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Fermions in worldline holography

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We analyze the worldline holographic framework for fermions. Worldline holography is based on the observation that in the worldline approach to quantum field theory, sources of a quantum field theory over Mink_4 naturally form a field theory over AdS_5 to all orders in the elementary fields and in the sources. Schwinger's proper time of the worldline formalism automatically appears with the physical four spacetime dimensions in an AdS_5 geometry. The worldline holographic effective action in general and the proper-time profiles of the sources in particular solve a renormalization group equation. By taking into account sources up to spin one, we reconstruct seminal holographic models. Considering spin two confirms AdS_5 as a consistent background.

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I. INTRODUCTION

Strong interactions offer an immensely rich phenomenology. Most of the time they overtax the computational abilities of the day and thus motivate us to put more energy into the development of new methods. In this context, for the last few decades, the holographic idea [1–4]—including the AdS/CFT correspondence—promises progress and, for example, has been applied to quantum chromodynamics (QCD) [5–7], extensions of the Standard Model [8,9], condensed-matter physics [10], and the Schwinger effect [11–13]. All concrete instances of such correspondences discovered to date, however, hold for theories with a particle content that is different from QCD. For the time being, extrapolated "bottom-up" AdS/QCD descriptions are considered, and they capture the hadron spectrum thought-provokingly accurately [5,14]. Yet, they lack a derivation from first principles, and this is the motivation for delving into the fundamental reasons for which such an approach could be tenable. Considerable work has been done in this area already [15–17].

We managed to show [13,18–22] that a quantum field theory over Mink_4 readily turns into a field theory for its sources over AdS_5 in the framework of the worldline formalism [23–25] for quantum field theory. Schwinger's proper time naturally takes the role of the fifth dimension of the AdS_5 geometry.[1] Schwinger's proper time sets a length scale (inverse energy scale), and this is the interpretation of the fifth dimension in holography [2–7] as well. Divergences occurring in a theory necessitate regularization. In the worldline formalism they are naturally taken care of by proper-time regularization, i.e., the introduction of a minimal positive proper time. This proper-time regularization corresponds to the UV-brane regularization [2–7].

References [18,19] demonstrated how such an AdS_5 formulation comes about to all orders in the sources and the elementary fields—matter and gauge. Analyzing the consequences of regulator independence of worldline holography in Ref. [18] identified it as a renormalization group framework. In fact, we can define [21] worldline holography as a variational solution to a Wilson (gradient) flow [26][2] and, from there, using the exact same computational steps, obtain the identical result. Consequently, worldline holography was regulator independent all along, but we can and will define it using this requirement henceforth.

In the past we concentrated on scalar elementary matter, as it provides the least impeded view of the underlying structure of worldline holography, due to the minimal number of internal degrees of freedom. There, among other things, worldline holography maps a free scalar theory onto a theory of arbitrarily high spins on AdS_{d+1} [27], as was previously conjectured [28].

Here, we turn to fermionic elementary matter, especially as it makes up the matter part of the Standard Model. Considering fermions overturns neither the worldline formalism [24,25] nor, as we shall see below, the worldline holographic framework in any way, but it makes its phenomenology richer.

In Sec. II we present worldline holography for fermionic elementary matter. In Sec. III we provide explicit computations in the free case. We derive the worldline holographic answer for (pseudo)scalar and (axial-)vector sources, which calls for a comparison with [5–7]. In Sec. III A we confirm the self-consistency of the AdS background within our framework. In Sec. IV we use our framework to study the renormalization of quantum electrodynamics (QED). Section V summarizes our findings.

II. WORLDLINE HOLOGRAPHY

As an introduction we present the general framework for fermions in worldline holography. We start with one

[1]These worldline dualities are also available for different pairs of spaces including the nonrelativistic case [19–21].

[2]We thank Roman Zwicky and Luigi Del Debbio for encouraging us to investigate this point.

massless fermion flavor and a vector source V combined with the gauge field G in the "covariant derivative" $\mathbb{D} = \partial - \mathrm{i}\mathbb{V}$, where $\mathbb{V} = G + V$. The generating functional for vector-current correlators is given by

$$Z = \langle \mathrm{e}^{w} \rangle = \int [\mathrm{d}G] \mathrm{e}^{w - \frac{i}{4e^2}\int \mathrm{d}^4 x G_{\mu\nu}^2}, \tag{1}$$

where

$$w = \ln \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] \mathrm{e}^{i\int \mathrm{d}^4 x \bar{\psi} \mathrm{i} \not{\mathbb{D}} \psi} = \tag{2}$$

$$= \ln \mathrm{Det} \mathrm{i}\not{\mathbb{D}} = \mathrm{Tr} \ln \mathrm{i}\not{\mathbb{D}} = \frac{1}{2} \mathrm{Tr} \ln \not{\mathbb{D}}^2, \tag{3}$$

and $\not{\mathbb{D}} = \gamma^\mu \mathbb{D}_\mu$.[3] The γ^μ stand for the anticommuting Dirac matrices $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where $\eta^{\mu\nu}$ represents the flat (inverse) metric. $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with all γ^μ, $\{\gamma^\mu, \gamma^5\} = 0$. The first step in the derivation of the worldline representation for this determinant is replacing the logarithm by an exponential proper-time integral representation [24,25],

$$\ln \not{\mathbb{D}}^2 = -\int_{\varepsilon>0}^{\infty} \frac{\mathrm{d}T}{T} \mathrm{e}^{-T\not{\mathbb{D}}^2}, \tag{4}$$

where we introduced the regulator $\varepsilon > 0$. This, however, requires an operator with a positive-definite spectrum, which $\not{\mathbb{D}}$ is not. For this reason, we continue with the last version of (3),

$$\not{\mathbb{D}}^2 = \mathbb{D}^2 - \frac{\mathrm{i}}{2} \sigma^{\mu\nu} [\mathbb{D}_\mu, \mathbb{D}_\nu]_-, \tag{5}$$

where $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_-$. The first addend in (5) corresponds to the kinetic operator for scalar elementary matter [18,19,24,25]; the second is an additional potential term, i.e., one without open derivatives, also referred to as spin factor. In the worldline formalism[4] [23–25] after a Wick rotation, w can be expressed as [13,18,21,22]

$$w = \int \mathrm{d}^4 x_0 \int_{\varepsilon>0}^{\infty} \frac{\mathrm{d}T}{2T^3} \mathfrak{L} \equiv \iint_{\varepsilon}^{\infty} \mathrm{d}^5 x \sqrt{g} \mathfrak{L}, \tag{6}$$

$$\mathfrak{L} = -\frac{\mathcal{N}}{(4\pi)^2} \int_{\mathrm{P}} [\mathrm{d}y] \mathrm{tr}_\gamma \mathbb{P} \mathrm{e}^{-\int_0^T \mathrm{d}\tau \{\frac{\dot{y}^2}{4} + \mathrm{i}\dot{y}\cdot\mathbb{V}(x_0+y) + \frac{\mathrm{i}}{2}\sigma^{\mu\nu}[\mathbb{D}_\mu, \mathbb{D}_\nu]_-\}}, \tag{7}$$

where the line element for the five-dimensional metric g reads

$$\mathrm{d}s^2 = g_{MN} \mathrm{d}x^M \mathrm{d}x^N = + \frac{\mathrm{d}T^2}{4T^2} + \frac{\mathrm{d}x_0 \cdot \mathrm{d}x_0}{T} \tag{8}$$

