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Scale magnetic effect in quantum electrodynamics and the Wigner-Weyl formalism

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The scale magnetic effect (SME) is the generation of electric current due to a conformal anomaly in an external magnetic field in curved spacetime. The effect appears in a vacuum with electrically charged massless particles. Similarly to the Hall effect, the direction of the induced anomalous current is perpendicular to the direction of the external magnetic field $\boldsymbol{B}$ and to the gradient of the conformal factor τ, while the strength of the current is proportional to the beta function of the theory. In massive electrodynamics the SME remains valid, but the value of the induced current differs from the current generated in the system of massless fermions. In the present paper we use the Wigner-Weyl formalism to demonstrate that in accordance with the decoupling property of heavy fermions the corresponding anomalous conductivity vanishes in the large-mass limit with $m^2 \gg |e\boldsymbol{B}|$ and $m \gg |\nabla\tau|$.

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I. INTRODUCTION

Anomalous transport phenomena have attracted the attention of the scientific community in recent years [1,2]. Anomalous transport is associated with quantum anomalies 3]] which break the original symmetries of classical systems due to quantum fluctuations. The axial anomaly and the mixed axial-gravitational anomaly are suggested to lead to various transport phenomena, such as the chiral magnetic [4,5], chiral separation [6,7], and chiral vortical effects [8,9] which generate both electric (vector) and axial (pseudovector) currents as well as energy flows [10,11] in usual or chirally imbalanced matter. These currents and flows may be directed along the axis of a background magnetic field or along a vorticity vector in the case where the matter is rotating. Anomalous transport appears both in solid state [12] and particle [13] physics contexts.

The basic rule is that anomalous symmetry breaking may be associated with a certain (anomalous) transport law that cannot otherwise appear in a classical system with an unbroken classical symmetry. Besides the axial anomalies, certain theories may also exhibit the conformal anomaly associated with the breaking of the classical conformal invariance at the quantum level. In Ref. [14], it was indeed shown that the anomalous breaking of conformal (scale) symmetry in conformally invariant gauge theories should also lead to the emergence of two new transport phenomena—the scale magnetic effect (SME) and the scale electric effect (SEE)—which generate electric current in an electromagnetic field background in curved spacetime. The SME is a stationary phenomenon which induces an electric current perpendicularly to the direction of the external magnetic field in a static curved space. The SEE is a nonstationary effect which is realized in an external electric field in a time-dependent gravitational background. The generated electric currents are proportional to the beta function of the corresponding theory. The explicit expressions for the SME and SEE are given, respectively, in Eqs. (5) and (6) below. Both of these effects appear in the theory with vanishing fermion masses or with nearly vanishing fermion masses that are much smaller than the energy scales associated with both the external electric/magnetic field and the variations of the gravitational field.

The aim of the present article is to consider the opposite limit when the fermion mass is much larger than the energy scales given by the external electromagnetic field and the gradient of the gravitational field. First, we notice that the SME and SEE phenomena are associated with the contribution of the conformal anomaly to the trace of the energy-momentum tensor in a classical electromagnetic field background [14]. In the (classically conformal) massless case this trace is entirely given by an anomalous contribution which originates from the change of the integration measure with respect to the Weyl transformation [15]. In the massive case the trace of the energy-momentum tensor also contains a nonanomalous extra contribution emerging due to the explicit breaking of the conformal symmetry in the classical Lagrangian. In our article we calculate the nonanomalous contribution to the scale magnetic effect in QED using the Wigner-Weyl formalism. We demonstrate that in the limit of large fermion mass the nonanomalous contribution cancels precisely the anomalous contribution coming from the integration measure over the fermionic fields. Therefore, in agreement with decoupling theorems, the scale magnetic effect is strongly suppressed for sufficiently massive fermions.

The structure of the paper is as follows. The next three sections are devoted to brief reviews of the scale magnetic and electric effects (Sec. II), relevant features of QED in a curved spacetime (Sec. III), and the basics of the Wigner-Weyl approach (Sec. IV). In Sec. V we derive the nonanomalous part of the electric current of the SME in QED with massive fermions, and demonstrate that it cancels precisely the contribution of the integration measure. The last section is devoted to a discussion of our results.

II. SCALE MAGNETIC/ELECTRIC EFFECTS

A. Massless fermions

Let us consider massless QED with one species of Dirac fermion ψ in $(3+1)$ spacetime dimensions:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}i\gamma^{\mu}D_{\mu}\psi, \tag{1}$$

where γ^{μ} are the Dirac matrices, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the field-strength tensor of the gauge field A_{μ}, and $D_{\mu} = \partial_{\mu} + ieA_{\mu}$ is the covariant derivative.

In massless QED the trace of the classical stress-energy tensor $T^{\mu\nu}$ is identically zero, $(T^{\mu}_{\mu})_{\mathrm{cl}} \equiv 0$, because of the conformal invariance of the theory at the classical level. The theory does not contain a characteristic energy scale, thus implying the invariance of the classical action $S = \int d^4x\mathcal{L}$ under the scale transformations $x \to \lambda^{-1}x$, $A_{\mu} \to \lambda A_{\mu}$, and $\psi \to \lambda^{3/2}\psi$. However, quantum corrections make the electric charge e dependent on the renormalization energy scale, $e = e(\mu)$. The apparent noninvariance of the quantum theory on the energy scale is explicitly manifested in the nonzero beta function of the theory:

$$\beta(e) = \frac{\mathrm{d}e}{\mathrm{d}\ln\mu}. \tag{2}$$

As a result, in the background of the classical electromagnetic field A^{cl}_{μ} the trace of the stress-energy tensor becomes nonzero due to quantum corrections [3]:

$$\langle T^{\alpha}_{\alpha}(x)\rangle = \frac{\beta(e)}{2e}F^{\mathrm{cl},\mu\nu}(x)F^{\mathrm{cl}}_{\mu\nu}(x). \tag{3}$$

Below, we study the effects in classical background gauge fields only and therefore we omit hereafter the superscript "cl" in A^{cl}_{μ} and $F^{\mathrm{cl}}_{\mu\nu}$.

The simplest way to reveal anomalous transport effects emerging due to the conformal (scale) anomaly (3) is to consider the following conformally flat metric:

$$g_{\mu\nu}(x) = e^{2\tau(x)}\eta_{\mu\nu}, \tag{4}$$

where $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ is the flat Minkowski metric. Starting from Eq. (3), and assuming that the conformal factor in Eq. (4) is small $|\tau| \ll 1$, one can show that in the curved background (4) the conformal (scale) anomaly generates an anomalous electric current in the presence of the background of a magnetic field $\boldsymbol{B}$,

$$\boldsymbol{J} = \frac{2\beta(e)}{e}\boldsymbol{\nabla}\tau(x) \times \boldsymbol{B}(x), \tag{5}$$

which is proportional to the gradient of the local scale factor $\tau(x)$ of the conformally flat metric (4). The anomalous generation of the electric current by the background magnetic field (5) is the SME proposed in Ref. [14].

In the electric field background $\boldsymbol{E}$ the anomalous generation of the electric current resembles Ohm's law [14],

$$\boldsymbol{J} = \sigma(x)\boldsymbol{E}(x), \tag{6}$$

with the essential difference being that the metric-dependent anomalous electric conductivity

$$\sigma(t,\boldsymbol{x}) = -\frac{2\beta(e)}{e}\frac{\partial\tau(t,\boldsymbol{x})}{\partial t} \tag{7}$$

may take *negative* values. Equations (6) and (7) determine the SEE. It was suggested that the SEE describes the negative vacuum conductivity associated with the Schwinger pair production in an expanding de Sitter universe. Earlier, a negative electric conductivity was indeed found for fermionic [16] and bosonic [17] Schwinger effects.

Notice that the classical electric current induced by the external electromagnetic field in the conformal background (4) is identically zero. The scale magnetic [Eq. (5)] and scale electric [Eq. (6)] effects are related to each other as they originate from the same Lorentz-covariant expression [14]. The corresponding currents are proportional to the beta function (2). Below, we rederive the SME current (5) using a straightforward calculation based on a truncated Wigner expansion. This approach will also allow us to identify possible effects of a nonzero fermion mass on the anomalous current.

