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Cubic interaction vertices and one-loop self-energy in the stable string bit model

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We provide a formalism to calculate the cubic interaction vertices of the stable string bit model, in which string bits have s spin degrees of freedom but no space to move. With the vertices, we obtain a formula for one-loop self-energy, i.e., the $\mathcal{O}(1/N^2)$ correction to the energy spectrum. A rough analysis shows that, when the bit number M is large, the ground state one-loop self-energy ΔE_G scale as $M^{5-s/4}$ for even s and $M^{4-s/4}$ for odd s. Particularly, in $s=24$, we have $\Delta E_G \sim 1/M$, which resembles the Poincaré invariant relation of $1+1$ dimension $P^- \sim 1/P^+$. We calculate analytically the one-loop correction for the ground energies with $M=3$ and $s=1, 2$. We then numerically confirm that the large M behavior holds for $s \leq 4$ cases.

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I. INTRODUCTION

In the string bit model [1], a string is a chain comprised of pointlike entities called string bits. While the chain is discretized, it behaves like a continuous string when the bit number M is large enough.

The string bit model is an implementation of 't Hooft's idea of holography [2–4]. In Lorentz invariant theory, spacetime can be described by light cone coordinates with transverse dimensions $\mathbf{x}=(x^2,\ldots,x^{D-1})$ and the $\pm$ dimensions $x^\pm=(x^0\pm x^1)/\sqrt{2}$. In the string bit model, the x^- coordinate of string bits is missing, and hence the Lorentz invariance is not *present a priori*. String bits enjoy the dynamic of Galilean symmetry, under which the $+$-component momentum $P^+=(P^0+P^1)/\sqrt{2}$ is identified as mM, where m is the mass of one string bit. When M is large enough and P^+ is fixed, P^+ can be considered as a continuous variable, and its conjugate x^- can be interpreted as the missing coordinate. The Lorentz invariance can be therefore regained, and string theory emerges.

With 't Hooft's large N limit [5,6], the type II-B superstring was formulated in Ref. [7] as a string bit model. In the model, a superstring-bit creation operator, which was an adjoint representation of $U(N)$ color group, has up to s spin indices and moves in transverse space. A more drastic form of holography was studied in recent papers [8–11], where string bits have no transverse coordinate and hence no space to move. However, new compactified bosonic coordinates can be generated from spin degrees of freedom of string bits. If suitable dynamics is chosen, these spin degrees of freedom are converted to one-dimensional spin waves, which then act as compactified bosonic coordinates. The $1/N$ perturbation of the latter model was studied in Ref. [11], where the cubic interaction vertices and their application to the calculation of the one-loop self-energy were discussed.

Following the main idea of Ref. [11], we continue the work in the following way:

(i) A more detailed study of the cubic interaction vertices is performed. We present a systematic way to build conjugates of energy eigenfunctions, determine the sign factors of the vertices, and (anti) symmetrize the vertices, which are denoted as V_{qpr} and W_{rpq} and shown as Fig. 1, over the indices p and q. We then show that the interaction vertices can be calculated by finding the vacuum expectation values of ladder operators. These are necessary for the use of interaction vertices in our calculation of observables.

(ii) The calculation of the one-loop self-energy is improved, and its large M behavior for the ground states is analyzed. We assemble the ingredients necessary to calculate the one-loop self-energy. The one-loop self-energies of ground states, ΔE_G, are studied, and their large M behavior is analyzed. We calculate ΔE_G analytically for the $M=3$, $s=1$ and $M=3$, $s=2$ cases. A qualitative analysis shows that ΔE_G scales as $M^{5-s/4}$ for even s and $M^{4-s/4}$ for odd s. The scaling behavior is consistent with Lorentz invariance in $1+1$ dimensions when $s=24$, the critical Grassmann dimension, and the protostring model [11] emerges.

(iii) ΔE_G is determined numerically for higher M and s. We confirm the large M behavior of ΔE_G for $s \leq 4$. We also verify that ΔE_G increases exponentially with respect to s when M is fixed. We generalize the Hamiltonian of the model by adding $\mathcal{O}(1/N)$ terms $s\xi\Delta H$ and numerically show that, for the $s=2$ case,

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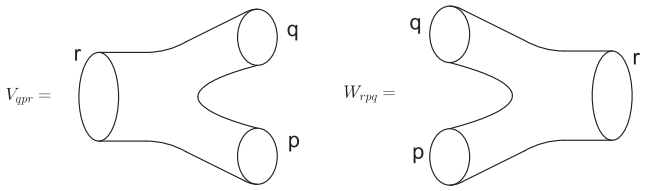

FIG. 1. The vertex V_{qpr} is the amplitude of splitting a large string r into two small strings p and q, while the vertex W_{rpq} is the amplitude of joining p and q into r.

the Hamiltonian is bounded from below with respect to M only when $\xi \geq 1$. Our analysis suggests that this is true for all the even s cases. The result shows that the $s\xi\Delta H$ generalization is necessary for building a physical string bit model.

The rest of this paper is organized as follows. In Sec. II, we review some results of stable string bit models obtained by Ref. [11]. Specifically, we introduce the Hamiltonian of the model, solve for the energy spectrum of the model at $N = \infty$, and summarize the three chains overlap calculation. In Sec. III, we provide a systematic approach to build conjugate eigenfunctions, which will be used in the calculation of the $1/N$ expansion. In Sec. IV, the cubic interaction vertices are studied by $1/N$ perturbation. In Sec. V, we use the cubic interaction vertices to calculate one-loop self-energies. Numerical results for the one-loop self-energy are analyzed in Sec. VI. The main text is closed with a conclusion section. Finally, several Appendixes are included for technical details.

II. STABLE STRING BIT MODEL

The purpose of this section is to review some results of stable string bit models obtained in Ref. [11] and introduce useful notations. These results are necessary for setting up the $1/N$ expansion of the model. Meanwhile, some modifications specific to this paper are incorporated. To be clear, the modifications are as follows. In Sec. II A, we add an $\mathcal{O}(1/N)$ term $\xi\Delta H$ to the Hamiltonian of the model. In Sec. II B, the diagonalization of the Hamiltonian at $N = \infty$ is done via different intermediate variables.

A. Hamiltonian

The superstring-bit creation operator is

$$(\bar{\phi}_{a_1\cdots a_n})^{\beta}_{\alpha}, \quad a_i = 1,\ldots,s, \quad n = 0,\ldots,s, \quad \alpha,\beta = 1,\ldots,N. \tag{2.1}$$

where a_i are totally antisymmetric spin indices and α, β are color indices of $U(N)$. $\bar{\phi}$ is bosonic when n is even and fermionic when n is odd. In Fock space, a closed string is represented by a color singlet trace operator acting on the vacuum state, that is of the form $\mathrm{Tr}\bar{\phi}\cdots\bar{\phi}|0\rangle$. The number of $\bar{\phi}$ in the trace operator is the eigenvalue of the bit number operator $M = \sum_n \frac{1}{n!}\mathrm{Tr}\bar{\phi}_{a_1\cdots a_n}\phi_{a_1\cdots a_n}$.

