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Fermion zero mode associated with instantonlike self-dual solution to lattice Euclidean gravity

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We prove the existence of the lattice fermion zero mode associated with the self-dual lattice gravity solution.

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I. INTRODUCTION

It is known that in continuum relativistic gauge theories coupled with fermions some of the currents conserved in classical mechanics become nonconserved ones in quantum mechanics due to vacuum quantum fluctuations; the divergences of some currents are equal to a certain local functions of the gauge field called "anomaly," which are generally not zero.

Consider, for example, four-dimensional (4D) Euclidean Yang-Mills theory. The following formulas are well known:

$$\partial_\mu(i\Psi^\dagger\gamma^\mu\gamma^5\Psi) = -2\sum_{N:|\epsilon_N|<\Lambda\to\infty}\Psi_N^\dagger\gamma^5\Psi_N = \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\lambda\rho}\mathrm{tr}F_{\mu\nu}F_{\lambda\rho}, \tag{1.1}$$

$$i\gamma^\mu\nabla_\mu\Psi_N = \epsilon_N\Psi_N, \quad \nabla_\mu = \partial_\mu + ieA_\mu, \qquad [\nabla_\mu,\nabla_\nu] = -ieF_{\mu\nu}.$$

Let us integrate the last equation in Eq. (1.1) over space. We obtain

$$\sum_{N_0:\epsilon_{N_0}=0}\int \mathrm{d}^{(4)}x\Psi_{N_0}^\dagger\gamma^5\Psi_{N_0} = -\frac{e^2}{32\pi^2}\varepsilon^{\mu\nu\lambda\rho}\int \mathrm{d}^{(4)}x\mathrm{tr}F_{\mu\nu}F_{\lambda\rho} = q \tag{1.2}$$

since the modes Ψ_N and $\gamma^5\Psi_N$ are mutually orthogonal for $\epsilon_N \neq 0$. Here, $q = 0, \pm 1, \ldots$ is a topological charge of the Yang-Mills field instanton. Now, the Atiyah-Singer index theorem is obtained if we substitute for γ^5 its decomposition $\gamma^5 \equiv (1/2)(1+\gamma^5) - (1/2)(1-\gamma^5)$ into the left-hand side of Eq. (1.2):

$$n_+ - n_- = q. \tag{1.3}$$

Here, n_+ (n_-) is the number of right (left) fermion zero modes associated with the instanton with the topological charge q.

It is seen from this consideration that the existence of fermion zero modes associated with the instanton in Yang-Mills theory is provided by the existence of anomaly in divergence of the corresponding fermion axial current (1.1).

But the problem of the anomalies and their connection with fermion zero modes in lattice gauge theories is qualitatively more complicated one (see Ref. [1]). Note that all lattice theories under consideration possess the common fundamental property: lattice theories transform into corresponding continuum relativistic theories at the naive long-wavelength limit.

Let us consider first the Yang-Mills instanton in a lattice theory. The configuration of the Yang-Mills instanton field is smooth at each region of space-time. This property is very important for the validity of the second equality in Eq. (1.1). Indeed, this equality is obtained correctly only for long-wavelength (as compared to fermion field wavelengths) gauge fields. Therefore, the second equality in Eq. (1.1) is valid in the lattice theories in the naive long-wavelength limit. This property of the Yang-Mills theory implies very important physical consequences. In particular, it follows from here that the irregular ultrashort (doubled) fermion quanta with low energy also exist [1] in addition to soft regular long-wavelength fermion quanta.

The gravity equivalent of the second equality in Eq. (1.1) fails since the gravitational instanton field configuration is singular near the center of the instanton (see Sec. III) in lattice gravity theory. Therefore, a proof of the existence of the lattice fermion zero mode associated with the instanton would be significantly different from the above exposed method. We use here the method that has been successful in solving lattice pure gravity self-dual equations with given boundary conditions [2] (lattice instanton).

The method makes it possible to establish the main result of the paper: a proof of the existence of the fermion zero mode associated with a lattice gravitational instanton. Here, we emphasize that the lattice approach developed in

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Ref. [2] and used here cannot be extended to usual continuum field theories since any finite space-time region contains an innumerable set of variables in such theories.

The following extremely important point is in order. The Eguchi-Hanson continuous solution (see Refs. [3,4]) is valid for the manifold M with a boundary as $r \to \infty$, which is the cotangent bundle of the complex plane, $P_1(\mathbb{C}) \approx S^2$:

$$M = T^*(P_1(\mathbb{C})),$$
$$\partial M = SO(3) = S^3/Z_2. \tag{1.4}$$

The manifold M is smooth. On the other hand, the discrete analog of the Eguchi-Hanson solution [2] and Dirac zero mode exist on a triangulation of manifold $\mathbb{R}^4$. which can be considered as S^3 of extra-large r including its interior. This triangulation is designated as $\mathfrak{K}'$ (see Sec. III), $\partial\mathfrak{K}' \approx S^3$. Evidently, the topologies of the manifold M and simplicial complex $\mathfrak{K}'$ are different.

The organization of the paper is as follows. In Secs. II and III, the early obtained results that are necessary here, the definition of lattice gravity theory and self-dual solution on the lattice, are shortly outlined. In Sec. IV, the asymptotic behavior on a long-wavelength limit of the fermion zero mode is studied. In Sec. V, the existence of the lattice fermion zero mode associated with the self-dual solution is proved with the help of the method used for the proof of existence lattice self-dual solution [2].

II. LATTICE GRAVITY MODEL

Let's introduce some designations:

$$\gamma^\alpha = \begin{pmatrix} 0 & i\sigma^\alpha \\ -i\sigma^\alpha & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha = 1,2,3,$$
$$\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b], \qquad a,b,\ldots = 1,2,3,4,$$
$$\sigma^{\alpha 4} = \frac{i}{2}\begin{pmatrix} \sigma^\alpha & 0 \\ 0 & -\sigma^\alpha \end{pmatrix}, \qquad \sigma^{\alpha\beta} = \frac{i\varepsilon_{\alpha\beta\gamma}}{2}\begin{pmatrix} \sigma^\gamma & 0 \\ 0 & \sigma^\gamma \end{pmatrix}, \tag{2.1}$$

$\sigma^\alpha$are Pauli matrices.

It is necessary to sketch out the model of lattice gravity, which is used here. A detailed description of the model is given in Refs. [1,5,6].

The orientable four-dimensional simplicial complex and its vertices are designated as $\mathfrak{K}$ and $a_\mathcal{V}$, and the indices $\mathcal{V} = 1,2,\ldots,\mathfrak{N} \to \infty$ and $\mathcal{W}$ enumerate the vertices and 4-simplices, correspondingly. We assume here that $\mathfrak{K} \approx \mathbb{R}^4$ in a topological sense. It is necessary to use the local enumeration of the vertices $a_\mathcal{V}$ attached to a given 4-simplex: all five vertices of a 4-simplex with index $\mathcal{W}$ are enumerated as $a_{\mathcal{W}i}$, $i = 1,2,3,4,5$. The later notations with extra index $\mathcal{W}$ indicate that the corresponding quantities belong to the 4-simplex with index $\mathcal{W}$. The Levi-Cività symbol within pairs different indices $\varepsilon_{\mathcal{W}ijklm} = \pm 1$ depending on whether the order of vertices $s^4_\mathcal{W} = a_{\mathcal{W}i}a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m}$ defines the positive or negative orientation of 4-simplex $s^4_\mathcal{W}$.