and $\sqrt{g}$ represents the square root of the absolute of its determinant. "·" stands for the contraction with $\eta^{\mu\nu}$. The Wick rotation turns the Minkowski $\eta_{\mu\nu}$ in the convention mostly plus to the Euclidean all plus. Simultaneously, Eq. (8) turns from an $\mathrm{AdS}_{4,1}$ (frequently simply referred to as AdS_5; the pair of indices indicates the metric signature) into an $\mathrm{AdS}_{5,0}$ (also referred to as H_5 or EAdS_5) line element. The isometry group of the five-dimensional AdS space is the conformal group of the corresponding four-dimensional flat space. T represents Schwinger's proper time. One factor of T^{-1} came from exponentiating the logarithm in (4), while another factor of T^{-2} arose when taking the functional trace. The Lagrangian density $\mathcal{L}$ consists of a path integral over all closed paths over the proper-time interval $[0;T]$, i.e., with $x(0) = x(T)$, where $x(\tau) = x_0 + y(\tau)$. The normalization cancels the free part, $\mathcal{N}\int_{\mathrm{P}}[\mathrm{d}y]\mathrm{e}^{-\frac{1}{4}\int_0^T \mathrm{d}\tau \dot{y}^2} = 1$. The $\mathrm{d}^4 x_0$ integral translates otherwise equivalent paths to every position in space. The translations are the zero modes of the kinetic operator ∂_τ^2, where $\dot{y} \equiv \partial_\tau y$. Separating them from the rest of the path integral also serves to make momentum conservation manifest. The choice of the representant loop for each equivalence class modulo translations is conventional; the center-of-mass convention, for example, is defined through $\int_0^T \mathrm{d}\tau y = 0$ and the starting-point convention through $y(0) = 0 = y(T)$.[5] tr_γ indicates that the finite-dimensional trace over the γ matrices remains to be taken.[6] $\mathbb{P}$ signifies that the exponential is path ordered, which is required due to the noncommutative nature of the γ matrices. With the path ordering already in place, we can consider non-Abelian flavor (f) and color (c) groups right away as well, $\mathrm{tr}_\gamma \to \mathrm{tr}_{\gamma,f,c}$.

A rewrite of the free kinetic term of the worldline action, $\int_0^T \mathrm{d}\tau \frac{(\partial_\tau y)^2}{4} = \int_0^1 \mathrm{d}\hat{\tau} \frac{(\partial_{\hat{\tau}} y)^2}{4T}$, where $\hat{\tau} = \tau/T$, shows that small values of T confine y to short relative distances, i.e., to the UV regime. Therefore, the proper-time regularization $T \geq \varepsilon > 0$ is a UV regularization and corresponds to the UV-brane regularization in holography [5–7].

A. Volume elements

Taking stock, in the worldline formalism, w automatically takes the form of an action (6) over AdS_5. In e^w, however, there are all powers of w. Thus, we have to show that this also holds for all other contributions to Z. The nth power is given by

[3]Throughout the manuscript, we omit field-independent normalization terms.

[4]The worldline form (6) of the functional determinant (3) is the particle dual of the determinant's wave(-function or field) representation as a Feynman functional integral (2) in the sense of the particle-wave duality.

[5]For details and more intermediate steps, see [19,21,24,25].

[6]We have chosen to retain the γ-matrix representation of the anticommutativity of the fermions. Alternatively, an antiperiodic integration over Grassmann variables can be used to this effect [24,25].

$$w^n = \prod_{j=1}^{n} \int \mathrm{d}^4 x_j \int_{\varepsilon}^{\infty} \frac{\mathrm{d}T_j}{2T_j^3} \mathfrak{L}(x_j, T_j). \tag{9}$$

The source-free part only depends on the positions of the single contributions $\mathfrak{L}(x_j, T_j)$ relative to one another. As before, we separate off an absolute coordinate $x_0 = \mathfrak{x}_0(\{x_j\})$. It can be chosen as any linear combination of the x_j, like the center of mass $\frac{1}{n}\sum_{j=1}^{n} x_j$, for example. This splits the $4n$ integrations into 4 over the absolute, $\mathrm{d}^4 x_0$, and $4(n-1)$ over the relative coordinates, $\mathrm{d}^{4(n-1)}\Delta$,

$$\int \prod_{j=1}^{n} \mathrm{d}^4 x_j \int \mathrm{d}^4 x_0 \delta^{(4)}[x_0 - \mathfrak{x}_0(\{x_k\})] = \int \mathrm{d}^4 x_0 \int \mathrm{d}^{4(n-1)}\Delta. \tag{10}$$

Accordingly, we define an overall proper time $T = \mathfrak{T}(\{T_j\})$ and proper-time fractions $t_j = T_j/T$. Without the introduction of additional and thus artificial dimensionful scales, on dimensional grounds, we always have that $\mathfrak{T}(\{T_j\}) = T \times \mathfrak{T}(\{t_j\})$. A choice that is symmetric under the pairwise exchange of the T_j makes the corresponding symmetry of w^n manifest from the beginning. (Otherwise, one could and would have to use the corresponding symmetry of w^n to make the symmetry visible again.) The sum of all individual proper times $\mathfrak{T} = \sum_{j=1}^{n} T_j$ is the arguably simplest choice satisfying these requirements. Implementing this change of variables with the help of

$$1 = \int \mathrm{d}T \delta[T - \mathfrak{T}(\{T_j\})] \prod_{j=1}^{n} \left[\int \mathrm{d}t_j \delta\left(t_j - \frac{T_j}{T}\right) \right] \tag{11}$$

yields

$$\prod_{j=1}^{n} \int_{\varepsilon}^{\infty} \frac{\mathrm{d}T_j}{2T_j^3} = \int \mathrm{d}T \left[\prod_{j=1}^{n} \int_{\varepsilon}^{\infty} \frac{\mathrm{d}T_j}{2T_j^3} \int \mathrm{d}t_j \delta\left(t_j - \frac{T_j}{T}\right) \right] \times \tag{12}$$

$$\times \delta[T - \mathfrak{T}(\{T_l\})] = \int \frac{\mathrm{d}T}{2T^3} T^{-2(n-1)} \int_{\frac{\varepsilon}{T}}^{\infty} \left(\prod_{j=1}^{n} \frac{\mathrm{d}t_j}{2t_j^3} \right) 2\delta[1 - \mathfrak{T}(\{t_l\})]. \tag{13}$$

Reexpressing w^n in terms of the new variables, we obtain

$$w^n = \int \mathrm{d}^4 x_0 \int \frac{\mathrm{d}T}{2T^3} \int \frac{\mathrm{d}^{4(n-1)}\Delta}{T^{2(n-1)}} \int_{\frac{\varepsilon}{T}}^{\infty} \left[\prod_{j=1}^{n} \frac{\mathrm{d}t_j}{2t_j^3} \times \mathfrak{L}(x_0 + x_j - x_0, Tt_j) \right] 2\delta[1 - \mathfrak{T}(\{t_l\})]. \tag{14}$$

Here $x_j - x_0$ is a function only of the relative coordinates Δ and not of the absolute coordinate x_0. Finally, we convert to dimensionless relative coordinates $\hat{\Delta} = \Delta/\sqrt{T}$, such that

$$w^n = \int \mathrm{d}^4 x_0 \int \frac{\mathrm{d}T}{2T^3} \int \mathrm{d}^{4(n-1)}\hat{\Delta} \int_{\frac{\varepsilon}{T}}^{\infty} \left[\prod_{j=1}^{n} \frac{\mathrm{d}t_j}{2t_j^3} \times \mathfrak{L}(x_0 + \widehat{x_j - x_0}\sqrt{T}, Tt_j) \right] 2\delta[1 - \mathfrak{T}(\{t_l\})], \tag{15}$$

which shows that w^n takes the form of a Lagrangian density integrated over AdS_5 for all n.

