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Quantum stability of nonlinear wave type solutions with intrinsic mass parameter in QCD

Youngman Kim,[1,*] Bum-Hoon Lee,[2,3,†] D. G. Pak,[4,3,5,‡] Chanyong Park,[4,6,§] and Takuya Tsukioka[7,‖]

[1] *Rare Isotope Science Project, Institute for Basic Science, Daejeon 305-811, Korea*
[2] *Asia Pacific Center of Theoretical Physics, Pohang 790-330, Korea*
[3] *CQUEST, Sogang University, Seoul 121-742, Korea*
[4] *Asia Pacific Center of Theoretical Physics, Pohang, 790-330, Korea*
[5] *Chern Institute of Mathematics, Nankai University, Tianjin 300071, China*
[6] *Department of Physics, Postech, Pohang 790-784, Korea*
[7] *School of Education, Bukkyo University, Kyoto 603-8301, Japan*

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The problem of the existence of a stable vacuum field in pure QCD is revised. Our approach is based on using classical stationary nonlinear wave type solutions with an intrinsic mass scale parameter. Such solutions can be treated as quantum-mechanical wave functions describing massive spinless states in quantum theory. We verify whether nonlinear wave type solutions can form a stable vacuum field background within the framework of the effective action formalism. We demonstrate that there is a special class of stationary generalized Wu-Yang monopole solutions that are stable against quantum gluon fluctuations.

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I. INTRODUCTION

The origin of quark/color confinement and the mass gap in quantum chromodynamics represents the principal problem in the foundations of the theory of strong interactions [1]. One of the most attractive mechanisms of quark confinement is based on the dual Meissner effect in color superconductors by means of monopole condensation [2–5]. If such a stable monopole condensate is generated, it will immediately imply confinement [6–8], which has been confirmed in lattice simulations [9–13]. The theoretical foundation of the confinement mechanism with the dual Meissner effect encounters several obstacles. Among them, the realization of physical monopole solutions in standard QCD and the quantum stability of monopole condensation represent a long-standing problem since the late 1970s when the Savvidy-Nielsen-Olesen vacuum instability was found [14,15]. So far, neither a regular monopole solution nor a strict construction of a stable color magnetic condensate has been discovered in the framework of the basic standard theory of QCD. This causes serious doubts that the known Copenhagen "spaghetti" vacuum and other models of the QCD vacuum can provide a rigorous microscopic description of the vacuum structure [16–21].

In the present paper we elaborate the idea that classical stationary nonlinear wave type solutions can be treated in a quantum-mechanical sense and describe physical states in quantum theory. The idea that stationary nonsolitonic wave solutions correspond to particles or quasiparticles was proposed a long time ago [22–24]. Our goal is to find a proper regular stationary solution which will be stable against quantum gluon fluctuations within the formalism of the effective action in the one-loop approximation. Such a stable field configuration can serve as a structural element in the further construction of a true QCD vacuum. There is a wide class of known stationary nonlinear wave solutions [25–30] which possess nontrivial features: the presence of mass scale parameters, nonvanishing longitudinal components of color fields along the propagation direction, color magnetic charge, and a vanishing classical spin density operator. This gives a hint that some of these classical solutions describe quantum states corresponding to massive spinless quasiparticles, which might lead to the formation of a stable vacuum condensate. Surprisingly, we show that there is a special class of stationary spherically symmetric monopole solutions which possess quantum stability.

The paper is organized as follows. In Sec. II we give an overview of the main critical points in the vacuum stability problem and outline possible ways to construct a stable vacuum field configuration. The quantum stability of nonlinear plane-wave solutions is considered in Sec. III. A careful analysis shows that, in spite of several attractive properties of such solutions, the nonlinear plane waves are unstable against vacuum gluon fluctuations. In Sec. IV we consider the quantum stability of a recently proposed stationary monopole solution [30] which represents a system of a static Wu-Yang monopole interacting with off-diagonal components of the gluon field. We prove that

*ykim@ibs.re.kr
†bhl@sogang.ac.kr
‡dmipak@gmail.com
§chanyong.park@apctp.org
‖tsukioka@bukkyo-u.ac.jp

such a generalized monopole solution provides a stable vacuum field background in the effective action of QCD in the one-loop approximation. Conclusions and a discussion of our results are presented in the last section. An additional qualitative analysis of the quantum stability of the stationary monopole field is given in the Appendix.

II. VACUUM STABILITY PROBLEM

Let us consider the structure of the QCD effective action in the presence of constant homogeneous classical fields and expose the critical issues of vacuum instability for this simple case. In order to study the vacuum structure in quantum field theory, it is suitable to apply a quantization scheme based on the functional integral formalism and calculate the quantum effective action with a properly chosen classical background field. The background field satisfying the classical equations of motion corresponds to a vacuum-averaged value of the quantum field operator in the presence of a source, or in the adiabatic limit when the external source vanishes at time $t \to +\infty$. A nontrivial vacuum structure can be retrieved from the behavior of the effective potential and from the structure of the effective action. In general, the effective potential admits several local minima, and only the lowest and stable one determines a true physical vacuum. Moreover, the symmetry properties of the vacuum state determine fundamental properties of the theory, such as the type of symmetry breaking, possible phase transitions, etc. The knowledge of the analytic structure of the effective action represents an important step which verifies whether a nontrivial classical vacuum in the theory corresponds to a physical vacuum at the quantum level. As usual, the presence of an imaginary part of the effective action indicates vacuum instability.

We concentrate mainly on the structure of the effective action in the case of pure $SU(2)$ QCD. For the case of a constant homogeneous classical background field the effective action can be calculated in a complete form in the one-loop approximation. We start with a classical Lagrangian of Yang-Mills theory,

$$\mathcal{L}_0 = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a, \tag{1}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu.$$

The space-time indices μ, ν and those for colors a, b, c run through 0, 1, 2, 3 and 1, 2, 3, respectively. We work with the convention $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and $\epsilon^{123} = 1$.

An initial gauge potential A^a_μ is split into classical ($\mathcal{B}^a_\mu$) and quantum (Q^a_μ) parts,

$$A^a_\mu = \mathcal{B}^a_\mu + Q^a_\mu. \tag{2}$$

One should stress that the classical gauge potential $\mathcal{B}^a_\mu$ must be a solution to the classical Euler-Lagrange equations of motion. Only in that case can the external classical field $\mathcal{B}^a_\mu$ be treated as a vacuum-averaged value of the quantum operator A^a_μ in a consistent manner with the effective action formalism. One should note that a static homogeneous classical gauge potential $\mathcal{B}^a_\mu$ cannot provide a constant field strength unless the gauge symmetry is broken. The field $\mathcal{B}^a_\mu$ is defined as a vacuum-averaged value of the quantum operator A^a_μ in the limit of vanishing source $J^a_\mu \to 0$ during the time evolution ($t \to +\infty$),

$$\mathcal{B}^a_\mu = \langle 0|A^a_\mu|0\rangle_{J\to 0}, \tag{3}$$

where $|0\rangle$ is a vacuum state. It is clear that due to the gauge and Lorentz invariances the vacuum-averaged value of the gauge potential must be identically zero, i.e., $\mathcal{B}^a_\mu \equiv 0$. A partial solution to this problem was suggested by proposing the "spaghetti" vacuum model where the vacuum is represented by a statistical ensemble of vortex domains, which leads to a zero mean value of the gauge field. However, in such cases one encounters two principal obstacles: (i) the statistical field ensemble does not represent an exact solution to the classical equations of motion, and (ii) at the microscopic scale each domain or a single vortex causes instability due to a nonvanishing contribution to the imaginary part of the effective action. Thus, a statistical ensemble does not provide a microscopic theory of the vacuum structure on the firm basis of standard quantum field theory.

With these preliminaries, let us write down the main equations which allow to retrieve the analytic structure of the effective action for an arbitrary background gauge field configuration. It is convenient to choose a covariant Lorenz gauge-fixing condition for the quantum gauge potential,

$$(\mathcal{D}_\mu Q^\mu)^a = 0, \tag{4}$$

where $\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu + g\epsilon^{acb}\mathcal{B}_{\mu c}$ is a covariant derivative including the background gauge field potential $\mathcal{B}^a_\mu$. Applying a standard functional technique, one can express the one-loop correction to the classical action in terms of functional determinants,

$$\begin{aligned} S^{1\,\mathrm{loop}}_{\mathrm{eff}} &= -\frac{i}{2}\ln \mathrm{Det}[K^{ab}_{\mu\nu}] + i\ln \mathrm{Det}[M^{ab}_{\mathrm{FP}}], \\ K^{ab}_{\mu\nu} &= -g_{\mu\nu}(\mathcal{D}^\rho \mathcal{D}_\rho)^{ab} - 2\epsilon^{acb}\mathcal{F}^c_{\mu\nu}, \\ M^{ab}_{\mathrm{FP}} &= -(\mathcal{D}^\rho \mathcal{D}_\rho)^{ab}, \end{aligned} \tag{5}$$

where $\mathcal{F}^a_{\mu\nu}$ is a background field strength and the operators $K^{ab}_{\mu\nu}$, M^{ab}_{FP} correspond to one-loop contributions of gluons and Faddeev-Popov ghosts. One should stress that Eq. (5) represents an exact one-loop result for an arbitrary configuration of the background gauge field $\mathcal{B}^a_\mu$. One can obtain similar expressions for the one-loop functional determinants when using an initial temporal gauge for the quantum gauge potential and an additional Coulomb-type gauge condition, which fixes the residual symmetry.