B. Massive fermions

QED with one species of massive Dirac fermion is described by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^{\mu}D_{\mu} - m)\psi. \tag{8}$$

With the help of the perturbative methods the trace of the corresponding energy-momentum tensor can be represented in the operator form [18]:

$$T^\alpha_\alpha(x) = \frac{\beta(e)}{2e} F^{\mu\nu}(x) F_{\mu\nu}(x) + (1+\gamma_m) m\bar{\psi}\psi + \text{discontinuous terms}, \tag{9}$$

where $\gamma_m = 3\alpha_{\rm QED}/2\pi + \cdots$ is the mass anomalous dimension, $\alpha_{\rm QED} = e^2/4\pi$ is the fine-structure constant, and the ellipsis denotes higher-order corrections in $\alpha_{\rm QED}$. It is however more illuminating to derive Eq. (9) using the Fujikawa method in the path integral formalism [15] which attributes a leading $O(\alpha^1_{\rm QED})$ part of the first term in Eq. (9) to the contribution from the integration measure over the fermionic fields. The remaining terms appear due to virtual photons and nonanomalous contributions originating from the classically nonconformal mass term of the action (8), which include higher-order $O(\alpha^n_{\rm QED})$ terms with $n \geq 1$.

The electric current generated by the scale magnetic effect depends on the expectation value of the trace of the energy-momentum tensor in the background of the classical gauge field $A^{\rm cl}_\mu$ [14]. In the massive theory this trace includes both anomalous and nonanomalous parts coming from the quantum measure and the classical action, respectively (9). The anomalous part gives the known contribution to the current (5). As for the nonanomalous part, we may only say that in the heavy mass limit ($m \to \infty$) the fermions should decouple from the dynamics of the model [19] and therefore the contribution from the nonanomalous part should cancel the anomalous term (5).

Let us consider QED [Eq. (8)] in the classical electromagnetic field background with the field strength $F^{\rm cl}_{\mu\nu}$ in addition to the dynamical photons. Using symmetry properties as well as dimensional arguments one finds that the leading terms of the local derivative expansion in the dimensional regularization give $\langle (1+\gamma_m) m\bar{\psi}(x)\psi(x)\rangle = C_1 m^4 + C_2 F^{{\rm cl},\mu\nu}(x) F^{\rm cl}_{\mu\nu}(x) + O(m^{-2})$, where the constants C_1 and C_2 may, in principle, contain a divergent dependence on the parameter $\epsilon = D - 4$ of the dimensional regularization as well as a dependence on the dimensional parameter μ through the combination $\log(\mu/m)$. The first term in this expression is irrelevant for the dynamics of the model. The factor in front of the field-dependent term is then fixed by the decoupling theorem:

$$\langle (1+\gamma_m) m\bar{\psi}(x)\psi(x)\rangle = \text{const} - \frac{\beta(e)}{2e} F^{{\rm cl},\mu\nu}(x) F^{\rm cl}_{\mu\nu}(x) + O(m^{-2}). \tag{10}$$

This expression is valid in any regularization in the limit, when both the classical electromagnetic field and gradients of the metric are smaller than the corresponding power of the fermion mass m.

Equation (9) is consistent with the existing calculations of the triangle correlator $\langle TJJ\rangle$ of the fermionic stress tensor T and two external electric currents J which may alternatively be used to compute the electric current generated by the scale electric and magnetic effects. The $\langle TJJ\rangle$ correlator vanishes in the limit when the large fermionic mass m exceeds the external momenta associated with the vertices of the triangle diagram [20,21], thus indicating that the induced electric current should also vanish in the $m \to \infty$ limit. It was proposed that the dependence on the mass m enters the anomalous relation (3) through the modified effective β function [21],

$$\beta(p^2, m^2, M^2) = -e p^2 \frac{d\Pi_R(p^2, m^2, M^2)}{dp^2}, \tag{11}$$

determined via the renormalized photon self-energy

$$\Pi_R(p^2, m^2, M^2) = \Pi(p^2, m^2) - \Pi(p^2 = -M^2, m^2). \tag{12}$$

It is worth mentioning that this definition of the beta function differs slightly from the standard textbook definition, as the latter is determined by the dependence of the polarization operator on the mass scale M rather than on the value of the momentum p. The two definitions coincide in the ultraviolet limit $M \to \infty$. Therefore, the concrete form of Eq. (11) depends on the details of the regularization scheme [21,22].

Notice that in an alternative heat kernel approach the correlator $\langle TJJ\rangle$ can be computed using different calculation schemes [22] which give different expressions in the domain $p^2 \ll m^2$. However, all computations share the same qualitative feature: the correlator $\langle TJJ\rangle$ tends to zero in the infrared region $p^2 \ll m^2$ [20–22].

In the present paper we develop the Wigner-Weyl approach to compute the electric current generated by the scale magnetic effect. In agreement with the decoupling theorem, we explicitly demonstrate that the electric current of massive fermions contains both anomalous and nonanomalous contributions which cancel each other exactly in the large-mass limit.

III. QED IN CURVED SPACETIME

A. Fermionic Lagrangian in curved spacetime

A Dirac fermion field ψ with mass m in a curved background is described by the action

$$S = \int d^4x \sqrt{-g}\mathcal{L}, \tag{13}$$

with the following Lagrangian [23]:

$$\mathcal{L} = \frac{i}{2}[\bar{\psi} e^\mu_a \gamma^a \nabla_\mu \psi - (\nabla_\mu \bar{\psi}) e^\mu_a \gamma^a \psi] - m\bar{\psi}\psi, \tag{14}$$

where $\nabla_\mu \bar{\psi} \equiv (\nabla_\mu \psi)^\dagger \gamma^0$ and γ^a are the standard, coordinate-independent Dirac matrices. The vierbein (tetrad) field $e^\mu_a \equiv e^\mu_a(x)$ is related to the spacetime metric $g_{\mu\nu}$ as follows: $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, where η_{ab} is the metric of the flat

Minkowski spacetime and $e^\mu_a = g^{\mu\nu}\eta_{ab}e^b_\nu$. The raising/lowering of the curved spacetime indices (denoted by the greek letters $\mu, \nu, \ldots$) and of the flat indices (denoted by the latin letters $a, b, \ldots$) of the vierbein $e_a{}^\mu$ are done with respect to the curved metric $g_{\mu\nu}$ and the flat metric η_{ab}, respectively. For example, $e^{\mu a} = g^{\mu\nu}e_\nu{}^a = g^{\mu\nu}\eta^{ab}e_{\nu b}$, etc.

The covariant derivative

$$\nabla_\mu = D_\mu + \Gamma_\mu, \qquad D_\mu = \partial_\mu + ieA_\mu \tag{15}$$

enforces the invariance of the Lagrangian (14) with respect to the U(1) gauge transformations

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \qquad \bar\psi(x) \to e^{-i\alpha(x)}\bar\psi(x),$$
$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x), \tag{16}$$

and local Lorentz transformations in the curved spacetime $x^\mu \to x^{\mu'} = \Lambda^\mu_\nu x^\nu$. The latter is done with the help of the (matrix) spin connection

$$\Gamma_\mu = -\frac{i}{4}\omega^{ab}_\mu \sigma_{ab}, \tag{17}$$

where

$$\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b] \tag{18}$$

is the generator of the Lorentz transformations and

$$\omega_\mu{}^{ab} = e_\nu{}^a \Gamma^\nu{}_{\sigma\mu} e^{\sigma b} + e_\nu{}^a \partial_\mu e^{\nu b}, \tag{19}$$

with the Christoffel symbols

$$\Gamma^\nu{}_{\alpha\mu} = \frac{1}{2} g^{\nu\beta}(\partial_\alpha g_{\beta\mu} + \partial_\mu g_{\alpha\beta} - \partial_\beta g_{\alpha\mu}). \tag{20}$$

A variation of the Lagrangian (14) with respect to $\bar\psi$ leads to the covariant Dirac equation:

$$(ie^\mu{}_a\gamma^a\nabla_\mu - m)\psi = 0. \tag{21}$$

Below, we consider the conformally flat metric (4). The corresponding components of the vierbein can be chosen as $e_\mu{}^a = e^\tau\delta^a_\mu$, so that $e^{\mu a} = e^{-\tau}\eta^{\mu a}$ and

$$e^\mu{}_a = e^{-\tau}\delta^\mu_a. \tag{22}$$

The metric determinant in Eq. (13) is $g \equiv \det g_{\mu\nu} = -e^{8\tau}$.

