Asymptotic freedom in certain $SO(N)$ and $SU(N)$ models

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We calculate the β-functions for $SO(N)$ and $SU(N)$ gauge theories coupled to adjoint and fundamental scalar representations, correcting longstanding, previous results. We explore the constraints on N resulting from requiring asymptotic freedom for all couplings. When we take into account the actual allowed behavior of the gauge coupling, the minimum value of N in both cases turns out to be larger than realized in earlier treatments. We also show that in the large N limit, both models have large regions of parameter space corresponding to total asymptotic freedom.

I. INTRODUCTION

The discovery of asymptotic freedom (AF) in 1973 [1,2] heralded a new era in particle physics. There was immediate interest in the extent to which AF persists following the inclusion in a renormalizable gauge theory of fermion and scalar multiplets. For fermions alone the question is easily answered, but for scalars, or both fermions and scalars, it becomes nontrivial. A pioneering and remarkably comprehensive analysis was performed very early by Cheng *et al.* (CEL) [3]. Under certain assumptions, a search for models of this type was carried out recently by Giudice *et al.* [4], who labeled such models *totally asymptotically free* (TAF). Other studies of this sort include Refs. [5,6], who consider relatively low-scale "unification' to a semisimple group that is TAF.

Another important question arises once scalar multiplets are introduced, being the nature and consequences of spontaneous symmetry breaking (SSB) in such AF theories, for example as to whether one can have an AF theory with SSB to an Abelian subgroup. CEL also address this issue, concluding that having enough scalar multiplets to achieve this is incompatible with AF. This explicit goal no longer seems essential; however, a fully AF theory remains desirable.

In a series of recent papers [7–10], we have addressed some other aspects of these issues in the context of a gauge theory with scalar multiplets coupled to renormalizable, classically scale invariant gravity. Our motivation in that work was twofold. Firstly, to demonstrate examples of such theories that are AF and hence may be termed ultraviolet (UV) complete; secondly, to show that in such theories, SSB may occur via a variation on the perturbative dimensional transmutation mechanism first elucidated by Coleman and Weinberg [11].

Here we return to the AF issue, but in a class of theories with a more complicated scalar sector than we have previously considered, namely two distinct scalar representations

transforming according to the adjoint and the fundamental representations, with gauge groups $SO(N)$ and $SU(N)$.[1] In contrast to Refs. [4–6], we restrict our attention to grand unification in a simple group, even though this model is incomplete and does not contain the Standard Model (SM).

We assume the presence of a fermion sector contributing to the gauge β-functions, but that concomitant Yukawa couplings are sufficiently small that they are all asymptotically free. As usual [3], they will then make negligible contributions to the β-functions of the quartic scalar couplings. We review the flat space CEL calculations, where we find a number of significant differences from their β-functions. In the light of these changes, we reconsider the results for the minimum value of N consistent with AF in the case of both gauge groups. Here we find some differences from previous results. For example, CEL correctly point out that the optimal situation for AF of the scalar self-couplings occurs for the *minimum* of the (absolute) value of the gauge β-function coefficient (b_g), which they choose to approximate by zero. However, as we point out, this approximation can be inadequate to establish the actual minimum value of N, and the *genuine* minimum of b_g should be used in each case. This model for the $SU(N)$ case has been previously considered in Ref. [12], with whose β-functions we agree.[2] We believe our treatment of this $SO(N)$ model is new.

We gain further insight into the "minimum value of N" issue by considering the large N limit of these theories with appropriate rescaling of the scalar self-couplings. We shall discuss the extension of these results to renormalizable gravity elsewhere [13].

The organization of the remainder of the paper is as follows: In Secs. II and III, we give the beta-functions for the

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[1]The $SU(N)$ case for such scalars was considered by CEL, but we find some differences in our results both for the β-functions and for the minimum allowed value of N.

[2]The authors of Ref. [12] did not mention their disagreements with CEL. Our minimum value of N differs from theirs.

$SO(N)$ and $SU(N)$ models, respectively, and discuss the minimum value of N consistent with TAF, comparing with earlier determinations. In Sec. IV, we take up the large N limits of these models and determine the ultraviolet stable FPs (UVFPs) for various associated fermionic content. After the Conclusions, Sec. V, we add two appendices deriving from the large N models. In Appendix A, we indicate how the analytic solutions for the UVFPs can be obtained. In Appendix B, we discuss the possible existence of an infrared fixed point (IRFP) for the gauge couplings at two loops in certain cases.

II. THE $SO(N)$ MODEL

The scalar potential of the theory is

$$V(\Phi, \chi) = \frac{1}{2}\lambda_1 (\mathrm{Tr}\Phi^2)^2 + \lambda_2 \mathrm{Tr}\Phi^4 + \frac{1}{8}\lambda_3(\chi_i\chi_i)^2$$
$$+ \frac{1}{2}\lambda_4\chi_i\chi_i\mathrm{Tr}\Phi^2 + \frac{1}{4}\lambda_5\chi_i\Phi_{ik}\Phi_{kj}\chi_j. \quad (2.1)$$

Here $\Phi = R^a\phi^a$, where $[a = 1, 2\ldots N(N-1)/2]$ represents a real adjoint representation, and $\chi_i[i = 1, 2\ldots N]$ is a real multiplet in the defining (fundamental) representation, and R^a are the associated antisymmetric $N \times N$ matrices normalized as usual so that

$$\mathrm{Tr}R^a R^b \equiv T(R)\delta^{ab}, \quad \text{where } T(R) = \frac{1}{2}. \quad (2.2)$$

Thus, $\mathrm{Tr}[\Phi^2] = \phi^a\phi^a/2$.