A. A constant Abelian magnetic field

Let us first consider the simple case of the Savvidy vacuum [14], based on a classical solution for the constant homogeneous magnetic Abelian-type field defined by the gauge potential $\mathcal{B}^a_\mu = g_{\mu 2}\delta^{a3}xH$. The gauge field strength $\mathcal{F}^a_{\mu\nu}$ has only one nonvanishing magnetic component, $\mathcal{F}^3_{12} = H$. In this case, the expression for the one-loop correction to the effective action (5) can be simplified as

$$S^{\text{1 loop}}_{\text{eff}} = i\sum_{S_z=\pm 1} 2\text{Tr}\ln[-\mathcal{D}^\mu\mathcal{D}_\mu + 2gHS_z], \qquad (6)$$

where $S_z = \pm 1$ is a spin projection onto the z axis of the gluon which is treated as a massless vector particle in the Nielsen-Olesen approach [15]. It is clear that the operator inside the logarithmic function is not positively defined for $S_z = -1$. This gives rise to an imaginary part of the effective action and implies the Nielsen-Olesen unstable “tachyon” mode [15]. An important issue is that the origin of the vacuum instability is due to a specific interaction structure of the non-Abelian gauge theory, namely, the anomalous magnetic moment interaction of the vector gluon with the magnetic field H. Note that the contribution of the Faddeev-Popov ghosts does not induce an imaginary part since the interaction of spin-zero ghost fields with the magnetic field has no such anomalous magnetic moment interaction. The functional determinants in Eq. (6) can be calculated using the Schwinger proper-time method. With this, the effective Lagrangian can be expressed in a compact integral form [31–38],

$$\mathcal{L}^{\text{1 loop}}_{\text{eff}} = \frac{1}{16\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^{(2-\varepsilon)}}\frac{gH/\mu^2}{\sinh(gHs/\mu^2)}\left(\mathrm{e}^{-\frac{2gHs}{\mu^2}} + \mathrm{e}^{\frac{2gHs}{\mu^2}}\right), \qquad (7)$$

where ε is the ultraviolet cutoff parameter and μ^2 is a mass scale parameter corresponding to the subtraction point. The second exponential term in the last equation leads to a severe infrared divergence, which is a reflection of the same anomalous magnetic moment interaction term in Eq. (6). One can perform an infrared regularization by changing the proper-time variable to a pure imaginary one, $s \to it$ [38],

$$\mathcal{L}^{\text{1 loop}}_{\text{eff}} = -\frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}t}{t^{(2-\varepsilon)}}\frac{gH/\mu^2}{\sin(gHt/\mu^2)}\cos(2gHt/\mu^2). \qquad (8)$$

This removes the infrared divergence, but now one encounters an ambiguity in choosing contours of the integral due to the appearance of an infinite number of poles at $t = \pi k\mu^2/gH$, $(k = 0, 1, 2, \ldots)$. We define the integration path $t = 0 - i\delta$ with an infinitesimal number factor δ. One can verify that a total residue contribution from the poles exactly reproduces the Nielsen-Olesen imaginary part of the effective Lagrangian [15],

$$\text{Im}\mathcal{L} = \frac{1}{8\pi}g^2H^2. \qquad (9)$$

Note that a color electric field causes the vacuum instability due to the Schwinger mechanism of charged particle-antiparticle pair creation in the external electric field. Moreover, in pure gluodynamics it has been shown that a homogeneous chromoelectric field E leads to a negative imaginary part of the effective one-loop Lagrangian [39],

$$\text{Im}\mathcal{L} = -\frac{11}{96\pi}g^2E^2. \qquad (10)$$

One concludes that a constant homogeneous color magnetic and electric field of Abelian type is unstable. The physical meaning of such an instability is the gluon pair creation in the chromomagnetic field and the gluon pair annihilation in the case of the chromoelectric background field [39].

B. Non-Abelian constant field configuration

It has been established that $SU(N)$ Yang-Mills theory admits two types of constant homogeneous field configurations [40]. The first type is represented by Abelian-type gauge potentials, which correspond to the Cartan subalgebra of the Lie algebra $\mathfrak{su}(N)$. The constant homogeneous fields of the second type originate from the non-Abelian structure of the gauge field strength due to the noncommutativity of the Lie-algebra-valued gauge potentials [40],

$$\vec{F}_{\mu\nu} = \vec{A}_\mu \times \vec{A}_\nu. \qquad (11)$$

Contrary to the case of Abelian constant color magnetic fields, the non-Abelian magnetic field admits a spherically symmetric configuration. It was observed that the symmetrization of the QCD Hamiltonian might help to cure the Nielsen-Olesen instability [41]. After the discovery of the Savvidy-Nielsen-Olesen vacuum instability, some attempts have been undertaken to construct a stable vacuum made of constant non-Abelian gauge fields. The results of studies of such a vacuum have led to a vacuum instability of the same origin, i.e., the presence of the anomalous magnetic moment interaction [42,43].

Let us briefly review the known results, with the purpose of finding a way to resolve the problem of vacuum stability. We consider the following isotropic homogeneous field configuration of non-Abelian type defined by the classical gauge potential,

$$\mathcal{B}^a_0 = 0, \qquad \mathcal{B}^a_m = \phi(t)\delta^a_m. \qquad (12)$$

Throughout this paper, we use latin indices m, n to denote the space components of the four-vectors. The function $\phi(t)$ may require a time dependence to include the case with nonvanishing constant color electric field as well.

We can find eigenvalues of the operators $K^{ab}_{\mu\nu}$, $M^{ab}_{\rm FP}$ in the weak-field approximation, assuming that $\phi(t)$ is a slowly varying function. In the momentum-space representation one has

$$\begin{aligned} K^{ab}_{mn} &= \delta_{mn}(\delta^{ab}(k^2+2\phi^2) - 2i\phi\epsilon^{acb}k_c) \\ &\quad - 2\phi^2(\delta^b_m\delta^a_n - \delta^a_m\delta^b_n), \\ K^{ab}_{0n} &= 2\epsilon^{ab}{}_n b, \\ M^{ab}_{\rm FP} &= \delta^{ab}(k^2+2\phi^2) - 2i\phi\epsilon^{acb}k_c = -K^{ab}_{00}, \end{aligned} \quad (13)$$

where the time derivative term $b \equiv \partial_0\phi$ corresponds to components of a color electric field in the temporal gauge $\mathcal{B}^a_0 = 0$. In the weak-field approximation the fields ϕ and b are treated as constant fields. To find the eigenvalues of the operators $K^{ab}_{\mu\nu}$, $M^{ab}_{\rm FP}$, let us first calculate the corresponding matrix determinants with respect to Lorentz and color indices. After some calculations, we obtain

$$\begin{aligned} \det K^{ab}_{\mu\nu} &= L_1L_2L_3L_4, \\ \det M^{ab}_{\rm FP} &= (2\phi^2+k^2)((2\phi^2+k^2)^2 - 4\phi^2\vec{k}^2), \end{aligned} \quad (14)$$

with

$$\begin{aligned} L_1 &= k^4 - 4\phi^2\vec{k}^2, \\ L_2 &= (2\phi^2+k^2)(k^2(4\phi^2+k^2)(6\phi^2+k^2) \\ &\quad - 4\phi^2(2\phi^2+k^2)\vec{k}^2) + 8k^2(6\phi^2+k^2)b^2, \\ L_3 &= 2\phi^2(k^2-2\phi|\vec{k}|)(3k^2-2\phi|\vec{k}|) + 8\phi^4(k^2-\phi|\vec{k}|) \\ &\quad + k^2(k^2-2\phi|\vec{k}|)^2 + 8(k^2-\phi|\vec{k}|)b^2, \\ L_4 &= 2\phi^2(k^2+2\phi|\vec{k}|)(3k^2+2\phi|\vec{k}|) + 8\phi^4(k^2+\phi|\vec{k}|) \\ &\quad + k^2(k^2+2\phi|\vec{k}|)^2 + 8(k^2+\phi|\vec{k}|)b^2. \end{aligned}$$

In the particular case with a constant pure magnetic field background, $b = 0$, our result reduces exactly to the known expressions obtained earlier in Ref. [42], where it was shown that all eigenvalues corresponding to the operators L_i are real. Explicit expressions for all 12 eigenvalues of the operators L_i in the case of a pure magnetic background field ($b = 0$) were obtained in Ref. [42].