B. Contractions

In order to be a genuine action over AdS_5, the contractions of all spacetime indices have to be performed with (inverse) AdS metrics. We start demonstrating this by extracting the dependence on $x_j - x_0$ and y_j from the sources by means of a translation operator,

$$\mathbb{V}(y_j + x_j) = \mathrm{e}^{(y_j + x_j - x_0)\cdot\partial_{x_0}} \mathbb{V}(x_0). \tag{16}$$

The relation holds for any function of $y_j + x_j$, i.e., here the vector $\mathbb{V}_\mu$ but also the field tensor $\mathrm{i}[\mathbb{D}_\mu, \mathbb{D}_\nu]_-$. The combinations $x_j - x_0$ depend only on the relative coordinates Δ. Putting everything into the Lagrangian density yields

$$\begin{aligned} \mathfrak{L}(x_j, T_j) &= \mathfrak{L}[x_0 + (x_j - x_0), Tt_j] \\ &= -\frac{\mathcal{N}}{(4\pi)^2} \int_{\mathrm{P}} [\mathrm{d}y_j] \mathrm{e}^{-\int_0^1 \mathrm{d}\hat{\tau}_j \frac{(\partial_{\hat{\tau}_j} y_j^\mu) g_{\mu\nu} (\partial_{\hat{\tau}_j} y_j^\nu)}{4t_j}} \\ &\quad \times \mathrm{tr}_{\gamma,f,c} \mathbb{P} \exp\Big(-\int_0^1 \mathrm{d}\hat{\tau}_j \mathrm{e}^{[y_j^\lambda + (x_j - x_0)^\lambda]\cdot\frac{\partial}{\partial x_0^\lambda}} \\ &\quad \times \{ \mathrm{i}(\partial_{\hat{\tau}_j} y_j^\rho) \mathbb{V}_\rho(x_0) + \frac{\mathrm{i}}{2} t_j \check{\sigma}^{\rho\sigma} [\mathbb{D}_\rho(x_0), \mathbb{D}_\sigma(x_0)]_- \} \Big), \end{aligned} \tag{17}$$

where $\hat{\tau}_j = \tau_j/T_j = \tau_j/(Tt_j)$. The width of the $[\mathrm{d}y]$ integration is set by the four-dimensional part of the metric g, Eq. (8). Consistently, the normalization $\mathcal{N}$ compensates for the volume elements $\sqrt{g^{(4)}}$, the absolute of the determinant of the four-dimensional part of g. Consequently, after carrying out the $[\mathrm{d}y]$ integration, every pair of y^μ-s generates an inverse metric $y_j^\mu y_k^\nu \xrightarrow{\int[\mathrm{d}y]} g^{\mu\nu}$ times a function of the proper times τ_j and τ_k, which are integrated out subsequently. The $\mathrm{d}\Delta$ integration has a flat measure and the consistent nonunit volume element,

$$\int \mathrm{d}^{4(n-1)}\hat{\Delta} = \int \frac{\mathrm{d}^{4(n-1)}\Delta}{T^{2(n-1)}} = \int \mathrm{d}^{4(n-1)}\Delta \left(\sqrt{g^{(4)}}\right)^{n-1}. \tag{18}$$

$g_{\mu\nu}^{(4)}$ alone describes a flat space but has a nonunit normalization. Accordingly, we introduced γ matrices with the same normalization, $\check{\gamma}^\mu = \sqrt{T}\gamma^\mu$, and therefore, $\{\check{\gamma}^\mu, \check{\gamma}^\nu\} = 2g^{\mu\nu}$ as well as $\check{\sigma}^{\mu\nu} = T\sigma^{\mu\nu}$. All the above taken together shows that the last two lines in

$$w^n = \int \frac{\mathrm{d}T}{2T} \int_{\frac{\varepsilon}{T}}^{\infty} \left(\prod_{j=1}^{n} \frac{\mathrm{d}t_j}{2t_j^3} \right) 2\delta[1 - \mathfrak{T}(\{t_l\})] \times \int \mathrm{d}^4 x_0 \sqrt{g^{(4)}} \int \mathrm{d}^{4(n-1)} \Delta (\sqrt{g^{(4)}})^{n-1} \times \prod_{k=1}^{n} \mathfrak{L}[x_0 + (x_k - x_0), Tt_k], \tag{19}$$

belong to a field theory over the space with the metric $g^{(4)}_{\mu\nu}$. (T and the t_j can be seen as external parameters in the four-dimensional context, and T is exclusively present in the metric g.) Consequently, after carrying out the $[\mathrm{d}y]$ and $\mathrm{d}\Delta$ integrations as well as the trace over the γ matrices, the remaining coordinate will be x_0 and all spacetime indices will be contracted with (inverse) metrics g [18,19].

Integrating out the gauge field G does not change this, as we can identically rewrite the action in the exponent of the integration measure using g instead of η,

$$\int \mathrm{d}^4 x \sqrt{\eta} \eta^{\mu\kappa} \eta^{\nu\lambda} G_{\mu\nu} G_{\kappa\lambda} = \int \mathrm{d}^4 x \sqrt{g^{(4)}} g^{\mu\kappa} g^{\nu\lambda} G_{\mu\nu} G_{\kappa\lambda}, \tag{20}$$

which holds already at the level of the integrand. Consequently, after the $[\mathrm{d}G]$ integration is also carried out, all contractions are still with g, which accounts for all powers of T. In summary, Eq. (1) can be expressed as an action over AdS_5 for its sources to all orders and to all orders in the elementary fields.

C. Fifth-dimensional components and renormalization

As explained in [18], asking for the independence from the unphysical value of the UV regulator corresponds to a Wilson-Polchinski renormalization condition [29] and is achieved by completing the five-dimensional field theory. Here we compile the essentials. An expansion of the effective action in powers of gradients and sources after carrying out the $[\mathrm{d}y]$, $\hat{\Delta}$, t_j, and $\hat{\tau}_j$ integrations yields symbolically[7]

$$Z_\varepsilon = \int \mathrm{d}^4 x_0 \int_\varepsilon^\infty \mathrm{d}T \sqrt{g} \sum_{n_\partial, n_V} \#_{n_\partial, n_V} \times (g^{\circ\circ})^{\frac{n_\partial + n_V}{2}} (\partial_\circ)^{n_\partial} [V_\circ(x_0)]^{n_V}. \tag{21}$$

The addends are to represent all possible occurring combinations. Each of the derivatives, generally, only acts on some of the sources and never to the right of the sources. The $\#_{n_\partial, n_V}$ are dimensionless numerical coefficients,[8] and "$\circ$" signifies that only four-dimensional contractions are carried out.

Z_ε depends on the proper-time regulator $\varepsilon > 0$, the value of which, however, has *a priori* no physical meaning. Consequently, the physical effective action $\ln Z_\varepsilon^{\mathrm{phys}}$ should be regulator independent; i.e., we are looking for a solution to

$$\varepsilon \partial_\varepsilon \ln Z_\varepsilon^{\mathrm{phys}} \stackrel{!}{=} 0, \tag{22}$$

which is a Wilson-Polchinski renormalization condition.

Equation (21) is already an action over AdS_5 [13,18,19, 21,22], albeit without fifth-dimensional components. The group of isometries of AdS_5 is the conformal group over Mink_4, including the invariance under scale transformations. Scale invariance would make the value of ε irrelevant. In order to have the AdS isometries at our disposal, we have to complete the field theory by adding the missing components.