In the conformally flat metric (4) the spin connection part (17) in the Lagrangian (14) takes the following form:

$$e^\mu{}_a\gamma^a\Gamma_\mu = \gamma^a\Gamma_a(x), \qquad \Gamma_a = \frac{3}{2}e^{-\tau}\partial_a\tau, \tag{23}$$

where we took into account that $\omega^{ab}_\mu = \delta^a_\mu\partial^b\tau - \delta^b_\mu\partial^a\tau$.

Generally, the curved background affects the fermionic fields via the volume element $\sqrt{-g}$ in the action (13), the vierbein field e^μ_a, and the spin connection Γ_μ in the Lagrangian (14) and Eq. (15). However, in the conformal background (4) the contribution from the spin connection drops out from the Lagrangian (14) because the spin connection (23) is a real-valued vector $\Gamma^*_\mu \equiv \Gamma_\mu$ proportional to the identity matrix in the spinor space. Then the Lagrangian (24) gets simplified:

$$\mathcal{L} = \frac{i}{2}[\bar\psi e^\mu{}_a\gamma^a D_\mu\psi - (D_\mu\bar\psi)e^\mu{}_a\gamma^a\psi] - m\bar\psi\psi, \tag{24}$$

where the electromagnetic covariant derivative D_μ is given in Eq. (15).

B. Partition function and electric current

We consider fermions in the fixed curved spacetime given by the metric $g_{\mu\nu}$ subjected to the fixed background of the external electromagnetic field A_μ. The fermionic partition function

$$\mathcal{Z}[A,g] = \int D\bar\psi D\psi e^{iS}$$
$$\equiv \int D\bar\psi D\psi \exp\left\{ i\int d^4x \bar\psi(x)\mathcal{D}[A,g,m]\psi(x)\right\}$$
$$= \mathrm{const}\cdot \det \mathcal{D}[A,g,m] \tag{25}$$

is proportional to the determinant of the fermionic operator $\mathcal{D}$ which enters the Lagrangian density (14) in the action (13):

$$\sqrt{-g(x)}\mathcal{L}(x) = \bar\psi(x)\mathcal{D}[A,g,m]\psi(x). \tag{26}$$

In this article we consider the scale magnetic effect which arises in thermal equilibrium in a static gravitational field with a time-independent metric [Eq. (4)] in the background of an external magnetic field $\boldsymbol{B} \neq 0$. The electric field is zero, $\boldsymbol{E} = 0$. Since we consider the system in thermal equilibrium it is convenient to perform a Wick rotation of the time coordinate $x_0 \to x_4 = -ix_0$ and formulate the theory in Euclidean four-dimensional spacetime. The operator $\mathcal{D}$ in Euclidean space can be explicitly calculated with the help of Eqs. (15), (22), (24), and (26):

$$\mathcal{D} = -\frac{i}{2}\sum_{\mu=1}^4 \gamma_\mu\left[e^{3\tau(x)}\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x^\mu}e^{3\tau(x)}\right]$$
$$- e^{3\tau(x)}\sum_{\mu=1}^4 \gamma_\mu e A_\mu(x) - ie^{4\tau(x)}m, \tag{27}$$

where γ_μ are the Euclidean Dirac matrices. Correspondingly, we have

$$\mathcal{Z}[A,g] = \int D\bar{\psi}D\psi \exp\left\{-\int d^4x\bar{\psi}(x)\mathcal{D}[A,g,m]\psi(x)\right\}. \tag{28}$$

The local electric current induced by the external gauge field A_μ in the curved spacetime background $g_{\mu\nu}$ is given by the following variational derivative:

$$J^\mu(x;A,g) = -\frac{\delta \log \mathcal{Z}[A,g]}{\delta A_\mu(x)}. \tag{29}$$

In flat space with the Euclidean metric $\eta_{\mu\nu} = \delta_{\mu\nu}$ the integration measure over the fermion field $D_\eta\bar{\psi}D_\eta\psi$ in Eq. (28) is independent of the gauge field. In the curved background with the conformal metric (4) the integration measure $D_g\bar{\psi}D_g\psi$ acquires a dependence on the external gauge field [15]:

$$D_g\bar{\psi}D_g\psi = D_\eta\bar{\psi}^\tau D_\eta\psi^\tau \cdot \exp\left\{\frac{\beta^{1\,\text{loop}}_{\text{QED}}}{2e}\int d^4x\tau(x)F^{\mu\nu}(x)F_{\mu\nu}(x)\right\}, \tag{30}$$

where $\psi^\tau(x) = e^{\frac{3}{2}\tau(x)}\psi(x)$, while

$$\beta^{1\,\text{loop}}_{\text{QED}} = \frac{e^3}{12\pi^2}, \tag{31}$$

is the one-loop QED beta function. Equation (30) originates from the transformation of the measure given in Ref. [15] under the Weyl transformations

$$\psi(x) \to e^{-3\tau(x)/2}\psi(x), \tag{32}$$

$$g_{\mu\nu}(x) \to e^{2\tau(x)}g_{\mu\nu}(x). \tag{33}$$

Since we consider the response of the virtual fermions on the background electromagnetic field in the vacuum, the background field is assumed to be induced by an external electric current located outside of the considered region of space. In this case we have two contributions to the induced electric current,

$$J^\mu(x;A,g) = J^\mu_{\text{measure}}(x;A,g) + J^\mu_{\text{action}}(x;A,g), \tag{34}$$

given, respectively, by the one-loop anomalous contribution from the fermionic integration measure

$$J^\mu_{\text{measure}}(x;A,g) \equiv -\frac{2\beta^{(1)}(e)}{e}F^{\mu\nu}(x)\partial_\nu\tau(x), + \frac{2\beta^{(1)}(e)}{e}\tau(x)\partial_\nu F^{\mu\nu}(x). \tag{35}$$

Here, the second line is proportional to the external current that creates the given external field. We assume that it is localized outside the region of observations. The remaining contribution comes from the classical action:

$$J^\mu_{\text{action}}(x;A,g) = -\frac{\delta \log \mathcal{Z}_\eta[A,g]}{\delta A_\mu(x)}, \tag{36}$$

where

$$\mathcal{Z}_\eta[A,g] = \int D_\eta\bar{\psi}^\tau D_\eta\psi^\tau e^{-\int d^4x\bar{\psi}(x)\mathcal{D}[A,g,m]\psi(x)} = \int D_\eta\bar{\psi}^\tau D_\eta\psi^\tau e^{-\int d^4x\bar{\psi}^\tau(x)\mathcal{D}[e^\tau A,\eta,e^\tau m]\psi^\tau(x)}. \tag{37}$$