The operator L_1 is decomposed into the product of two eigenvalues,

$$\lambda_{1,2} = k^2 \pm 2\phi|\vec{k}|. \quad (15)$$

It is easy to verify that the expression for L_2 is non-negative for any values of ϕ, b, $\vec{k}$, and k. The operator L_3 has one real and two complex eigenvalues, and L_4 has eigenvalues which are complex conjugate to the eigenvalues of the operator L_3. In general, the complex and negative eigenvalues of the operators L_1, L_3, and L_4 cause vacuum instability.

One may observe that Eq. (15) implies negative eigenvalues for small momentum $\vec{k}$ of the virtual gluon inside the loop. We recall that the Nielsen-Olesen unstable mode originates from the anomalous magnetic moment interaction term gHS_z in Eq. (6), which does not depend on the internal momentum $\vec{k}$. So, in the case of a symmetric field configuration one has no instability in the limit of zero momentum $\vec{k}$. So, the symmetric non-Abelian magnetic field configuration makes the instability problem more soft, even though the source of the negative eigenvalues remains the same as for the Nielsen-Olesen unstable mode.

The instability of the vacuum coming from the non-Abelian gauge field is somewhat puzzling since one expects that the dynamics of a non-Abelian gauge field should provide a consistent quantum vacuum in pure QCD. Thus, one should observe one essential weak point in the above consideration: the constant non-Abelian gauge field does not represent a classical solution. Due to this the standard method based on the formalism of functional integration cannot be applied self-consistently to the derivation of the one-loop effective action. This raises a question of whether a non-Abelian-type magnetic field can be realized as a strict solution, and if so, whether such a solution can provide a stable vacuum. Note that to find a stable physical vacuum one should go beyond the one-loop approximation, since at the one-loop level a quartic self-interaction term in the initial Yang-Mills Lagrangian is omitted and does not affect the final result. However, the confinement phenomenon is certainly provided by the self-interaction of gluons. So, the quartic interaction term should be an essential part of the nonperturbative dynamics. The evaluation of an exact two-loop effective action in QCD represents a hard unresolved problem. To go beyond the one-loop approximation one can implement nonperturbative effects in the structure of the classical solution used as a background field in the effective action. We conclude that one should look for a proper nonperturbative and essentially non-Abelian solution of the classical equations of motion, which can lead to a consistent description of the stable vacuum.

III. QUANTUM INSTABILITY OF NONLINEAR PLANE WAVES

Stationary nonlinear wave type solutions can be treated as quantum-mechanical wave functions which describe possible states in quantum theory. In particular, we are interested in classical solutions that are stable against quantum gluon fluctuations. A known class of nonlinear plane-wave solutions with a mass scale and zero spin

[25–30] is of primary interest in our search of possible stable vacuum fields, since one expects that a system of massive spinless particles can form a stable condensate in the classical theory. The presence of spinless states can help in removing the Nielsen-Olesen instability. We consider a special plane-wave solution in $SU(2)$ Yang-Mills theory which possesses a spherically symmetric configuration in the rest frame [25–30]. A simple ansatz for nonvanishing components of the gauge potential reads

$$\mathcal{B}^a_m = \delta^a_m \phi(u), \tag{16}$$

where $u \equiv k_0 t$. Substituting the ansatz into the Yang-Mills equations, we obtain an ordinary differential equation,

$$k_0^2 \frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} + 2g^2\phi^3 = 0. \tag{17}$$

One has the following nonvanishing components for the color electric and magnetic fields:

$$F^1_{10} = F^2_{20} = F^3_{30} = -\partial_t \phi,$$
$$F^a_{mn} = g\epsilon_{mn}{}^a \phi^2. \tag{18}$$

The solution to Eq. (17) is given by the Jacobi elliptic function

$$\phi(u) = \frac{M}{g}\,\mathrm{sn}[Mt, -1], \tag{19}$$

which is a double-periodic analytic function with a periodicity $T_0 = 4K[-1] \simeq 5.244\ldots$, ($M = 1$), and $K[-1]$ is a complete elliptic integral of the first kind. The solution contains a mass scale parameter M due to the conformal invariance of the equations of motion.

The one-loop effective potential in a constant color electric and magnetic field possesses a local minimum for a nonzero value of the magnetic field and for a vanishing electric field. The presence of the electric field in the solution (18) can lead to the instability of the vacuum due to the Schwinger pair-creation effect. However, since the electric field of the solution is represented by a periodic function, the time dependence may change the stability properties of the vacuum field. Another advantage of treating the stationary plane-wave solutions as a quantum-mechanical wave function describing the vacuum state is that the time averaging leads naturally to a vanishing of the vacuum expectation value of the gauge potential, $\langle 0|A^a_\mu|0\rangle = 0$, whereas the averaged magnetic field remains nonzero.

Now we can study the structure of the functional determinants in Eq. (13). It turns out that the matrix operator K^{ab}_{mn} gains complex eigenvalues. The presence of complex eigenvalues complicates the analysis of the structure of the effective action, since in that case one needs to know the analytic structure of the full effective action in the presence of color magnetic and electric fields [44]. Due to this, we consider the structure of the one-loop effective action in the temporal gauge, $Q^a_0 = 0$ [the background gauge field satisfies the temporal condition due to the structure of the ansatz (16)], which significantly simplifies the analysis of possible unstable modes. In the temporal gauge, one has a known residual gauge symmetry under the gauge transformations with space-dependent gauge parameters. To fix this symmetry one can impose an additional Coulomb constraint, $\partial^m Q^a_m = 0$. Therefore, the calculation of the Faddeev-Popov ghost determinant becomes more difficult since one should introduce secondary ghosts. However, since all ghost fields correspond to interactions of spinless particles with the magnetic field they do not cause vacuum instability, and we do not need to calculate ghost contributions when studying the imaginary part of the effective action. With this, one can perform a functional integration over the quantum field Q^a_μ and obtain the following expression for the matrix operator K^{ab}_{mn}:

$$K^{ab}_{mn} = \delta_{mn}\delta^{ab}(\partial_t^2 - \partial_l^2 + 2g^2\phi^2(t)) + 2\epsilon^{ab}{}_c \mathcal{F}^c_{mn}$$
$$- g\phi(t)(\epsilon^{ab}{}_m \partial_n + \epsilon^{ab}{}_n \partial_m + 2\epsilon^{acb}\delta_{mn}\partial_c). \tag{20}$$

Since the field $\phi(t)$ does not depend on space coordinates, one can easily perform a Fourier transformation with respect to the space coordinates. After performing the Wick rotation $t \to i\tau$, one arrives at the following expression for the operator K^{ab}_{mn} in the momentum-space representation:

$$K^{ab}_{mn} = \delta_{mn}\delta^{ab}(-\partial_\tau^2) + \delta^{ab}\delta_{mn}(\vec{k}^2 + 2\phi^2)$$
$$+ \epsilon^{ab}{}_c(2\phi^2\epsilon^c_{mn} + 2\phi\delta_{mn} i\vec{k}^c - \phi\delta^c_m i\vec{k}_n + \phi\delta^c_n i\vec{k}_m)$$
$$\equiv \delta_{mn}\delta^{ab}(-\partial_\tau^2) + \hat{K}^{ab}_{mn}. \tag{21}$$

One can find the eigenvalues $\hat{L}_i$ of the matrix operator $\hat{K}^{ab}_{mn}$ since the field ϕ does not depend on the space components of the momentum,

$$\hat{L}_1 = \vec{k}^2,$$
$$\hat{L}_{2,3} = \vec{k}^2 + 4\phi^2 \pm \phi|\vec{k}|,$$
$$\hat{L}_{4,5} = \vec{k}^2 + 5\phi^2 \pm \sqrt{\phi^4 + 6\phi^2\vec{k}^2},$$
$$\hat{L}_{6,7} = \vec{k}^2 \pm \phi|\vec{k}|,$$
$$\hat{L}_{8,9} = \vec{k}^2 \pm 2\phi|\vec{k}|. \tag{22}$$

With this, one has finally nine ordinary second-order differential equations for eigenfunctions of the initial kinetic operator K^{ab}_{mn},

$$\left(-\frac{d^2}{d\tau^2}+\hat{L}_q\right)\psi_q=\lambda_q\psi_q, \qquad (q=1,2,\ldots,9). \tag{23}$$

The differential equations containing the operators $\hat{L}_q$, $(q = 6, 7, 8, 9)$ might have negative eigenvalues since the respective differential operators are not positively defined at small momenta $\vec{k}$. Let us rewrite Eq. (23) in the case $q = 6, 7, 8, 9$ in the following form:

$$-\frac{d^2\psi}{d\tau^2}+(k^2+\alpha k\phi(\tau))\psi=\lambda\psi, \tag{24}$$

where $k\equiv|\vec{k}|$ and $\alpha=\pm1,\pm2$. Note that the classical solution $\phi(\tau)$ is identical to the original solution $\phi(t)$ in Eq. (19), since by definition the classical field $\mathcal{B}_m^a$ corresponds to a vacuum-averaged value of the quantum operator A_μ^a in the real Minkowski space-time. We recall that the Wick rotation $t\rightarrow i\tau$ provides a causal structure for the Green function, and it does not mean that one should treat the classical field $\mathcal{B}_m^a$ as a solution of the equations of motion in the Euclidean space-time.