Then again, the original four-dimensional theory has no fifth-dimensional polarizations. We can only remove them again if the $\check{\mathcal{V}}_T = 0$ is an allowed gauge condition. This means that the five-dimensional extension—in which the value of ε is irrelevant—has to have five-dimensional local invariance under the flavor group. This fixes the form of the five-dimensional completion. Since Z_ε is already locally invariant under four-dimensional transformations,[9]

$$\mathcal{Z} = \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g} \sum_{n_\partial, n_V} \#_{n_\partial, n_V} \times (g^{\bullet\bullet})^{\frac{n_\partial + n_V}{2}} (\nabla_\bullet)^{n_\partial} [\mathcal{V}_\bullet(x_0, T)]^{n_V} \tag{23}$$

is invariant under five-dimensional transformations. Here "$\bullet$" stands for the five-dimensional contractions, and ∇ for the AdS covariant derivative. Equation (23) features the full AdS_5 isometries including scale invariance. Consequently, it is independent of the value of ε, if $\mathcal{V}(x_0, T)$ transforms like a five-dimensional vector. ($\mathcal{V}$ does *not* depend explicitly on ε.) Imposing $\mathcal{V}_T = 0$ gauge at the level of the action would still manifestly preserves scale invariance because scale transformations do not mix the tensor components.

[7]The interaction part in (7) consists of a Wilson line part and a field tensor part. Therefore, it is locally invariant under the flavor transformation $V^\mu \to \Omega[V^\mu + \mathrm{i}\Omega^\dagger(\partial^\mu \Omega)]\Omega^\dagger$, which brings about hidden local symmetry [30]. Consequently, Z_ε can also be expressed purely in terms of covariant derivatives [31,32],

$$Z_\varepsilon = \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g} \sum_n \#_n (g^{\circ\circ})^n (D_\circ)^{2n},$$

where $D = \partial - \mathrm{i}V$. Furthermore, the proper-time regularization preserves this symmetry; which, for example, a momentum cutoff would not.

[8]There are only contributions from $n_\partial + n_V$ even.

[9]This is manifest in the expansion shown in footnote [7]. There the completion would proceed by replacing all flavor covariant derivatives by flavor and generally covariant derivatives.

(The full symmetry would also be intact, but modulo a local flavor transformation.)

So far, $\mathcal{Z}$ is, however, only some functional of just any source configuration $\mathcal{V}$. It is a variational principle that makes it the (effective) action of a field theory, thus singling out a special field configuration (or configurations) as saddle point(s), $\check{\mathcal{V}}$. Through the boundary condition

$$\check{\mathcal{V}}_\mu(x_0, T = \varepsilon) = V_\mu(x_0), \tag{24}$$

the four-dimensional polarizations are handed on to the five-dimensional solution $\check{\mathcal{V}}$, and the normalization, which makes it the source for exactly *once* the vector current, is preserved.[10]

Moreover, Eq. (24) coincides with the previous findings of worldline holography [13,19,21], i.e., that the worldline formalism induces a Wilson (gradient) flow of the sources in the fifth dimension with this boundary condition.

Furthermore, Eq. (24) localizes the bare source configuration at the UV end of the fifth dimension, i.e., at small values of the proper time T corresponding to short four-dimensional distances. In conjunction with the requirement (22) that the effective action do not depend on the unphysical value of the UV regulator ε, this is a Wilson-Polchinski renormalization condition [29].

Finally, holography is the concept of extrapolating the sources from their boundary values (from the UV brane) into the bulk, and the effective action for the four-dimensional side of the holographic duality is described by the five-dimensional action evaluated on its saddle point [2–7]. As a matter of fact, there the computational steps are well-nigh identical, albeit, in parts, with a different reasoning.

Taking all the above into account, the desired cutoff-independent effective action is obtained by evaluating $\mathcal{Z}$ on the saddle-point configuration with the boundary condition (24) and in $\check{\mathcal{V}}_T = 0$ gauge,

$$\begin{aligned}\check{\mathcal{Z}} = \iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{g}\sum_{n_\partial,n_V}\#_{n_\partial,n_V} \\ \times (g^{\bullet\bullet})^{\frac{n_\partial+n_V}{2}}(\nabla_\bullet)^{n_\partial}[\check{\mathcal{V}}_\circ(x_0,T)]^{n_V}.\end{aligned} \tag{25}$$

Hence, worldline holography identifies Schwinger's proper time as the fifth dimension [13,18,19,21,22] and fixes the fifth-dimensional profile of the sources as a solution to the renormalization group equation (22).

[10]A quantized version of $\ln\mathcal{Z}$ also bears the necessary isometries to be a solution of the renormalization condition (22), barring anomalies. The saddle point is then the leading contribution. The subleading correction is the fluctuation determinant. (The worldline formalism also relates this to the Gutzwiller trace formula [33,34], which also describes quantum systems through classical attributes.) In certain cases a distinction between "quantum" and "classical" turns out to be irrelevant because the sum over the quantum contributions from all bulk fields cancels [35].

III. FREE CASE

In order to obtain more insight into the worldline holographic formalism, let us turn to the free case. To this end, we switch off the coupling to the gauge bosons G in (1) by setting to zero the coupling to the gauge bosons G, which is tantamount to analyzing w with $\mathbb{V} \to V$.[11]

For the sake of clarity, above we studied the vector, a rank-one source. In an expansion in the rank of the sources, however, we would thereby have omitted several other sources, namely the scalar S, the pseudoscalar P, and the axial vector A. These are also the sources needed for a comparison to other holographic frameworks [5–9].

With those sources in place,

$$w = \mathrm{Tr}\ln(i\partial\!\!\!/ + \Gamma), \tag{26}$$

where

$$\Gamma = V\!\!\!/ + \gamma^5 A\!\!\!/ + S + i\gamma^5 P. \tag{27}$$

We again would like to use (4) for which we need an operator with a positive-definite spectrum. We choose the approach

$$\mathcal{O} = (\mathcal{O}\mathcal{O})^{1/2} = [(\mathcal{O}/\mathcal{O}^\dagger)(\mathcal{O}^\dagger\mathcal{O})]^{1/2} \tag{28}$$

such that

$$\mathrm{Tr}\ln\mathcal{O} = \frac{1}{2}\mathrm{Tr}\ln(\mathcal{O}\mathcal{O}^\dagger) + \frac{1}{2}\mathrm{Tr}\ln(\mathcal{O}/\mathcal{O}^\dagger). \tag{29}$$

In what follows, we analyze w maximally up to the fourth order in the fields and/or gradients. For $\mathcal{O} = \mathrm{i}\partial\!\!\!/ + \Gamma$, this result does not contain terms with an odd number of γ^5 matrices, which would come from the second addend in the previous equation [40,41]. (See also Appendix.) Hence, we only retain the first term,

$$\begin{aligned}Z_\varepsilon \supset -\frac{1}{(4\pi)^2}\iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{g}\mathcal{N}\int_{\mathrm{P}}[\mathrm{d}y]\mathrm{tr}_{f,\gamma}\mathbb{P} \\ \exp\Bigg[-\int_0^T \mathrm{d}\tau\Bigg(\frac{\dot{y}^2}{4} + \gamma_{\mathrm{R}}\Big\{\mathrm{i}\dot{y}^\mu L_\mu + \frac{1}{2}\sigma^{\mu\nu}L_{\mu\nu} \\ + \Phi\Phi^\dagger - \gamma^\mu D_\mu\Phi\Big\} + \gamma_{\mathrm{L}}\{L \leftrightarrow R\;\&\;\Phi \leftrightarrow \Phi^\dagger\}\Bigg)\Bigg].\end{aligned} \tag{30}$$