The Hamiltonian H to be studied in this paper reads

$$H = \sum_{i=1}^{5} H_i + s\xi\Delta H, \tag{2.2}$$

where expressions of H_i and ΔH are given in Eqs. (A3) and (A6). The H_is make a $\mathcal{O}(1)$ contribution to H, while ΔH makes only $\mathcal{O}(1/N)$ contribution and hence does not affect the large N limit. We note that H is a generalization of the $s = 1$ Hamiltonian in Refs. [8,10]. The H_i parts have been proposed in Refs. [9,11]; ΔH is the new term added by this paper, and its derivation is given in Appendix A 1.

Let us now consider the action of H on trace states space, which is defined as follows. We introduce s Grassmann coordinates θ^a, $a = 1,\ldots,s$ and then define a superbit creation operator

$$\psi(\theta) = \sum_{k=0}^{s} \frac{1}{k!}\bar{\phi}_{c_1\cdots c_k}\theta^{c_1}\cdots\theta^{c_k}$$

and a single trace operator

$$T(\theta_1,\ldots,\theta_k) = \mathrm{Tr}\psi(\theta_1)\psi(\theta_2)\cdots\psi(\theta_k),$$

where θ_i are s-component Grassmann variables. The trace states space, i.e., color singlet subspace of Fock space, is then spanned by states like

$$T(\theta_1,\ldots,\theta_K)T(\eta_1,\ldots,\eta_L)\cdots|0\rangle,$$

where $|0\rangle$ is the vacuum state. The action of each H_i and ΔH on trace states is given in Appendix A. To summarize the results, let us define

$$\bar{h}_{kl} = 2\left(s - 2\theta^a_k\frac{d}{d\theta^a_k}\right) + 2\theta^a_k\frac{d}{d\theta^a_l} + 2\theta^a_l\frac{d}{d\theta^a_k} - 2i\theta^a_k\theta^a_l - 2i\frac{d}{d\theta^a_k}\frac{d}{d\theta^a_l} + 2s\xi - 2s\delta_{k,l}, \tag{2.3}$$

$$\bar{h} = \sum_{k=1}^{M}(\bar{h}_{k,k+1} - 2s\xi). \tag{2.4}$$

Then, the actions of H on single and double trace states can be written as.[1]

[1]The actions of each H_i on single and double trace states are shown in Appendix A.

$$HT(\theta_1,\ldots,\theta_M)|0\rangle = \bar{h}T(\theta_1\cdots\theta_M)|0\rangle + \frac{1}{N}\sum_{k=1}^{M}\sum_{l\neq k+1}\bar{h}_{kl}T(\theta_l\cdots\theta_k) \times T(\theta_{k+1}\cdots\theta_{l-1})|0\rangle \tag{2.5}$$

$$\begin{aligned} HT(\theta_1\cdots\theta_K)T(\eta_1\cdots\eta_L)|0\rangle &= (\bar{h}_\theta+\bar{h}_\eta)T(\theta_1\cdots\theta_K)T(\eta_1\cdots\eta_L) \\ &+\frac{1}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\bar{h}_{kl}T(\theta_{k+1}\cdots\theta_k\eta_l\cdots\eta_{l-1})|0\rangle \\ &+\frac{1}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\bar{h}_{lk}T(\theta_k\cdots\theta_{k-1}\eta_{l+1}\cdots\eta_l)|0\rangle \\ &+\frac{1}{N}\text{fission terms}. \end{aligned} \tag{2.6}$$

Note that in Eqs. (2.6), the $-2s\delta_{kl}$ term of $\bar{h}_{kl}$ should be zero even if $k=l$, as they label different variables.

While $\bar{h}_{kl}$ acts on the trace states, to solve for energy eigenstates, it is helpful to convert $\bar{h}_{kl}$ to an equivalent form acting on the wave function of an energy eigenstate at $N=\infty$. The wave function ψ_r is defined as follows. It follows from Eq. (2.5) that, at $N=\infty$, H evolves single trace states to single trace states. Therefore, we can express a single trace energy state as

$$T_r|0\rangle = \int d^s\theta_1\cdots d^s\theta_M T(\theta_1,\ldots,\theta_M)\psi_r(\theta_1,\ldots,\theta_M)|0\rangle, \tag{2.7}$$

where ψ_r is the wave function. Since $T(\theta_1,\ldots,\theta_M)$ is invariant under the cyclic permutation $\theta_i\to\theta_{i+1}$, we can constrain ψ_r by

$$\psi_r(\theta_1,\ldots,\theta_M) = (-)^{s(M-1)}\psi_r(\theta_2,\ldots,\theta_M,\theta_1) \tag{2.8}$$

without loss of generality. The sign factor follows from the fact that the measure $d^s\theta_1\cdots d^s\theta_M$ is changed by a factor $(-)^{s(M-1)}$ under the cyclic transformation $\theta_i\to\theta_{i+1}$. Now, the action of $\bar{h}_{kl}$ on $T_r|0\rangle$ is

$$\begin{aligned} \bar{h}_{kl}T_r|0\rangle &= \int d\theta\bar{h}_{kl}T(\theta)\psi_r(\theta)|0\rangle \\ &= \int d\theta T(\theta)h_{kl}\psi_r(\theta)|0\rangle, \end{aligned} \tag{2.9}$$

where we have performed an integration by parts in the last step and

$$\begin{aligned} h_{kl} = &-2\left(s-2\theta_k^a\frac{d}{d\theta_k^a}\right)-2\theta_k^a\frac{d}{d\theta_l^a}-2\theta_l^a\frac{d}{d\theta_k^a} \\ &-2i\theta_k^a\theta_l^a-2i\frac{d}{d\theta_k^a}\frac{d}{d\theta_l^a}+2s\xi+2s\delta_{k,l}. \end{aligned} \tag{2.10}$$

Note that, in the derivation of h_{kl}, the $k=l$ case needs special treatment. Likewise, the action of $\bar{h}$ on $T(\theta)$ is equivalent to the action on $\psi_r(\theta)$ by

$$h = \sum_{k=1}^{M}(h_{k,k+1}-2s\xi). \tag{2.11}$$

B. Diagonalizing Hamiltonian at $N=\infty$

Now, let us solve for the energy spectrum of the model at $N=\infty$. A single trace energy eigenstate is determined by an eigenfunction ψ_r satisfying the equation

$$h\psi_r(\theta_1,\ldots,\theta_M) = E_r\psi_r(\theta_1,\ldots,\theta_M). \tag{2.12}$$

To solve the eigenvalue problem Eq. (2.12), we need to find the lowering and raising eigenoperators of h. This has been done by Ref. [11]. Here, we repeat the procedure with different sets of intermediate variables.

From (2.10), we see that each term of h contains only variables or derivatives of the same θ^a. It implies the variables can be separated, and we only need to solve the equation of one variable. We therefore drop the spin index a in the following calculation.

We introduce Fourier transforms [8,10]

$$\begin{aligned} \alpha_n &= \frac{1}{\sqrt{M}}\sum_{k=1}^{M}\theta_k e^{-2\pi ikn/M}, \\ \beta_n &= \frac{1}{\sqrt{M}}\sum_{k=1}^{M}\frac{d}{d\theta_k}e^{-2\pi ikn/M}, \end{aligned} \tag{2.13a}$$

$$\begin{aligned} \theta_k &= \frac{1}{\sqrt{M}}\sum_{n=0}^{M-1}\alpha_n e^{2\pi ikn/M}, \\ \frac{d}{d\theta_k} &= \frac{1}{\sqrt{M}}\sum_{n=0}^{M-1}\beta_n e^{2\pi ikn/M}, \\ n &= 0,\ldots M-1, \qquad k=1,\ldots M, \end{aligned} \tag{2.13b}$$

which satisfy

$$\{\alpha_n,\beta_m\} = \delta_{m+n,M}+\delta_{m,0}\delta_{n,0}. \tag{2.14}$$

In Ref. [11], instead of θ_k and $\frac{d}{d\theta_k}$, the diagonalization was done via the Grassmann variables $S_k=\theta_k+\frac{d}{d\theta_k}$, $\tilde{S}_k=i(\theta_k+\frac{d}{d\theta_k})$, and their Fourier transforms. Such different choices should not affect the eigenoperators and the energy spectrum.