An element of the group Spin(4) and an element of the Clifford algebra,

$$\Omega_{\mathcal{W}ij} = \Omega^{-1}_{\mathcal{W}ji} = \exp(\omega_{\mathcal{W}ij}), \qquad \omega_{\mathcal{W}ij} \equiv \frac{1}{2}\sigma^{ab}\omega^{ab}_{\mathcal{W}ij},$$
$$\hat{e}_{\mathcal{W}ij} = \hat{e}^\dagger_{\mathcal{W}ij} \equiv e^a_{\mathcal{W}ij}\gamma^a \equiv -\Omega_{\mathcal{W}ij}\hat{e}_{\mathcal{W}ji}\Omega^{-1}_{\mathcal{W}ij}, \tag{2.2}$$

are assigned for each oriented 1-simplex $a_{\mathcal{W}i}a_{\mathcal{W}j}$. The Dirac spinors $\Psi_\mathcal{V}$ and $\Psi^\dagger_\mathcal{V}$, each of the components of which assumes values in a complex Grassman algebra, are assigned to each vertex $a_\mathcal{V}$. In the case of the Euclidean signature, the spinors $\Psi_\mathcal{V}$ and $\Psi^\dagger_\mathcal{V}$ are independent variables and are interchanged under the Hermitian conjugation.

Thus, the used representation realizes automatically the separation of a total gauge group into two subgroups: $\mathrm{Spin}(4) \approx \mathrm{Spin}(4)_{(+)} \otimes \mathrm{Spin}(4)_{(-)}$. For example,

$$\frac{1}{2}\sigma^{ab}\omega^{ab}_{\mathcal{W}ij} = \frac{i\sigma^\alpha}{2}\begin{pmatrix} \omega^\alpha_{(+)\mathcal{W}ij} & 0 \\ 0 & \omega^\alpha_{(-)\mathcal{W}ij} \end{pmatrix},$$
$$\omega^\alpha_{(\pm)\mathcal{W}ij} \equiv \left\{\pm\omega^{\alpha 4}_{\mathcal{W}ij} + \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\omega^{\beta\gamma}_{\mathcal{W}ij}\right\}. \tag{2.3}$$

The underwritten lattice instanton solution and fermion zero mode are described in terms of the subgroup $\mathrm{Spin}(4)_{(+)}$.

The considered lattice action has the form

$$\mathfrak{A} = \mathfrak{A}_g + \mathfrak{A}_\Psi, \tag{2.4}$$

$$\mathfrak{A}_g = -\frac{1}{5\cdot 24\cdot 2\cdot l_P^2}\sum_\mathcal{W}\sum_{i,j,k,l,m}\varepsilon_{\mathcal{W}ijklm} \times \mathrm{tr}\gamma^5\Omega_{\mathcal{W}mi}\Omega_{\mathcal{W}ij}\Omega_{\mathcal{W}jm}\hat{e}_{\mathcal{W}mk}\hat{e}_{\mathcal{W}ml}, \tag{2.5}$$

$$\mathfrak{A}_\Psi = -\frac{1}{5\cdot 24^2}\sum_\mathcal{W}\sum_{i,j,k,l,m}\varepsilon_{\mathcal{W}ijklm} \times \mathrm{tr}\gamma^5\hat{\Theta}_{\mathcal{W}mi}\hat{e}_{\mathcal{W}mj}\hat{e}_{\mathcal{W}mk}\hat{e}_{\mathcal{W}ml},$$
$$\hat{\Theta}_{\mathcal{W}ij} = \frac{i}{2}\gamma^a(\Psi^\dagger_{\mathcal{W}i}\gamma^a\Omega_{\mathcal{W}ij}\Psi_{\mathcal{W}j} - \Psi^\dagger_{\mathcal{W}j}\Omega_{\mathcal{W}ji}\gamma^a\Psi_{\mathcal{W}i}) \equiv \Theta^a_{\mathcal{W}ij}\gamma^a. \tag{2.6}$$

This action is invariant relative to the gauge transformations

$$\tilde{\Omega}_{\mathcal{W}ij} = S_{\mathcal{W}i}\Omega_{\mathcal{W}ij}S^{-1}_{\mathcal{W}j}, \qquad \tilde{e}_{\mathcal{W}ij} = S_{\mathcal{W}i}e_{\mathcal{W}ij}S^{-1}_{\mathcal{W}i},$$
$$\tilde{\Psi}_{\mathcal{W}i} = S_{\mathcal{W}i}\Psi_{Ai}, \qquad \tilde{\Psi}^{\dagger}{}_{\mathcal{W}i} = \Psi^{\dagger}_{\mathcal{W}i}S^{-1}_{\mathcal{W}i}$$
$$S_{\mathcal{W}i} \in \mathrm{Spin}(4). \tag{2.7}$$

The action (2.4) reduces to the continuum action of gravity in a four-dimensional Euclidean space in the limit of slowly varying fields, minimally connected with a Dirac field.

Consider a certain 4D subcomplex of complex $\mathfrak{K}$ with the trivial topology of the four-dimensional disk. Realize geometrically this subcomplex in $\mathbb{R}^4$. Suppose that the geometric realization is an almost smooth four-dimensional surface [7]. Thus, each vertex of the subcomplex acquires the coordinates x^μ that are the coordinates of the vertex image in $\mathbb{R}^4$:

$$x^\mu_{\mathcal{W}i} = x^\mu_{\mathcal{V}} \equiv x^\mu(a_{\mathcal{W}i}) \equiv x^\mu(a_{\mathcal{V}}), \qquad \mu = 1,2,3,4. \tag{2.8}$$

We stress that these coordinates are defined only by their vertices rather than by the higher-dimension simplices to which these vertices belong; moreover, the correspondence between the vertices from the considered subset and the coordinates (2.8) is one to one.

The four vectors

$$\mathrm{d}x^\mu_{\mathcal{W}ji} \equiv x^\mu_{\mathcal{W}i} - x^\mu_{\mathcal{W}j}, \qquad i = 1,2,3,4 \tag{2.9}$$

are linearly independent, and

$$\begin{vmatrix} \mathrm{d}x^1_{\mathcal{W}m1} & \mathrm{d}x^2_{\mathcal{W}m1} & \ldots & \mathrm{d}x^4_{\mathcal{W}m1} \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{d}x^1_{\mathcal{W}m4} & \mathrm{d}x^2_{\mathcal{W}m4} & \ldots & \mathrm{d}x^4_{\mathcal{W}m4} \end{vmatrix} \gtrless 0, \tag{2.10}$$

depending on whether the frame $(X^{\mathcal{W}}_{m1}, \ldots, X^{\mathcal{W}}_{m4})$ is positively or negatively oriented. Here, the differentials of coordinates (2.9) correspond to one-dimensional simplices $a_{\mathcal{W}j}a_{\mathcal{W}i}$ so that if the vertex $a_{\mathcal{W}j}$ has coordinates $x^\mu_{\mathcal{W}j}$, then the vertex $a_{\mathcal{W}i}$ has the coordinates $x^\mu_{\mathcal{W}j} + \mathrm{d}x^\mu_{\mathcal{W}ji}$.

In the continuous limit, the holonomy group elements (2.2) are close to the identity element so that the quantities ω^{ab}_{ij} tend to zero being of the order of $O(\mathrm{d}x^\mu)$. Thus, one can consider the following system of equations for $\omega_{\mathcal{W}m\mu}$:

$$\omega_{\mathcal{W}m\mu}\mathrm{d}x^\mu_{\mathcal{W}mi} = \omega_{\mathcal{W}mi}, \qquad i = 1,2,3,4. \tag{2.11}$$

In this system of linear equations, the indices $\mathcal{W}$ and m are fixed, the summation is carried out over the index μ, and the index runs over all its values. Since the determinant (2.10) is nonzero, the quantities $\omega_{\mathcal{W}m\mu}$ are defined uniquely. Suppose that a one-dimensional simplex $X^{\mathcal{W}}_{mi}$ belongs to four-dimensional simplices with indices $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_r$. Introduce the quantity

$$\omega_\mu\left(\frac{1}{2}(x_{\mathcal{W}m} + x_{\mathcal{W}i})\right) \equiv \frac{1}{r}\{\omega_{\mathcal{W}_1 m\mu} + \cdots + \omega_{\mathcal{W}_r m\mu}\}, \tag{2.12}$$

which is assumed to be related to the midpoint of the segment $[x^\mu_{\mathcal{W}m}, x^\mu_{\mathcal{W}i}]$. Recall that the coordinates $x^\mu_{\mathcal{W}i}$ as well as the differentials (2.9) depend only on vertices but not on the higher-dimensional simplices to which these vertices belong. According to the definition, we have the following chain of equalities:

$$\omega_{\mathcal{W}_1 mi} = \omega_{\mathcal{W}_2 mi} = \ldots = \omega_{\mathcal{W}_r mi}. \tag{2.13}$$

It follows from Eqs. (2.9) and (2.11)–(2.13) that

$$\omega_\mu\left(x_{\mathcal{W}m} + \frac{1}{2}\mathrm{d}x_{\mathcal{W}mi}\right)\mathrm{d}x^\mu_{\mathcal{W}mi} = \omega_{\mathcal{W}mi}. \tag{2.14}$$

The value of the field element ω_μ in Eq. (2.14) is uniquely defined by the corresponding one-dimensional simplex.