Here we switched to the basis

$$L = V + A, \tag{31}$$

[11]In low-energy scattering processes these should actually also be the kinematically dominant diagrams, justified by the observation that there the contributions with the lowest number of exchanged gauge bosons dominate [36–39].

$$R = V - A, \tag{32}$$

$$\gamma_{L/R} = (1 \mp \gamma^5)/2, \tag{33}$$

$$\Phi = S + \mathrm{i}P, \tag{34}$$

and introduced a flavor-covariant derivative

$$D_\mu \Phi = \partial_\mu \Phi - \mathrm{i} L_\mu \Phi + \mathrm{i}\Phi R_\mu. \tag{35}$$

$L_{\mu\nu}$ $(R_{\mu\nu})$ stands for the field tensor for L_μ (R_μ). The expansion to the level of the (flavor-covariant) kinetic terms for all source fields yields

$$Z_\varepsilon \supset -\frac{4}{(4\pi)^2} \times \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{g} \mathrm{tr}_f \left[\frac{1}{2} g^{\mu\nu} (D_\mu \sqrt{T}\Phi)^\dagger (D_\nu \sqrt{T}\Phi) - |\sqrt{T}\Phi|^2 + \frac{1}{12} g^{\mu\kappa} g^{\nu\lambda} (L_{\mu\nu} L_{\kappa\lambda} + R_{\mu\nu} R_{\kappa\lambda})\right], \tag{36}$$

after carrying out the $[\mathrm{d}y]$ as well as $\mathrm{d}\tau$ integrations and dropping total derivatives.

According to the recipe detailed in Sec. II,

$$\mathcal{Z} \supset -\frac{4}{(4\pi)^2} \times \iint_\varepsilon^\infty \mathrm{d}^5 x \sqrt{\mathfrak{g}} \mathrm{tr}_f \left[\frac{1}{2} \mathfrak{g}^{MN} (\mathcal{D}_M \Phi)^\dagger (\mathcal{D}_N \Phi) - \#|\Phi|^2 + \frac{1}{12} \mathfrak{g}^{MK} \mathfrak{g}^{NJ} (\mathcal{L}_{MN} \mathcal{L}_{KJ} + \mathcal{R}_{MN} \mathcal{R}_{KJ})\right], \tag{37}$$

back in Minkowski space and where $\sqrt{T}\Phi \to \Phi$ [19]. The retention of the coefficients during the five-dimensional completion was needed to ensure the local invariance that allows us to gauge away the unphysical fifth polarization. This holds for all fields but the spin-zero mass term, which is neither influenced by the introduction of fifth polarizations nor fifth gradients. Consequently, its coefficient is not thus protected. (In this context, it is important to remember that also in a purely four-dimensional utilization of the worldline formalism, additional conditions must be identified that are not automatically transferred by the formalism to ensure the correct renormalization of the mass [42,43].) Because of the way that S is coupled to the elementary fermions in (27), a finite mass m of these fermions corresponds to a constant value $S = m$. Accordingly, in order to have a consistent framework, $\breve{\Phi} = m\sqrt{T}$ must be an admissible classical solution for Φ in Eq. (37). For 4D homogeneous solutions, the classical equation of motion reads

$$(2\partial_T T^{-1} \partial_T + \# T^{-3})\breve{\Phi} = 0 \tag{38}$$

and possesses power-law solutions T^α, where $\alpha = 1 \pm \frac{1}{2}$ if $\# = 3/2$. This is exactly the prediction of the holographic dictionary [3,4] for the fifth-dimensional mass of the scalar, which always includes the second independent solution $\propto T^{3/2}$.[12] Consistently, inserting the solution $\propto T^{1/2}$ into the (flavor-covariant) spin-zero kinetic term generates a mass term for the axial vector $\propto m^2 \mathcal{A}^2$, but not the vector [5,6]. For the free theory, the part of the action for the vector $\mathcal{V}$ does not contain any scale.

In order to see what we can expect for an interacting gauge theory in its confining phase, let us represent the part of (1) in which all sources are connected to a single matter loop by a confining term in the worldline action. (See also the discussion in [19].) We can consider the Gaussian model from [21],

$$S_{\mathrm{Gauss}} = \frac{1}{4} \int_0^T \mathrm{d}\tau (\dot{y}^2 + c^2 y^2), \tag{39}$$

or an area law for the area of the corresponding loop,

$$S_{\mathrm{area}} = \frac{1}{4} \int_0^T \mathrm{d}\tau \dot{y}^2 + \mathrm{const} \times \mathrm{area}. \tag{40}$$

As already discussed above, if the kinetic term sets the length scale, the typical length will be $O(\sqrt{T})$. Then y^2 as well as the area are $O(T)$, and, to logarithmic accuracy, we expect a (warp) factor $\mathrm{e}^{-\mathrm{const}' \times T}$ in the effective action. Let us check our expectations for the first case (39). Carrying out the path integral yields[13]

$$\mathcal{N} \int [\mathrm{d}y] \mathrm{e}^{-S_{\mathrm{Gauss}}} = \prod_{n=1}^\infty \left[1 + \frac{c^2 T^2}{4(2\pi)^2 n^2}\right]^{-d} \tag{41}$$

$$= \left[\frac{\sinh(cT/4)}{cT/4}\right]^{-4} = \mathrm{e}^{-cT + O(\ln T)}. \tag{42}$$

Taking stock, (37) with $\# = 3/2$ and a confining potential/warp factor closely resembles the soft-wall model [5].

A. Self-consistency of the AdS geometry

An effective action like (21), particularly in the covariant form given in footnote 7, can also be obtained for a generally curved background metric $\mathfrak{g}$,

[12]This second solution is associated with spontaneous chiral symmetry breaking [6] and thus should not contribute in the free case. The $T^{1/2}$ solution corresponds to the tachyon (squared) profile [44] for a free theory of elementary matter with the explicit mass m.

[13]There are other subleading differences between the effective warp factors of the different addends due to different powers of the modified worldline propagator on the ground floor.

$$Z_\varepsilon = \int_\varepsilon^\infty \frac{\mathrm{d}T}{2T^3}\int \mathrm{d}^4x_0\sqrt{\mathfrak{g}}\sum_n \#_n(T\mathfrak{g}^{\circ\circ})^n(\nabla_\circ[\mathfrak{g}])^{2n}$$
$$= \iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{\bar{\mathfrak{g}}}\sum_n \sharp_n(\bar{\mathfrak{g}}^{\circ\circ})^n(\nabla_\circ[\mathfrak{g}])^{2n}, \tag{43}$$

where $\bar{\mathfrak{g}}$ stands for the five-dimensional Fefferman-Graham [45] embedding of $\mathfrak{g}$,

$$\mathrm{d}s^2 = \bar{\mathfrak{g}}_{MN}\mathrm{d}x^M\mathrm{d}x^N = \natural\left(\frac{\mathrm{d}T^2}{4T^2} + \frac{\mathfrak{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}{T}\right), \tag{44}$$

$\sharp_n = \#_n\natural^{n-5/2}$, and $\nabla[\mathfrak{g}]$ for the Levi-Civita connection. The $\#_n$ are the DeWitt-Gilkey-Seeley coefficients [46].