The Hermiticity of the Hamiltonian implies that $\theta_k^\dagger = \frac{d}{d\theta_k}$, $\frac{d}{d\theta_k}^\dagger = \theta_k$, from which it follows that

$$\alpha_n^\dagger = \beta_{M-n}, \qquad \beta_n^\dagger = \alpha_{M-n},$$
$$\{\alpha_n, \alpha_m^\dagger\} = \{\beta_n, \beta_m^\dagger\} = \delta_{m,n}, \qquad 0 \le n, m \le M-1.$$

We now express h in terms of α_n and β_n as

$$h = 2\sum_{n=1}^{M-1}\left[(\alpha_n\alpha_{M-n} + \beta_n\beta_{M-n})\sin\frac{2n\pi}{M} + 2\left(1-\cos\frac{2n\pi}{M}\right)(\alpha_n\beta_{M-n} + \alpha_{M-n}\beta_n)\right] - 2M \tag{2.15}$$

and seek for eigenoperators of h,

$$F_k = r_k\alpha_k + \beta_k, \qquad [h, F_k] = \epsilon_k F_k, \tag{2.16}$$

where r_k and ϵ_k are constants. Substituting (2.15) into (2.16) yields

$$\epsilon_k^\pm = \pm 8\sin\frac{k\pi}{M}, \qquad r_k^\pm = \tan\frac{k\pi}{M} \pm \sec\frac{k\pi}{M}.$$

We then normalize the coefficients of F_k to obtain the lowering and raising operators for $k \ge 1$,

$$F_k = s_k\alpha_k + c_k\beta_k, \quad \bar{F}_k = c_k\alpha_k - s_k\beta_k, \quad k = 1,\ldots,M-1, \tag{2.17a}$$

where $c_k = \cos(\frac{\pi}{4} - \frac{k\pi}{2M})$ and $s_k = \sin(\frac{\pi}{4} - \frac{k\pi}{2M})$. It follows from (2.17a) that

$$F_k^\dagger = \bar{F}_{M-k} = c_k\alpha_{M-k} + s_k\beta_{M-k}, \qquad 1 \le k \le M-1, \tag{2.17b}$$

The zero modes need special treatment:

$$F_0 = F_M = e^{i\pi/4}\beta_0, \qquad F_0^\dagger = \bar{F}_0 = \bar{F}_M = e^{-i\pi/4}\alpha_0. \tag{2.17c}$$

The phase factors are chosen so that the expression of h_{kl} in terms of eigenoperators will have a simple form; see Eq. (4.11). A direct calculation shows that the eigenoperators satisfy the following anticommutation relations:

$$\{F_k, F_l\} = \{F_k^\dagger, F_l^\dagger\} = 0, \quad \{F_k, F_l^\dagger\} = \delta_{kl}, \quad 0 \le k, l \le M-1. \tag{2.18}$$

To obtain the energy spectrum, we need to find the ground energy E_G and the ground eigenfunction ψ_G, which is annihilated by all the lowering operators. Since the zero mode does not change energy eigenvalues, there are degeneracies in ground state. To eliminate the ambiguity, we require the ground eigenfunction to be annihilated by the zero mode F_0 as well. The ground eigenfunction can be [10]

$$\psi_G^{s=1} = \prod_{k=1}^{\lfloor (M-1)/2\rfloor}(c_k - s_k\alpha_{M-k}\alpha_k), \tag{2.19}$$

where $\lfloor (M-1)/2\rfloor$ indicates the integral part of $(M-1)/2$. To verify $F_m\psi_G^{s=1} = 0$, one only needs to check that

$$F_k(c_k - s_k\alpha_{M-k}\alpha_k) = F_{M-k}(c_k - s_k\alpha_{M-k}\alpha_k) = 0, \qquad 1 \le k \le M-1, \tag{2.20}$$

$$[F_k, c_l - s_l\alpha_{M-l}\alpha_l] = 0, \qquad k \ne l, k \ne M-l. \tag{2.21}$$

Acting h on the ground eigenfunction, we obtain the ground energy

$$E_G^{s=1} = -4\sum_{k=1}^{M-1}\sin\frac{k\pi}{M} = -4\cot\frac{\pi}{2M} = -\frac{8M}{\pi} + \frac{2\pi}{3M} + \mathcal{O}(M^{-3}). \tag{2.22}$$

We can now build general eigenfunctions for arbitrary s case. The ground eigenfunction and energy are

$$\psi_G = \psi_G^{(1)}\psi_G^{(2)}\cdots\psi_G^{(s)}, \qquad E_G = -4s\cot\frac{\pi}{2M}, \tag{2.23}$$

where each $\psi_G^{(a)}$ has the form of (2.19). A general energy eigenfunction ψ_r and its corresponding energy can be written as

$$\psi_r = \left(F_{r_{1,1}}^{(1)}F_{r_{1,2}}^{(1)}\cdots F_{r_{2,1}}^{(2)}F_{r_{2,2}}^{(2)}\cdots F_{r_{s,1}}^{(s)}F_{r_{s,2}}^{(s)}\cdots\right)^\dagger$$
$$\psi_G \equiv F_{\{r\}}^\dagger\psi_G, \tag{2.24a}$$

$$E_r = -4s\cot\frac{\pi}{2M} + 8\sum_{a,i}\sin\frac{r_{a,k}\pi}{M}, \tag{2.24b}$$

where we have defined $F_{\{r\}}$ as a string of eigenoperators and we choose $0 \le r_{a,1} < r_{a,2} < \cdots \le M-1$ as a convention. To build a physical state, the modes $r_{a,i}$ (2.24a) need to satisfy the cyclic constraint (2.8). Under the cyclic permutation $\theta_k^a \to \theta_{k+1}^a$, $F_k^{a\dagger}$ transforms as $F_k^{a\dagger} \to e^{-2ik\pi/M}F_k^{a\dagger}$. It then follows from Eq. (2.8) that the modes must satisfy

$$\sum_{a,k} r_{a,k} = \begin{cases} nM & \text{for even } s(M-1) \\ (n+\frac{1}{2})M & \text{for odd } s(M-1) \end{cases}, \quad n = 0, 1, 2, \ldots \tag{2.25}$$

Since the zero modes do not change the energy, the ground energy eigenstate has at least 2^s degeneracies. This is the consequence of H commuting with supersymmetry operators Q^a, as defined in Eq. (A8). The constraint (2.25) has a profound impact on the energy spectrum of the model. When s is even, all the ground states are allowed by (2.25) and are hence physical. But when s is odd, the ground state is allowed only when M is odd. It then follows that the lowest single trace state for even M is the one corresponding to $F^{a\dagger}_{M/2}\psi_G$.