C. The case of massless fermions

One can see that even for a vanishing mass there may be an extra contribution to the electric current given by

$$J^\mu_{\text{action}}(x;A,g)|_{m=0} \equiv e^{\tau(x)}\mathrm{Tr}\left[G_\eta\frac{\delta\mathcal{D}[A^\tau,\eta,0]}{\delta A^\tau_\mu(x)}\right], \tag{38}$$

where $A^\tau(x) = e^{\tau(x)}A(x)$ and the Green function

$$G_\eta(x,y) = \frac{1}{Z_\eta[A,g]}\int D_\eta\bar{\psi}^\tau D_\eta\psi^\tau\psi^\tau(x)\bar{\psi}^\tau(y) \times \exp\left\{-\int d^4x\bar{\psi}^\tau(x)\mathcal{D}[e^\tau A,\eta,0]\psi^\tau(x)\right\} \tag{39}$$

satisfies the relation

$$\mathcal{D}[A^\tau,\eta,0]G_\eta(x,y) = \delta^{(4)}(x-y). \tag{40}$$

The field $A^\tau(x)$ gives rise to the "magnetic" field $\partial_{[i}A^\tau_{j]}\epsilon^{ijk0}$ and to the "electric" field $\partial_{[0}A^\tau_{k]}$. According to the results of Ref. [24], in such systems the vacuum current proportional to the first power of the "magnetic" or "electric" field is also proportional to the topological invariant in momentum space, which vanishes for the system under consideration. The terms linear in the first derivatives of the "magnetic" or "electric" field might appear with a dimensionless coefficient. If it exists, such a term would have the form $J^k_{\text{action}}(x) = \text{const}e^{\tau(x)}\partial_i\partial^{[i}A^{\tau,k]}(x)$, i.e., it should be proportional to the electric current that creates the given external field. Essentially, it is the renormalization of this current due to the quantum fluctuations as well as due to the contribution from the second line in Eq. (35). In our consideration we assume that such a current is localized far outside the region of observations. This assumption ensures that in the relevant order the extra contribution to the SME current is absent. The terms proportional to the second power of the "magnetic" field in the conformal limit are suppressed as $1/\Lambda$, where Λ is the ultraviolet cutoff. The same conclusion is valid for the higher-order corrections. Overall, the component J^k_{action} vanishes to all orders for the system of massless fermions.

D. The general case

Below, we consider the case of massive fermions where the current $J^k_{\rm action}$ remains nonvanishing. We will explore the following relation:

$$J^\mu_{\rm action}(x;A,g) \equiv {\rm Tr}\left[G\frac{\delta\mathcal{D}[A,g,m]}{\delta A_\mu(x)}\right], \qquad (41)$$

where

$$G(x,y) = \frac{1}{Z_\eta[A,g]}\int D_\eta\bar\psi^\tau D_\eta\psi^\tau e^{-3(\tau(x)+\tau(y))/2}\psi^\tau(x)\bar\psi^\tau(y) \times e^{-\int d^4x\bar\psi^\tau(x)e^{-3\tau(x)/2}\mathcal{D}[A,g]e^{-3\tau(x)/2}\psi^\tau(x)} \qquad (42)$$

is the fermionic Green function satisfying the relation

$$e^{3(\tau(y)-\tau(x))/2}\mathcal{D}[A,g]G(x,y) = \delta^{(4)}(x-y), \qquad (43)$$

which is equivalent to

$$\mathcal{D}[A,g]G(x,y) = \delta^{(4)}(x-y). \qquad (44)$$

According to Eqs. (15), (22), and (24), the variation of the local operator $\mathcal{D}$ with respect to the gauge potential A_μ in Eq. (41) gives the following ultra-local two-point operator:

$$\left(\frac{\delta\mathcal{D}}{\delta A_\mu(x)}\right)(y,z) = -\sqrt{-g(x)}e^\mu{}_a(x)\gamma^a\mathbb{1}_{x,y}\mathbb{1}_{x,z} = -ee^{-3\tau(x)}\gamma^\mu\mathbb{1}_{x,y}\mathbb{1}_{x,z}, \qquad (45)$$

where we denoted for compactness

$$\mathbb{1}_{x,y} = \delta^{(4)}(x-y). \qquad (46)$$

The first prefactor "e" in the last line of Eq. (45) is the electric charge. It is then convenient to rewrite the induced electric current (41) in the following compact form:

$$J^\mu(x) = -ee^{-3\tau(x)}{\rm Tr}_{y,z}[G(y,z)\cdot\gamma^\mu\mathbb{1}_{x,y}\mathbb{1}_{x,z}], \qquad (47)$$

where the exponential prefactor corresponds to a trivial conformal volume factor coming from the fact that the electric current has the dimension $[{\rm mass}]^3$.

Technically, our aim is to calculate the electric current (47) with the help of the Green function (42) determined by Eqs. (44) and (27). To this end we will use the Wigner-Weyl formalism described in the next section.

IV. WIGNER-WEYL FORMALISM

Let us very briefly review the basic features of Wigner functions and Weyl symbols in quantum mechanics that we will use later in the quantum field theory. A pedagogical overview of the Wigner-Weyl quantization formalism may be found, for example, in Refs. [25–27]. We choose the system of units $\hbar = c = 1$ and work in $(3+1)$-dimensional spacetime.

Let $\hat A$ be an operator that is a function of the position operator $\hat{\boldsymbol x}$ and the momentum operator $\hat{\boldsymbol p}$ which obey the standard commutation rule:

$$[\hat x^k,\hat p^l] = i\delta^{kl}. \qquad (48)$$

The Weyl symbol $\tilde A$ of the operator $\hat A$ is a function of the three-dimensional coordinate $\boldsymbol x$ and momentum $\boldsymbol p$ and is given by the following Wigner transformation [28–30]:

$$\tilde A(\boldsymbol x,\boldsymbol p) = \int d^3r e^{-i\boldsymbol{pr}}\langle\boldsymbol x-\boldsymbol r/2|\hat A(\hat{\boldsymbol x},\hat{\boldsymbol p})|\boldsymbol x+\boldsymbol r/2\rangle, \qquad (49)$$

which is expressed via the matrix elements $\langle\boldsymbol x|\hat A|\boldsymbol x'\rangle$ of the operator $\hat A$ in the basis of wave functions $|\boldsymbol x\rangle$ labeled by the coordinate $\boldsymbol x$. The Wigner transformation maps operators to functions.

The Wigner function[1]

$$W(\boldsymbol x,\boldsymbol p) = \int d^3r e^{-i\boldsymbol{pr}}\langle\boldsymbol x-\boldsymbol r/2|\hat\rho|\boldsymbol x+\boldsymbol r/2\rangle \qquad (50)$$

is the Wigner transform (49) of the density matrix operator $\hat\rho$. For pure states $\hat\rho = |\psi\rangle\langle\psi|$, the Wigner function (50) can be directly expressed via the wave functions $\psi(\boldsymbol x) = \langle\boldsymbol x|\psi\rangle$ as follows:

$$W(\boldsymbol x,\boldsymbol p) = \int d^3r e^{-i\boldsymbol{pr}}\psi(\boldsymbol x-\boldsymbol r/2)\psi^*(\boldsymbol x+\boldsymbol r/2). \qquad (51)$$

The Wigner-Weyl formalism has many useful features. The trace of the two operators $\hat A$ and $\hat B$ is given by a convolution of their Weyl symbols over the whole phase space:

$${\rm Tr}(\hat A\,\hat B) = \int\frac{d^3xd^3p}{(2\pi)^3}\tilde A(\boldsymbol x,\boldsymbol p)\tilde B(\boldsymbol x,\boldsymbol p). \qquad (52)$$

Therefore, the expectation value of an operator $\hat A$ can be expressed as a convolution of the Weyl symbol of the operator $\hat A$ and the Wigner function W:

$$\langle\hat A\rangle \equiv {\rm Tr}(\hat\rho\,\hat A) = \int\frac{d^3xd^3p}{(2\pi)^3}W(\boldsymbol x,\boldsymbol p)\tilde A(\boldsymbol x,\boldsymbol p). \qquad (53)$$

Weyl symbols of certain operators are easy to calculate. For our purposes (which will become evident below), let us consider the following operator:

$$\hat K(\hat{\boldsymbol x},\hat{\boldsymbol p}) = \hat A(\hat{\boldsymbol p}) + \frac{1}{2}[B(\hat{\boldsymbol x})\boldsymbol b\hat{\boldsymbol p} + \boldsymbol b\hat{\boldsymbol p}B(\hat{\boldsymbol x})] + \hat C(\hat{\boldsymbol x}), \qquad (54)$$

[1] We rescale the Wigner function in Eq. (50) by the factor $(2\pi)^3$ compared to the standard definition [25] in order to keep the conventional form of the phase-space volume in Eq. (53) and thereafter.

where $\boldsymbol{b}$ is a fixed vector, and the operator $\hat{A}$ is a function of a momentum operator $\hat{\boldsymbol{p}}$ only while the operators $\hat{B}$ and $\hat{C}$ depend only on the coordinate operator $\hat{\boldsymbol{x}}$. The Weyl symbol of the operator (54) is given by the sum of the corresponding functions,

$$\tilde{K}(\boldsymbol{x},\boldsymbol{p}) = A(\boldsymbol{p}) + B(\boldsymbol{x})\boldsymbol{b}\boldsymbol{p} + C(\boldsymbol{x}), \tag{55}$$

so that the Weyl transformation (49) for the particular form of the operator (54) amounts to the simple substitutions $\hat{\boldsymbol{x}} \to \boldsymbol{x}$ and $\hat{\boldsymbol{p}} \to \boldsymbol{p}$. For more complex operators this is not the case.