Suppressing in each case a factor of $(16\pi^2)^{-1}$, the flat space β-functions are

$$\beta_{g^2} = -b_g(g^2)^2, \quad b_g \equiv \frac{21N-43}{6} - \frac{4}{3}T_F,$$

$$\beta_{\lambda_1} = \left(\frac{N(N-1)}{2}+8\right)\lambda_1^2 + 2(2N-1)\lambda_1\lambda_2 + 6\lambda_2^2 + N\lambda_4^2$$
$$+ \lambda_4\lambda_5 - 6(N-2)g^2\lambda_1 + 9g^4,$$

$$\beta_{\lambda_2} = (2N-1)\lambda_2^2 + 12\lambda_1\lambda_2 + \frac{1}{8}\lambda_5^2 - 6(N-2)g^2\lambda_2 + \frac{3(N-8)}{2}g^4,$$

$$\beta_{\lambda_3} = (N+8)\lambda_3^2 + \frac{N(N-1)}{2}\lambda_4^2 + \frac{N-1}{16}\lambda_5^2 + \frac{N-1}{2}\lambda_4\lambda_5$$
$$- 3(N-1)g^2\lambda_3 + \frac{3(N-1)}{4}g^4,$$

$$\beta_{\lambda_4} = 4\lambda_4^2 + \frac{1}{8}\lambda_5^2 + \lambda_5\left[\frac{N-1}{4}\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3\right]$$
$$+ \lambda_4\left[\left(\frac{N(N-1)}{2}+2\right)\lambda_1 + (2N-1)\lambda_2\right.$$
$$\left.+ (N+2)\lambda_3 - \frac{3(3N-5)}{2}g^2\right] + \frac{3}{2}g^4,$$

$$\beta_{\lambda_5} = \frac{N}{4}\lambda_5^2 + \lambda_5\left[2\lambda_1 + (N-1)\lambda_2 + 2\lambda_3 + 8\lambda_4 - \frac{3(3N-5)}{2}g^2\right]$$
$$+ 3(N-4)g^4. \quad (2.3)$$

Here

$$\mathrm{Tr}R_F^a R_F^b \equiv T_F\delta^{ab}, \quad (2.4)$$

where the fermions transform according to the representation R_F, and the coefficient of T_F in Eq. (2.3) reflects use of two-component or Majorana fermions. We obtained these results both by direct calculation and by use of the RG equation for the effective potential (in the Landau gauge) in the manner explained in the Standard Model context in Refs. [14,15]. The disagreements with CEL are in the coefficients of the following terms:

$$\beta_{\lambda_1}: \lambda_4\lambda_5; \qquad \beta_{\lambda_2}: \lambda_5^2; \qquad \beta_{\lambda_3}: \lambda_4\lambda_5, \lambda_5^2;$$
$$\beta_{\lambda_4}: \lambda_1\lambda_5, \lambda_2\lambda_5, \lambda_3\lambda_5, \lambda_5^2, g^4; \qquad \beta_{\lambda_5}: \lambda_4\lambda_5, \lambda_5^2, g^4.$$

To analyze the RG behavior of the couplings, it is convenient to introduce rescaled couplings $x_i = \lambda_i/g^2$, whereupon the "reduced" β-functions are

$$\bar\beta_{x_1} = \left(\frac{N(N-1)+16}{2}\right)x_1^2 + 6x_2^2 + 2(2N-1)x_1x_2$$
$$+ Nx_4^2 + x_4x_5 + (b_g - 6(N-2))x_1 + 9,$$

$$\bar\beta_{x_2} = (2N-1)x_2^2 + 12x_1x_2 + \frac{1}{8}x_5^2 + (b_g - 6(N-2))x_2$$
$$+ \frac{3(N-8)}{2},$$

$$\bar\beta_{x_3} = (N+8)x_3^2 + \frac{N(N-1)}{2}x_4^2$$
$$+ \frac{N-1}{16}x_5^2 + \frac{N-1}{2}x_4x_5$$
$$+ (b_g - 3(N-1))x_3 + \frac{3(N-1)}{4},$$

$$\bar\beta_{x_4} = 4x_4^2 + \frac{1}{8}x_5^2 + x_5\left[\frac{N-1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right]$$
$$+ x_4\left[\left(\frac{N(N-1)}{2}+2\right)x_1 + (2N-1)x_2\right.$$
$$\left.+ (N+2)x_3 + b_g - \frac{3(3N-5)}{2}\right] + \frac{3}{2},$$

$$\bar\beta_{x_5} = \frac{N}{4}x_5^2 + x_5\left[2x_1 + (N-1)x_2 + 2x_3 + 8x_4\right.$$
$$\left.+ b_g - \frac{3(3N-5)}{2}\right] + 3(N-4). \quad (2.5)$$

In Eq. (2.5),

$$\bar{\beta}_{x_i} \equiv \frac{dx_i}{du}, \quad \text{where } du \equiv g^2(t)dt. \tag{2.6}$$

We now proceed to find and classify the fixed points (FPs) of this system by setting all the reduced β-functions to zero. As long as one has $b_g > 0$, it is clear that any such FP (for finite x_i) corresponds to TAF. In fact, there are several FP solutions of this system of equations but, it turns out, only one is UV stable in all the ratios x_i. By UV stable, we mean that the matrix $S_{ij} \equiv \partial \bar{\beta}_{x_i}/\partial x_j$ has only negative eigenvalues at the FP, so that all ratios x_i flow toward the FP asymptotically. We shall refer to such a point as a UVFP, even though the original couplings are all TAF.

If any of the eigenvalues is zero, then one would have to go beyond the *linear* approximation to determine whether the associated flat direction is in fact a minimum. Should that test fail, one would have to go beyond the *one-loop* approximation unless one can identify an exact symmetry ensuring that such a flat direction persists to all orders in perturbation theory. (It turns out in the models considered in this paper, such flat directions do not arise, so this issue is moot.) For flat directions, there may also be nonperturbative effects such as instantons that lift the degeneracy but which we have not investigated presently.

For $SO(N)$, there will be a minimum value of N consistent with the existence of a UVFP, and this minimum value of N is generically a monotonically increasing function of b_g. For this reason, CEL set $b_g = 0$ in order to obtain the minimum of N consistent with a UVFP. However, this reasoning results in incorrect results when we consider that, in fact, b_g changes by discrete finite steps obtained by varying the fermion representations of the model.

If we assume a fermion content consisting of n_F fundamental (N-dimensional) two-component (or Majorana) representations, then

$$b_g = \frac{21N - 43 - 4n_F}{6}. \tag{2.7}$$

Note that for AF we require $N > 2$. The minimum values of b_g are obtained by taking n_F as large as possible consistent with $b_g > 0$. These minima, $b_g^{\min}$, are shown in Table I.