Next, we assume that the fields ω_μ smoothly depend on the points belonging to the geometric realization of each four-dimensional simplex. In this case, the following formula is valid up to $O((\mathrm{d}x)^2)$ inclusive,

$$\Omega_{\mathcal{W}mi}\Omega_{\mathcal{W}ij}\Omega_{\mathcal{W}jm} = \exp\left[\frac{1}{2}\mathfrak{R}_{\mu\nu}(x_{\mathcal{W}m})\mathrm{d}x^\mu_{\mathcal{W}mi}\mathrm{d}x^\nu_{\mathcal{W}mj}\right], \tag{2.15}$$

where

$$\mathfrak{R}_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] \equiv \frac{1}{2}\sigma^{ab}\mathfrak{R}^{ab}_{\mu\nu},$$
$$\mathfrak{R}^{ab} \equiv \mathfrak{R}^{ab}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{2.16}$$

In exact analogy with Eq. (2.11), let us write out the following relations for a tetrad field without explanations:

$$\hat{e}_{\mathcal{W}m\mu}\mathrm{d}x^\mu_{\mathcal{W}mi} = \hat{e}_{\mathcal{W}mi} \to e^a = e^a_\mu \mathrm{d}x^\mu. \tag{2.17}$$

Using Eqs. (2.2), (2.9), and (2.11), we can rewrite the 1-form (2.6) as

$$\hat{\Theta}_{\mathcal{W}ij} = \gamma^a \frac{i}{2}[\Psi^\dagger\gamma^a\mathcal{D}_\mu\Psi - (\mathcal{D}_\mu\Psi)^\dagger\gamma^a\Psi]\mathrm{d}x^\mu_{Aij}$$
$$\equiv \Theta^a\gamma^a, \tag{2.18}$$

to within $O(\mathrm{d}x)$; here,

$$\mathcal{D}_\mu\Psi = \partial_\mu\Psi + \omega_\mu\Psi, \qquad \Psi = \begin{pmatrix}\phi \\ \eta\end{pmatrix}, \tag{2.19}$$

and the smooth field $\Psi(x)$ takes the values $\Psi(x_{\mathcal{W}i}) = \Psi_{\mathcal{W}i}$.

Applying formulas (2.15)–(2.18) to the discrete action (2.4) and changing the summation to integration, we obtain in the continuum limit the well-known gravity action:

$$\mathfrak{A} = \int \varepsilon_{abcd}\left\{-\frac{1}{l_P^2}\mathfrak{R}^{ab}\wedge e^c\wedge e^d - \frac{1}{6}\Theta^a\wedge e^b\wedge e^c\wedge e^d\right\}. \tag{2.20}$$

Thus, in the naive continuum limit, the action (2.4) proves to be equal to the gravity action in the Palatini form minimally coupled to a Dirac field with Euclidean signature.

Another way of constructing Dirac fermions on simplicial complexes is stated in Ref. [8].

III. LATTICE GRAVITATIONAL INSTANTON

Let us consider first the instanton field configuration far apart from the instanton center where the continuous limit is valid (Eguchi-Hanson solution).

The designations

$$\sigma^a \equiv (\sigma^1,\sigma^2,\sigma^3,i),\qquad \mathrm{d}x^\mu = (\,\mathrm{d}\theta,\ \mathrm{d}\varphi,\ \mathrm{d}\psi,\ \mathrm{d}r),\qquad a=1,2,3,4,$$
$$\partial_\mu \equiv \begin{pmatrix}\partial_\theta\\ \partial_\varphi\\ \partial_\psi\\ \partial_r\end{pmatrix},\quad \varsigma^1\equiv\begin{pmatrix}\sin\psi\\ -\sin\theta\cos\psi\\ 0\\ 0\end{pmatrix},\quad \varsigma^2\equiv\begin{pmatrix}\cos\psi\\ \sin\theta\sin\psi\\ 0\\ 0\end{pmatrix},\quad \varsigma^3\equiv\begin{pmatrix}0\\ -\cos\theta\\ -1\\ 0\end{pmatrix},\quad \varsigma^4\equiv\begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix} \tag{3.1}$$

for the row and column matrices are used. We have

$$\mathrm{d}x^\mu e_\mu^a = (\,\mathrm{d}\theta,\quad \mathrm{d}\varphi,\quad \mathrm{d}\psi,\quad \mathrm{d}r\,)\times(\tfrac{1}{2}r\varsigma^1,\quad \tfrac{1}{2}r\varsigma^2,\quad \tfrac{1}{2}rg\varsigma^3,\quad g^{-1}\varsigma^4\,),$$
$$g = \sqrt{1-\frac{a^4}{r^4}} \tag{3.2}$$

for the Eguchi-Hanson self-dual solution to continuous Euclidean gravity [3,4].

The 4×4 matrix, which is the inverse of that in (3.2), is of the form

$$e_a^\mu = 2\times\begin{pmatrix} r^{-1}\sin\psi & -(r\sin\theta)^{-1}\cos\psi & r^{-1}\cot\theta\cos\psi & 0\\ r^{-1}\cos\psi & (r\sin\theta)^{-1}\sin\psi & -r^{-1}\cot\theta\sin\psi & 0\\ 0 & 0 & -(rg)^{-1} & 0\\ 0 & 0 & 0 & g/2\end{pmatrix}. \tag{3.3}$$

For the instanton gravitational field, we have

$$\frac{1}{2}\sigma^{ab}\omega_\mu^{ab} = \frac{i\sigma^\alpha}{2}\begin{pmatrix}\omega^\alpha_{(+)\mu} & 0\\ 0 & 0\end{pmatrix}, \tag{3.4}$$

$$\frac{i}{2}\mathrm{d}x^\mu(\omega^\alpha_{(+)\mu}\sigma^\alpha) = \frac{i}{2}(\,\mathrm{d}\theta,\quad \mathrm{d}\varphi,\quad \mathrm{d}\psi,\quad \mathrm{d}r\,)(g\varsigma^1\sigma^1 + g\varsigma^2\sigma^2 + (2-g^2)\varsigma^3\sigma^3). \tag{3.5}$$

Now, we describe the lattice self-dual gravitational field configuration [2,9].

The following notations are used below: $\mathfrak{k}\subset\mathfrak{K}$ means a finite subcomplex containing the center of the instanton with the boundary $\partial\mathfrak{k}\approx S^3$; $\mathfrak{K}'\subset\mathfrak{K}$ is an extra-large but finite subcomplex with the boundary $\partial\mathfrak{K}'\approx S^3$ containing the center of the instanton and vertices $a_\mathcal{V}\in\mathfrak{K}'$, $\mathcal{V}=1,2,\ldots\mathfrak{N}'\gg 1$ so that the long-wavelength limit is valid and the continuous solution (3.2)–(3.5) approximates correctly the exact lattice solution in a wide vicinity of $\partial\mathfrak{K}'$; the hypersurface $\partial\mathfrak{K}'$ is given by the equation $r=R=\mathrm{Const}\to\infty$. The Euler characteristics $\chi(\mathfrak{k})=\chi(\mathfrak{K}')=\chi(\mathfrak{K})=1$.