The Weyl transform of a product of two operators $\hat{D}$ and $\hat{G}$ can be expressed in terms of the Wigner transformations of these operators via the Groenewold formula [31]:

$$\widetilde{DG}(\boldsymbol{x},\boldsymbol{p}) = \tilde{D}(\boldsymbol{x},\boldsymbol{p}) \star \tilde{G}(\boldsymbol{x},\boldsymbol{p}), \tag{56}$$

where the Moyal (star) product [32]

$$\star \equiv e^{\frac{i}{2}\overleftrightarrow{\partial}_{\boldsymbol{xp}}} = 1 + \frac{i}{2}\overleftrightarrow{\partial}_{\boldsymbol{xp}} - \frac{1}{8}\overleftrightarrow{\partial}^2_{\boldsymbol{xp}} + \cdots \tag{57}$$

is essentially an exponentiation of the Poisson bracket kernel which can be expanded in powers of the double-derivative operator [30]

$$\overleftrightarrow{\partial}_{\boldsymbol{xp}} = \overleftarrow{\partial}_{\boldsymbol{x}}\overrightarrow{\partial}_{\boldsymbol{p}} - \overleftarrow{\partial}_{\boldsymbol{p}}\overrightarrow{\partial}_{\boldsymbol{x}} \tag{58}$$

that acts on both the left and right sides (for example, $f\overleftrightarrow{\partial}_{\boldsymbol{xp}}g = \partial_{\boldsymbol{x}}f\partial_{\boldsymbol{p}}g - \partial_{\boldsymbol{p}}f\partial_{\boldsymbol{x}}g$, etc).

Stationary systems may be described by the time-independent Wigner function (51). Nonstationary processes may be treated with the help of the time-dependent Wigner function $W(\boldsymbol{x},\boldsymbol{p};t)$ in which the time variable enters in a different way compared to the spatial coordinates. The time evolution of the Wigner function is determined by the Hamiltonian of the system H via the Moyal (star) bracket (58) [32]:

$$i\frac{\partial}{\partial t}W(\boldsymbol{x},\boldsymbol{p};t) = H \star W(\boldsymbol{x},\boldsymbol{p};t) - W(\boldsymbol{x},\boldsymbol{p};t) \star H. \tag{59}$$

Before going into further details we would like to highlight why the Wigner approach is a particularly useful method for our problem. We calculate the quantum average (47) of a current j which is given by the trace of the product $\langle j\rangle = \mathrm{Tr}[GB]$ of the Green function G and the simple operator B. The trace can be calculated as the convolution (52) of the corresponding Weyl symbols $\tilde{G}$ and $\tilde{B}$. We will see that the Weyl symbol $\tilde{B}$ may be easily obtained by the Weyl transformation (49) of the operator B itself, while the Weyl symbol $\tilde{G}$ of the Green function G may be calculated using the Groenewold formula (56) applied to the identity $\mathbb{1} = DG$, where D is an operator which possesses the Weyl symbol $\tilde{D}$ of a simple functional form. Since $\tilde{\mathbb{1}} \equiv 1$, the Groenewold equation (56) transforms to $1 = \tilde{D} \star \tilde{G}$ which can be solved iteratively with respect to the Green function $\tilde{G}$ in terms of the gradient expansion (57) of the exponentiated double-derivative operator (58). This strategy—which was applied for the first time to Euclidean quantum field theory in Ref. [24] and is common for noncommutative field theories [33]—will be realized in the next section.

V. THE NONANOMALOUS CONTRIBUTION TO THE ELECTRIC CURRENT

A. Closed form of the electric current

In order to determine the nonanomalous contribution J_{action} to the generated electric current we apply the Wigner-Weyl formalism to the vacuum of QED in nontrivial gravitational and electromagnetic backgrounds. We are interested in stationary effects in thermal equilibrium in three spatial dimensions $\boldsymbol{x} = (x_1, x_2, x_3)$, which may be formulated in Euclidean four-dimensional space in which the fourth "time" coordinate plays the role of the imaginary time x_4. In order to ensure the validity of the Wick rotation of our Euclidean results back to Minkowski spacetime, we assume that the electric field is vanishing (it would otherwise be imaginary in Euclidean space) and that the metric is a time-independent quantity.

The Wigner-Weyl formalism may naturally be generalized to four-dimensional Euclidean space in which the four-component coordinate operator $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_4)$ is conjugated with the four-component momentum operator $\hat{p}$. This technique—which utilizes the Groenewold formula (56) and the derivative expansion of the Moyal product (57) at the level of the Green functions and the Weyl symbols of the corresponding operators—has been worked out in detail in Ref. [24]. Below, we describe the essential details of the approach.

In coordinate space the momentum operator $\hat{p}$ takes the familiar form of the derivative operator $\hat{p}_\mu = -i\partial_{x_\mu}$, with $\mu = 1, \ldots, 4$, and the fermionic operator (27) takes the following form:

$$\hat{\mathcal{D}}(\hat{x},\hat{p}) = \frac{1}{2}[e^{3\tau(\hat{x})}\hat{\not{p}} + \hat{\not{p}}e^{3\tau(\hat{x})}] - e\not{A}(\hat{x}) - ie^{4\tau(\hat{x})}m, \tag{60}$$

where we used the standard "slashed" notation $\not{a} = \sum_{\mu=1}^4 \gamma_\mu a_\mu$.

The Weyl symbol of the fermionic operator $\hat{\mathcal{D}}$ is given by the Wigner transformation (49):

$$\tilde{\mathcal{D}}(x,p) = e^{3\tau(x)}[\not{p} - e\not{A}(x)] - ie^{4\tau(x)}m, \tag{61}$$

where we used the fact that the operator (60) matches the general form of the operator (54) with the known Weyl symbol (55). Then, the Weyl symbol for the fermionic Green function (42) is formally given by the Wigner transform (49):

$$\tilde{G}(x,p) = \int d^4 r e^{-ipr} G(x - r/2, x + r/2). \quad (62)$$

This expression assumes that the Green function G is defined in a background of a classical U(1) gauge field A_μ in a fixed gauge. Therefore, one can suggest that Eq. (62) does not require the explicit introduction of a parallel gauge transport in the form of the Schwinger line $P(x,y)$ which is an exponentiated gauge field integrated along an open contour joining the two points of the Green function (62). Below, we explicitly demonstrate that the inclusion of the Schwinger line does not affect the final result for the electric current in the magnetic field background. The presence of the Schwinger line is more suitable for systems with dynamical gauge fields [34].

Notice that, in addition to the gauge transport line, one could also expect the appearance of the spin connection transport. However, in the background of the conformally flat metric (4) the spin connection term does not enter the Lagrangian (24) and therefore the parallel spin transport is trivial.

