges Carr, Paris, 1892), Vol. 2, Chap. 12.

[78] G. F. Torres del Castillo and I. Rubalcava-García, The Jones vector as a spinor and its representation on the Poincaré sphere, Rev. Mex. Fis. **57**, 406 (2011).

[79] H. Bondi and F. A. E. Pirani, Energy conversion by gravitational waves, Nature (London) **332**, 212 (1988).

[80] F. Bloch, Nuclear induction, Phys. Rev. **70**, 7 (1946).

[81] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. R. Soc. A **392**, 45 (1984).

[82] B. Simon, Holonomy, the Quantum Adiabatic Theorem, and Berry's Phase, Phys. Rev. Lett. **51**, 2167 (1983).

[83] Y. Aharonov and J. Anandan, Phase Change During a Cyclic Quantum Evolution, Phys. Rev. Lett. **58**, 1593 (1987).

[84] C. Bouchiat and G. W. Gibbons, Nonintegrable quantum phase in the evolution of a spin 1 system: A physical consequence of the nontrivial topology of the quantum state space, J. Phys. II (France) **49**, 187 (1988).

[85] S. Pancharatnam, Generalised theory of interference and its applications, in *Collected Works of S. Pancharatnam* (OUP, New York, 1975).

[86] M. V. Berry, The adiabatic phase and pancharatnam's phase for polarized light, J. Mod. Opt. **34**, 1401 (1987).

[87] G. Morandi, *The Role of Topology in Classical and Quantum Physics*, Springer Lectures Notes in Physics, Monographs m7 (Springer, New York, 1992).

[88] A. Lasenby, Black holes and gravitational waves, *Talks Given at the Royal Society Workshop on "Black Holes"*, Chicheley Hall, UK, 2017 (KIAA, Beijing, 2017).

[89] L. P. Grishchuk and A. G. Polnarev, Gravitational wave pulses with "velocity coded memory", Zh. Eksp. Teor. Fiz. **96**, 1153 (1989) Sov. Phys. JETP **69**, 653 (1989).