Equation (24) includes the momentum k as a free positive parameter, and the quantum vacuum stability of the classical solution will occur if all of the eigenvalues of Eq. (24) are non-negative for all values of "k" and for $\alpha=\pm1,\pm2$. It is convenient to rewrite Eq. (24) as follows:

$$-\frac{d^2\psi}{d\tau^2}+V_0(1-\phi(\tau))\psi=E\psi, \tag{25}$$

with $V_0\equiv\alpha k$ and $E\equiv\lambda-k^2+\alpha k$. The equation represents a Schrödinger-type equation for a quantum-mechanical problem in one-dimensional space parametrized by $\tau\geq0$, and $\psi(\tau)$ is a wave function describing quantum fluctuations of the virtual gluon. One can make another analogy, namely, that Eq. (25) describes the behavior of the electron in a one-dimensional crystal with a periodic potential. It is known that such an electron in a crystal is not localized and can move freely in the entire crystal volume. The electron wave function is expressed by the periodic Bloch function and the energy spectrum forms a band structure (see, for example, Ref. [45]). To check whether Eq. (25) has negative eigenvalues, it is enough to estimate a lowest energy bound in the first energy band. As a qualitative estimation, we first consider a Schrödinger equation with a periodic rectangular potential,

$$V(\tau)=\begin{cases}+1, & nT\leq\tau\leq(2n+1)\frac{T}{2},\\ -1, & (2n-1)\frac{T}{2}\leq\tau\leq nT,\end{cases} \tag{26}$$

where $n=0,\pm1,\pm2,\cdots$. Analytic expressions for a solution of the Schrödinger equation with the potential $V(\tau)$ and the dispersion relation can be obtained by solving the equation on a finite interval $(0\leq\tau\leq T)$ [45]. Taking the shift in the potential height into account and setting $T=1$, one can find an eigenvalue corresponding to the lowest energy level in the first band, which turns out to be negative: $E_{\text{lowest}}\simeq-0.04$.

The numerical analysis of Eq. (25) shows that for $\alpha=1$ there are no negative eigenvalues for any momentum k, and the eigenvalue λ approaches zero from positive values when $k\rightarrow0$. For the case $\alpha=\pm2$ the numeric solutions of Eq. (25) imply negative eigenvalues for the momentum k in the range $(0\leq k\leq0.733)$, with the lowest eigenvalue $\lambda_{\text{lowest}}=-0.0361$ at $k_0=0.482$. Note that the scale parameter M in the nonlinear plane-wave solution $\phi(x)=M\text{sn}[Mx,-1]$ leads to a rescaling of the eigenvalue λ and does not affect the stability properties, as it should be due to the conformal invariance of the original classical Yang-Mills theory. We conclude that, despite several attractive properties, the nonlinear plane-wave solutions cannot provide a stable vacuum field configuration.

IV. A STABLE SPHERICALLY SYMMETRIC MONOPOLE FIELD BACKGROUND

Let us first describe the main properties of the stationary spherically symmetric monopole solution [30]. Due to the conformal invariance of Yang-Mills theory, the static soliton solutions do not exist, in agreement with Derrick's theorem. It is somewhat unexpected that pure QCD admits a regular stationary monopole-like solution [30]. The solution is described by a simple ansatz which generalizes the static Wu-Yang monopole solution [in spherical coordinates (r,θ,φ)],

$$A_\varphi^1=-\psi(r,t)\sin\theta, \qquad A_\theta^2=\psi(r,t), \qquad A_\varphi^3=\frac{1}{g}\cos\theta, \tag{27}$$

where $\psi(r,t)$ is an arbitrary function and all other components of the gauge potential vanish. In the case where $\psi(r,t)=0$, the ansatz describes a Wu-Yang monopole solution which is singular at the origin $r=0$. The case $\psi(r,t)=1$ corresponds to a pure gauge field configuration. For a nontrivial function $\psi(r,t)$, the ansatz (27) describes a system of a static Wu-Yang monopole dressed in an off-diagonal gluon field. Substituting the ansatz into the equations of motion, we obtain a single partial differential equation,

$$\partial_t^2\psi-\partial_r^2\psi+\frac{1}{r^2}\psi(g^2\psi^2-1)=0. \tag{28}$$

Equation (28) was obtained in the past by using a spherically symmetric "hedgehog" ansatz describing generalized $SU(2)$ Wu-Yang monopole field configurations $(a=1,\ 2,\ 3)$,

$$A^a_m = -\epsilon^{abc}\hat{n}^b\partial_m\hat{n}^c\left(\frac{1}{g} - \psi(t,r)\right), \tag{29}$$

where $\hat{n} = \vec{r}/r$ [46–50]. Note that the "hedgehog" ansatz (29) is related to the ansatz (27) by an appropriate singular gauge transformation [51]. We prefer to use the ansatz (27) in the so-called Abelian gauge [51] since such a representation allows us to interpret our monopole solution as a static Wu-Yang monopole interacting with dynamic off-diagonal gluons represented by the field $\psi(r,t)$. Note that the ansatz in the Abelian gauge admits a generalization to the case of $SU(N)$ stationary Wu-Yang monopole solutions, and it is suitable as a description of a stationary system of monopoles and antimonopoles located at different points.

It has been shown that Eq. (28) admits a wide class of time-dependent solutions, including nonstationary solitonic propagating solutions in the effective two-dimensional space-time (r,t) [46–50]. Surprisingly, a stationary regular Wu-Yang type monopole solution with a finite energy density everywhere was missed in previous studies. We will show that such a solution provides a stable vacuum configuration in pure $SU(2)$ QCD.

Let us consider a classical Hamiltonian written in terms of the field $\psi(r,t)$,

$$H = \int \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \sin\theta((\partial_t\psi)^2 + (\partial_r\psi)^2 + \frac{1}{2g^2r^2}(g^2\psi^2 - 1)^2) \equiv 4\pi\int \mathrm{d}r\mathcal{E}(r,t), \tag{30}$$

where $\mathcal{E}$ is an effective energy density in one-dimensional space. One has the following nonvanishing field-strength components:

$$F^2_{r\theta} = \partial_r\psi, \qquad F^1_{r\varphi} = -\partial_r\psi\sin\theta,$$
$$F^3_{\theta\varphi} = g^2\left(\psi^2 - \frac{1}{g^2}\right)\sin\theta,$$
$$F^2_{t\theta} = \partial_t\psi, \qquad F^1_{t\varphi} = -\partial_t\psi\sin\theta, \tag{31}$$

where the radial component of the field strength $F^3_{\theta\varphi}$ describes spherically symmetric monopole configuration with a nonvanishing color magnetic flux through a sphere with a center at the origin $r = 0$ [30]. The color magnetic charge of the monopole depends on time and the radius of the sphere.

One can find an asymptotic behavior of the stationary solution, which approaches a standing spherical wave in the leading order of the Fourier series expansion,

$$\psi(r,t) \simeq a_0 + A_0\cos(Mr)\sin(Mt) + \mathcal{O}\left(\frac{1}{r}\right), \tag{32}$$

where a_0 and A_0 are parameters characterizing the mean value and amplitude of the standing spherical wave in the asymptotic region. The mass scale parameter M corresponds to the conformal symmetry of the original Yang-Mills equations.