C. Three chains overlap

We have constructed the energy eigenfunctions for $N = \infty$. To obtain the $1/N$ expansion results, we also need to calculate the overlap among three chains: one large chain of M bits and two small chains of K bits and $L = M - K$ bits. The calculation can be done by establishing the relation among the eigenoperators of large chain and two small chains. Here, we recap the results of Ref. [11].

Let us only consider the $s = 1$ case. let $F^{(K)}_m$ and $F^{(L)}_n$ be lowering operators of L-bit and K-bit chains. Define a set of operators

$$f_0 = F^{(L)}_0\sqrt{\frac{L}{M}} + F^{(K)}_0\sqrt{\frac{K}{M}}, \tag{2.26a}$$

$$f_n = F^{(L)}_l, \qquad 1 \le n \le L-1, \tag{2.26b}$$

$$f_{n+L} = F^{(K)}_k, \qquad 1 \le n \le K-1, \tag{2.26c}$$

$$f_{M-1} = e^{-i\pi/4}\left(F^{(L)}_0\sqrt{\frac{K}{M}} - F^{(K)}_0\sqrt{\frac{L}{M}}\right), \tag{2.26d}$$

which satisfy the anticommutation relationship $\{f_n, f_m\} = \{f^\dagger_n, f^\dagger_m\} = 0$ and $\{f_n, f^\dagger_m\} = \delta_{nm}$. Note that f_0 equals F_0 of the large chain [11]. We then express the large chain operators in terms of f and $f^\dagger$ as

$$F_m = \sum_{n=0}^{M-1}(f_n C_{mn} + f^\dagger_n S_{mn}), \quad 0 \le m \le M-1. \tag{2.27}$$

The anticommutation relation among F_m and $F^\dagger_m$ requires

$$CS^T + SC^T = 0, \qquad CC^\dagger + SS^\dagger = I. \tag{2.28}$$

The matrix elements of C and S are given by

$$C_{0n} = C_{n0} = \delta_{0,n}, \qquad S_{0,n} = S_{n,0} = 0, \qquad 0 \le n < M$$

and [11]

$$C_{mn} = -\frac{1}{\sqrt{ML}}\frac{1-e^{-2\pi imL/M}}{1-e^{-2\pi i(n/L-m/M)}}\cos\left(\frac{n\pi}{2L} - \frac{m\pi}{2M}\right), \quad 1 \le n < L \tag{2.29a}$$

$$C_{m,L+n-1} = \frac{1}{\sqrt{MK}}\frac{1-e^{-2\pi imL/M}}{1-e^{-2\pi i(n/K-m/M)}}\cos\left(\frac{n\pi}{2K} - \frac{m\pi}{2M}\right), \quad 1 \le n < K \tag{2.29b}$$

$$C_{m,M-1} = -\frac{1}{\sqrt{LK}}\frac{1-e^{-2\pi imL/M}}{1-e^{2i\pi m/M}}\cos\left(\frac{m\pi}{2M} - \frac{\pi}{4}\right), \tag{2.29c}$$

$$S_{mn} = -\frac{1}{\sqrt{ML}}\frac{1-e^{-2\pi imL/M}}{1-e^{2\pi i(n/L+m/M)}}\cos\left(\frac{n\pi}{2L} + \frac{m\pi}{2M}\right), \quad 1 \le n < L \tag{2.29d}$$

$$S_{m,L+n-1} = \frac{1}{\sqrt{MK}}\frac{1-e^{-2\pi imL/M}}{1-e^{2\pi i(n/K+m/M)}}\cos\left(\frac{n\pi}{2K} + \frac{m\pi}{2M}\right), \quad 1 \le n < K \tag{2.29e}$$

$$S_{m,M-1} = -\frac{1}{\sqrt{LK}}\frac{1-e^{-2\pi imL/M}}{1-e^{2\pi im/M}}\cos\left(\frac{m\pi}{2M} + \frac{\pi}{4}\right), \tag{2.29f}$$

where $1 \le m \le M-1$ in Eqs. (2.29). When M is large, the determinate of C can be approximated as [11]

$$\det CC^\dagger \sim \frac{0.9290}{(KLM)^{1/6}}\left(\frac{L}{M}\right)^{(M/K-L/M)/3-2/3} \times \left(\frac{K}{M}\right)^{(M/L-K/M)/3-2/3}. \tag{2.30}$$

We then express the ground eigenfunction of the large chain as

$$\psi^{(M)}_G = \exp\left(\frac{1}{2}\sum_{kl} f^\dagger_k D_{kl} f^\dagger_l\right)\psi^{(K)}_G\psi^{(L)}_G[\det(I + DD^\dagger)]^{-1/4}, \tag{2.31}$$

where $\psi^{(K)}_G$ and $\psi^{(L)}_G$ are ground eigenfunction for two small chains. The constraints $F_m\psi_G = 0$ imply

$$C_{mn}D_{nl} + S_{ml} = 0. \tag{2.32}$$

From the above construction, it is clear that the first rows and columns of the matrices C, S, and D are trivial. One can therefore write them as $C = (1) \oplus C'$, $S = (0) \oplus S'$, and $D = (0) \oplus D'$ where C', S', and D' are nontrivial matrices of dimension $(M-1)\times(M-1)$.

TABLE I. Gradings of functions and operators.

T_r	ψ_r	$\bar{\psi}_r$	ψ_G	$\bar{\psi}_G$	$F_{\{r\}}$
g_r	$g_r - sM$	g_r	0	sM	$g_r - sM$

With (2.28) and (2.32), we can simplify (2.31):

$$\det(I + DD^\dagger) = \frac{\det[C(I + DD^\dagger)C^\dagger]}{\det[CC^\dagger]} = \frac{\det[CC^\dagger + SS^\dagger]}{\det(CC^\dagger)} = |\det C|^{-2},$$
$$\psi_G^{(M)} = |\det C|^{1/2} \exp\left(\frac{1}{2}\sum_{kl} f_k^\dagger D_{kl} f_l^\dagger\right)\psi_G^{(K)}\psi_G^{(L)}. \tag{2.33}$$

III. CONJUGATE EIGENFUNCTION

We have built energy eigenfunctions of the model at $N = \infty$ in Sec. II B. To calculate $1/N$ expansion results, we also need to find functions that conjugate to the energy eigenfunctions. For convenience, we call these functions conjugate eigenfunctions. In this section, we will construct conjugate eigenfunctions systematically.

A conjugate eigenfunction $\bar{\psi}_r$ is a function of θ_i that satisfies the normalization condition [11]

$$\int d^s\theta_1 \cdots d^s\theta_M \bar{\psi}_r(\theta_1, \ldots, \theta_M)\psi_s(\theta_1, \ldots, \theta_M) = \delta_{rs} \tag{3.1}$$

and the completeness relation

$$\sum_r \psi_r(\theta_1, \ldots, \theta_M)\bar{\psi}_r(\eta_1, \ldots, \eta_M) = \tilde{\delta}(\theta - \eta), \tag{3.2}$$

where the delta function $\tilde{\delta}(\theta - \eta)$ is understood to be symmetrized under cyclic constraint like (2.8).[2] We stress that, once there is a complete set of $\bar{\psi}_r$ and ψ_r fulfilling the normalization condition, the completeness relation is satisfied automatically.