It has been proven in Ref. [2] that there exists the solution of the system of equations and boundary conditions

$$\delta\mathfrak{A}_g/\delta\omega^\alpha_{(\pm)\mathcal{W}mi}=0,\qquad \delta\mathfrak{A}_g/\delta e^a_{\mathcal{W}mi}=0,$$
$$\omega^\alpha_{(-)\mathcal{W}mi}=0\leftrightarrow\Omega_{(-)\mathcal{W}ij}=1, \tag{3.6}$$

$$\frac{i\sigma^\alpha}{2}\omega^\alpha_{(+)}\to U^{-1}\mathrm{d}U\quad\text{as}\quad r\to\infty, \tag{3.7}$$

where

$$U=\exp\left(-\frac{i\sigma^3}{2}\varphi\right)\exp\left(\frac{i\sigma^2}{2}\theta\right)\exp\left(-\frac{i\sigma^3}{2}\psi\right), \tag{3.8}$$

and

$$\Omega_{(+)\mathcal{W}ij} = -1, \qquad s^4_{\mathcal{W}} \in \mathfrak{k},$$
$$e^a_{\mathcal{V}_1\mathcal{V}_2} = \phi^a_{\mathcal{V}_2} - \phi^a_{\mathcal{V}_1}, \qquad a_{\mathcal{V}_1}a_{\mathcal{V}_2} \in \mathfrak{k}, \qquad a_{\mathcal{V}_1}a_{\mathcal{V}_2} \notin \partial\mathfrak{k} \tag{3.9}$$

on $\mathfrak{k}$ ($a_{\mathcal{V}_1}a_{\mathcal{V}_2}$ is a 1-simplex). The solution of Eqs. (3.6)–(3.9) is the lattice analog of the Eguchi-Hanson self-dual solution. It is denoted as $\Omega_{(\mathrm{inst})\mathcal{V}_1\mathcal{V}_2}$, $e^a_{(\mathrm{inst})\mathcal{V}_1\mathcal{V}_2}$.

IV. ASYMPTOTIC BEHAVIOR OF FERMION ZERO MODE ASSOCIATED WITH GRAVITATIONAL INSTANTON

To begin with, we define the lattice variant of the (right) neutrino action. For that purpose. it is necessary to extract from the quantity (2.6) the part interacting with the field $\Omega_{(+)\mathcal{W}ij}$ only [10]:

$$\mathfrak{A}_{(+)} = -\frac{1}{5\cdot 6\cdot 24}\sum_{\mathcal{W}}\sum_{i,j,k,l,m} \varepsilon_{\mathcal{W}ijklm}\varepsilon_{abcd} \times \Theta^a_{(+)\mathcal{W}mi} e^b_{\mathcal{W}mj} e^c_{\mathcal{W}mk} e^d_{\mathcal{W}ml},$$
$$\Theta^a_{(+)\mathcal{W}ij} = \frac{1}{2}(\eta^\dagger_{\mathcal{W}i}\sigma^a\Omega_{(+)\mathcal{W}ij}\phi_{\mathcal{W}j} + \phi^\dagger_{\mathcal{W}j}\Omega_{(+)\mathcal{W}ji}(\sigma^a)^\dagger\eta_{\mathcal{W}i}). \tag{4.1}$$

It is convenient to write the continuous variant of the introduced fermion lattice action (4.1) in the form

$$\mathfrak{A}_{(+)} = \int \mathrm{d}^{(4)}x |(\det e^b_\lambda)| \left\{\frac{1}{2} e^\mu_a [\eta^\dagger \sigma^a \mathcal{D}_{(+)\mu}\phi + \text{c.c.}]\right\},$$
$$\mathcal{D}_{(+)\mu} \equiv \partial_\mu + \frac{i}{2}\sigma^\alpha\omega^\alpha_{(+)\mu}. \tag{4.2}$$

The set of independent fermion variables is described by $\{\phi, \eta, \phi^\dagger, \eta^\dagger\}$.

The actions (4.1), (4.2) can be interpreted as the lattice and continuous variants of (right) neutrino actions, correspondingly.

Further, it is believed that the gravitational fields in Eq. (4.1) are the lattice instanton solutions (3.6)–(3.8), and the gravitational fields in Eq. (4.2) are the corresponding fields in the long-wavelength limit (3.2)–(3.5).

At the limit $r \to \infty$, we have $g = 1$. Let us introduce the designations for the case $g = 1$:

$$\frac{i}{2}\sigma^\alpha\omega^{\alpha(0)}_{(+)\mu} \equiv \frac{i}{2}\sigma^\alpha\omega^\alpha_{(+)\mu}|_{g=1} = \frac{i}{2}(\varsigma^1\sigma^1 + \varsigma^2\sigma^2 + \varsigma^3\sigma^3)_\mu,$$
$$\mathcal{D}^{(0)}_{(+)\mu} \equiv \partial_\mu + \frac{i}{2}\sigma^\alpha\omega^{\alpha(0)}_{(+)\mu}.$$

So, we have

$$\mathcal{D}_{(+)\mu} = \mathcal{D}^{(0)}_{(+)\mu} + \frac{i}{2}((1-g)\varsigma^1\sigma^1 + (1-g)\varsigma^2\sigma^2 + (1-g^2)\varsigma^3\sigma^3)_\mu. \tag{4.3}$$

It is easy to see that [the definition of $U \in \mathrm{SU}(2)$ is given in (3.8)]

$$\mathcal{D}^{(0)}_{(+)\mu} = U^{-1}\hat{\partial}_\mu U. \tag{4.4}$$

Combining Eqs. (4.3) and (4.4), we rewrite the Dirac-Weyl operator in (4.2) as follows:

$$\sigma^a e^\mu_a \mathcal{D}_{(+)\mu} = U^{-1}(U\sigma^a U^{-1})e^\mu_a\left\{\partial_\mu + \frac{i}{2}U((1-g)\varsigma^1\sigma^1 + (1-g)\varsigma^2\sigma^2 + (1-g^2)\varsigma^3\sigma^3)_\mu U^{-1}\right\}U. \tag{4.5}$$

One can obtain the row matrix

$$U\sigma^a U^{-1} \equiv \sigma^b A^a_b = (\sigma^1, \sigma^2, \sigma^3, i)\begin{pmatrix} (\cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi) & -(\cos\theta\cos\varphi\sin\psi + \sin\varphi\cos\psi) & -\sin\theta\cos\varphi & 0 \\ (\cos\theta\sin\varphi\cos\psi + \cos\varphi\sin\psi) & (-\cos\theta\sin\varphi\sin\psi + \cos\varphi\cos\psi) & -\sin\theta\sin\varphi & 0 \\ \sin\theta\cos\psi & -\sin\theta\sin\psi & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.6}$$

as a result of direct calculations. Further, according to Eqs. (3.3) and (4.6),

$$A^a_b e^\mu_a = 2\begin{pmatrix} -r^{-1}\sin\varphi & -r^{-1}\cot\theta\cos\varphi & (r\sin\theta)^{-1}\cos\varphi(\cos^2\theta + g^{-1}\sin^2\theta) & 0 \\ r^{-1}\cos\varphi & -r^{-1}\cot\theta\sin\varphi & (r\sin\theta)^{-1}\sin\varphi(\cos^2\theta + g^{-1}\sin^2\theta) & 0 \\ 0 & -r^{-1} & r^{-1}(1-g^{-1})\cos\theta & 0 \\ 0 & 0 & 0 & g/2 \end{pmatrix}. \tag{4.7}$$

Using the aforementioned formulas, we transform the operator (4.5) into the form

$$\sigma^a e^\mu_a \mathcal{D}_{(+)\mu} = U^{-1} i \Big\{ \frac{2}{r}\begin{pmatrix} -l_3 & l_- \\ l_+ & l_3 \end{pmatrix} + g\frac{\partial}{\partial r} + \left[\frac{2}{r\sin\theta}(\cos^2\theta + g^{-1}\sin^2\theta)\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix} + \frac{2i}{r}(1-g^{-1})\cos\theta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\left(-i\frac{\partial}{\partial\psi}\right) + \frac{1+2g-3g^2}{rg}\Big\} U, \tag{4.8}$$

$$l_3 = -i\frac{\partial}{\partial\varphi}, \qquad l_\pm = e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right),$$
$$[l_\pm, l_3] = \mp l_\pm, \qquad [l_+, l_-] = 2l_3. \tag{4.9}$$