Note that in the case $N = 3$ (mod 4), it in fact *is* possible to have $b_g = 0$. However, in that case the two-loop

correction to β_g is necessarily positive in the absence of Yukawa couplings [16] (which we have been ignoring throughout) and therefore this case fails to be AF.

With $b_g = 0$, the minimum value of N such that a UVFP results is $N = 10$. However, this is *not* sustained when the actual value $b_g = 1/2$ is used from Table I. For $b_g \neq 0$, the minimum value of N for a UVFP is $N = 12$. With $N = 12$ and $b_g = 1/6$, we then find such a FP with

$$x_1 = 0.262953, \qquad x_2 = 0.111668, \qquad x_3 = 0.376914,$$
$$x_4 = 0.104270, \qquad x_5 = 0.581883. \tag{2.8}$$

III. THE $SU(N)$ MODEL

In this case we have the scalar potential

$$V(\Phi, \chi) = \frac{1}{2}\lambda_1 (\text{Tr}\Phi^2)^2 + \lambda_2 \text{Tr}\Phi^4 + \frac{1}{2}\lambda_3 (\chi_i^\dagger \chi^i)^2$$
$$+ \lambda_4 \chi_i^\dagger \chi^i \text{Tr}\Phi^2 + \lambda_5 \chi_i^\dagger \Phi^i{}_k \Phi^k{}_j \chi^j. \tag{3.1}$$

Again $\Phi = R^a \phi^a$, where now $a = 1, 2 \ldots N^2 - 1$. $\chi^i [i = 1, 2 \ldots N]$ is now a *complex* multiplet in the defining (fundamental) representation, and R^a are no longer (all) antisymmetric; they are again normalized as usual so that

$$\text{Tr} R^a R^b = \frac{1}{2}. \tag{3.2}$$

Thus, $\text{Tr}[\Phi^2] = \phi^a \phi^a / 2$.

As indicated in our Introduction (Sec. I), this model was examined in Chap. 9 of Ref. [12], with β-functions given in Eq. (9.26) in a slightly different notation. We have however checked that their flat-space results are in agreement with ours below.[3] Our gravitational corrections differ from theirs, but we shall discuss these elsewhere [13].

Comparing with the corresponding expression in CEL, on the face of it the definition of the λ_4 terms differ by a factor of 4. However, in comparing results for the β-functions, it seems clear that CEL have used *our* definition above in the actual calculations. Nevertheless, there still remain significant differences in the results. Ours are as follows:

TABLE I. Minimum value of b_g in the class of $SO(N)$ models.

N	3 (mod 4)	4 (mod 4)	5 (mod 4)	6 (mod 4)
$b_g^{\min}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

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[3]Reference [12] does in fact have an error, presumably inadvertent, in their formula for β_{f_3}, in which the coefficient of $g^2 f_3$ should be $3(3N^2 - 1)/N$, the same as given in their formula for β_{f_4}.

$$\beta_{g^2} = -b_g(g^2)^2, \qquad b_g \equiv \frac{21N-1}{3} - \frac{4}{3}T_F,$$

$$\beta_{\lambda_1} = (N^2+7)\lambda_1^2 + \frac{4(2N^2-3)}{N}\lambda_1\lambda_2 + \frac{12(N^2+3)}{N^2}\lambda_2^2$$
$$+ 2N\lambda_4^2 + 4\lambda_4\lambda_5 - 12Ng^2\lambda_1 + 18g^4,$$

$$\beta_{\lambda_2} = \frac{4(N^2-9)}{N}\lambda_2^2 + 12\lambda_1\lambda_2 + \lambda_5^2 - 12Ng^2\lambda_2 + 3Ng^4,$$

$$\beta_{\lambda_3} = 2(N+4)\lambda_3^2 + (N^2-1)\lambda_4^2 + \frac{(N-1)(N^2+2N-2)}{2N^2}\lambda_5^2$$
$$+ \frac{2(N^2-1)}{N}\lambda_4\lambda_5 - \frac{6(N^2-1)}{N}g^2\lambda_3$$
$$+ \frac{3(N-1)(N^2+2N-2)}{2N^2}g^4,$$

$$\beta_{\lambda_4} = 4\lambda_4^2 + \lambda_4\left[(N^2+1)\lambda_1 + \frac{2(2N^2-3)}{N}\lambda_2 + 2(N+1)\lambda_3\right]$$
$$+ \lambda_5^2 + \lambda_5\left[\frac{N^2-1}{N}\lambda_1 + \frac{2(N^2+3)}{N^2}\lambda_2 + 2\lambda_3\right]$$
$$- \frac{3(3N^2-1)}{N}g^2\lambda_4 + 3g^4,$$

$$\beta_{\lambda_5} = \frac{N^2-4}{N}\lambda_5^2 + \lambda_5\left[2\lambda_1 + \frac{2(N^2-6)}{N}\lambda_2 + 2\lambda_3 + 8\lambda_4\right.$$
$$\left. - \frac{3(3N^2-1)}{N}g^2\right] + 3Ng^4. \tag{3.3}$$

Assuming, as indicated above, that CEL actually used our definition of λ_4, we disagree with them only in the coefficients of the following terms:

$$\beta_{\lambda_4}: g^4; \qquad \beta_{\lambda_5}: \lambda_4\lambda_5, g^4. \tag{3.4}$$

As before, the form for b_g above in Eq. (3.5) assumes that the fermions are two-component (or Majorana). For example, if we have an arbitrary number n_F of fermions in the N-dimensional representation, then $T_F = 1/2$ and

$$b_g = \frac{21N-1}{3} - \frac{2n_F}{3}. \tag{3.5}$$

However the N-dimensional representation of $SU(N)$ gives nonzero triangle anomalies for $N \geq 3$, so, in that case, n_F above is necessarily even. Using Eq. (3.5), the results for $b_g^{\min}$ are shown in Table II. (One can achieve $b_g = 0$ in the case $N = 5 \pmod 4$, but we eschew this as before because of the effect of two-loop corrections.)

TABLE II.   Minimum value of b_g in the class of $SU(N)$ models.