To construct $\bar{\psi}_r$ explicitly, it is convenient to define operators $\tilde{F}_k^\pm$ as conjugate to F_k under integration by parts,

$$\int d^M\theta \psi(\theta)[F_k\chi(\theta)] = \int d^M\theta [\tilde{F}_k^\pm \psi(\theta)]\chi(\theta), \tag{3.3}$$

where the $+$ superscript is chosen if $\psi(\theta)$ is Grassmann even and $-$ is chosen otherwise. It then follows from Eqs. (2.17) that

$$\tilde{F}_0^\pm = \mp e^{i\pi/4}\beta_0, \quad \tilde{F}_k^\pm = \pm(s_k\alpha_k - c_k\beta_k), \quad 1 \le k \le M-1, \tag{3.4}$$

$$\tilde{F}_0^{\dagger\pm} = \pm e^{-i\pi/4}\alpha_0,$$
$$\tilde{F}_k^{\dagger\pm} = \pm(c_k\alpha_{M-k} - s_k\beta_{M-k}), \qquad 1 \le k \le M-1. \tag{3.5}$$

In the remainder of this paper, we may suppress the superscript $\pm$ if there is no danger of ambiguity.

In the $s = 1$ case, we claim that the conjugate to the ground eigenfunction $\psi_G^{s=1}$ is

$$\bar{\psi}_G^{s=1} = (-i)^{\lfloor M/2\rfloor}\alpha_0 \prod_{i=1}^{\lfloor M/2\rfloor}(-s_i + c_i\alpha_{M-i}\alpha_i) \quad \text{for odd } M \tag{3.6a}$$

$$\bar{\psi}_G^{s=1} = (-i)^{M/2+1} \prod_{i=1}^{\lfloor (M-1)/2\rfloor}(-s_i + c_i\alpha_{M-i}\alpha_i)\alpha_0\alpha_{M/2} \quad \text{for even } M. \tag{3.6b}$$

In Appendix B, we verify that $\bar{\psi}_G^{s=1}$ satisfies the normalization condition (3.1). The function conjugate to the general eigenfunction (2.24a) can be built by acting on $\bar{\psi}_G$ with a string of $\tilde{F}_k^{(a)}$ as

$$\bar{\psi}_r = \tilde{F}_{r_{1,1}}^{(1)}\tilde{F}_{r_{1,2}}^{(1)} \cdots \tilde{F}_{r_{2,1}}^{(2)}\tilde{F}_{r_{2,2}}^{(2)} \cdots \tilde{F}_{r_{s,1}}^{(s)}\tilde{F}_{r_{s,2}}^{(s)} \cdots \bar{\psi}_G \equiv \tilde{F}_{\{r\}}\bar{\psi}_G, \tag{3.7a}$$

$$\bar{\psi}_G = (-)^{s(s-1)M(M-1)/4}\psi_G^{(s)}\psi_G^{(s-1)} \cdots \psi_G^{(1)}, \tag{3.7b}$$

where all the $\tilde{F}$ s pick $\tilde{F}^+$ if $\bar{\psi}_G$ is Grassmann even and $\tilde{F}^-$ otherwise. The normalization condition (3.1) can be easily verified,

$$\int d^M\theta \bar{\psi}_r\psi_r = \int d^M\theta \tilde{F}_{\{r\}}\bar{\psi}_G F_{\{r\}}^\dagger \psi_G = \int d^M\theta \bar{\psi}_G F_{\{r\}}F_{\{r\}}^\dagger \psi_G = \int d^M\theta \bar{\psi}_G\psi_G = 1,$$

where we used (3.3) in the second equality and (2.18) in the third equality. In the last equality, the sign factor of $\bar{\psi}_G$ cancels

[2] To be specific, it means that

$$\int d^M\theta f(\theta_1, \ldots, \theta_M)\tilde{\delta}(\theta - \eta) = \frac{1}{M}\sum_{k=0}^{M-1}(-)^{ks(M-1)} f(\eta_{k+1}, \ldots, \eta_{k+M}).$$

the sign introduced by the rearrangement of the measure from $d^s\theta_1\cdots d^s\theta_M$ to $(\prod_{i=1}^M d\theta_i^{(1)})\cdots(\prod_{i=1}^M d\theta_i^{(s)})$.

By analogy with (2.31), for the $s=1$ case, the overlap of conjugate eigenfunctions among the large chain and two small chains is given by

$$\bar\psi_G^{(M)} = |\det C|^{1/2}\exp\left(\frac{1}{2}\sum_{ij}\tilde f_i D_{ij}^\dagger \tilde f_j\right)\bar\psi_G^{(K)}\bar\psi_G^{(L)}, \quad (3.8)$$

where $\tilde f$ picks $\tilde f^+$ if M is even and $\tilde f^-$ if M is odd and all the notations follow the ones of Sec. II C.

Let us conclude this section by discussing the grading of energy eigenstates and eigenfunctions. We define

$$g_r \equiv \mathrm{g}(T_r) = \text{grading of } T_r.$$

Now, we can write the trace operator $T(\theta)$ as a linear combination of $\bar\psi_r$. Let $T(\theta)=\sum_t X_t\bar\psi_t(\theta)$, where X_t is independent of θ; then,

$$\begin{aligned} T_r &= \int d^s\theta_1\cdots d^s\theta_M T(\theta)\psi_r(\theta)\\ &= \sum_t (-)^{sM\mathrm{g}(X_t)}X_t\int d^s\theta_1\cdots d^s\theta_M\bar\psi_t(\theta)\psi_r(\theta)\\ &= (-)^{sM\mathrm{g}(X_r)}X_r, \end{aligned}$$

where the sign factor comes from the commutation of the measure and X_t. It implies that X_r differs from T_r only by a sign factor. So, we have $\mathrm{g}(X_r)=\mathrm{g}(T_r)$ and

$$T(\theta)=\sum_r (-)^{sMg_r}T_r\bar\psi_r(\theta). \quad (3.9)$$

Finally, from Eqs. (3.9), (3.1), and (2.23), we obtain the gradings (modulo 2) of functions and operators as Table I. These results will be used in the next section.

IV. CUBIC INTERACTION VERTICES

Let $T_p|0\rangle$, $T_q|0\rangle$, and $T_r|0\rangle$ be energy eigenstates of strings with K, L, and $M=K+L$ bits, respectively; then, the interaction vertices V_{qpr} and W_{rpq} are defined as [11]

$$HT_r|0\rangle = E_rT_r|0\rangle + \frac{1}{N}\sum_{K=1}^{M-1}\sum_{p,q}T_pT_q|0\rangle V_{qpr}, \quad (4.1a)$$

$$HT_pT_q|0\rangle = (E_p+E_q)T_pT_q|0\rangle + \frac{1}{N}\sum_r T_r|0\rangle W_{rpq}+\cdots. \quad (4.1b)$$

The vertex V_{qpr} represents the amplitude of breaking one large string into two small strings, and the vertex W_{rpq} represents the amplitude of joining two small strings into one large string. Without loss of generality, we can (anti) symmetrize the vertices over indices p and q as

$$V_{pqr}=(-)^{g_pg_q}V_{qpr}, \qquad W_{rqp}=(-)^{g_pg_q}W_{rpq}. \quad (4.2)$$

In this section, we shall find that

$$V_{qpr} = M|\det C|^{s/2}\prod_{a=1}^s\left\langle F^a_{\{qp\}}, F^{a\dagger}_{\{r\}}\right\rangle_V, \quad (4.3a)$$

$$W_{rpq} = KL|\det C|^{s/2}\prod_{a=1}^s\left\langle F^a_{\{qp\}}, F^{a\dagger}_{\{r\}}\right\rangle_W^\dagger. \quad (4.3b)$$