We see that the operator in curly brackets in (4.8) does not depend on the variable ψ. Therefore, it is natural to take the simplest ansatz for the zero mode in the form

$$\phi = U^{-1}\tilde\phi, \qquad \eta = U^{-1}\tilde\eta,$$
$$(\partial/\partial\psi)\tilde\phi = 0, \qquad (\partial/\partial\psi)\tilde\eta = 0. \tag{4.10}$$

Thus, the operator (4.8) for the zero mode problem for $r \gg a$ reduces effectively to

$$\sigma^a e^\mu_a \mathcal{D}_{(+)\mu} = U^{-1} i\left[\frac{2}{r}\begin{pmatrix} -l_3 & l_- \\ l_+ & l_3 \end{pmatrix} + \frac{\partial}{\partial r}\right]U, \tag{4.11}$$

and the effective action describing the fermion zero mode configuration takes the form

$$\mathfrak{A}_{(+)} = \int r^3 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}\psi \times \left\{\frac{i}{2}\tilde\eta^\dagger\left[\frac{2}{r}\begin{pmatrix} -l_3 & l_- \\ l_+ & l_3 \end{pmatrix} + \frac{\partial}{\partial r}\right]\tilde\phi + \mathrm{c.c.}\right\} \tag{4.12}$$

because of

$$|(\det e^b_\lambda)|\mathrm{d}^{(4)}x = r^3\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}\psi \tag{4.13}$$

for the instanton field solution.

Note that the frequently used operator $2\mathbf{ls}$ in hydrogen atom physics has the form

$$2\mathbf{ls} = \begin{pmatrix} l_3 & l_- \\ l_+ & -l_3 \end{pmatrix},$$

and it differs from that in Eq. (4.12).

The action stationarity condition relative to variable $\tilde\eta^\dagger$ gives the zero mode equation

$$\left\{\frac{2}{r}\begin{pmatrix} -l_3 & l_- \\ l_+ & l_3 \end{pmatrix} + \frac{\partial}{\partial r}\right\}\tilde\phi_0 = 0. \tag{4.14}$$

The stationarity condition of the action (4.12) relative to variable $\tilde\phi$ yields

$$\frac{2}{r}\begin{pmatrix} -l_3 & l_- \\ l_+ & l_3 \end{pmatrix}\tilde\eta_0 = \left(\frac{3}{r} + \frac{\partial}{\partial r}\right)\tilde\eta_0. \tag{4.15}$$

Equations (4.14) and (4.15) imply that the functions $\tilde\phi_0$ and $\tilde\eta_0$ are the eigenfunctions of the operator

$$\begin{pmatrix} -l_3 & l_- \\ l_+ & l_3 \end{pmatrix} \tag{4.16}$$

with the common eigenvalue λ. Otherwise, the action (4.12) would be equal to zero identically since the operator (4.16) is Hermitian.

Let us consider the ansatz

$$\tilde\phi_0 = f(r)\left[\exp\left(-\frac{i\sigma^3}{2}\varphi\right)\begin{pmatrix} h(\theta)/\sqrt{\sin\theta} \\ k(\theta)/\sqrt{\sin\theta} \end{pmatrix}\right]. \tag{4.17}$$

The equation

$$\begin{pmatrix} -l_3 & l_- \\ l_+ & l_3 \end{pmatrix}\left[\exp\left(-\frac{i\sigma^3}{2}\varphi\right)\begin{pmatrix} h(\theta)/\sqrt{\sin\theta} \\ k(\theta)/\sqrt{\sin\theta} \end{pmatrix}\right] = \lambda\left[\exp\left(-\frac{i\sigma^3}{2}\varphi\right)\begin{pmatrix} h(\theta)/\sqrt{\sin\theta} \\ k(\theta)/\sqrt{\sin\theta} \end{pmatrix}\right] \tag{4.18}$$

is satisfied when and only when

$$\frac{\mathrm{d}k}{\mathrm{d}\theta} = -\left(\lambda - \frac{1}{2}\right)h, \qquad \frac{\mathrm{d}h}{\mathrm{d}\theta} = \left(\lambda - \frac{1}{2}\right)k. \tag{4.19}$$

To have the acceptable boundary conditions at $\theta = 0, \pi$, one must consider only the eigenvalues $\lambda = (n + 1/2)$, $n = 0, \pm 1, \ldots$. Equation (4.14) shows that the eigenvalues are acceptable only for $n \geq 1$. Otherwise, the mode ϕ_0 would be non-normalizable. On the other hand, the function η_0 would be $\mathrm{O}(r^{2(n-1)})$ as $r \to \infty$ for $n \geq 2$ according to Eq. (4.15); i.e., it would be non-normalizable. Therefore, the only acceptable eigenvalue is $\lambda = 3/2$. Then, Eqs. (4.14) and (4.15) give

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{3}{r}\right)f=0\rightarrow f\sim\frac{\mathrm{Const}}{r^3}, \tag{4.20}$$

$$\frac{\partial}{\partial r}\tilde{\eta}_0=0 \quad \text{as} \quad r\rightarrow\infty. \tag{4.21}$$

There are only two solutions,

$$\begin{pmatrix} h \\ k \end{pmatrix}^{(1)}=\sqrt{2}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}, \qquad \begin{pmatrix} h \\ k \end{pmatrix}^{(2)}=\sqrt{2}\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, \tag{4.22}$$

for $\lambda=3/2$. Combining Eqs. (3.8), (4.10), (4.17), (4.20), (4.21), and (4.22), we obtain two asymptotic solutions:

$$\begin{aligned}
\phi_0^{(1)}&=\frac{\mathrm{Const}}{r^3}\exp\left(\frac{i\sigma^3}{2}\psi\right)\begin{pmatrix}\sqrt{\tan(\theta/2)}\\ \sqrt{\cot(\theta/2)}\end{pmatrix},\\
\eta_0^{(1)}&=\mathrm{Const}\cdot\exp\left(\frac{i\sigma^3}{2}\psi\right)\begin{pmatrix}\sqrt{\tan(\theta/2)}\\ \sqrt{\cot(\theta/2)}\end{pmatrix},\\
\phi_0^{(2)}&=\frac{\mathrm{Const}}{r^3}\exp\left(\frac{i\sigma^3}{2}\psi\right)\begin{pmatrix}\sqrt{\cot(\theta/2)}\\ -\sqrt{\tan(\theta/2)}\end{pmatrix},\\
\eta_0^{(2)}&=\mathrm{Const}\cdot\exp\left(\frac{i\sigma^3}{2}\psi\right)\begin{pmatrix}\sqrt{\cot(\theta/2)}\\ -\sqrt{\tan(\theta/2)}\end{pmatrix}.
\end{aligned} \tag{4.23}$$

It is known that two spinors ϕ and $(i\sigma^2\phi)^*$ transform identically under the gauge transformations $\mathrm{Spin}(4)_{(+)}$ [11]. Here, the upper index * means complex conjugation. But we have

$$\phi_0^{(2)}=(i\sigma^2\phi_0^{(1)})^*.$$

This equality leads to the conclusion that there is only one independent smooth fermion zero mode associated with the lattice gravitational instanton. Therefore, any linear combination of the solutions (4.23) can be considered as asymptotic behavior of the zero mode.

It will be proved that the corresponding lattice solution is normalizable.

Note that there is a great number of other solutions of the equation $\sigma^a e_a^\mu \mathcal{D}_{\mathrm{inst}(+)\mu}\phi_0=0$. We give an example of a series of the operator (4.18) eigenfunctions with eigenvalues $\lambda\neq 3/2$:

$$\phi_0{}'=f'(r)\exp\left(\frac{i\sigma^3}{2}\psi\right)\left[e^{im\varphi}(\sin\theta)^m\begin{pmatrix}\sqrt{\cot(\theta/2)}\\ -\sqrt{\tan(\theta/2)}\end{pmatrix}\right],$$
$$\lambda=(m+3/2), \qquad m=1,2,\ldots$$

As was shown above, the eigenvalues $\lambda\neq 3/2$ are not acceptable.