The Weyl symbol of the Green function (62) can be calculated explicitly with the help of the Groenewold formula (56) applied to Eq. (44):

$$1 = \tilde{\mathcal{D}}(x,p) \star \tilde{G}(x,p), \quad (63)$$

with $\tilde{\mathcal{D}}$ given explicitly in Eq. (61). The star product in Eq. (63) is a straightforward generalization of Eqs. (57) and (58) to four-dimensional Euclidean space:

$$\star \equiv e^{\frac{i}{2}\overleftrightarrow{\partial}} = 1 + \frac{i}{2}\overleftrightarrow{\partial} - \frac{1}{8}\overleftrightarrow{\partial}^2 + \cdots, \quad (64)$$

with

$$\overleftrightarrow{\partial} \equiv \overleftrightarrow{\partial}_{xp} = \sum_{\mu=1}^{4} (\overleftarrow{\partial}_{x_\mu}\overrightarrow{\partial}_{p_\mu} - \overleftarrow{\partial}_{p_\mu}\overrightarrow{\partial}_{x_\mu}). \quad (65)$$

The nonanomalous electric current (47) can be calculated with the help of a four-dimensional generalization of the convolution formula (52):

$$J^\mu_{\rm action}(x) = -e e^{-3\tau(x)} \int \frac{d^4 s\, d^4 p}{(2\pi)^4} {\rm tr}[\tilde{G}(s,p)\tilde{\mathbb{1}}_x(s,p)\gamma^\mu], \quad (66)$$

where the trace goes over the spinor indices only. The Wigner transform $\tilde{\mathbb{1}}_x(s,p)$ of the product $\mathbb{1}_x(y,z) \equiv \mathbb{1}_{x,y}\mathbb{1}_{x,z}$ of the unit operators (46) can be calculated straightforwardly with the help of Eq. (49):

$$\tilde{\mathbb{1}}_x(s,p) = \delta^{(4)}(x-s), \quad (67)$$

where s is the four-dimensional spacetime coordinate and p is the four-dimensional momentum. Substituting Eq. (67) into Eq. (66), we get the compact expression for the nonanomalous electric current via the Wigner transform of the fermionic propagator $\tilde{G}(x,p)$:

$$J^\mu_{\rm action}(x) = -e e^{-3\tau(x)} \int \frac{d^4 p}{(2\pi)^4} {\rm tr}[\tilde{G}(x,p)\gamma^\mu]. \quad (68)$$

Now let us briefly demonstrate that the inclusion of the parallel gauge transport

$$P(x,y) = \exp\left[ie \int_x^y dx_\mu A^\mu(x)\right] \quad (69)$$

in the definition of the Weyl symbol (62) for the fermionic Green function (42) gives us the gauge-invariant symbol

$$\tilde{G}_{\rm inv}(x,p) = \int d^4 r e^{-ipr} P(x - r/2, x + r/2) \times G(x - r/2, x + r/2), \quad (70)$$

which does not affect the result for the anomalous current (68). To this end, we choose the contour connecting the points $x - r/2$ and $x + r/2$ in the form of a straight line,

$$\bar{x}^\mu(t) = x^\mu + \left(t - \frac{1}{2}\right) r^\mu, \quad (71)$$

parametrized by the parameter $t \in [0,1]$. Taking the gauge potential of the magnetic field B in the symmetric gauge, $A^\mu = (-eBx^2/2, eBx^1/2, 0, 0)$, we calculate the Schwinger line (69) with the straight open contour (71), and then we get the following expression for the gauge-invariant Weyl-symbol (70):

$$\tilde{G}_{\rm inv}(x,p) = \int d^4 r e^{-ipr} e^{i(x_1 r_2 - x_2 r_1)eB/2} G(x - r/2, x + r/2) \equiv \tilde{G}(x, p - \bar{p}(x)), \quad (72)$$

where the standard Weyl symbol $\tilde{G}(x,p)$ is given in Eq. (62) and $\bar{p}(x) = (Bx^2/2, -Bx^1/2, 0, 0)$. The next step is to calculate the anomalous current (68) using the invariant Weyl symbol $\tilde{G}_{\rm inv}$ of Eq. (72) instead of the standard symbol $\tilde{G}$. However, by shifting the integration variable $p \to p + \bar{p}(x)$ we find that both definitions of the Weyl symbol (62) and (70) lead to the same current of Eq. (68). Therefore, the Schwinger line (the parallel gauge transport) may indeed be ignored in the definition of the Weyl symbol in the background of the classical magnetic field.

Below, we explicitly calculate the induced electric current (68) in the leading order in the derivative expansion of the star product (64). The derivative series also corresponds to a semiclassical expansion in the powers of the Planck constant $\hbar$. The latter is evident from the form of the exponential operator (64) in which the Planck constant is reinstated: $\star = \exp\{i\frac{\hbar}{2}\overleftrightarrow{\partial}\}$.

B. Electric current in the leading order

The electric current is given in Eq. (68) where the Wigner transform of the fermionic propagator $\tilde{G}(x,p)$ is completely determined by Eqs. (61), (63), (64), and (65). The Groenewold equation (63) for $\tilde{G}$ can be solved iteratively in terms of the series

$$\tilde{G} = \tilde{G}^{(0)} + \tilde{G}^{(1)} + \tilde{G}^{(2)} + \cdots \tag{73}$$

Notice that the iterative solution is the derivative expansion (64) as each power of the double-faced derivative (65) gives one power of a spatial derivative of either the conformal factor $\tau(x)$ or electromagnetic field $A_\mu(x)$.

The nth-order term $\tilde{G}^{(n)}(x,p)$ is a local function of x and p proportional to the products of derivatives over x of the form $(\partial^{l_1}\tau(x))\ldots(\partial^{l_L}\tau(x))(\partial^{m_M}A(x))\ldots(\partial^{m_1}A(x))$, where the sum over the positive integers l_i and m_i is equal to the order of the expansion, $l_1+\cdots+l_L+m_1+\cdots+m_M=n$. The electric current generated by the scale magnetic effect is given by the first-order (linear) response both in the conformal factor τ and in the electromagnetic field, $\partial_\alpha\tau\partial_\beta A_\gamma$, so that the effect appears in the second-order term $\tilde{G}^{(2)}$ in the expansion (73).

The zeroth-order term in the expansion (73) is the usual (algebraic) inverse of the Weyl symbol (61) of the fermionic operator $\hat{\mathcal{D}}$:

$$\begin{aligned}\tilde{G}^{(0)}(x,p) &= \tilde{\mathcal{D}}^{-1}(x,p) \equiv 1/\tilde{\mathcal{D}}(x,p)\\ &= \frac{[\not{p} - e\not{A}(x)] + ie^{\tau(x)}m}{[p-eA(x)]^2 + e^{2\tau(x)}m^2}e^{-3\tau(x)}.\end{aligned} \tag{74}$$

By expanding the Groenewold equation (63) in powers of the double derivative $\overleftrightarrow{\partial}$, we express—via the Weyl symbol (61) and its inverse (74)—the first-order term

$$\tilde{G}^{(1)} = -\frac{i}{2}\tilde{\mathcal{D}}^{-1}(\tilde{\mathcal{D}}\overleftrightarrow{\partial}\tilde{\mathcal{D}}^{-1}), \tag{75}$$

and then the second-order term

$$\tilde{G}^{(2)} = \tilde{G}^{(2)}_I + \tilde{G}^{(2)}_{II}, \tag{76a}$$

$$\tilde{G}^{(2)}_I = -\frac{1}{4}\tilde{\mathcal{D}}^{-1}\{\tilde{\mathcal{D}}\overleftrightarrow{\partial}[\tilde{\mathcal{D}}^{-1}(\tilde{\mathcal{D}}\overleftrightarrow{\partial}\tilde{\mathcal{D}}^{-1})]\}, \tag{76b}$$

$$\tilde{G}^{(2)}_{II} = \frac{1}{8}\tilde{\mathcal{D}}^{-1}(\tilde{\mathcal{D}}\overleftrightarrow{\partial}^{\,2}\tilde{\mathcal{D}}^{-1}). \tag{76c}$$