Several notations are used in (4.3) for convenience. $F^a_{\{qp\}}\equiv F^a_{\{q\}}F^a_{\{p\}}$, and the superscript a indicates that only operators of spin index a are involved. The brackets $\langle\cdot,\cdot\rangle_{V,W}$ stand for vacuum expectation values of operators

$$\langle F^a_{\{qp\}}, F^{a\dagger}_{\{r\}}\rangle_{V,W} \equiv \left\langle F^a_{\{qp\}}h^a_{(V,W)}F^{a\dagger}_{\{r\}}\exp\left(\frac{1}{2}f_k^{a\dagger}D_{kl}f_l^{a\dagger}\right)\right\rangle, \quad (4.4a)$$

$$h^a_V\equiv\frac{1}{2}(h^a_{K,1}+h^a_{M,K+1}), \qquad h^a_W = h^a_{K,K+1}+h^a_{M,1}, \quad (4.4b)$$

where the matrix D and operators f_k are defined as (2.32) and (2.26) and h_{kl} is given by (2.10). The vacuum of (4.4a) is the state annihilated by all lowering operators of L-bit and K-bit systems, i.e., $F_i^{(K)}|0\rangle = F_i^{(L)}|0\rangle = 0$. In the following, we first mark remarks on the interaction vertices in Sec. IV A and then give all the technical details of the derivation of (4.3) in Sec. IV B.

A. Remarks on vertices

The form of vertices in (4.3) can be interpreted as follows. The prefactor M of V_{qpr} shows that, when a large chain splits into two small chains of K and L bits, there are M ways to choose the break points, and each way contributes equally to V_{qpr}. Likewise, the prefactor KL of W_{rpq} shows that, when two small chains join into a large chain, there are $K\times L$ ways to choose the joint points, and each way contributes equally to W_{rpq}. The operator $h^a_V=\frac{1}{2}(h^a_{K,1}+h^a_{M,K+1})$ reflects the fact that, to break one M-bit string into K-bit and L-bit strings, one needs to connect bit 1 to bit K and bit $(K+1)$ to bit M. Similarly, the operator $h_W=h^a_{K,K+1}+h^a_{M,1}$ reflects the fact that, to join back the above two small strings into one, one needs to connect bit K to bit $(K+1)$ and bit M to bit 1. The difference of factor 2 between h^a_V and h^a_W is because that, when joining two strings, one can inverse the labels of the first small string as $1+i\leftrightarrow K-i$ to obtain a different large string.

B. Derivation of V_{qpr} and W_{rpq}

Now, let us derive the formula (4.3). Acting the Hamiltonian to the zeroth order energy eigenstate $T_r|0\rangle$ and using (2.7) and (2.5), we have

$$\begin{aligned} HT_r|0\rangle &= E_rT_r|0\rangle + \frac{1}{N}\int d\theta \sum_{i=1}^{M}\sum_{j=i+2}^{M+i} \bar{h}_{ij}T(\theta_j\cdots\theta_i)T(\theta_{i+1}\cdots\theta_{j-1})|0\rangle\psi_r(\theta_1,\ldots,\theta_M) \\ &= E_rT_r|0\rangle + \frac{1}{N}\sum_{i=1}^{M}\sum_{K=1}^{M-1}\int d\theta \bar{h}_{i,i+K+1}T(\theta_{i+K+1}\cdots\theta_i)T(\theta_{i+1}\cdots\theta_{i+K})|0\rangle\psi_r \\ &= E_rT_r|0\rangle + \frac{1}{N}\sum_{i=1}^{M}\sum_{K=1}^{M-1}\int d\theta \sum_{p,q}(-)^{s(Kg_p+Lg_q)}\bar{h}_{i,i+K+1}T_q\bar{\psi}_qT_p\bar{\psi}_p|0\rangle\psi_r, \end{aligned} \tag{4.5}$$

where in the second equality we renamed the indices as $j \to i+K+1$ and in the last equality we used (3.9). Comparing (4.5) with (4.1a), we arrive at

$$\tilde{V}_{qpr} = (-)^{s(Kg_q+Lg_p)}\sum_{i=1}^{M}\int d\theta \bar{h}_{i,i+K+1}\bar{\psi}_q(\theta_{i+K+1}\cdots\theta_i)\bar{\psi}_p(\theta_{i+1}\cdots\theta_{i+K})\psi_r. \tag{4.6}$$

The vertex is decorated with a tilde because we have not yet applied the constraint (4.2) to it. Note that the sign factor is changed due to the reorder of T_p and T_q.

The action of H on the double trace produces both fusion and fission terms:

$$\begin{aligned} HT_pT_q|0\rangle &= (E_p+E_q)T_pT_q|0\rangle + \frac{1}{N}\sum_r T_r\int d\theta d\eta \sum_{k,l}(-)^{sL(g_p-sK)}\bar{h}_{kl}\bar{\psi}_r(\theta_{k+1}\cdots\theta_k\eta_l\cdots\eta_{l-1})\psi_p(\theta)\psi_q(\eta)|0\rangle \\ &\quad + \frac{1}{N}\sum_r T_r\int d\theta d\eta \sum_{k,l}(-)^{sL(g_p-sK)}\bar{h}_{lk}\bar{\psi}_r(\theta_k\cdots\theta_{k-1}\eta_{l+1}\cdots\eta_l)\psi_p(\theta)\psi_q(\eta)|0\rangle + \frac{1}{N}\text{fission terms}. \end{aligned}$$

Comparing the above with (4.1b), we have $\tilde{W}_{rpq} = W^{(1)}_{rpq} + W^{(2)}_{rpq}$, where

$$\begin{aligned} W^{(1)}_{rpq} &= (-)^{sL(g_p-sK)}\int d\theta d\eta \sum_{k,l}\bar{h}_{kl}\bar{\psi}_r(\theta_{k+1}\cdots\theta_k\eta_l\cdots\eta_{l-1})\psi_p(\theta_1\cdots\theta_K)\psi_q(\eta_1\cdots\eta_L), \\ W^{(2)}_{rpq} &= (-)^{sL(g_p-sK)}\int d\theta d\eta \sum_{k,l}\bar{h}_{lk}\bar{\psi}_r(\theta_k\cdots\theta_{k-1}\eta_{l+1}\cdots\eta_l)\psi_p(\theta_1\cdots\theta_K)\psi_q(\eta_1\cdots\eta_L). \end{aligned}$$

Note that so far the derivation of V and W follows the one of Ref. [11] except that we changed the notation slightly and determined the sign factors of the vertices, which are overlooked by Ref. [11] in Eqs. (21) and (27).