N	2 (mod 4)	3 (mod 4)	4 (mod 4)	5 (mod 4)
$b_g^{\min}$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$

The corresponding reduced β-functions ($x_i \equiv \lambda_i/g^2$) are

$$\bar{\beta}_{x_1} = (N^2+7)x_1^2 + \frac{4(2N^2-3)}{N}x_1x_2 + \frac{12(N^2+3)}{N^2}x_2^2$$
$$+ 2Nx_4^2 + 4x_4x_5 + (b_g - 12N)x_1 + 18,$$

$$\bar{\beta}_{x_2} = \frac{4(N^2-9)}{N}x_2^2 + 12x_1x_2 + x_5^2 + (b_g - 12N)x_2 + 3N,$$

$$\bar{\beta}_{x_3} = 2(N+4)x_3^2 + (N^2-1)x_4^2 + \frac{(N-1)(N^2+2N-2)}{2N^2}x_5^2$$
$$+ \frac{2(N^2-1)}{N}x_4x_5$$
$$+ \left(b_g - \frac{6(N^2-1)}{N}\right)x_3 + \frac{3(N-1)(N^2+2N-2)}{2N^2},$$

$$\bar{\beta}_{x_4} = 4x_4^2 + x_4\left[(N^2+1)x_1 + \frac{2(2N^2-3)}{N}x_2 + 2(N+1)x_3\right]$$
$$+ x_5^2 + x_5\left[\frac{N^2-1}{N}x_1 + \frac{2(N^2+3)}{N^2}x_2 + 2x_3\right]$$
$$+ \left(b_g - \frac{3(3N^2-1)}{N}\right)x_4 + 3,$$

$$\bar{\beta}_{x_5} = \frac{N^2-4}{N}x_5^2 + x_5\left[2x_1 + \frac{2(N^2-6)}{N}x_2 + 2x_3 + 8x_4\right.$$
$$\left. + \left(b_g - \frac{3(3N^2-1)}{N}\right)\right] + 3N. \tag{3.6}$$

For this model, using the approximation $b_g = 0$, the smallest value of N required to have all couplings AF was given as $N_{\min} = 7$ in Ref. [3], using incorrect β-functions, and as $N_{\min} = 8$ in Ref. [12], using the same β-functions as ours. For $N = 8$, the actual minimum value is $b_g^{\min} = 1$, for which we find the model is *not* AF. For $N = 9$, we have $b_g^{\min} = 4/3$, for which the model is AF with its UVFP at

$$x_1 = 0.386000, \qquad x_2 = 0.293121, \qquad x_3 = 0.502429,$$
$$x_4 = 0.195158, \qquad x_5 = 0.398832. \tag{3.7}$$

IV. THE LARGE N LIMIT

Let us consider the large N limit of this class of theories. Of course, as shown many years ago by 't Hooft [17] for $SU(N)$, the relevant graphs in the large N limit are *planar*; summing these graphs to obtain the full leading N approximation has proved elusive, even for the pure

Yang-Mills theory, and despite the fact that there must exist a classical master equation [18]. Consequently, to salvage perturbative believability, our results will still require the relevant couplings to be small. Nevertheless, the results have features of interest.

Let us begin by considering the $SU(N)$ case. [The results for $SO(N)$ turn out to be essentially the same and will be given below.] Because the gauge contribution to b_g naturally grows as N, $\tilde{b}_g \equiv b_g/N$ remains finite as $N \to \infty$. Then, as 't Hooft showed [17], defining a rescaled gauge coupling $\tilde{g}^2 \equiv Ng^2$, its β-function satisfies

$$\beta_{\tilde{g}^2} = -\tilde{b}_g(\tilde{g}^2)^2. \tag{4.1}$$

Thus, in the limit $N \to \infty$, $g \to 0$ for fixed $\tilde{g}^2$, $\beta_{\tilde{g}^2}$ remains finite. Similarly, if we rescale the couplings λ_i in a certain way, the resulting $\beta_{\tilde{\lambda}_i}$ will have finite limits in terms of rescaled couplings $\tilde{\lambda}_i$. This requires

$$\lambda_1 = \tilde{\lambda}_1/N^2, \qquad \lambda_2 = \tilde{\lambda}_2/N, \qquad \lambda_3 = \tilde{\lambda}_3/N,$$
$$\lambda_4 = \tilde{\lambda}_4/N^{p_4}, \qquad \lambda_5 = \tilde{\lambda}_5/N, \tag{4.2}$$

for $3/2 \le p_4 \le 2$. This ambiguity in the rescaling of λ_4 reflects a nonuniformity of the limiting behavior. For $3/2 < p_4 < 2$, all dependence on $\tilde{\lambda}_4$ drops out except in $\beta_{\tilde{\lambda}_4}$, and we find

$$\beta_{\tilde{\lambda}_1} = \tilde{\lambda}_1^2 + 8\tilde{\lambda}_1\tilde{\lambda}_2 + 12\tilde{\lambda}_2^2 + 18\tilde{g}^4 - 12\tilde{g}^2\tilde{\lambda}_1,$$
$$\beta_{\tilde{\lambda}_2} = 4\tilde{\lambda}_2^2 + 3\tilde{g}^4 - 12\tilde{g}^2\tilde{\lambda}_2,$$
$$\beta_{\tilde{\lambda}_3} = 2\tilde{\lambda}_3^2 + \frac{1}{2}\tilde{\lambda}_5^2 + \frac{3}{2}\tilde{g}^4 - 6\tilde{g}^2\tilde{\lambda}_3,$$
$$\beta_{\tilde{\lambda}_4} = \tilde{\lambda}_4(\tilde{\lambda}_1 + 4\tilde{\lambda}_2 + 2\tilde{\lambda}_3 - 9\tilde{g}^2),$$
$$\beta_{\tilde{\lambda}_5} = 2\tilde{\lambda}_2\tilde{\lambda}_5 + \tilde{\lambda}_5^2 + 3\tilde{g}^4 - 9\tilde{g}^2\tilde{\lambda}_5. \tag{4.3}$$

Inasmuch as $\beta_{\tilde{\lambda}_4}$ is linear in $\tilde{\lambda}_4$, it differs from the others and from the finite N, Eq. (3.3), β-functions. Consequently, it has a FP at $\lambda_4 = 0$, independent of the values of the other couplings. It turns out that, when one forms the reduced β-functions in terms of the ratios $\tilde{y}_i \equiv \tilde{\lambda}_i/\tilde{g}^2$, $y_4 = 0$ is in fact a UVFP for Eq. (4.3).