V. EXISTENCE OF LATTICE FERMION ZERO MODES

We must solve lattice equations

$$\delta\mathfrak{A}_{(+)}/\delta\phi_{\mathcal{V}}=0, \qquad \delta\mathfrak{A}_{(+)}/\delta\eta_{\mathcal{V}}^\dagger=0 \tag{5.1}$$

as well as their complex conjugate equations [12] for the action (4.1) taken on the self-dual gravitational solution (3.6)–(3.9) $\Omega_{(\mathrm{inst})\mathcal{V}_1\mathcal{V}_2}$, $e^a_{(\mathrm{inst})\mathcal{V}_1\mathcal{V}_2}$ with the boundary conditions (4.23) as $r\rightarrow\infty$.

To solve the problem, we use the method that has been successful in solving lattice pure gravity self-dual equations with given boundary conditions [2]. The method can be applied to lattice theory, but it is fundamentally unacceptable in the case of continuous theories. The reason is that the number of variables (degrees of freedom) associated with finite space-time volume is finite in any lattice theory, while the number of variables per volume is infinite (uncountable) in continuous theories.

Introduce the following Lagrange function on $\mathfrak{K}'$ [see the text between Eqs. (3.5) and (3.6)] depending on the variables $\{\phi_{\mathcal{V}},\eta_{\mathcal{V}},\phi_{\mathcal{V}}^\dagger,\eta_{\mathcal{V}}^\dagger\}$, $\mathcal{V}=1,\ldots,\mathfrak{N}'$:

$$\begin{aligned}
\mathcal{L}=&-\frac{1}{5\cdot 6\cdot 24}\sum_{\mathcal{W}:s^4_{\mathcal{W}}\in\mathfrak{K}'}\sum_{i,j,k,l,m}\varepsilon_{\mathcal{W}ijklm}\varepsilon_{abcd}\\
&\times\Theta^a_{(+)\mathcal{W}mi}e^b_{\mathcal{W}mj}e^c_{\mathcal{W}mk}e^d_{\mathcal{W}ml}-\lambda^{(\phi)}\Phi^{(\phi)}-\lambda^{(\eta)}\Phi^{(\eta)}\\
&-\left\{\sum_{\mathcal{V}:a_{\mathcal{V}}\in\partial\mathfrak{K}'}\sum_{s=1,2}(\lambda^{(\phi)}_{\mathcal{V},s}\Phi^{(\phi)}_{\mathcal{V},s}+\lambda^{(\eta)}_{\mathcal{V},s}\Phi^{(\eta)}_{\mathcal{V},s})+\mathrm{c.c.}\right\}.
\end{aligned} \tag{5.2}$$

Here, $\{\lambda\}$ are Lagrange multipliers.

The constraints

$$\begin{aligned}
\Phi^{(\phi)}_{\mathcal{V},s}&=\left(\frac{\phi^s_{\mathcal{V}}}{\phi^s_0(x_{\mathcal{V}})}-\frac{\phi^s_{\mathcal{V}_0}}{\phi^s_0(x_{\mathcal{V}_0})}\right),\\
\Phi^{(\eta)}_{\mathcal{V},s}&=\left(\frac{\eta^s_{\mathcal{V}}}{\eta^s_0(x_{\mathcal{V}})}-\frac{\eta^s_{\mathcal{V}_0}}{\eta^s_0(x_{\mathcal{V}_0})}\right),\\
a_{\mathcal{V}}&\in\partial\mathfrak{K}', \qquad a_{\mathcal{V}_0}\notin\partial\mathfrak{K}'
\end{aligned} \tag{5.3}$$

fix the boundary conditions (4.23) near the hypersurface $\partial\mathfrak{K}'$. Here, $x_{\mathcal{V}}$ and $x_{\mathcal{V}_0}$ are the coordinate values at the vertices $a_{\mathcal{V}}$ and $a_{\mathcal{V}_0}$, correspondingly [see Eq. (2.8)]; $a_{\mathcal{V}_0}$ is a fixed vertex from the immediate neighborhood of hypersurface $\partial\mathfrak{K}'$ so that there is a 1-simplex $a_{\mathcal{V}_0}a_{\mathcal{V}}$ for some vertex $a_{\mathcal{V}}\in\partial\mathfrak{K}'$; index $s=1,2$ enumerates the components of Weyl spinors $\phi^\dagger$, ϕ.

The constraints

$$\begin{aligned}
\Phi^{(\phi)}&=\left(\sum_{\mathcal{V}:a_{\mathcal{V}}\in(\mathfrak{K}'\backslash\partial\mathfrak{K}')}v_{\mathcal{V}}\sum_{s=1,2}|\phi^s_{\mathcal{V}}|^2-1\right),\\
\Phi^{(\eta)}&=\left(\sum_{\mathcal{V}:a_{\mathcal{V}}\in(\mathfrak{K}'\backslash\partial\mathfrak{K}')}v_{\mathcal{V}}\sum_{s=1,2}|\eta^s_{\mathcal{V}}|^2-1\right)
\end{aligned} \tag{5.4}$$

mean that the fermion field configurations are normalizable on $\mathfrak{K}'$,

$$v_{\mathcal{V}} = \frac{1}{5}\sum_{\mathcal{W}:a_{\mathcal{V}}\in\mathcal{W}} v_{\mathcal{W}}, \tag{5.5}$$

$$v_{\mathcal{W}} = \frac{1}{4!}\varepsilon_{\mathcal{W}ijklm}e^1_{\mathcal{W}mi}e^2_{\mathcal{W}mj}e^3_{\mathcal{W}mk}e^4_{\mathcal{W}ml}. \tag{5.6}$$

The expression (5.6) means the oriented volume of the 4-simplex $s_{\mathcal{W}}$, factor $1/4!$ is required since the volume of a four-dimensional parallelepiped with generatrices $e^1_{\mathcal{W}mi}$, $e^2_{\mathcal{W}mj}$, $e^3_{\mathcal{W}mk}$, and $e^4_{\mathcal{W}ml}$ is 4! times larger than the volume of a 4-simplex with the same generatrices. The expression $v_{\mathcal{V}}$ in Eq. (5.5) is the sum of the volumes $v_{\mathcal{W}}$ for that $\mathcal{W}$-4-simplices which contain the vertex $a_{\mathcal{V}}$, the factor $1/5$ is necessary due to the fact that all five vertices of each simplex are taken into account independently in Eq. (5.4). So, the volume (5.5) is the specific volume per vertex. In the long-wavelength limit, the constraints (5.4) transform into (the same is true for $\Phi^{(\eta)}$)

$$\Phi^{(\phi)} = \left(\int_{\mathfrak{K}'}\left(\sum_{s=1,2}|\phi^s(x)|^2\right)e^1\wedge e^2\wedge e^3\wedge e^4 - 1\right),$$
$$e^a = e^a_\mu \mathrm{d}x^\mu.$$

Since the subcomplex $\mathfrak{K}'$ contains a finite number $\mathfrak{N}'$ of vertices, the Lagrange function (5.2) depends on a finite number of classical variables $\{\phi^\dagger_{\mathcal{V}},\phi_{\mathcal{V}}\}$, $\mathcal{V}=1,\ldots,\mathfrak{N}'<\infty$. For "not-patologic" complexes $\mathfrak{K}$, the estimation

$$\mathfrak{N}' \sim R^4 \tag{5.7}$$

is valid.