Now, let us simplify $\tilde{V}$ and $\tilde{W}$. We denote the integral with index i in (4.6) as $\tilde{V}^{(i)}_{qpr}$. It can be shown as follows that all the M integrals $\tilde{V}^{(i)}_{qpr}$ are the same. For the integral with index i, we can rename all integration variables as $\theta_j \to \theta_{j+1}$ and then use the cyclic constraint (2.8) to bring ψ_r and the measure to their original form. The value of the integral is invariant under both changes, but $\tilde{V}^{(i)}_{qpr}$ is changed to $\tilde{V}^{(i+1)}_{qpr}$. It implies that $\tilde{V}^{(i)}_{qpr}$ is independent of i, and we can choose $i = M$ for every integral to give

$$\tilde{V}_{qpr} = (-)^{s(Kg_q+Lg_p)}M\int d\theta\bar{\psi}_q(\theta_{K+1}\cdots\theta_M)\bar{\psi}_p(\theta_1\cdots\theta_K)h_{M,K+1}\psi_r(\theta_1,\ldots,\theta_M).$$

To find the vertex satisfying the constraint (4.2), we let $V_{qpr} = \frac{1}{2}(\tilde{V}_{qpr} + (-)^{g_pg_q}\tilde{V}_{pqr})$, where $\tilde{V}_{pqr}$ can be obtained by exchanging $p \leftrightarrow q$, $K \leftrightarrow L$:

$$
\begin{aligned}
(-)^{g_p g_q}\tilde{V}_{pqr} &= (-)^{g_p g_q + s(Kg_q + Lg_p)} M \int d\theta \bar{\psi}_p(\theta_{L+1}\cdots\theta_M)\bar{\psi}_q(\theta_1\cdots\theta_L)h_{M,L+1}\psi_r \\
&= (-)^{g_p g_q + s(Kg_q + Lg_p)} M \int d\theta \bar{\psi}_p(\theta_1\cdots\theta_K)\bar{\psi}_q(\theta_{K+1}\cdots\theta_M)h_{K,1}\psi_r \\
&= (-)^{s(Kg_q + Lg_p)} M \int d\theta \bar{\psi}_q(\theta_{K+1}\cdots\theta_M)\bar{\psi}_p(\theta_1\cdots\theta_K)h_{K,1}\psi_r.
\end{aligned}
$$

We therefore have

$$
V_{qpr} = (-)^{s(Kg_q + Lg_p)} M \int d\theta \bar{\psi}_q(\theta_{K+1}\cdots\theta_M)\bar{\psi}_p(\theta_1\cdots\theta_K)h_V\psi_r(\theta_1,\ldots,\theta_M), \tag{4.7}
$$

where h_V is given by (4.4b).

We perform a similar calculation for the $\tilde{W}$ vertex. All the integrals of $W^{(1)}$ and $W^{(2)}$ are independent of the indices k and l. So, we can simply replace the sums over k and l with the factor $K \times L$. We then rename $\eta_1, \ldots, \eta_L$ to $\theta_{K+1}, \ldots, \theta_M$ and fix the indices as $k = K, l = K+1$ for $W^{(1)}$ and $k = 1$, $l = M$ for $W^{(2)}$ to give

$$
\tilde{W}_{rpq} = (-)^{sL(g_p - sK)} KL \int d\theta \bar{\psi}_r(\theta_1\cdots\theta_M)(h_{K,K+1} + h_{M,1})\psi_p(\theta_1\cdots\theta_K)\psi_q(\theta_{K+1}\cdots\theta_M). \tag{4.8}
$$

Exchanging $p \leftrightarrow q$ and $K \leftrightarrow L$, we have

$$
\tilde{W}_{rqp} = (-)^{sK(g_q - sL)} KL \int d\theta \bar{\psi}_r(\theta_1\cdots\theta_M)(h_{L,L+1} + h_{M,1})\psi_q(\theta_1\cdots\theta_L)\psi_p(\theta_{L+1}\cdots\theta_M). \tag{4.9}
$$

Renaming the integral variables as $\{\theta_1, \ldots, \theta_L\} \to \{\theta_{K+1}, \ldots, \theta_M\}$, $\{\theta_{L+1}, \ldots, \theta_M\} \to \{\theta_1, \ldots, \theta_K\}$, under which $h_{L,L+1} + h_{M,1}$ becomes $h_{M,1} + h_{K,K+1}$, and then applying the property that $\bar{\psi}_r(\theta_1\cdots\theta_M)$ is invariant under the cyclic permutation $\theta_k \to \theta_{k+1}$,[3] we obtain that $\tilde{W}_{rqp} = (-)^{g_p g_q}\tilde{W}_{rpq}$, which implies that $W_{rpq} = (\tilde{W}_{rpq} + (-)^{g_p g_q}\tilde{W}_{rqp}) = \tilde{W}_{rpq}$.

Let us now get rid of the integral in the expression of V. For simplicity, we consider the $s = 1$ case. We use (2.24a) and (3.7a) to write $\psi_r = F^\dagger_{\{r\}}\psi^{(M)}_G$, $\bar{\psi}_r = \tilde{F}_{\{r\}}\bar{\psi}^{(M)}_G$ and similarly for states p and q. We then use (2.33) to express $\psi^{(M)}_G$ in terms of $\psi^{(L)}_G$ and $\psi^{(K)}_G$. By a little algebra, we arrive at

$$
V^{s=1}_{qpr} = (-)^{L(g_p - K)} M |\det C|^{1/2} \int d\theta \bar{\psi}^{(L)}_G \bar{\psi}^{(K)}_G F_{\{qp\}} h_V F^\dagger_{\{r\}} \exp\left(\frac{1}{2}\sum_{kl} f^\dagger_k D_{kl} f^\dagger_l\right)\psi^{(K)}_G\psi^{(L)}_G. \tag{4.10}
$$

The ground eigenfunctions $\psi^{(L)}_G$ and $\psi^{(K)}_G$ are annihilated by any lowering eigenoperators of the small chains. Their conjugates $\bar{\psi}^{(L)}_G$ and $\bar{\psi}^{(K)}_G$ can be annihilated by any raising eigenoperators of the small chains, as Eq. (B3) shows. Therefore, the rhs of (4.10) can be interpreted as a vacuum expectation value of the operator $F_{\{q\}}F_{\{p\}}h_V F^\dagger_{\{r\}} \exp\left(\frac{1}{2}\sum_{kl} f^\dagger_k D_{kl} f^\dagger_l\right)$. We therefore have

$$
V^{s=1}_{qpr} = (-)^{L(g_p - K)} M |\det C|^{1/2} \left\langle F_{\{qp\}} h_V F^\dagger_{\{r\}} \exp\left(\frac{1}{2}\sum_{kl} f^\dagger_k D_{kl} f^\dagger_l\right)\right\rangle,
$$

where the vacuum is understood to be the state annihilated by all $F^{(K)}_i$ and $F^{(L)}_i$. We perform a similar calculation for W_{rpq} and find

[3]One can show that $\bar{\psi}_r(\theta_1\cdots\theta_M)$ is invariant under the cyclic permutation $\theta_k \to \theta_{k+1}$ as follows. From Eq. (3.6a) and (3.6b), we see that $\bar{\psi}^{s=1}_G \to (-)^{M-1}\bar{\psi}^{s=1}_G$ as $\theta_k \to \theta_{k+1}$. It then follows that $\bar{\psi}_G$ transforms as $\bar{\psi}_G \to (-)^{s(M-1)}\bar{\psi}_G$. From the cyclic constraint (2.25), we see that $\tilde{F}_{\{r\}}$ transforms in the same way as $\bar{\psi}_G$. Therefore, $\bar{\psi}_r = \tilde{F}_{\{r\}}\bar{\psi}_G$ is invariant.