At the extreme values, $p_4 = 3/2$ or $p_4 = 2$, other terms survive. In the case, $p_4 = 3/2$, there are quadratic terms in $\tilde{\lambda}_4$ that survive in $\beta_{\tilde{\lambda}_1}$ and $\beta_{\tilde{\lambda}_3}$, to wit,

$$\beta_{\tilde{\lambda}_1} = \tilde{\lambda}_1^2 + 8\tilde{\lambda}_1\tilde{\lambda}_2 + 12\tilde{\lambda}_2^2 + 2\tilde{\lambda}_4^2 + 18\tilde{g}^4 - 12\tilde{g}^2\tilde{\lambda}_1,$$
$$\beta_{\tilde{\lambda}_3} = 2\tilde{\lambda}_3^2 + \frac{1}{2}\tilde{\lambda}_5^2 + \tilde{\lambda}_4^2 + \frac{3}{2}\tilde{g}^4 - 6\tilde{g}^2\tilde{\lambda}_3. \tag{4.4}$$

The remaining three β-functions are the same as in Eq. (4.3). It turns out that the UVFP remains at $\lambda_4 = 0$

in this case, so the presence of these additional terms does not change the values of the UVFP from the case $3/2 < p_4 < 2$, Eq. (4.3). They will however affect the running of the couplings away from the FP.

For $p_4 = 2$, all $\beta_{\tilde{\lambda}_i}$ for $i \ne 4$, are unchanged, whereas $\beta_{\tilde{\lambda}_4}$ becomes

$$\beta_{\tilde{\lambda}_4} = \tilde{\lambda}_5(\tilde{\lambda}_1 + 2\tilde{\lambda}_2 + 2\tilde{\lambda}_3) + \tilde{\lambda}_5^2 + 3\tilde{g}^4$$
$$+ \tilde{\lambda}_4(\tilde{\lambda}_1 + 4\tilde{\lambda}_2 + 2\tilde{\lambda}_3 - 9\tilde{g}^2). \tag{4.5}$$

In fact, this, together with the other β-functions from Eq. (4.3), are an excellent approximation to the large-N behavior of the exact equations, Eq. (3.3). The other choices for p_4 do not appear to be physically relevant.

For $p_4 = 2$, the reduced β-functions in terms of $\tilde{y}_i \equiv \tilde{\lambda}_i/\tilde{g}^2$, are

$$\bar{\beta}_{\tilde{y}_1} = \tilde{y}_1^2 + 12\tilde{y}_2^2 + 18 - (12 - \tilde{b}_g - 8\tilde{y}_2)\tilde{y}_1,$$
$$\bar{\beta}_{\tilde{y}_2} = 4\tilde{y}_2^2 + 3 - (12 - \tilde{b}_g)\tilde{y}_2,$$
$$\bar{\beta}_{\tilde{y}_3} = 2\tilde{y}_3^2 + \frac{1}{2}\tilde{y}_5^2 + \frac{3}{2} - (6 - \tilde{b}_g)\tilde{y}_3,$$
$$\bar{\beta}_{\tilde{y}_4} = \tilde{y}_5(\tilde{y}_1 + 2\tilde{y}_2 + 2\tilde{y}_3) + \tilde{y}_5^2 + 3$$
$$+ \tilde{y}_4(\tilde{y}_1 + 4\tilde{y}_2 + 2\tilde{y}_3 - (9 - \tilde{b}_g)),$$
$$\bar{\beta}_{\tilde{y}_5} = \tilde{y}_5^2 + 3 - (9 - \tilde{b}_g - 2\tilde{y}_2)\tilde{y}_5. \tag{4.6}$$

Solving simultaneously the equations $\bar{\beta}_{\tilde{y}_i} = 0$, we find several FPs, one of which is UV stable. The values of this UVFP for various values of $\tilde{b}_g$ are given in Table III. For $\tilde{b}_g \gtrsim 0.84798$, there are no real FPs.

The results for $SO(N)$ are precisely analogous to those above. Given the definitions of the self-couplings λ_i in Eq. (2.1), the β-functions for the gauge and self-couplings are

TABLE III. UVFPs for $SU(\infty)$.

$\tilde{b}_g$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$	$\tilde{y}_5$
0.	2.64270	0.275255	0.289413	0.970346	0.371374
1/3	2.94605	0.284989	0.312552	1.20422	0.389234
1/2	3.15683	0.290153	0.325788	1.39047	0.398894
3/4	3.67495	0.298306	0.348280	1.94791	0.414424
0.84798	4.36728	0.301646	0.358128	2.99190	0.420885

$$\beta_{\tilde{g}^2} = -\tilde{b}_g \tilde{g}^4,$$

$$\beta_{\tilde{\lambda}_1} = \frac{1}{2}\tilde{\lambda}_1^2 + 6\tilde{\lambda}_2^2 + 9\tilde{g}^4 + (4\tilde{\lambda}_2 - 6\tilde{g}^2)\tilde{\lambda}_1,$$

$$\beta_{\tilde{\lambda}_2} = 2\tilde{\lambda}_2^2 + \frac{3}{2}\tilde{g}^4 - 6\tilde{g}^2\tilde{\lambda}_2,$$

$$\beta_{\tilde{\lambda}_3} = \tilde{\lambda}_3^2 + \frac{1}{16}\tilde{\lambda}_5^2 + \frac{3}{4}\tilde{g}^4 - 3\tilde{g}^2\tilde{\lambda}_3,$$

$$\beta_{\tilde{\lambda}_4} = \tilde{\lambda}_5\left[\frac{1}{4}\tilde{\lambda}_1 + \frac{1}{2}\tilde{\lambda}_2 + \frac{1}{2}\tilde{\lambda}_3\right] + \frac{1}{8}\tilde{\lambda}_5^2 + \frac{3}{2}\tilde{g}^4$$
$$+ \tilde{\lambda}_4\left[\frac{1}{2}\tilde{\lambda}_1 + 2\tilde{\lambda}_2 + \tilde{\lambda}_3 - \frac{9}{2}\tilde{g}^2\right],$$

$$\beta_{\tilde{\lambda}_5} = \frac{1}{4}\tilde{\lambda}_5^2 + 3\tilde{g}^4 + \left(\tilde{\lambda}_2 - \frac{9}{2}\tilde{g}^2\right)\tilde{\lambda}_5. \tag{4.7}$$

The $\tilde{\lambda}_i$ above are defined as in Eq. (4.2), with $p_4 = 2$.