As seen above, the independence (22) from ε can be achieved by means of the completion to a five-dimensional action

$$\mathcal{Z} = \iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{\bar{\mathfrak{g}}}\sum_n \sharp_n(\bar{\mathfrak{g}}^{\bullet\bullet})^n(\nabla_\bullet[\bar{\mathfrak{g}}])^{2n}, \tag{45}$$

and its subsequent evaluation on its saddle point for the boundary condition,

$$\breve{\bar{\mathfrak{g}}}_{\mu\nu}(x_0, T=\varepsilon) = \frac{\natural}{\varepsilon}\mathfrak{g}_{\mu\nu}(x_0), \tag{46}$$

in the gauge where

$$\breve{\bar{\mathfrak{g}}}_{TT} \overset{!}{=} \natural g_{TT}, \qquad \breve{\bar{\mathfrak{g}}}_{T\nu} \overset{!}{=} 0, \tag{47}$$

with g_{TT} from (8). This corresponds to the absence of deviations from g with fifth-dimensional polarizations,

$$h_{TN} \overset{!}{=} 0 \quad \forall\, N. \tag{48}$$

The two leading terms [47] correspond to a negative cosmological constant and an Einstein-Hilbert term,

$$\mathcal{Z} \supset -\frac{1}{3(4\pi)^2}\iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{\bar{\mathfrak{g}}}(R[\bar{\mathfrak{g}}]+12). \tag{49}$$

As a consequence, the corresponding Einstein equations admit an AdS_5 solution with the squared AdS curvature radius

$$\natural = \frac{(5-1)(5-2)}{12} = 1. \tag{50}$$

Taking into account the boundary (46) and gauge conditions (47), the solution is $\breve{\mathfrak{g}} = g$. Therefore, to this order, an AdS background is a self-consistent prediction of the formalism.

At higher orders, AdS, being a space of constant curvature, is still a saddle-point solution, although generally with a different curvature radius. The AdS_5 isometry group does not depend on the value of the curvature radius and is always the conformal group over Mink_4. (Analogously, Mink_4 is Poincaré invariant for every value of the speed of light.) As a consequence, the value of the AdS radius is of secondary importance. For one thing,

$$\mathcal{Z} = \iint_\varepsilon^\infty \mathrm{d}^5x\breve{\bar{\mathfrak{g}}}^{1/2}\sum_{n_\partial,n_V}\sharp_{n_\partial,n_V}$$
$$\times(\breve{\bar{\mathfrak{g}}}^{\bullet\bullet})^{\frac{n_\partial+n_V}{2}}(\nabla_\bullet[\breve{\bar{\mathfrak{g}}}])^{n_\partial}[\mathcal{V}_\bullet(x_0,T)]^{n_V}, \tag{51}$$

where $\sharp_{n_\partial,n_V} = \#_{n_\partial,n_V}\natural^{(n_\partial+n_V-5)/2}$, is identical to (23), which itself does not depend on $\natural$. Likewise, the covariant derivatives are independent from the curvature radius as is the (1,3) Riemann tensor.

IV. HOLOGRAPHIC 2-LOOP CHARGE RENORMALIZATION OF QED

Interpreting the vector source V as a (background) gauge field Z_ε is the QED effective action in the background-field formalism. There is a logarithmic divergence in the leading term

$$Z_\varepsilon = \#_{2,2}\iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{g}g^{\mu\kappa}g^{\nu\lambda}V_{\mu\nu}V_{\kappa\lambda}, \tag{52}$$

where $V_{\mu\nu}$ represents the (presently Abelian) field-strength tensor. The divergence appears in the $\mathrm{d}T$ integration, where there is a factor of T^{-3} from the volume element and two factors of T, one from each metric $g^{\mu\nu}$, which makes an overall $\mathrm{d}T/T$. To two loops for $N_f\times N_c$ quarks [25],

$$\#_{2,2} = 2\frac{N_fN_c}{(4\pi)^2}\left(-\frac{1}{3} - \frac{e^2}{(4\pi)^2}\right). \tag{53}$$

The full five-dimensional action

$$\mathcal{Z} = \#_{2,2}\iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{g}g^{MK}g^{NL}\mathcal{V}_{MN}\mathcal{V}_{KL} \tag{54}$$

is independent of ε. Here capital indices run over all five dimensions. The corresponding saddle-point equations are given by

$$g^{NL}\nabla_N\breve{\mathcal{V}}_{KL} = 0. \tag{55}$$

In the axial gauge $\breve{\mathcal{V}}_T = 0$ these equations of motion also imply Lorenz gauge $\partial\cdot\breve{\mathcal{V}} = 0$. Then the remaining transverse components (here in 4D momentum space) must obey

$$\left(\partial_T^2 - \frac{p^2}{4T}\right)\tilde{\breve{\mathcal{V}}}^\perp = 0. \tag{56}$$

The normalizable solution with (24) is given by

$$\tilde{\check{\mathcal{V}}}^{\perp} = \tilde{V}^{\perp}(p)\frac{\sqrt{p^2T}K_1(\sqrt{p^2T})}{\sqrt{p^2\varepsilon}K_1(\sqrt{p^2\varepsilon})}, \tag{57}$$

where Bessel's K_n is defined in Eqs. 9.6.1. ff. in [48], and for which (see Eq. 9.6.28 in [48])

$$\partial_T\tilde{\check{\mathcal{V}}}^{\perp} = \tilde{V}^{\perp}(p)\frac{p^2K_0(\sqrt{p^2T})/2}{\sqrt{p^2\varepsilon}K_1(\sqrt{p^2\varepsilon})}. \tag{58}$$

Putting this solution back into the 4D Fourier transformed action (54), we obtain a surface term,

$$\check{\mathcal{Z}} = 4\#_{2,2}\int \mathrm{d}^4x_0\eta^{\nu\lambda}[\check{\mathcal{V}}^{\perp}_{\nu}\partial_T\check{\mathcal{V}}^{\perp}_{\lambda}]^{\infty}_{\varepsilon} \tag{59}$$

$$= 4\#_{2,2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\eta^{\nu\lambda}[\tilde{\check{\mathcal{V}}}^{\perp}_{\nu}\partial_T\tilde{\check{\mathcal{V}}}^{\perp *}_{\lambda}]^{\infty}_{\varepsilon} \tag{60}$$

$$= -2\#_{2,2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\eta^{\nu\lambda}\tilde{V}^{\perp}_{\nu}\tilde{V}^{\perp *}_{\lambda}p^2K_0(\sqrt{p^2\varepsilon}), \tag{61}$$

where the tilde marks the Fourier transform and the asterisk the complex conjugate. Making use of Eq. 9.6.13. from [48],

$$\check{\mathcal{Z}} = \#_{2,2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\underbrace{\eta^{\nu\lambda}\tilde{V}^{\perp}_{\nu}\tilde{V}^{\perp *}_{\lambda}p^2}_{\hat{=}|\tilde{V}_{\mu\nu}|^2/2}\{\ln(p^2\varepsilon) + O[(p^2\varepsilon)^0]\}. \tag{62}$$

In our conventions, where the coupling e is absorbed in the field, the prefactor of the kinetic term equals $-(4e^2)^{-1}$.

To two loops, the β function describing the running of the coupling with the scale μ is given by

$$\frac{\mathrm{d}e}{\mathrm{d}\ln\mu} = \beta_1e^3 + \beta_2e^5. \tag{63}$$

Integrating (63) and solving for $e^{-2}(\mu) - e^{-2}(\mu_0)$[14] yields

$$e^{-2}(\mu) - e^{-2}(\mu_0) = -2(\beta_1 + \beta_2e^2)\ln\frac{\mu}{\mu_0} + \ldots, \tag{64}$$

where the ellipsis stands for terms of $O(e^4)$, and this order also depends on the three-loop coefficient. The comparison of the divergent pieces yields

$$2\#_{2,2}\ln(p^2\varepsilon) = -e^{-2} = (\beta_1 + \beta_2e^2)2\ln\frac{\mu}{\mu_0}. \tag{65}$$

[14]As usual, we subtract the bare contribution from the induced term.