The second-order term (76) can further be rewritten as follows:

$$\tilde{G}^{(2)}_I = -\frac{1}{4}(R_\mu\partial_{p_\mu} - C_\mu\partial_{x_\mu})[(C_\nu R_\nu - R_\nu C_\nu)\tilde{\mathcal{D}}^{-1}], \tag{77a}$$

$$\begin{aligned}\tilde{G}^{(2)}_{II} = \frac{1}{8}[&R_{\mu\nu}(C_\mu C_\nu + C_\nu C_\mu - C_{\mu\nu})\\ &+ C_{\mu\nu}(R_\mu R_\nu + R_\nu R_\mu - R_{\mu\nu})\\ &- S_{\mu\nu}(2R_\mu C_\nu + 2C_\nu R_\mu - S_{\mu\nu} - S_{\nu\mu})]\tilde{\mathcal{D}}^{-1},\end{aligned} \tag{77b}$$

where

$$R_\mu = \tilde{\mathcal{D}}^{-1}\partial_{x_\mu}\tilde{\mathcal{D}}, \qquad C_\mu = \tilde{\mathcal{D}}^{-1}\partial_{p_\mu}\tilde{\mathcal{D}}, \tag{78a}$$

$$R_{\mu\nu} = \tilde{\mathcal{D}}^{-1}\partial_{x_\mu}\partial_{x_\nu}\tilde{\mathcal{D}}, \qquad C_{\mu\nu} = \tilde{\mathcal{D}}^{-1}\partial_{p_\mu}\partial_{p_\nu}\tilde{\mathcal{D}}, \tag{78b}$$

$$S_{\mu\nu} = \tilde{\mathcal{D}}^{-1}\partial_{p_\mu}\partial_{x_\nu}\tilde{\mathcal{D}}. \tag{78c}$$

In deriving Eq. (77) we used the following identities:

$$\partial_{x_\mu}\tilde{\mathcal{D}}^{-1} = -R_\mu\tilde{\mathcal{D}}^{-1}, \qquad \partial_{p_\mu}\tilde{\mathcal{D}}^{-1} = -C_\mu\tilde{\mathcal{D}}^{-1}, \tag{79a}$$

$$\partial_{x_\mu}\partial_{x_\nu}\tilde{\mathcal{D}}^{-1} = (R_\mu R_\nu + R_\nu R_\mu - R_{\mu\nu})\tilde{\mathcal{D}}^{-1}, \tag{79b}$$

$$\partial_{p_\mu}\partial_{p_\nu}\tilde{\mathcal{D}}^{-1} = (C_\mu C_\nu + C_\nu C_\mu - C_{\mu\nu})\tilde{\mathcal{D}}^{-1}, \tag{79c}$$

$$\partial_{p_\mu}\partial_{x_\nu}\tilde{\mathcal{D}}^{-1} = (C_\mu R_\nu + R_\nu C_\mu - S_{\mu\nu})\tilde{\mathcal{D}}^{-1}. \tag{79d}$$

Notice that the quantities (78) are matrices in spinor space and therefore in general they do not commute with each other. Moreover, $S_{\mu\nu} \neq S_{\nu\mu}$, while $R_{\mu\nu} \equiv R_{\nu\mu}$ and $C_{\mu\nu} \equiv C_{\nu\mu}$.

Using the relations

$$\partial_{x_\mu}R_\nu = R_{\mu\nu} - R_\mu R_\nu, \qquad \partial_{p_\mu}C_\nu = C_{\mu\nu} - C_\mu R_\nu, \tag{80a}$$

$$\partial_{p_\mu}R_\nu = S_{\mu\nu} - C_\mu R_\nu, \qquad \partial_{x_\mu}C_\nu = S_{\nu\mu} - R_\mu C_\nu, \tag{80b}$$

we rewrite Eq. (77a) as follows:

$$\begin{aligned}\tilde{G}^{(2)}_I = -\frac{1}{4}[&R_\nu(C_{\mu\nu} - \{C_\mu, C_\nu\})R_\mu - R_\nu(S_{\nu\mu} - \{R_\mu, C_\nu\})C_\mu\\ &+ C_\nu(R_{\mu\nu} - \{R_\mu, R_\nu\})C_\mu - C_\nu(S_{\mu\nu} - \{C_\mu, R_\nu\})R_\mu\\ &- R_\nu[C_\mu, R_\mu]C_\nu + C_\nu[C_\mu, R_\mu]R_\nu + [R_\mu, C_\nu]S_{\mu\nu}\\ &- R_\mu R_\nu C_{\mu\nu} - C_\mu C_\nu R_{\mu\nu}],\end{aligned} \tag{81}$$

where

$$[A,B] = AB - BA, \qquad \{A,B\} = AB + BA \tag{82}$$

are, respectively, the commutator and anticommutator.

The second-order correction $\tilde{G}^{(2)}$ is now equal to the sum (76a) of the two terms $\tilde{G}_I^{(2)}$ and $\tilde{G}_{II}^{(2)}$ given, respectively, in Eqs. (77b) and (81). Since these terms do not contain external derivatives, we should expand them in powers of the derivatives and eventually keep the terms containing the product $\partial_\mu \tau \partial_\nu A_\alpha$. As one can see from the explicit definitions (61) and (78), the relevant terms enter the S and R quantities:

$$S_{\mu\nu}(x,p) = 3C_\mu(x,p)\partial_\nu \tau(x), \tag{83a}$$

$$R_\mu(x,p) = \left[3 - \frac{im}{\tilde{\mathcal{D}}(x,p)}\right]\partial_\mu \tau(x) - C_\nu(x,p)\partial_\mu A_\nu(x), \tag{83b}$$

$$R_{\mu\nu}(x,p) = -3\partial_{\{\mu,}\tau(x)\partial_{\nu\}}A_\alpha(x)C_\alpha(x,p). \tag{83c}$$

In Eq. (83c) all irrelevant terms with double derivatives are not shown. The symmetrization with respect to the Lorentz indices is denoted by the curly brackets.

Equation (83) indicates that every term in the second-order corrections to the Weyl symbol of the Green function [Eqs. (77b) and (81)] contains the required combination of the derivatives $\partial_\mu \tau \partial_\nu A_\alpha$. Since we are looking for the terms bilinear in τ and A_μ, we keep the mentioned combination while setting τ and A_μ to zero in the prefactors of these terms. Denoting the latter with the superscript "(0)," one then immediately gets the following expressions for the S and R Lorentz structures (83):

$$S^{(0)}_{\mu\nu}(x,p) = 3P_\mu(p)\partial_\nu \tau(x), \tag{84a}$$

$$R^{(0)}_\mu(x,p) = [3 - imP_0(p)]\partial_\mu \tau(x) - P_\nu(p)\partial_\mu A_\nu(x), \tag{84b}$$

$$R^{(0)}_{\mu\nu}(x,p) = -3\partial_{\{\mu,}\tau(x)\partial_{\nu\}}A_\alpha(x)P_\alpha(p), \tag{84c}$$

where

$$P_\mu(p) \equiv C^{(0)}_\mu(p) = \lim_{\tau\to 0}\lim_{A\to 0} C_\mu(x,p) \tag{85}$$

is the vector C_μ given by the second expression in Eq. (78a) in flat space in the absence of an external electromagnetic field.

Using the explicit form of the Weyl symbol $\tilde{\mathcal{D}}$ of the fermionic operator (61) and the second relation in Eq. (78a), we explicitly get for Eq. (85)

$$P_\mu(p) = \frac{1}{p^2 + m^2}(\not{p} + im)\gamma_\mu, \tag{86}$$

where γ_μ are Euclidean gamma matrices for $\mu = 1, \ldots, 4$ and $\gamma_0 \equiv \mathbb{1}$ is a unit matrix.[2] In addition, we notice that because the Weyl symbol $\tilde{\mathcal{D}}$ is a linear function of the momentum p, the second relation in Eq. (61) gives us $C_{\mu\nu} \equiv 0$.