A local solution near the origin $r = 0$ is given by the Taylor series expansion

$$\psi = \frac{1}{g} + \sum_{k=1} c_{2k}(t)r^{2k}, \tag{33}$$

where all coefficient functions $c_{2k>2}(t)$ are expressed in terms of one arbitrary function $c_2(t)$ defining the initial conditions. The presence of the first term $1/g$ indicates a nonperturbative origin of the solution. One can verify that such a term regularizes the singularity of the Wu-Yang monopole and provides a finite energy density. To find a stationary solution one can impose initial conditions by choosing the function $c_2(t)$ in its simplest form, $c_2(t) = \tilde{c}_0 + \tilde{c}_{20}\sin(Mt)$. We will choose the initial profile function $c_2(t)$ in terms of the Jacobi elliptic function (19),

$$c_2(t) = c_0 + c_{20}\mathrm{sn}[Mt, -1], \tag{34}$$

where the set of the parameters c_0, c_{20}, and M provides a unique general solution within a consistent Cauchy problem for the differential equation (28). The choice of the initial profile function $c_2(t)$ [Eq. (34)] provides additional control over the consistency of the numerical calculation to verify that the numerical solution matches the asymptotic solution (32) given precisely by the ordinary sine function $\sin(Mt)$ (in the leading order of the Fourier series decomposition). A subclass of stationary solutions is classified by one independent parameter: c_0 or c_{20}.

A simple dimensional analysis implies that the energy corresponding to the Hamiltonian (30) is proportional to the scale parameter M. Due to this, the energy vanishes in the limit $M \to 0$. This might cause some doubts about existence of a solution. However, one should stress that standard arguments on the existence of solitonic solutions based on Derrick's theorem [22] cannot be applied to the case of stationary solutions which satisfy a variational principle of extremal value of the classical action, not the energy functional. In addition, in the case of a pure Yang-Mills theory the action is invariant under conformal transformations, and its first variational derivative with respect to the scale parameter M equals zero identically. So the parameter M represents a moduli space parameter of solutions related by conformal transformations (dilatations) $r \to Mr$, $t \to Mt$. Without loss of generality one can fix the value of M to an arbitrary number which determines the unit of the space-time coordinates.

In order to solve Eq. (28) numerically we choose special values for the parameters: $g = 1$, $M = T_0/(2\pi)$, and

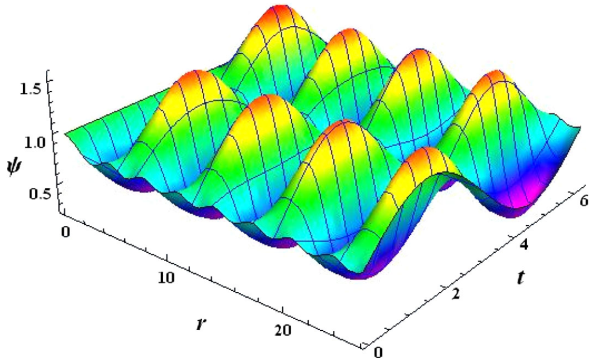


FIG. 1. Stationary spherically symmetric monopole solution in the numeric domain $(0 \le r \le 8\pi, 0 \le t \le 2\pi)$, with $c_0 = -0.251$, $a_0 = 0.84175$, and $A_0 = 0.6405$.

$c_0 = -0.251$. The parameter c_{20} is fixed by the requirement that a numeric solution should match the asymptotic solution (32). The mean value a_0 and amplitude A_0 of the oscillating asymptotic solution are extracted from the numeric solution, which is depicted in Fig. 1.

Note that at large distance r_f after space-time averaging over the ring $(r_f \le r \le r_f + 2\pi, 0 \le t \le 2\pi)$, one gains a partial screening effect for the monopole charge. The obtained numeric solution implies an averaged monopole charge at $r_f = 30$,

$$g_{\rm m} = \frac{1}{4\pi}\int \mathrm{d}\theta\mathrm{d}\varphi H_{\theta\varphi} = \frac{1}{4\pi}\int \mathrm{d}\theta\mathrm{d}\varphi(\langle\psi^2\rangle - 1)\sin\theta \\ = 0.195\cdots, \tag{35}$$

with

$$\langle\psi^2\rangle = \frac{1}{4\pi^2}\int_{r_f}^{r_f+2\pi}\mathrm{d}r\int_0^{2\pi}\mathrm{d}t\psi^2(r,t).$$

The space-time-averaged magnetic flux of the radial color magnetic field $H^3_{\theta\varphi}$ through a sphere does not vanish in general and depends on the radius of the sphere. There is a special nontrivial solution with the parameters $a_0 = 0.1\cdots$, $A_0 = 1.989\cdots$ which corresponds to a totally screened averaged monopole charge.

With a given numeric monopole solution one can verify the quantum stability of the monopole field in a similar manner as we considered in the previous section. One should solve the following Schrödinger-type eigenvalue equation for possible unstable modes [the space indices $(m, n = 1, 2, 3)$ correspond to the spherical coordinates (r, θ, φ), respectively]:

$$K^{ab}_{mn}\Psi^b_n(r,\theta,\varphi,t) = \lambda\Psi^a_m(r,\theta,\varphi,t), \tag{36}$$

where $\Psi^b_n(r,\theta,\varphi,t)$ are the wave functions describing the quantum gluon fluctuations, and K^{ab}_{mn} is a differential matrix operator corresponding to the one-loop gluon contribution to the effective action in the temporal gauge,

$$K^{ab}_{mn} = -\delta^{ab}g_{mn}\partial_t^2 - g_{mn}(\mathcal{D}_n\mathcal{D}_n)^{ab} + 2\epsilon^{ab}{}_c\mathcal{F}^c_{mn}. \tag{37}$$

The Schrödinger-type equation (36) represents a system of nine nonlinear partial differential equations which should be solved on a three-dimensional numeric domain with sufficiently high numerical accuracy. An additional technical difficulty in the numerical calculation is that one must solve the equations while changing the size of the numeric domain in the radial direction in the limit $r \to \infty$ to verify that all eigenvalues remain positive. Fortunately, the numerical analysis of the solutions corresponding to the lowest eigenvalue is simplified drastically due to the factorization property of the original equation (36) and a special feature of the class of ground-state solutions, as we will see below.

Equation (36) in component form admits factorization, and it can be written as two decoupled systems of partial differential equations as follows (for brevity of notation we set $g = 1$ since the coupling constant can be absorbed by the monopole function ψ):

(I)

$$\begin{aligned}
&(\hat\Delta\Psi)^2_2 - \frac{2}{r^2}\partial_\theta\Psi^2_1 + \frac{1}{r^2}((\psi^2-1)\Psi^2_2 - 2\psi^2\Psi^1_3 + 2\csc^2\theta(\Psi^2_2+\Psi^1_3) + 2\cot\theta\psi\Psi^3_3) = \lambda\Psi^2_2,\\
&(\hat\Delta\Psi)^1_3 - \frac{2}{r^2}\psi\partial_\theta\Psi^3_3 + \frac{1}{r^2}(\psi^2(-2\Psi^2_2+\Psi^1_3) - \Psi^1_3 + 2\csc^2\theta(\Psi^2_2+\Psi^1_3) + 2\cot\theta\Psi^2_1) = \lambda\Psi^1_3,\\
&(\hat\Delta\Psi)^2_1 + \frac{2}{r^2}\partial_\theta\Psi^2_2 + \frac{1}{r^2}((\cot^2\theta+\psi^2)\Psi^2_1 + 2\cot\theta(\Psi^2_2+\Psi^1_3) + 2\psi\Psi^3_3 + 2\Psi^2_1) - \frac{2}{r}\partial_r\psi\Psi^3_3 = \lambda\Psi^2_1,\\
&(\hat\Delta\Psi)^3_3 + \frac{2}{r^2}\psi\partial_\theta\Psi^1_3 + \frac{1}{r^2}(2\psi\Psi^2_1 + 2\cot\theta\psi(\Psi^2_2+\Psi^1_3) + 2\psi^2\Psi^3_3 + \csc^2\theta\Psi^3_3) - \frac{2}{r}\partial_r\psi\Psi^2_1 = \lambda\Psi^3_3,
\end{aligned} \tag{38}$$