Upon identification of $\ln(p^2\varepsilon) \leftrightarrow 2\ln\frac{\mu}{\mu_0}$ we obtain

$$\beta_1 + \beta_2e^2 = 2\#_{2,2} = 4\frac{N_fN_c}{(4\pi)^2}\left(-\frac{1}{3} - \frac{e^2}{(4\pi)^2}\right), \tag{66}$$

which are the known β-function coefficients.[15]

In the worldline formalism there are no subdivergences in the two-loop contribution [25] to the coefficient $\#_{2,2}$. (This does not only hold for proper-time regularization, but also other four-dimensional regularization schemes like Pauli-Villars.) The absence of subdivergences is known to persist for the quenched contributions to all loops [43]. For higher unquenched orders the analysis is still pending.

Nonholographic renormalization of QED was treated in the worldline formalism before [42,43]. There, obtaining the two-loop term in the analog of (62) required knowledge of the counterterm from mass renormalization, where the mass was used as an infrared regulator. Here, we work with massless elementary matter. Hence, there is no mass renormalization. Asking for the integrability of the saddle-point solution led to an infrared finite result and the known two-loop contribution.

We never forced ε to be small. [Equation (62) only presents the behavior of $\check{\mathcal{Z}}$ if ε were small.] Above, ε was introduced to regularize the UV divergence of Z. Then, we had in mind to sent the regulator to zero at the end of the calculation. At nonzero ε the renormalization condition (22) makes ε a scale. If we want to keep ε in its original role as a regulator, we can introduce counterterms for the divergent pieces. In (62), for instance,

$$\check{\mathcal{Z}} = \#_{2,2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\eta^{\nu\lambda}\tilde{V}^{\perp}_{\nu}\tilde{V}^{\perp *}_{\lambda}p^2 \times \{\ln(\mu^2\varepsilon) + \ln(p^2/\mu^2) + O[(p^2\varepsilon)^0]\}, \tag{67}$$

the first addend inside the braces, which diverges when $\varepsilon \to 0$, must be compensated by the introduction of a counterterm which can also contain additional finite parts. Now μ^2 plays the role of the scale, and ε remains the regulator.

V. SUMMARY

We studied the worldline holographic framework for fermionic elementary matter. Worldline holography maps a d-dimensional quantum field theory onto a $d+1$-dimensional field theory for the sources of the former, to all

[15]In nonholographic worldline computations a finite mass was needed as an IR regulator, and its behavior under renormalization had to be determined in an additional computation to achieve this result [42,43]. Here, we considered the massless case, chose the integrable, i.e., the IR-finite solution, and did not need any additional input.

orders in the elementary fields and sources. The $d+1$-dimensional metric is the Fefferman-Graham embedding [45] of the d-dimensional one. For Mink_d this results in AdS_{d+1}. Worldline holography is the solution to a Wilson-Polchinski renormalization condition (22), which guarantees the independence of physical quantities from the ultraviolet regulator. (Infrared scales can be handled analogously [18].) As a consistency check we holographically derived the QED beta-function coefficient to two loops in Sec. IV. In Sec. III, we explicitly determined the worldline holographic dual for a fermionic field theory on Mink_d with sources up to spin one and found a theory akin to the seminal holographic model [5]. Turning to spin 2 in Sec. III A allowed us to confirm that AdS_{d+1} is a self-consistent solution of the worldline holographic framework.

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APPENDIX: γ^5-ODD TERMS

By inspection of the source (27), up to fourth order in the fields and gradients taken together, one expects terms with one γ^5, $O[(\partial - \mathrm{i}V)^3 A]$, and three γ^5, $O[(\partial - \mathrm{i}V)A^3]$, to occur and to be contained in the $\mathcal{O}/\mathcal{O}^\dagger$ part of (29) [40,41]. [To this order, there are no corresponding contributions from the pseudoscalar P, as its γ^5 would have to be balanced by four γ^μ from vectors and/or (an even number of) axial vectors, which would amount to order five.] These terms encode the axial anomaly of the theory [40,41].

Confusingly, also the $\mathcal{O}\mathcal{O}^\dagger$ part seems to yield such a contribution,

$$Z_\varepsilon \supset \frac{1}{(8\pi)^2}\iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{g}T^2\epsilon^{\mu\nu\kappa\lambda}\mathrm{tr}_f(L_{\mu\nu}L_{\kappa\lambda} - R_{\mu\nu}R_{\kappa\lambda}),$$

contrary to the conclusions in [40,41]. Concentrating on the Abelian part, in a momentum-space computation the term drops out once momentum conservation is imposed. In [41] this can be seen from their Feynman rules on the first line of Table 1. There, the trace over the products of the $\sigma_{\mu\nu}$ terms in the vertices yields an $\epsilon_{\mu\nu\kappa\lambda}$, which, however, is contracted with p and $-p$ and thus vanishes for symmetry reasons.

Independently, we have carried out a momentum-space Feynman-diagram computation, where we encounter

$$\mathrm{tr}[\gamma_\mu\tilde{\Gamma}(p)\gamma_\nu\tilde{\Gamma}^*(p)] \supset -4i\epsilon_{\mu\kappa\nu\lambda}(\tilde{A}_\kappa\tilde{V}^*_\lambda - \tilde{V}_\kappa\tilde{A}^*_\lambda), \tag{A1}$$

which is contracted with either $p^\mu p^\nu$ or $\eta^{\mu\nu}$ and thus vanishes in any case.

In coordinate space, though, also by inspection of (5) in [41], one again identifies the combination

$$\mathrm{tr}[(\mathcal{O}\mathcal{O}^\dagger)^2] \supset \mathrm{tr}(\gamma^5\sigma^{\mu\nu}\sigma^{\kappa\lambda}A_{\mu\nu}V_{\kappa\lambda}) \tag{A2}$$

up to a numerical prefactor. The authors of [41] show that the analogous contribution from $\mathcal{O}/\mathcal{O}^\dagger$ saturates the axial anomaly, such that $\mathcal{O}\mathcal{O}^\dagger$ should not contribute here.

In any case, these terms are purely topological, they do not contribute to the equations of motion, and they yield at most surface contributions. They are readily embedded in our five-dimensional setting. Using the five-dimensional Levi-Civita tensor

$$\mathcal{E}^{MNKLJ} = \epsilon^{MNKLJ}/\sqrt{g} = 2T^3\epsilon^{MNKLJ} \tag{A3}$$

and the T components of the (diagonal) fünfbein,

$$E^A_T = \delta^A_T\sqrt{g_{TT}} = \delta^A_T/2T, \tag{A4}$$

we get[16]

$$\mathcal{E}^{MNKL\underline{T}}E^A_{\underline{T}} = T^2\epsilon^{\mu\nu\kappa\lambda}\delta^A_T \tag{A5}$$

and can thus express all contractions in (A1) with five-dimensional objects.[17]

Consistently, Eq. (A4) satisfies $E^A_M E^B_N H_{AB} = g_{MN}$, where H_{AB} is a reference metric. [If chosen flat (and "unity") it gives rise to the last of the equalities in (A4).] This is also in line with the expression for the CP-even term (only coming in at higher orders)

$$T^4(\epsilon^{\mu\nu\kappa\lambda}V_{\mu\nu}V_{\kappa\lambda})^2$$
$$= \mathcal{E}^{\mu\nu\kappa\lambda\underline{T}}g_{\underline{T}\underline{T}}\mathcal{E}^{\alpha\beta\gamma\delta\underline{T}}V_{\mu\nu}V_{\kappa\lambda}V_{\alpha\beta}V_{\gamma\delta} \tag{A6}$$

$$\overset{5d}{\to}\mathcal{E}^{MNKLJ}g_{JE}\mathcal{E}^{ABCDE}\mathcal{V}_{MN}\mathcal{V}_{KL}\mathcal{V}_{AB}\mathcal{V}_{CD} \tag{A7}$$

$$= 4!g^{MA}g^{NB}g^{KC}g^{LD}\mathcal{V}_{MN}\mathcal{V}_{KL}\mathcal{V}_{[AB}\mathcal{V}_{CD]}, \tag{A8}$$

which ultimately can be expressed purely using (inverse) metrics. (In this passage, V is a placeholder for L or R, respectively.)