Defining once again, $\tilde{y}_i \equiv \tilde{\lambda}_i/\tilde{g}^2$, the reduced β-functions are

$$\bar{\beta}_{\tilde{y}_1} = \frac{1}{2}\tilde{y}_1^2 + 6\tilde{y}_2^2 + 9 + (4\tilde{y}_2 + \tilde{b}_g - 6)\tilde{y}_1,$$

$$\bar{\beta}_{\tilde{y}_2} = 2\tilde{y}_2^2 + \frac{3}{2} + (\tilde{b}_g - 6)\tilde{y}_2,$$

$$\bar{\beta}_{\tilde{y}_3} = \tilde{y}_3^2 + \frac{1}{16}\tilde{y}_5^2 + \frac{3}{4} + (\tilde{b}_g - 3)\tilde{y}_3,$$

$$\bar{\beta}_{\tilde{y}_4} = \tilde{y}_5\left[\frac{1}{4}\tilde{y}_1 + \frac{1}{2}\tilde{y}_2 + \frac{1}{2}\tilde{y}_3\right] + \frac{1}{8}\tilde{y}_5^2 + \frac{3}{2}$$
$$+ \tilde{y}_4\left[\frac{1}{2}\tilde{y}_1 + 2\tilde{y}_2 + \tilde{y}_3 + \tilde{b}_g - \frac{9}{2}\right],$$

$$\bar{\beta}_{\tilde{y}_5} = \frac{1}{4}\tilde{y}_5^2 + 3 + \tilde{y}_5\left(\tilde{y}_2 + \tilde{b}_g - \frac{9}{2}\right). \tag{4.8}$$

As with $SU(N)$, we find several FPs, of which one is UV stable. The values of this UVFP for various values of $\tilde{b}_g$ are given in Table IV. For $\tilde{b}_g \gtrsim 0.42399$, there are no real FPs.

A cursory comparison of Tables III and IV indicates that many of the rows for the UVFP $\tilde{y}_n$ are approximately the same provided, in Table IV, one doubles $\tilde{b}_g$ and halves $\tilde{y}_5$. Most entries then agree at least in their first two significant figures. This comes about because the leading term in b_g is proportional to $C(G)$, which, for $SO(N)$, is $N/2$, half that of $SU(N)$. To understand the factor of 2 in $\tilde{y}_5$, we must compare the normalization of λ_5 in the potentials, Eqs. (2.1) and (3.1). Recalling that χ_i is complex for $SU(N)$ and real for $SO(N)$, we would anticipate the couplings might correspond at large N if λ_5 were replaced by $\lambda_5/2$ in the potential for $SU(N)$.

On the other hand, if, as with $SO(10)$, one were to add a fermion in the smallest spinor representation of $SO(2n)$, for which $T(R) = 2^{(n-4)}$, then obviously the condition that $b_g > 0$ will be violated at some finite n. (In fact, one must have $n \leq 10$.) Thus, there would be no large-N scaling limit in such a case.

TABLE IV.   UVFPs for $SO(\infty)$.

$\tilde{b}_g$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$	$\tilde{y}_5$
0.	2.64270	0.284989	0.310944	1.17978	0.710102
1/6	2.94605	0.290153	0.325788	1.39047	0.741044
1/3	3.45350	0.295531	0.338224	1.64814	0.774966
5/12	4.08657	0.301141	0.354074	2.44429	0.793191
0.42399	4.36728	0.301646	0.355550	2.90078	0.794836

Equations (4.6) and (4.8) are sufficiently simple to be solvable analytically (as functions of $\tilde{b}_g$) for the FPs of the β-functions, in particular, for the UVFP. This is described in Appendix A. In practice, it is actually easier simply to solve for the FPs numerically. Knowing from the preceding which of the FPs is the candidate UVFP, one can easily check whether the eigenvalues of the stability matrix S_{ij} are all negative. In fact, since the UVFP occurs for positive $\tilde{y}_i$, we can be confident that it is unique [19].[4]

With reference to the first rows of Tables III and IV, it is clear that for large but finite N, $\tilde{b}_g$ is very small. One ought to wonder whether the two-loop corrections to the β-functions might not be equally large in certain cases. Such a possibility has been examined in the past [20,21] and leads to the idea that there may be a finite IRFP in g^2, a so-called CBZ FP. We elaborate on this possibility in Appendix B.

V. CONCLUSIONS

We have presented the flat space one-loop β-functions for both $SU(N)$ and $SO(N)$ gauge theories coupled to scalar multiplets in both the adjoint and fundamental representations. Both cases were originally studied in CEL; our results differ from theirs in a number of terms, as do our conclusions regarding the minimum values of N consistent with TAF, i.e., asymptotic freedom of all the couplings. In the $SU(N)$ case, our results for the β-functions agree with those presented in BOS (though not so, as we shall discuss elsewhere [13], when extended to renormalizable gravity). Instead of simply approximating the minimum allowed value of $b_g > 0$ by zero, we paid particular attention to the actual minimum for an essentially arbitrary choice of fermion representations (Tables I and II), except for spinor representations, for which there is no large N scaling limit that is still TAF.

One interesting result in the case of $SO(N)$ is that the smallest allowed value of N is greater than $N = 10$ (as it is for $b_g = 0$) when the actual $b_g^{\min} = 1/2$. The minimum may go even higher than $N = 12$ when additional scalars are included in order to have appropriate Yukawa couplings

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[4]The example given in Ref. [19] unfortunately uses the β-functions of Ref. [3] for the model we have treated here. As we have stated, some of those β-functions are incorrect, but, in their application, the qualitative conclusions of Ref. [19] remain unchanged.

to accommodate the SM fermion spectrum and to incorporate electroweak symmetry breaking.