Finally, we substitute the Lorentz structures (84c) and (85) into the second-order corrections to the Weyl symbol of the Green function [Eqs. (77b) and (81)], sum them up, and put the result into the definition of the induced electric current (68). Then, in order to keep only the second-order corrections, we set $\tau = 0$ in the volume prefactor of the current (68), and after algebraic manipulations we get the following compact expression for the generated nonanomalous electric current:

$$J_{\mathrm{action},\mu} = -\frac{ime^2}{4}\frac{\partial\tau}{\partial x^\alpha}\frac{\partial A_\beta}{\partial x^\nu}\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}(P_\mu P_{\nu\alpha\beta}), \tag{87}$$

where the trace is taken over the spinor space, while the tensor structure

$$P_{\nu\alpha\beta} = \{\{P_0, P_\nu\}, \{P_\alpha, P_\beta\}\} + \{[P_0, P_\alpha], [P_\nu, P_\beta]\} - \{\{P_0, P_\beta\}, \{P_\nu, P_\alpha\}\}, \tag{88}$$

is given in terms of the commutators and anticommutators (82) of the matrices (86).

The electric current (87) is invariant under the U(1) gauge transformations (16) since the expression under the integral in Eq. (87) is antisymmetric with respect to the interchange of the indices β and ν. The latter fact can be checked directly by manipulation of Eq. (88). Therefore, the derivative $\partial_\nu A_\beta$ in Eq. (87) always appears in the form of the gauge-invariant electromagnetic field tensor, $\partial_\nu A_\beta - \partial_\beta A_\nu \equiv F_{\nu\beta}$.

Substituting Eq. (86) into Eq. (88) and taking the trace over the spinor indices, we get the electric current (87),

$$J_{\mathrm{action},\mu} = -\alpha(m)e^2 F_{\mu\nu}\partial_\nu \tau, \tag{89}$$

where the prefactor $\alpha(m)$ is given by the integral

$$\alpha(m) = 4m^2\int\frac{d^4p}{(2\pi)^4}\frac{(p^2 + 2m^2)}{(p^2 + m^2)^4}, \tag{90}$$

which evaluates to a finite mass-independent quantity,

$$\alpha(m) = \frac{1}{6\pi^2}. \tag{91}$$

We note that despite the fact that Eq. (89) has a visibly covariant four-tensor form, our Euclidean derivation is

[2] For convenience, we complemented the four-vector (86) with the fifth $\mu = 0$ component. In Euclidean space our choice $\gamma_0 \equiv \mathbb{1}$ does not interfere with the γ_0 matrix of Minkowski space.

formally valid only for the spatial indices μ and ν which do not allow us to consider either a nonzero background electric field $\boldsymbol{E}$ or a time-dependent conformal metric factor $\tau = \tau(t)$. Restricting ourselves to the case of the pure magnetic field background $\boldsymbol{B} \neq 0$ in spatially inhomogeneous curved space $\tau = \tau(\boldsymbol{x})$, we obtain from Eqs. (89) and (91) the following nonanomalous contribution to the electric current in *massive* QED:

$$\boldsymbol{J}_{\text{action}} = -\frac{e^2}{6\pi^2}\boldsymbol{\nabla}\tau \times \boldsymbol{B}. \tag{92}$$

Taking into account the value of the one-loop QED beta function (31), we find that the part of the electric current (92) precisely cancels the one-loop anomalous part,

$$\boldsymbol{J}_{\text{measure}} = \frac{2\beta^{1\,\text{loop}}_{\text{QED}}}{e}\boldsymbol{\nabla}\tau(x) \times \boldsymbol{B}(x), \tag{93}$$

which comes from the integration measure (35). Therefore, in the limit of heavy fermions the electric current generated by the SME vanishes,

$$\boldsymbol{J} = \boldsymbol{J}_{\text{measure}} + \boldsymbol{J}_{\text{action}} = 0 + O(\partial^2/m^2), \tag{94}$$

where the second term denotes higher-derivative terms which are suppressed in the large-mass limit. These terms appear naturally in the derivative series (73) which iteratively define the Wigner transform of the fermionic propagator as the solution of the Groenewold equation (63).

VI. DISCUSSIONS AND CONCLUSION

In this paper we discussed the SME which generates a vacuum electric current in an external magnetic field in a curved spacetime [14]. The origin of the effect is the conformal anomaly which breaks, at the quantum level, a conformal symmetry in classically conformal gauge theories. This effect has already been considered in QED with massless fermions. In this paper we asked the natural question of what happens with the SME if the fermions are massive so that the conformal invariance is already explicitly broken at the classical level. In particular, we considered the limit when the fermion mass m is much larger than the scale of the external magnetic field $m^2 \gg |\boldsymbol{B}|$ and the scale of the spatial gradient of the conformal factor τ of the metric, $m \gg |\boldsymbol{\nabla}\tau|$. This limit is opposite to the classically conformal case considered in Ref. [14].

We demonstrated that the anomalous electric current generated by the massive fermions can conveniently be calculated by the Wigner-Weyl formalism which gives the derivative series inversely proportional to increasing powers of the fermion mass m. We have found that there are two contributions to the electric current generated by the SME.

The first contribution $\boldsymbol{J}_{\text{measure}}$ comes from the integration measure over the fermionic fields. This current is anomalous because the measure is not invariant under conformal (Weyl) transformations in the presence of the background electromagnetic field [15]. As a result, the anomalous contribution $\boldsymbol{J}_{\text{measure}}$ does not depend on the fermion mass because the quantum measure is independent of the details of the classical fermionic action. At small fermion masses the anomalous term provides the major contribution to the SME current. The current $\boldsymbol{J}_{\text{measure}}$ should also contain contributions induced by exchanges by virtual photons. These loop corrections were not considered in the present paper since they are suppressed by higher orders of the fine-structure constant α_{QED}.

The second contribution $\boldsymbol{J}_{\text{action}}$ originates from the classically nonvanishing terms in the trace of the energy-momentum tensor. These terms appear due to the explicit breaking of the scale invariance at the level of the classical Lagrangian. Despite the fact that our derivation involved integrals in unbounded momentum space, the second contribution to the electric current is a finite quantity in both the ultraviolet and infrared regimes. The absence of the ultraviolet divergences implies that the anomalous current does not require regularization and subsequent renormalization. Our result was obtained in the classical electromagnetic field background A_μ in the leading order of the Wigner expansion. Next, higher-order terms in the Wigner expansion would correspond to (spatial) derivative series in terms of the electromagnetic gauge field A_μ and conformal factor τ. Quantum fluctuations of the electromagnetic field on top of the classical magnetic background would generate perturbative series over the electromagnetic coupling e at each given order of the Wigner expansion. We expect that the perturbative series would generate standard ultraviolet divergences which will be absorbed into the renormalization of the gauge coupling e. Thus, due to the renormalizability of QED, we expect that in the leading (lowest-derivative) order of the Wigner expansion the quantum corrections would lead to a renormalization of the electric charge without qualitatively altering the expression for the anomalous electric current (92).

We have explicitly found that for massive fermions the electric currents J_{measure} and J_{action} (originating, respectively, from the anomalous symmetry breaking and from the explicit symmetry breaking) cancel each other in the leading order in the number of derivatives. Therefore, for sufficiently heavy fermions the SME should be strongly suppressed. This conclusion is in agreement with the decoupling theorem for massive particles [19].

It is clear that the apparent Euclidean covariant structure of the generated current (89) is common both for the scale magnetic (5) and scale electric (6) effects [14]. Therefore, we believe that our conclusion may also be valid for the scale electric effect, which has not been explicitly discussed in this paper.

Finally we notice that there is a potential possibility to observe the scale magnetic effect in tabletop laboratory experiments with Dirac and Weyl semimetals. These materials possess both crucial ingredients, as they host relativistic massless fermionic excitations subjected to a gravitational field background. The relativistic fermions emerge naturally due to topological properties of the electronic band structure of these materials [35], while the emergent gravity may be induced by elastic deformations of their crystal structure [36]. The latter effect is very common for many solid state systems [37]. Thus, elastically deformed topological materials may provide a useful experimental tool to study a plethora of properties [23] of relativistic quantum field theory in curved spacetime, including the scale magnetic effect.

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