The introduction of the vielbein as the "square root" of the metric is consistent, but its origin can be elucidated further. Consider the five-dimensional Chern-Simons term [49]

[16] The index T stands for the fifth component. In particular, this means that we do *not* sum over it, if it appears in pairs. We mark that by an underline when necessary.

[17] One could still contract (A5) with a constant vector n_A.

$$CS_5 = \epsilon^{MNKLJ}\mathrm{tr}_f\bigg(\mathcal{V}_{MN}\mathcal{V}_{KL}\mathcal{V}_J - \frac{1}{2}\mathcal{V}_{MN}\mathcal{V}_K\mathcal{V}_L\mathcal{V}_J + \frac{1}{10}\mathcal{V}_M\mathcal{V}_N\mathcal{V}_K\mathcal{V}_L\mathcal{V}_J\bigg). \tag{A9}$$

Under local transformations $\mathcal{V}_M \to \Omega[\mathcal{V}_M + \mathrm{i}\Omega^\dagger(\partial_M\Omega)]\Omega^\dagger$, it changes only by a total derivative. We are, however, working on a manifold with a boundary (at $T = \varepsilon$), which can render such otherwise cyclic components physical.[18] The gauge transformation parametrizes such degrees of freedom, the longitudinal components of $\mathcal{V}$. Making the latter explicitly visible using the Stückelberg trick, one lifts them to auxiliary fields and looks at whether they contribute and, if so, where. Here the Stückelberg trick amounts to

$$\mathcal{V}_M \to \bar{\mathcal{V}}_M = \mathcal{V}_M + \partial_M\Sigma, \tag{A10}$$

where the 5D longitudinal part $\partial_M\Sigma$ corresponds to $\mathrm{i}\Omega^\dagger(\partial_M\Omega)$.[19] Then, perturbatively,

$$CS_5[\bar{\mathcal{V}}] = CS_5[\mathcal{V}] + \epsilon^{MNKLJ}\mathrm{tr}_f[(\partial_M\Sigma)\mathcal{V}_{NK}\mathcal{V}_{LJ}] + \dots, \tag{A11}$$

where the ellipsis stands for higher orders in $\partial_M\Sigma$. For configurations without fifth polarizations or gradients, CS_5 vanishes exactly. On the contrary,

$$\iint_\varepsilon^\infty \mathrm{d}^5x\sqrt{g}\mathcal{E}^{MNKLJ}\mathrm{tr}_f[(\partial_M\Sigma)\mathcal{V}_{NK}\mathcal{V}_{LJ}]$$

$$= -\iint_\varepsilon^\infty \mathrm{d}^5x\epsilon^{MNKLJ}\partial_M\mathrm{tr}_f[\Sigma\mathcal{V}_{NK}\mathcal{V}_{LJ}] \tag{A12}$$

$$= +\int \mathrm{d}^4x\epsilon^{TNKLJ}\mathrm{tr}_f[\Sigma\mathcal{V}_{NK}\mathcal{V}_{LJ}]_{T=\varepsilon} \tag{A13}$$

$$\overset{4d}{\to} + \int \mathrm{d}^4x\underbrace{\epsilon^{T\nu\kappa\lambda\rho}}_{=\epsilon^{\nu\kappa\lambda\rho}}\mathrm{tr}_f[\Sigma(T=\varepsilon)V_{\nu\kappa}V_{\lambda\rho}] \tag{A14}$$

[18] Another example is the zero mode of the temporal gauge field in thermal field theory [50], where it is due to the compactification of the time direction.

[19] Since L_μ/R_μ couples to Φ only through the covariant derivative (35), a linear combination of the longitudinal components $\Sigma_{L/R}$ and the spin-zero fields Φ is cyclic.

is, in general, nonzero. The presence of the boundary makes the zero mode of Σ physical. All other terms depend only on derivatives of Σ. This term encodes the chiral anomaly [51]. Consistently, CS_5 also contains the Wess-Zumino-Witten term [52],

$$CS_5[\partial\Sigma]$$

$$= +\epsilon^{MNKLJ}\mathrm{tr}_f[(\partial_M\Sigma)(\partial_N\Sigma)(\partial_K\Sigma)(\partial_L\Sigma)(\partial_J\Sigma)] \tag{A15}$$

$$= -\epsilon^{MNKLJ}\partial_M\mathrm{tr}_f[\Sigma(\partial_N\Sigma)(\partial_K\Sigma)(\partial_L\Sigma)(\partial_J\Sigma)]. \tag{A16}$$

We end this appendix with an interesting observation. When doing the momentum-space Feynman-diagram computation, for instance, of the two-point function, when using the Feynman trick to exponentiate the denominators of the propagators, one reproduces the fifth-dimensional structure of the worldline framework,

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\frac{\not{p}+\frac{\not{q}}{2}}{(p+\frac{q}{2})^2}\tilde{\Gamma}(q)\frac{\not{p}-\frac{\not{q}}{2}}{(p-\frac{q}{2})^2}\tilde{\Gamma}(-q)\right] \tag{A17}$$

$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^\infty \mathrm{d}T_1\mathrm{d}T_2\mathrm{e}^{T_1(p+\frac{q}{2})^2+T_2(p-\frac{q}{2})^2} \times \mathrm{tr}\left[\left(\not{p}+\frac{\not{q}}{2}\right)\tilde{\Gamma}(q)\left(\not{p}-\frac{\not{q}}{2}\right)\tilde{\Gamma}(-q)\right] \tag{A18}$$

$$= \frac{1}{(4\pi)^2}\int_\varepsilon^\infty \frac{\mathrm{d}T}{T^2}\int_0^1 \mathrm{d}\hat{\tau}T\mathrm{e}^{T\hat{\tau}(1-\hat{\tau})q^2}T\hat{\tau}(1-\hat{\tau}) \times (-q^2\eta^{\mu\nu}/2 - q^\mu q^\nu)\mathrm{tr}[\gamma_\mu\tilde{\Gamma}(q)\gamma_\nu\tilde{\Gamma}(-q)]. \tag{A19}$$

One recognizes the sum of the Feynman parameters $T = T_1 + T_2$ as Schwinger's proper time. The combination $T\hat{\tau}(1-\hat{\tau})$ is known as the worldline propagator [24,25]. $\int_0^1 \mathrm{d}\hat{\tau}\mathrm{e}^{T\hat{\tau}(1-\hat{\tau})q^2}$ is also the basic second-order form factor in the heat-kernel expansion in the background-field formalism [53]. Hence, calling the present framework proper-time holography or even heat-kernel holography, would not seem amiss, but it was the worldline formalism that allowed us to make the connection to AdS_5 and renormalization to all orders.

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