For $SU(N)$, we found that the smallest value of N for which all couplings are AF is $N_{\min} = 9$, for which $b_g^{\min} = 4/3$. This is to be compared with $N_{\min} = 7$ in Ref. [3], using incorrect β-functions, and $N_{\min} = 8$ in Ref. [12], using correct β-functions but taking $b_g = 0$.

We also discussed the large N limit in both theories, with couplings appropriately rescaled so as to render the β-function coefficients finite. One result there is that there is an allowed *maximum* value of b_g for large N beyond which there is no real UVFP. It is about $0.85N$ for $SO(N)$ and $0.42N$ for $SU(N)$, so the allowed range of choices for the fermion representations is not nearly so restrictive as suggested by choosing N to be as small as permitted, and it may become much easier to accommodate the three generations of fermions in the SM. These results are, we believe, novel and interesting.

These calculations constitute part of our efforts to develop a UV complete, TAF theory coupled to renormalizable, scale-invariant gravity that is realistic, i.e., one that leads to the Standard Model plus Einstein-Hilbert gravity at low energies. We plan to extend our results here to incorporate gravitational couplings and to explore whether dimensional transmutation can generate both gauge symmetry breaking and a Planck mass term, along the lines of Ref. [10]. Then, for a realistic model, other scalar representations and the effect of Yukawa couplings must be considered. We showed in Ref. [10] how breaking of $SO(10)$ to $SU(5) \times U(1)$ can occur in a scale invariant model; one outstanding problem is how further breaking may be engineered, eventually to the Standard Model gauge group. The results in this paper suggest that it will require $N_{\min} \geq 12$ for $SO(N)$ and $N_{\min} \geq 9$ for $SU(N)$, and these minimum values may be even larger after adding additional scalars needed to account for fermion masses and to break down to the SM gauge symmetries. Renormalizable gravity makes relatively small changes to the flat space results near the UVFP, but there remains the issue of unitarity in such theories.

ACKNOWLEDGMENTS

APPENDIX A: ANALYTIC SOLUTIONS FOR THE LARGE-N FIXED POINTS

As mentioned in the text, Eqs. (4.6) and (4.8) are sufficiently simple that, given $\tilde{b}_N$, their FPs can be analytically determined. Although all FPs may be so determined, we shall focus on finding the UVFP.

Consider first the $SU(N)$ case, Eq. (4.6). Note that $\bar{\beta}_{\tilde{y}_2}$ is a function of $\tilde{y}_2$ only. It will have real zeros if and only if the discriminant of the quadratic is positive:

$$(6 - \tilde{b}_g/2)^2 - 12 > 0. \tag{A1}$$

Assuming $12 - \tilde{b}_g > 0$, then the two FPs occur for $\tilde{y}_2 > 0$, and it is easy to see that the smaller is the UVFP. We can input this value of $\tilde{y}_2$ into the other four β-functions to search for a UVFP in the other $\tilde{y}_n$. This enables us to solve explictly for the FPs in $\tilde{y}_1$ from $\bar{\beta}_{\tilde{y}_1} = 0$ and for $\tilde{y}_5$ from $\bar{\beta}_{\tilde{y}_5} = 0$, along with further constraints on $\tilde{b}_g$ arising from requiring the equations to have real roots. In each case, we can choose the root of the quadratic equation having negative slope for fixed values of the other $\tilde{y}_n$, giving us further candidates for the UVFP. Given $\tilde{y}_5$, we can then solve for $\tilde{y}_3$ from $\bar{\beta}_{\tilde{y}_3} = 0$, and choose the smaller root once again. So now we have candidate values for $\tilde{y}_k$, $\{k = 1, 2, 3, 5\}$. Finally, $\bar{\beta}_{\tilde{y}_4}$ is linear in $\tilde{y}_4$, so it has a unique root that can be expressed in terms of the solutions for the other $\tilde{y}_n$. In principle, it could be positive or negative, but it is a UVFP only if the coefficient is negative, i.e., only for

$$\tilde{y}_1 + 4\tilde{y}_2 + 2\tilde{y}_3 < 9 - \tilde{b}_g. \tag{A2}$$

Thus, the root for $\tilde{y}_4$ is also positive. Since each of the UVFPs is known as a function of $\tilde{b}_g$, this inequality may further restrict the range of $\tilde{b}_g$ within which there are real solutions for all the UVFPs. (See Table III.)

Thus we arrive at a unique candidate for the UVFP, within a restricted range of $\tilde{b}_g$. We cannot immediately conclude that this is a UVFP because the stability matrix $S_{m,n} \equiv \partial \bar{\beta}_{\tilde{y}_m}/\partial \tilde{y}_n$ at a FP has nonzero off-diagonal terms (except in the case of $\bar{\beta}_{\tilde{y}_2}$.) In the preceding, we only took into account the signs of the diagonal entries in each case. One must verify that the true eigenvalues at the putative UVFP have the signs of the diagonal entries. In fact, they do.

The solution for the $SO(N)$ case, Eq. (4.8) can be obtained in precisely the same manner. The only changes are in the numerical coefficients of the couplings.

APPENDIX B: THE CBZ INFRARED FIXED POINT

While $b_g > 0$ is required for AF of the gauge coupling, to obtain AF for the quartic scalar couplings as well it is optimal to employ the smallest possible value of b_g. This suggests the possible existence of a CBZ [20,21] infrared stable fixed point (IRFP); in other words, the basin of attraction of the UVFP at $g^2 = 0$ is finite.[5] Writing

$$\beta_{g^2} = -\frac{b_g}{16\pi^2} g^4 + 2 \frac{B}{(16\pi^2)^2} g^6, \tag{B1}$$

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[5]We thank a referee for a suggestion that inspired the following remarks.

we have in general (in the absence of Yukawa couplings) that

$$b_g = 2\left(\frac{11}{3}C_G - \frac{2}{3}T_F - \frac{1}{6}T_S\right) \tag{B2}$$

and

$$B = \frac{10}{3}C_G T_F + 2\sum C_{F_\alpha}T_{F_\alpha} + 2\sum C_{S_\beta}T_{S_\beta} + \frac{1}{3}C_G T_S - \frac{34}{3}C_G^2. \tag{B3}$$

Here $T_F = \sum T_{F_\alpha}$ and $T_S = \sum T_{S_\beta}$ where we label the irreducible fermion and scalar representations by α, β respectively.