(II)

$$
\begin{aligned}
&(\hat{\Delta}\Psi)^1_1 + \frac{2}{r^2}\partial_\theta\Psi^1_2 - \frac{2}{r^2}\psi\partial_\theta\Psi^3_1 + \frac{1}{r^2}((2+\cot^2\theta+\psi^2)\Psi^1_1 + 2\psi\Psi^3_2 - 2\cot\theta(\Psi^2_3 - \Psi^1_2))\\
&\quad - \frac{2}{r}\partial_r\psi\Psi^3_2 = \lambda\Psi^1_1,\\
&(\hat{\Delta}\Psi)^3_2 - \frac{2}{r^2}\partial_\theta\Psi^3_1 + \frac{2}{r^2}\psi\partial_\theta\Psi^1_2 + \frac{1}{r^2}(2\psi\Psi^1_1 + 2\cot\theta\psi(\Psi^1_2 - \Psi^2_3) + (2\psi^2 + \csc^2\theta)\Psi^3_2)\\
&\quad - \frac{2}{r}\partial_r\psi\Psi^1_1 = \lambda\Psi^3_2,\\
&(\hat{\Delta}\Psi)^1_2 - \frac{2}{r^2}\partial_\theta\Psi^1_1 - \frac{2}{r^2}\psi\partial_\theta\Psi^3_2 + \frac{1}{r^2}(-2\psi\Psi^3_1 + \psi^2(\Psi^1_2 + 2\Psi^2_3) + 2\csc^2\theta(\Psi^1_2 - \Psi^2_3) - \Psi^1_2)\\
&\quad + \frac{2}{r}\partial_r\psi\Psi^3_1 = \lambda\Psi^1_2,\\
&(\hat{\Delta}\Psi)^3_1 + \frac{2}{r^2}\partial_\theta\Psi^3_2 + \frac{2}{r^2}\psi\partial_\theta\Psi^1_1 + \frac{1}{r^2}(2\cot\theta\psi\Psi^1_1 + 2(1+\psi^2)\Psi^3_1 - 2\psi(\Psi^1_2 + \Psi^2_3) + 2\cot\theta\Psi^3_2)\\
&\quad + \frac{2}{r}\partial_r\psi(\Psi^1_2 + \Psi^2_3) = \lambda\Psi^3_1,\\
&(\hat{\Delta}\Psi)^2_3 + \frac{1}{r^2}(2\cot\theta(\psi\Psi^3_2 - \Psi^1_1) - 2\psi\Psi^3_1 + \psi^2(2\Psi^1_2 + \Psi^2_3) - 2\csc^2\theta(\Psi^1_2 - \Psi^2_3) - \Psi^2_3)\\
&\quad + \frac{2}{r}\partial_r\psi\Psi^3_1 = \lambda\Psi^2_3,
\end{aligned}
\tag{39}
$$

where

$$
\hat{\Delta}\Psi^a_m \equiv -\left(\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cot\theta}{r^2}\partial_\theta\right)\Psi^a_m.
$$

To numerically solve the systems of equations (I) and (II), we choose a rectangular three-dimensional domain $(0 \le t \le 2\pi, r_0 \le r \le L, 0 \le \theta \le \pi)$ and use a simple interpolating function for the monopole solution $\psi(r,t)$,

$$
\begin{aligned}
\psi^{\rm int} = 1 &- \frac{(1-a_0)r^2}{1+r^2}\\
&+ A_0(1 - {\rm e}^{-d_0 r^2})\cos(Mr + b_0)\sin(Mt),
\end{aligned}
\tag{40}
$$

where d_0 and b_0 are fitting parameters. The obtained numerical solution to the system of equations (I) [Eq. (38)] implies that the lowest eigenvalue is positive, $\lambda_{\rm I} = 0.0531$, and the corresponding eigenfunctions have the following properties: the functions Ψ^2_1 and Ψ^3_3 vanish identically, and remaining two functions are related by the constraint $\Psi^1_3 = -\Psi^2_2$. Thus, there is only one independent nonvanishing eigenfunction which can be chosen as Ψ^2_2. An important feature of the solution corresponding to the lowest eigenvalue is that the eigenfunction Ψ^2_2 does not depend on the polar angle; see Fig. 2. This allows one to simplify the system of equations (I) in the case of solutions corresponding to the lowest eigenvalues. One can easily verify that the system of equations (I) reduces to one partial differential equation in two space-time dimensions,

$$
\left(-\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r + \frac{1}{r^2}(3\psi^2 - 1)\right)\Psi^2_2 = \lambda\Psi^2_2. \tag{41}
$$

The last equation represents a simple Schrödinger-type equation for a quantum-mechanical problem. The equation does not admit negative eigenvalues if the parameter a_0 of the monopole solution satisfies the condition $a_0 \ge 1/\sqrt{3} \simeq 0.577\ldots$, which provides a totally repulsive quantum-mechanical potential in this equation.

The structure of the system of equations (II) admits similar factorization properties in the space of ground-state solutions. We have numerically solved the equations (II)

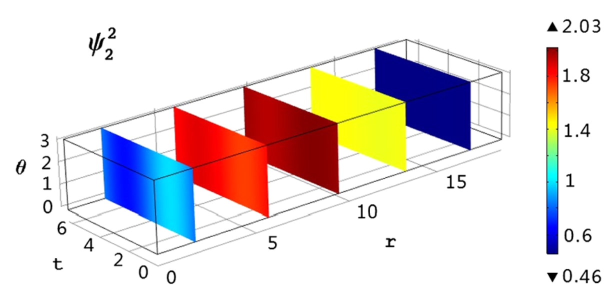
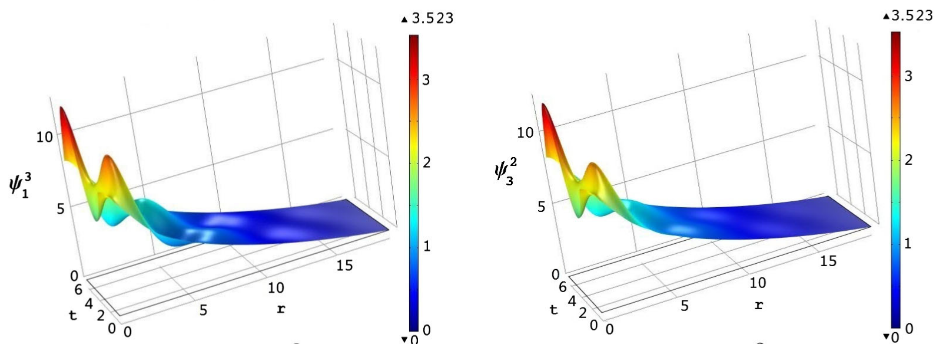


FIG. 2. Eigenfunction Ψ^2_2 for the ground state with the lowest eigenvalue $\lambda_I = 0.0531$, with $a_0 = 0.895$, $A_0 = 0.615$, $g = 1$, $M = 1$, $0 \le r \le 6\pi$, $0 \le t \le 2\pi$, and $0 \le \theta \le \pi$.



(a) Ψ_1^3 (b) Ψ_3^2

FIG. 3. Solutions to Eq. (42): the functions $\Psi_1^3(r,t)$ (a) and $\Psi_2^1(r,t) = \Psi_3^2(r,t)$ (b) corresponding to the eigenvalue $\lambda = 0.014218$, with $a_0 = 0.895$, $A_0 = 0.615$, $M = 1$, $0 \le r \le 6\pi$, and $0 \le t \le 2\pi$.

[Eq. (39)] with the same background monopole function $\psi(r,t)$ for various values of the parameters a_0, A_0, and M. In the special case with $a_0 = 0.895$, $A_0 = 0.615$, and $0 \le r \le 6\pi$ the obtained numeric solution for the ground state has a lowest eigenvalue $\lambda_{\rm II} = 0.0142$, which is less than $\lambda_{\rm I}$. None of the components of the solution are dependent on the polar angle, and they satisfy the following relationships: $\Psi_2^1 = \Psi_3^2$ and $\Psi_1^1 = \Psi_2^3 = 0$. There are two independent nonvanishing functions which can be chosen as Ψ_1^3 and Ψ_3^2. One can check that in the space of solutions corresponding to the lowest eigenvalue the system of equations (II) reduces to two coupled partial differential equations for two functions $\Psi_1^3(r,t)$ and $\Psi_3^2(r,t)$,

$$\left(-\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\Psi_1^3 + \frac{2}{r^2}((1+\psi^2)\Psi_1^3 - 2\psi\Psi_3^2) + \frac{4}{r}\partial_r\psi\Psi_3^2 = \lambda\Psi_1^3,$$
$$\left(-\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\Psi_3^2 + \frac{1}{r^2}((3\psi^2 - 1)\Psi_3^2 - 2\psi\Psi_1^3) + \frac{2}{r}\partial_r\psi\Psi_1^3 = \lambda\Psi_3^2. \qquad (42)$$

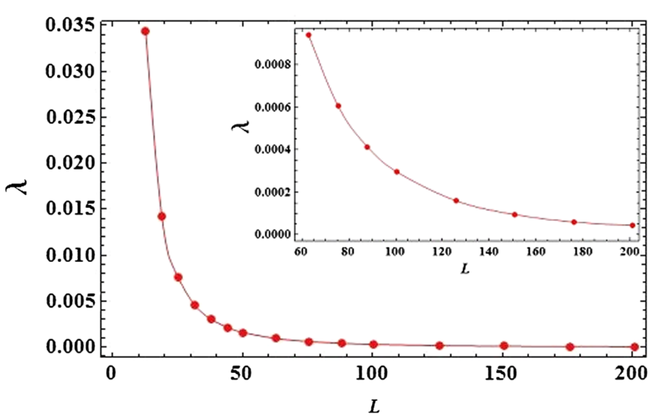


FIG. 4. Lowest eigenvalue dependence $\lambda(L)$ on the radial size L of the numerical domain.