The problem is as follows: the local maxima and minima of Lagrange function (5.2) constrained by the constraints (5.3) and (5.4) need to be studied. The simplicity of the constraints very much simplifies the problem: the constraints can be solved evidently. Thus, the constraints (5.3) give

$$\phi^s_{\mathcal{V}} = \left(\frac{\phi^s_0(x_{\mathcal{V}})}{\phi^s_0(x_{\mathcal{V}_0})}\right)\phi^s_{\mathcal{V}_0}, \qquad \eta^s_{\mathcal{V}} = \left(\frac{\eta^s_0(x_{\mathcal{V}})}{\eta^s_0(x_{\mathcal{V}_0})}\right)\eta^s_{\mathcal{V}_0},$$
$$a_{\mathcal{V}}\in\partial\mathfrak{K}', \qquad a_{\mathcal{V}_0}\notin\partial\mathfrak{K}'. \tag{5.8}$$

It is useful to divide the Lagrange function (5.2) into two terms:

$$\mathcal{L} = \mathcal{L}' + \partial\mathcal{L}. \tag{5.9}$$

Here, $\mathcal{L}'$ does not depend on the variables $(\phi_{\mathcal{V}_0},\eta_{\mathcal{V}_0},\phi^\dagger_{\mathcal{V}_0},\eta^\dagger_{\mathcal{V}_0})$, while $\partial\mathcal{L}$ is a homogeneous linear form for these variables. Evidently, $\partial\mathcal{L}$ depends only on the variables associated with vertices from the immediate neighborhood of hypersurface $\partial\mathfrak{K}'$.

Realization of the constraints (5.4) converts the Lagrange function (5.2) into a smooth function defined on the compact metric finite-dimensional manifold $\mathcal{C}$ without boundary. It is well known for this case that the Lagrange function is a bounded one and it has the local maximum (maxima) and minimum (minima) at some points $p_\xi\in\mathcal{C}$. Moreover, since the space $\mathcal{C}$ is without boundary, the total differentials of the Lagrange function at the points p_ξ are equal to zero.

It should be emphasized that the total differential of the Lagrange function must be calculated with respect to independent variables. The variables associated with the vertices $a_{\mathcal{V}}\in\partial\mathfrak{K}'$ are expressed evidently in terms of independent variables $\phi_{\mathcal{V}_0},\eta_{\mathcal{V}_0}$ according to Eqs. (5.8). Let $a_{\mathcal{V}_1}\notin\partial\mathfrak{K}'$ be a fixed vertex from the immediate neighborhood of hypersurface $\partial\mathfrak{K}'$. The constraints (5.4) will be resolved if we express, for example, the real component of $\phi^1_{\mathcal{V}_1}$ and $\eta^1_{\mathcal{V}_1}$ in terms of the rest of the independent variables,

$$\mathrm{Re}\phi^1_{\mathcal{V}_1} = \pm\frac{1}{\sqrt{v_{\mathcal{V}_1}}}\sqrt{1-\sum_{\mathcal{V}:a_{\mathcal{V}}\in(\mathfrak{K}'\setminus\partial\mathfrak{K}'),s=1,2}{}^{\prime}\; v_{\mathcal{V}}|\phi^s_{\mathcal{V}}|^2}, \tag{5.10}$$

and analogously for $\mathrm{Re}\eta^1_{\mathcal{V}_1}$. Here, the prime above the sum means that the variable $\mathrm{Re}\phi^1_{\mathcal{V}_1}$ is absent. Thus,

$$\frac{\partial\mathrm{Re}\phi^1_{\mathcal{V}_1}}{\partial\phi^s_{\mathcal{V}}} = \mp\left(\frac{v_{\mathcal{V}}}{v_{\mathcal{V}_1}}\right)\frac{\phi^s_{\mathcal{V}}}{\mathrm{Re}\phi^1_{\mathcal{V}_1}}. \tag{5.11}$$

Therefore, one should replace

$$\frac{\partial}{\partial\phi_{\mathcal{V}}} \to \frac{\partial}{\partial\phi_{\mathcal{V}}} \mp\left[\left(\frac{v_{\mathcal{V}}}{v_{\mathcal{V}_1}}\right)\frac{(\phi_{\mathcal{V}})^*}{\mathrm{Re}\phi^1_{\mathcal{V}_1}}\right]\frac{\partial}{\partial\mathrm{Re}\phi^1_{\mathcal{V}_1}}, \tag{5.12}$$

and so on.

Let us consider the stationarity condition for the Lagrange function (Lagrange multipliers can be put equal to zero) relative to the variable $\eta^\dagger_{\mathcal{V}}$:

$$\frac{\partial\mathcal{L}}{\partial\eta^\dagger_{\mathcal{V}}} \mp\left[\left(\frac{v_{\mathcal{V}}}{v_{\mathcal{V}_1}}\right)\frac{\eta_{\mathcal{V}}}{\mathrm{Re}(\eta^\dagger)^1_{\mathcal{V}_1}}\right]\frac{\partial\mathcal{L}}{\partial\mathrm{Re}(\eta^\dagger)^1_{\mathcal{V}_1}} = 0. \tag{5.13}$$

For $\mathcal{V}=\mathcal{V}_0$, we have the same equation, but it is convenient to divide the Lagrange function according to (5.9):

$$\frac{\partial\mathcal{L}'}{\partial\eta^\dagger_{\mathcal{V}_0}} + \frac{\partial(\partial\mathcal{L})}{\partial\eta^\dagger_{\mathcal{V}_0}} \mp\left[\left(\frac{v_{\mathcal{V}}}{v_{\mathcal{V}_1}}\right)\frac{\eta_{\mathcal{V}_0}}{\mathrm{Re}(\eta^\dagger)^1_{\mathcal{V}_1}}\right]\frac{\partial\mathcal{L}}{\partial\mathrm{Re}(\eta^\dagger)^1_{\mathcal{V}_1}} = 0. \tag{5.14}$$

Now, pass to the limit $R\to\infty$ in Eqs. (5.13)–(5.14).

We have

$$v_{\mathcal{V}}/v_{\mathcal{V}_1} \sim 1 \tag{5.15}$$

for the not-patologic complex. There is also the estimation

$$\frac{\eta_{\mathcal{V}}}{\mathrm{Re}(\eta^\dagger)^1_{\mathcal{V}_1}} \sim \mathrm{O}(R^0) \tag{5.16}$$

as a consequence of the boundary condition (4.21). Finally, the estimation

$$\frac{\partial \mathcal{L}}{\partial \mathrm{Re}(\eta^\dagger)^1_{\mathcal{V}_1}} \sim \mathrm{O}(R^{-3}) \tag{5.17}$$

is true since the quantity $\partial\mathcal{L}/\partial \mathrm{Re}(\eta^\dagger)^1_{\mathcal{V}_1}$ depends linearly only on a limited number (of the order of 1) of the variables $\phi_{\mathcal{V}}$ and due to the boundary condition (4.20).

The estimation of the second term on the left-hand side of Eq. (5.14) can be obtained if we take into account Eqs. (5.8) and the definition of the quantity $\partial\mathcal{L}$ [see Eq. (5.9)]. The angular dependence of the boundary variables is defined according to Eq. (5.8). Therefore, this quantity resides in the stationary point relative to the angular variations by definition of $\partial\mathcal{L}$. So. the derivative $\partial/\partial\eta^\dagger_{\mathcal{V}_0}$ comes to the derivative with respect to r acting into variables $\phi_{\mathcal{V}}$ for vertices $a_{\mathcal{V}}$ from the immediate neighborhood of hypersurface $\partial\mathfrak{K}'$, and $\partial\mathcal{L}$ is the sum of the quantities that are of the order of $\mathrm{O}(R^{-4})$, but the number of these quantities (the number of the vertices on $\partial\mathfrak{K}'$) is of the order of R^3. Thus,

$$\frac{\partial(\partial\mathcal{L})}{\partial\eta^\dagger_{\mathcal{V}_0}} \sim \mathrm{O}(R^{-1}). \tag{5.18}$$

Using the estimations (5.17) and (5.18), we obtain estimations

$$\frac{\partial \mathcal{L}}{\partial\eta^\dagger_{\mathcal{V}}} \sim \mathrm{O}(R^{-3}), \qquad \frac{\partial \mathcal{L}}{\partial\eta^\dagger_{\mathcal{V}_0}} \sim \mathrm{O}(R^{-1}). \tag{5.19}$$