It was first noted by Caswell [20] that, in a gauge theory with fermions (but no scalars), for $b_g = 0$, $B > 0$. It follows that for $b_g > 0$ but sufficiently small, there exists a perturbatively believable IRFP corresponding to

$$\frac{g_{\text{IR}}^2}{16\pi^2} = \frac{b_g}{2B}. \tag{B4}$$

In the case of a gauge theory with scalars (but no fermions) or with both scalars and fermions the corresponding result is less obvious, but a detailed examination of the possible quadratic Casimir operators confirms that the same result holds in these cases, too [22].

Given the proximity of the IRFP to the origin, it is clear that there is only a limited range of values, $0 < g < g_{\text{IR}}$, of g at some reference scale (the grand unified theory scale for instance), corresponding to AF. For $g > g_{\text{IR}}$, then g approaches a Landau pole in the UV, i.e., perturbation theory breaks down. In particular, at large N, for either $SO(N)$ or $SU(N)$, it is easy to see that $B \to kN^2$, where k is a constant. In the large N limit, we define $\tilde{b}_g \equiv b_g/N$, $\tilde{B} \equiv B/N^2$, and $\tilde{g}^2 \equiv Ng^2$, as in Sec. IV. Then

$$\frac{\tilde{g}_{\text{IR}}^2}{16\pi^2} = \frac{\tilde{b}_g}{2\tilde{B}}. \tag{B5}$$

It is thus clear that for very small $\tilde{b}_g$, corresponding to the first rows of Tables III and IV, the range of $\tilde{g}$ corresponding to AF is actually very limited. This may constrain model building involving renormalizable quantum gravity of the kind envisaged in Ref. [10], where it was important that the region of coupling constant space corresponding to dimensional transmutation and spontaneous symmetry breaking lay within the basin of attraction of the UVFP of coupling constant ratios corresponding to AF of all couplings. Conversely, should the IRFP of the gauge coupling be approached in the IR, the resulting theory would probably become strongly coupled, because the gravitational self-couplings increase in the IR. Then we would expect a QCD-type phase transition before the gauge coupling reaches its IRFP, unless all the other couplings also displayed CBZ behavior in the IR limit.

[1] D. J. Gross and F. Wilczek, Ultraviolet Behavior of Non-Abelian Gauge Theories, Phys. Rev. Lett. **30**, 1343 (1973).

[2] H. D. Politzer, Reliable Perturbative Results for Strong Interactions?, Phys. Rev. Lett. **30**, 1346 (1973).

[3] T. P. Cheng, E. Eichten, and L. F. Li, Higgs phenomena in asymptotically free gauge theories, Phys. Rev. D **9**, 2259 (1974).

[4] G. F. Giudice, G. Isidori, A. Salvio, and A. Strumia, Softened gravity and the extension of the standard model up to infinite energy, J. High Energy Phys. 02 (2015) 137.

[5] B. Holdom, J. Ren, and C. Zhang, Stable asymptotically free extensions (SAFES) of the standard model, J. High Energy Phys. 03 (2015) 028.

[6] G. M. Pelaggi, A. Strumia, and S. Vignali, Totally asymptotically free trinification, J. High Energy Phys. 08 (2015) 130.

[7] M. B. Einhorn and D. R. T. Jones, Naturalness and dimensional transmutation in classically scale-invariant gravity, J. High Energy Phys. 03 (2015) 047.

[8] M. B. Einhorn and D. R. T. Jones, Gauss-Bonnet coupling constant in classically scale-invariant gravity, Phys. Rev. D **91**, 084039 (2015).

[9] M. B. Einhorn and D. R. T. Jones, Induced gravity I: real scalar field, J. High Energy Phys. 01 (2016) 019.

[10] M. B. Einhorn and D. R. T. Jones, Induced gravity II: grand unification, J. High Energy Phys. 05 (2016) 185.

[11] S. R. Coleman and E. J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, Phys. Rev. D **7**, 1888 (1973).

[12] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (IOP Publishing, Bristol, 1992).

[13] M. B. Einhorn and D. R. T. Jones, One-loop corrections for $SO(N)$ and $SU(N)$ models in renormalizable gravity (to be published).

[14] M. B. Einhorn and D. R. T. Jones, Scale fixing by dimensional transmutation: Supersymmetric unified models and the renormalization group, Nucl. Phys. **B211**, 29 (1983).

[15] C. Ford, I. Jack, and D. R. T. Jones, The standard model effective potential at two loops, Nucl. Phys. **B387**, 373 (1992); C. Ford, I. Jack, and D. R. T. Jones, Erratum, Nucl. Phys. **B504**, 551 (1997).

[16] D. R. T. Jones, Asymptotic behavior of supersymmetric Yang-Mills theories in the two loop approximation, Nucl. Phys. **B87**, 127 (1975).

[17] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. **B72**, 461 (1974).

[18] E. Witten, The $1/N$ expansion in atomic and particle physics, NATO Sci. Ser. B **59**, 403 (1980).

[19] F. A. Bais and H. A. Weldon, Determining whether the Higgs self-couplings spoil asymptotic freedom, Phys. Rev. D **18**, 1199 (1978).

[20] W. E. Caswell, Asymptotic Behavior of Non-Abelian Gauge Theories to Two Loop Order, Phys. Rev. Lett. **33**, 244 (1974); D. R. T. Jones, Two loop diagrams in Yang-Mills theory, Conf. Proc. C **7406241**, 68 (1974).

[21] T. Banks and A. Zaks, On the phase structure of vector-like gauge theories with massless fermions, Nucl. Phys. **B196**, 189 (1982).

[22] A. D. Bond and D. F. Litim, Theorems for asymptotic safety of gauge theories, Eur. Phys. J. C **77**, 429 (2017); A. D. Bond and D. F. Litim, Erratum, Eur. Phys. J. C **77**, 525(E) (2017).