Exact numerical solution profiles for the functions Ψ_1^3 and Ψ_3^2 are shown in Fig. 3.

We have obtained that the lowest eigenvalue is positive when the asymptotic monopole amplitude A_0 is less than the critical value $a_{1\rm cr} \simeq 0.625$.

We conclude that a ground-state solution with the lowest eigenvalue satisfying the original eigenvalue equation (36) can be found by solving a simple system of partial differential equations [Eq. (42)]. Note that numerically solving the original eigenvalue equations (36) in three-dimensional space-time does not provide a high enough accuracy, especially in the case of a large radial size of the numerical domain. This causes difficulty in studying the positiveness of the eigenvalue spectrum in the limit of infinite space when the eigenvalues become very close to zero. Solving the reduced two-dimensional partial differential equations (42) can be performed easily using standard numerical packages with a high enough numerical accuracy and convergence. The obtained numerical accuracy for the eigenvalues $\lambda(L)$ in solving the two-dimensional equations (42) is 1.0×10^{-5}, which allows one to construct the eigenvalue dependence on the radial size L of the space-time domain in the range $6\pi \le L \le 64\pi$. We have proved that the lowest eigenvalue $\lambda(L)$ approaches zero with increasing L from positive values, as it is shown in Fig. 4. This implies that the ground-state solution describes the main mode of the standing spherical wave with a wave vector proportional to the inverse of the radial size of the box, $|\vec{p}| \simeq 1/L$. This completes the proof of the quantum stability of the spherically symmetric stationary monopole solution.

The stationary single monopole solution represents a simple example of a spherically symmetric vacuum field that has a nontrivial intrinsic microscopic structure determined by two parameters: the amplitude A_0 and frequency M of space-time oscillations of the monopole field. Quantum-mechanical consideration implies that the frequency of vacuum monopole field oscillations has a finite minimum value. One can estimate a lower bound on M

using the condition that the characteristic length $\lambda = 2\pi/M$ of the monopole field should be less than the hadron size. At the macroscopic scale, when the observation time is much larger than the period of oscillations of the stationary monopole solution, the vacuum-averaged value of the gauge potential $\langle 0|A^a_\mu|0\rangle$ vanishes, as it should be in the confinement phase. Contrary to this, the so-called vacuum gluon (monopole) condensate $H^2 \equiv \langle 0|\vec{F}^2_{\mu\nu}|0\rangle$ does not vanish after averaging over time, and it has an inhomogeneous distribution inside the hadron. Calculating an exact effective action in the case of inhomogeneous background vacuum fields represents an unresolved problem. In the weak-field approximation one can apply the known expression for the Savvidy renormalized one-loop effective potential [14,31–38]

$$V_{\text{eff}}(H) = \frac{1}{4}H^2 + \frac{11g^2(\mu)}{96\pi^2}H^2\left(\ln\frac{g(\mu)H}{\mu^2} - \frac{3}{2}\right), \quad (43)$$

where $g(\mu)$ is a renormalized coupling constant defined at the subtraction point $\mu^2 \simeq \Lambda_{\text{QCD}}$ $[\alpha_s = g^2(\mu)/(4\pi) \simeq 1]$. For qualitative estimates we replace the vacuum spherically symmetric monopole field H^2 with its mean value $\overline{H^2}$ obtained by averaging over space and time. The potential $V_{\text{eff}}(\bar{H})$ has a nontrivial minimum corresponding to a negative vacuum energy density at a nonzero value of the averaged monopole field, $\bar{H}_0 \simeq 0.138\ \mu^2$ [38]. The value $\bar{H}_0$ is consistent with the frequency and amplitude values $(M \simeq 1, A_0 \leq a_{1cr})$ corresponding to stable stationary monopole field configurations.

One should stress that the generation of a nontrivial vacuum originates from the magnetic moment interaction between the vacuum magnetic field and quantum gluon fluctuations. Such an interaction causes the vacuum energy to decrease for sufficiently small values of the vacuum monopole condensate parameter $\bar{H}$. In the case of the spherically symmetric monopole solution our numerical analysis confirms that for large values of the parameters M and A_0 (i.e., for large values of $\bar{H}_0$), the monopole field obtains quantum instability which prevents the generation of a stable monopole condensate.

V. DISCUSSION

We have demonstrated that there is a subclass of stationary spherically symmetric monopole solutions which possesses quantum stability for restricted values of the amplitude A_0 of the asymptotic monopole solution (32). Recently, it has been found that there is another stable stationary monopole-antimonopole solution in $SU(2)$ and $SU(3)$ QCD [52]. This gives hope that a true vacuum can be formed through condensation of such monopoles and/or monopole-antimonopole pairs.

The existence of stable monopole field configurations and the possible formation of a gauge-invariant vacuum monopole condensate may shed light on the origin of color confinement in QCD and give a partial answer to a simple but puzzling question: why do we have spontaneous symmetry breaking in the electroweak theory, while in QCD the color symmetry is preserved despite the similar gauge group structure in both theories? The vanishing vacuum-averaged value of the gluon field operator corresponding to the stationary monopole solution $\langle A^a_m\rangle$ testifies that there is no spontaneous symmetry breaking in QCD in the confinement phase. One can apply the ansatz (27) to electroweak gauge potentials corresponding to the group $SU(2) \times U_Y(1)$ of the Weinberg-Salam model to find similar stationary electroweak monopole solutions. One can consider the Higgs complex doublet Φ in the unitary gauge, and choose a simple Dirac monopole ansatz for the hypermagnetic field $\mathcal{B}_\mu$,

$$\Phi = \begin{pmatrix} 0 \\ \rho(r,t) \end{pmatrix}, \qquad \mathcal{B}_\mu = \cos\theta. \quad (44)$$

Direct substitution of the ansatz (27) and the last equations (44) into the equations of motion of the Weinberg-Salam model results in two equations for two functions $\psi(r,t)$ and $\rho(r,t)$,

$$\partial_t^2\psi - \partial_r^2\psi + \frac{1}{2}\psi\rho^2 + \frac{1}{r^2}(\psi^2 - 1) = 0,$$
$$\partial_t^2\rho - \partial_r^2\rho - \frac{2}{r^2}\partial_r\rho + \frac{1}{2r^2}\rho\psi^2 + \kappa\rho(\rho^2 - 1) = 0, \quad (45)$$

where κ is the coupling constant of the Higgs potential. In the special case of static field configurations, Eq. (45) reduces to the ordinary differential equations describing the known Cho-Maison monopole [53]. A simple numerical analysis of Eq. (45) shows that a nonstatic generalization of the Cho-Maison monopole exists; however, it has the same singularity at the origin $r = 0$. We conclude that there is a principal difference between the Weinberg-Salam model and QCD: the absence of a regular monopole solution in the Weinberg-Salam model implies that there is no generation of a stable monopole condensation like in QCD. This leads to nonvanishing vacuum-averaged values for the gauge bosons and, consequently, to the spontaneous symmetry breaking. Contrary to this, in the confinement phase of QCD the mean value of the monopole field $\langle 0|A^a_m|0\rangle$ averaged over the periodic space-time domain vanishes, so that the color symmetry is exact.

In conclusion, we have demonstrated that a classical stationary spherically symmetric monopole solution provides a stable vacuum field configuration in pure $SU(2)$ QCD. A generalization of our results to the case of $SU(3)$ QCD has been presented in a separate paper [54]. The possibility that a stationary classical solution can be related to vacuum structure is not very surprising, since it was noticed in the past that color magnetic flux tubes in the "spaghetti" vacuum should be vibrating due to quantum-mechanical considerations [55]. An unexpected result is that a stationary color monopole solution exists in pure QCD without any matter fields, and it possesses remarkable features such as a finite energy density, zero total spin, and the existence of an intrinsic mass scale parameter. This gives a strong indication of the generation of a stable vacuum condensate in QCD.