From (4.21) and (5.4), it follows that

$$\eta^\dagger_{\mathcal{V}} \sim \mathrm{O}(R^{-2}). \tag{5.20}$$

As a result of estimations (5.19) and (5.20), we obtain

$$\mathcal{L}' \sim \sum_{\mathcal{V}:a_{\mathcal{V}}\in(\mathfrak{K}'\backslash\partial\mathfrak{K}'),\mathcal{V}\neq\mathcal{V}_0} \eta^\dagger_{\mathcal{V}} \frac{\partial\mathcal{L}}{\partial\eta^\dagger_{\mathcal{V}}}$$
$$\sim \mathrm{O}(R^4)\cdot\mathrm{O}(R^{-2})\cdot\mathrm{O}(R^{-3}) \sim \mathrm{O}(R^{-1}). \tag{5.21}$$

Now, one should consider the stationary point of the Lagrange function (5.2) relative to the variables $\phi_{\mathcal{V}}$. For this purpose, it is enough to make a replacement $\eta^\dagger_{\mathcal{V}} \to \phi_{\mathcal{V}}$ in Eqs. (5.13) and (5.14). The estimation

$$\frac{\partial\mathcal{L}}{\partial\phi_{\mathcal{V}}} \sim \mathrm{O}(R^{-2}) \tag{5.22}$$

is true since the quantity $\partial\mathcal{L}/\partial\mathrm{Re}\phi^1_{\mathcal{V}_1}$ depends linearly only on a limited number (of the order of 1) of the variables $\eta^\dagger_{\mathcal{V}}$ and due to the estimation (5.20). Further, since the derivative $(\partial\eta^\dagger_{\mathcal{V}}/\partial r)$ is negligibly small in the neighborhood of hypersurface $\partial\mathfrak{K}'$ [see (4.21)], the quantity

$$\frac{\partial(\partial\mathcal{L})}{\partial\phi_{\mathcal{V}_0}}$$

is also negligibly small. Therefore, the estimation (5.22) is valid for all $a_{\mathcal{V}} \in \mathfrak{K}'\backslash\partial\mathfrak{K}'$.

Now, one should pass to the limit $R\to\infty$. It follows from the estimations (5.19) and (5.22) that the problem (5.1)–(5.2) possess a solution. Besides, according to Eq. (5.21), the action (4.1) is equal to zero in this solution. This means that the discussed solution really is a fermion zero mode.

VI. DISCUSSION: THE NUMBER OF LATTICE ZERO MODES AND THEIR NATURE

The important question is unanswered; how many lattice solutions for zero modes do exist? We cannot prove rigorously here that there exist two linearly independent lattice zero modes, but we formulate the conjecture.

Hypothesis.—There exist two linearly independent lattice fermion zero modes associated with the lattice gravitational instanton. One of them possesses the properties of a usual smooth mode, and another can be characterized as a singular mode [1].

We give here only some reasoning to justify the hypothesis.

Suppose that the subcomplex $\mathfrak{k}$ (center of the instanton) is large enough, i.e., the number of 4-simplices $s^4_{\mathcal{W}} \in \mathfrak{k}$ is a large number. According to Eqs. (4.1) and (3.9), the contribution to the fermion action that is associated with subcomplex $\mathfrak{k}$ is

$$\mathfrak{A}_{(+)(\mathfrak{k})} = \frac{1}{2\cdot5\cdot6\cdot24}\sum_{\mathcal{W}:s^4_{\mathcal{W}}\in\mathfrak{k}}\sum_{i,j,k,l,m}\varepsilon_{\mathcal{W}ijklm}\varepsilon_{abcd}$$
$$\times(\eta^\dagger_{\mathcal{W}m}\sigma^a\phi_{\mathcal{W}i} + \mathrm{c.c.})e^b_{\mathcal{W}mj}e^c_{\mathcal{W}mk}e^d_{\mathcal{W}ml}. \tag{6.1}$$

Let us consider two adjacent 4-simplices $s^4_{\mathcal{W}} = a_{\mathcal{W}i}a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m}$ and $s^4_{\mathcal{W}'} = a_{\mathcal{W}'i'}a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m}$ with common 3-simplex $s^3 = a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m}$ and different vertices $a_{\mathcal{W}i}$ and $a_{\mathcal{W}'i'}$. Evidently, $s^4_{\mathcal{W}}$ and $s^4_{\mathcal{W}'}$ have the opposite orientations. Therefore,

$$\varepsilon_{\mathcal{W}ijklm} = -\varepsilon_{\mathcal{W}'i'jklm}. \tag{6.2}$$

This implies that the contribution into Eq. (6.1) associated with 3-simplex $s^3 = a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m}$ vanishes for $\phi_{\mathcal{W}i} = \phi_{\mathcal{W}'i'}$. Note that there are no cavities in $\mathfrak{K}$ (and hence in $\mathfrak{k}$)

by definition. This means that each 3-simplex $s^3 = a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m} \in (\mathfrak{k}\backslash\partial\mathfrak{k})$ belongs to two and only two adjacent 4-simplices $s^4_{\mathcal{W}} = a_{\mathcal{W}i}a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m} \in \mathfrak{k}$ and $s^4_{\mathcal{W}'} = a_{\mathcal{W}'i'}a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m} \in \mathfrak{k}$. It follows from this consideration that the contribution (6.1) vanishes on the configuration

$$\phi_{\mathcal{V}} = \mathrm{Const} \quad \text{on} \quad \mathfrak{k}\backslash\partial\mathfrak{k}. \tag{6.3}$$

In other words, the configuration (6.3) satisfies Eq. (5.1) on $\mathfrak{k}\backslash\partial\mathfrak{k}$.

This consideration leads to the hypothesis that the configuration (6.3) is a part of the configuration of a regular zero mode on the instanton interior.

According to the hypothesis, the irregular zero mode (doubled fermion in the Wilson sense) does exist also.

The hypothesis is proved mathematically rigorously in the case of the Dirac zero modes for the Yang-Mills smooth instantons. The idea of the proof is based on the fact that the normal smooth fermion modes give the known anomaly contribution into the chiral current. But a trivial consequence of our definition of lattice Dirac fermions is the fact that the lattice fermion measure does not contain an axial anomaly. This means that the Dirac irregular modes compensate completely the contribution of the smooth fermion modes into anomaly. Since in the Yang-Mills theory the Dirac zero modes and anomalous contributions into axial current are inextricably connected (concerning the smooth modes, the statement is demonstrated in the Introduction), both normal and anomaly zero modes must exist. The detailed calculations are given in Ref. [1].

The problem formulated here as a hypothesis requires a detailed study as well as the physical consequences of the fermion zero mode existence.

For a final matter, we give some comments regarding the Wilson fermion doubling problem. It is well known that the lattice Dirac fermions possessing the chiral symmetry property possess also the Wilson fermion doubling property. The statement is valid for regular lattices [13,14] as well as for irregular lattices (simplicial complexes) [1]. However, there is a qualitative difference between the phenomena on regular and irregular lattices. In the case of the regular lattice, there are 16 doublers, and all of the quanta of all doublers propagate identically like free particles (in free theory). But there is a qualitative difference between the dynamics of soft regular and soft irregular quanta in the case of the irregular lattice. While the regular quanta propagate as free particles since they have lost the information about the lattice, the irregular doubled quanta cannot propagate in the space-time for the reason that the irregular quanta wave functions are determined essentially by the irregular "breathing" lattice [15]. Therefore, the irregular quanta cannot be observed directly but only by means of some physical effects taking place due to the existence of irregular quanta (more detailed comments on the question are contained in Ref. [1]). In a sense, the irregular quanta are not observable since they are not relevant for most of physics. We note also that the number 16 for the doublers for the cubic lattice is related with the cubic symmetry. Since general irregular lattices (simplicial complex) have no symmetries, the irregular quanta enumeration problem remains unsolved.

Note also that the zero modes differ from soft modes qualitatively; the zero mode is localized in the vicinity of the instanton and annihilates the Dirac operator precisely, while the soft mode is an eigenmode of the Dirac operator with nonzero eigenvalue.

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