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APPENDIX: VARIATIONAL ANALYSIS OF THE QUANTUM STABILITY OF THE STATIONARY MONOPOLE FIELD

To reveal the origin of the stability of our numerical solution, we analyze the eigenvalue spectrum of the Schrödinger-type equation (36). Since we are interested only in the lowest eigenvalue solution,we can approximately solve Eq. (36) by applying variational methods. Within the framework of the variational approach one has to minimize the following "energy" functional:

$$\mathcal{H} = \int \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}t r^2 \sin\theta \Psi^a_m K^{ab}_{mn} \Psi^b_n. \tag{A1}$$

The structure of the kinetic operator K^{ab}_{mn} and the finiteness condition of the functional $\mathcal{H}$ allow us to fix the singularities along the boundaries $\theta = 0$ and $\theta = \pi$ in the integral density in Eq. (A1). We factorize the angle dependence of the ground-state wave functions using the leading-order approximation in a Fourier series expansion for the functions f^a_m as

$$\Psi^3_1(r,\theta,\varphi,t) = f^3_1(r,t), \tag{A2}$$

and for the other functions

$$\Psi^a_m(r,\theta,\varphi,t) = f^a_m(r,t)\sin\theta. \tag{A3}$$

With this, one can perform the integration in Eq. (A1) over the angle variables (θ,φ) and obtain an effective "energy" functional,

$$\begin{aligned}\mathcal{H}^{\mathrm{eff}} &= \int \mathrm{d}r\mathrm{d}t r^2 f^a_m K^{ab}_{mn} f^b_n \\ &= \int \mathrm{d}r\mathrm{d}t r^2 f^a_m [g_{mn}\delta^{ab}\tilde{K}_0 + V^{ab}_{mn}(r,t)] f^b_n,\end{aligned} \tag{A4}$$

where $\tilde{K}_0 = -\partial^2_t - \partial^2_r - (2/r)\partial_r$, and V^{ab}_{mn} is an effective potential. The quadratic form

$$\langle f|V|f\rangle \equiv \sum_{m,n,a,b} f^a_m V^{ab}_{mn} f^b_n \tag{A5}$$

contains terms with radial dependences proportional to $(1/r^2)$ and $(1/r)$, which correspond to the centrifugal and Coulomb-like potentials, respectively. In the case of a pure Wu-Yang monopole it was shown that such a background field leads to vacuum instability due to the appearance of the attractive potential $(-1/r^2)$ in the respective eigenvalue equation for unstable modes [21]. In our case, in the presence of the stationary monopole solution, one can verify that due to the structure of the local solution near $r = 0$ in Eq. (33), the quadratic form containing the terms proportional to $(1/r^2)$ is positively defined for any smooth fluctuating functions $f^a_m(r,t)$ satisfying the finiteness condition of the "energy" functional. This provides a nonvanishing positive centrifugal potential in the corresponding Schrödinger equation which prevents the appearance of negative eigenmodes for a special class of background monopole solutions.

By variation of the functional $\mathcal{H}^{\mathrm{eff}}$ with respect to the functions $f^a_m(r,t)$, one obtains the following effective Schrödinger-type equation:

$$K^{ab}_{mn} f^b_n(r,t) = \lambda f^a_m(r,t). \tag{A6}$$

The obtained system of nine differential equations is explicitly factorized into four decoupled systems of partial differential equations:

(I)

$$\begin{aligned}&\tilde{K}_0 f^2_2 + \frac{1}{r^2}((5+2\psi^2)f^2_2 + (6-4\psi^2)f^1_3) = \lambda f^2_2, \\ &\tilde{K}_0 f^1_3 + \frac{1}{r^2}((6-4\psi^2)f^2_2 + (5+2\psi^2)f^1_3) = \lambda f^1_3,\end{aligned} \tag{A7}$$

(II)

$$\begin{aligned}&\tilde{K}_0 f^2_1 + \frac{1}{r^2}(3+\psi^2)f^2_1 + 2\psi f^3_3) - \frac{2\psi'}{r} f^3_3 = \lambda f^2_1, \\ &\tilde{K}_0 f^3_3 + \frac{2}{r^2}((1+\psi^2)f^3_3 + \psi f^2_1) - \frac{2\psi'}{x} f^2_1 = \lambda f^3_3,\end{aligned} \tag{A8}$$

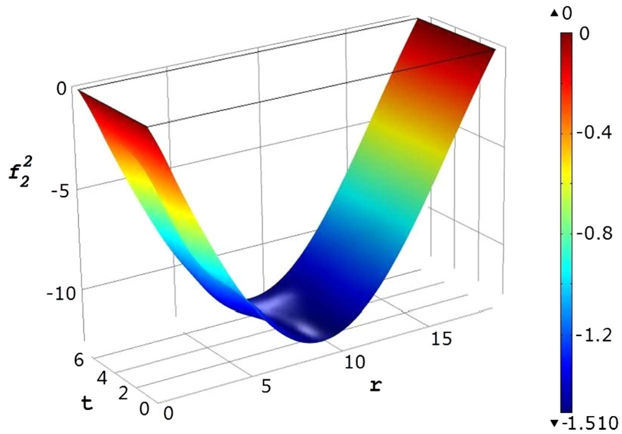

FIG. 5. Solution f_2^2 of system (I), $f_3^1(r,t) = -f_2^2(r,t)$, for $\lambda_I = 0.0586$, with $0 \le r \le 6\pi$ and $0 \le t \le 2\pi$.

(III)

$$\tilde{K}_0 f_2^1 + \frac{1}{r^2}((10+4\psi^2)f_2^1 - (12-8\psi^2)f_3^2 - 3\pi\psi f_1^3) + \frac{3\pi\psi'}{r} f_1^3 = \lambda f_2^1,$$
$$\tilde{K}_0 f_1^3 + \frac{1}{2r^2}(4(1+\psi^2)f_1^3 - \pi\psi(f_2^1+f_3^2)) + \frac{\pi\psi'}{2r}(f_2^1+f_3^2) = \lambda f_1^3,$$
$$\tilde{K}_0 f_3^2 + \frac{1}{4r^2}(2(5f_3^2 - 6f_2^1) + 4\psi^2(2f_2^1+f_3^2) - 3\pi\psi f_1^3) + \frac{3\pi\psi'}{4r} f_1^3 = \lambda f_3^2. \quad \text{(A9)}$$

The remaining system (IV) of two equations for the functions f_1^1 and f_2^3 is the same as the system (II) for the functions f_1^2 and f_3^3 with the replacements $f_1^2 \to f_1^1$ and $f_3^3 \to f_2^3$. The obtained equations represent Schrödinger-type equations for a charged particle with a positive centrifugal potential and oscillating Coulomb potential. It is clear that solutions $\psi(r,t)$ with small enough parameters a_0, A_0 will imply a positive eigenvalue spectrum, since a potential with a small enough depth and asymptotic behavior $[\mathcal{O}(1/r^\alpha)$ and $(\alpha \le 1)]$ does not lead to bound states in the case of space dimension $d \ge 3$. Substituting the interpolating function (40) into the Schrödinger equations, one can solve them and obtain the eigenvalue spectrum. Numerical analysis shows that a complete positive eigenvalue spectrum exists for solutions $\psi(r,t)$ with parameter values of a_0 in a finite range $0.89 \le a_0 \le 1$. We solve the systems (I)–(III) for the case of a monopole background field specified by the parameters $a_0 = 0.895$ and $A_0 = 0.615$. A typical profile function for the solutions to systems (I) and (II) has a weak dependence on time; see Fig. 5. The corresponding ground-state eigenvalues are close to each other: $\lambda_{\rm I} \simeq 0.0586$ and $\lambda_{\rm II} \simeq 0.0552$. The solution to system (III) has a lower eigenvalue ($\lambda_{\rm III} \simeq 0.0293$) and manifests larger time fluctuations, as it is shown in Fig. 6. Note that the principal lowest eigenvalue originates from the decoupled system of equations (III) for the functions f_2^1, f_1^3, and f_3^2, which is in qualitative agreement with the exact numerical solution of the original eigenvalue equation presented in Sec. IV.

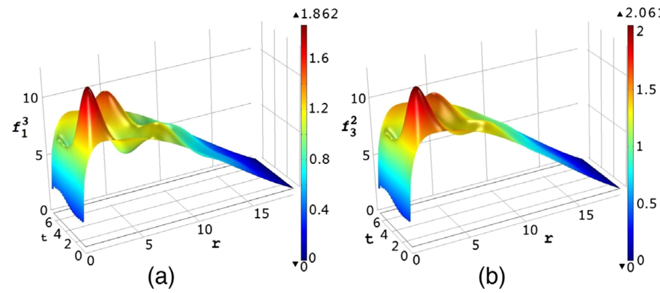

FIG. 6. Solution of system (III) for $\lambda = 0.0293$, with $0 \le r \le 6\pi$ and $0 \le t \le 2\pi$: (a) $f_1^3(r,t)$; (b) $f_2^1(r,t) = f_3^2(r,t)$.

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