$$W^{s=1}_{rpq} = (-)^{L(g_p-K)}KL|\det C|^{\frac{1}{2}}\left\langle \exp\left(\frac{1}{2}f_k D^\dagger_{kl} f_l\right) F_{\{r\}} h_W F^\dagger_{\{p\}} F^\dagger_{\{q\}}\right\rangle$$
$$= (-)^{L(g_p-K)}KL|\det C|^{\frac{1}{2}}\left\langle F_{\{qp\}} h_W F^\dagger_{\{r\}} \exp\left(\frac{1}{2}f^\dagger_k D_{kl} f^\dagger_l\right)\right\rangle^\dagger.$$

Note that $V^{s=1}_{qpr}$ and $W^{s=1}_{rpq}$ have the same sign factor $(-)^{L(g_p-K)}$. We shall see that physical observables, like one-loop self-energies, only depend on products like $W_{rpq}V_{qpr}$. It implies that the sign factors are unphysical and can be dropped in the calculation of physical observables. So, for arbitrary s, up to a common unphysical sign factor, we can express V and W as products of vacuum expectation values over spin index a. We therefore obtain the formula (4.3).

To calculate the vacuum expectation values, we need to express h_V and h_W in terms of eigenoperators. From Eqs. (C4), (C3), and (C5), we have

$$h_{(V,W)} = \frac{2}{M}\sum_{n,m=0}^{M-1}\left(A^{(V,W)\dagger}_{nm}F^\dagger_n F^\dagger_m + A^{(V,W)}_{nm}F_n F_m + 2A^{(V,W)}_{-n,m}F^\dagger_n F_m\right) + \frac{2}{M}\mu_{(V,W)}, \tag{4.11}$$

where

$$A^{(V)}_{nm} = \frac{1}{2}\left[1-\exp\left(2\pi i\frac{Kn}{M}\right)\right]\left[1-\exp\left(2\pi i\frac{Km}{M}\right)\right]\sin\frac{m-n}{2M}\pi$$
$$+\frac{1}{2}\left[\exp\left(2\pi i\frac{Kn}{M}\right)+\exp\left(2\pi i\frac{Km}{M}\right)\right]\left[1+\exp\left(\pi i\frac{m+n}{M}\right)\right]\sin\frac{m-n}{2M}\pi \tag{4.12a}$$

$$\mu_V = -\cot\frac{\pi}{2M}+\frac{1}{2}\left(\cot\frac{2K-1}{2M}\pi-\cot\frac{2K+1}{2M}\pi\right)+M\xi, \tag{4.12b}$$

$$A^{(W)}_{nm} = \left[1+\exp\left(\pi i\frac{n+m}{M}\right)\right]\left[1+\exp\left(2\pi iK\frac{m+n}{M}\right)\right]\sin\frac{m-n}{2M}\pi, \tag{4.12c}$$

$$\mu_W = -4\cot\frac{\pi}{2M}+2M\xi. \tag{4.12d}$$

V. ONE-LOOP SELF-ENERGY

One application of the interaction vertices is to calculate the one-loop self-energy, i.e., the $\mathcal{O}(1/N^2)$ correction to energy spectrum. In this section, we will first express the one-loop self-energy in terms of cubic interaction vertices [11]. We then apply the results of previous sections and obtain a formula for analytic and numerical computation.

For a finite N energy eigenstate, we use the ansatz

$$|E\rangle = T_r|0\rangle + T_pT_q|0\rangle C_{pq}+\cdots, \tag{5.1}$$

where the coefficients $C_{pq} = (-)^{g_pg_q}C_{qp}$ are c-numbers of order $1/N$. Imposing the eigenvalue equation $(H-E)|E\rangle = 0$ and using perturbation theory, we obtain [11]

$$C_{pq} = \frac{1}{E_r-E_p-E_q}\frac{1}{N}V_{qpr}+\mathcal{O}(N^{-2}), \tag{5.2}$$

$$\Delta E_r = \frac{1}{N^2}\sum_{K=1}^{M-1}\sum_{p,q}W_{rpq}\frac{1}{E_r-E_p-E_q}V_{qpr}, \tag{5.3}$$

where ΔE_r is the leading order correction to E_r, i.e., $E = E_r+\Delta E_r+\mathcal{O}(1/N^3)$. We stress that the vertices in (5.3) should be the ones satisfying the constraint (4.2); otherwise, it would lead to an incorrect ΔE_r.

We now apply the formulas of V and W to (5.3). Let us first consider the $s=1$ case. The zero modes require special treatment. Substitute (4.3a) and (4.3b) into (5.3), and write the sum over zero modes explicitly,

$$\Delta E^{s=1}_r = \frac{1}{N^2}\sum_{K=1}^{M-1}{\sum_{p,q}}'\frac{KLM|\det C|}{E_r-E_p-E_q}$$
$$\times\sum_{\lambda,\kappa=0,1}\left\langle F_{\{qp\}}F^\lambda_{0,K}F^\kappa_{0,L},F^\dagger_{\{r\}}\right\rangle^*_W$$
$$\times\left\langle F_{\{qp\}}F^\lambda_{0,K}F^\kappa_{0,L},F^\dagger_{\{r\}}\right\rangle_V,$$

where we wrote $F^{(K)}_0$ as $F_{0,K}$ for convenience and ${\sum_{p,q}}'$ indicates the sum over states without zero modes. We can replace $F^{(K)}_0$ and $F^{(L)}_0$ with f_0 and f_{M-1} given the following

reasoning. The sum over λ and κ produces four terms. For the term with $\lambda=\kappa=1$, we find $F_0^{(K)}F_0^{(L)}=e^{i\pi/4}f_0f_{M-1}$ by Eqs. (2.26a) and (2.26d). The phase is irrelevant. The $\lambda=1$, $\kappa=0$, and $\lambda=0$, $\kappa=1$ terms are quadratic forms of $F_0^{(K)}$ and $F_0^{(L)}$. One can easily verify that $F_0^{(K)*}F_0^{(K)}+F_0^{(L)*}F_0^{(L)}=f_0^*f_0+f_{M-1}^*f_{M-1}$. So, the sum over $F_0^{(K)}$ and $F_0^{(L)}$ can be replaced by the one over f_0 and f_{M-1}. We then have

$$\Delta E_r^{s=1}=\frac{1}{N^2}\sum_{K=1}^{M-1}{\sum_{p,q}}'\frac{KLM|\det C|}{E_r-E_p-E_q}\times\sum_{\lambda_i=0,1}\left\langle F_{\{qp\}}f_0^{\lambda_0}f_{M-1}^{\lambda_{M-1}},F_{\{r\}}^\dagger\right\rangle_W^*\times\left\langle F_{\{qp\}}f_0^{\lambda_0}f_{M-1}^{\lambda_{M-1}},F_{\{r\}}^\dagger\right\rangle_V.$$

For arbitrary s, $|\det C|$ is replaced by $|\det C|^s$, and each term inside the summation becomes a product over a. So, we have

$$\Delta E_r=\frac{1}{N^2}\sum_{K=1}^{M-1}{\sum_{p,q}}'\frac{KLM|\det C|^s}{E_r-E_p-E_q}\times\sum_{\lambda_{i,j}=0,1}\prod_{a=1}^{s}\left\langle F_{\{qp\}}^a f_{0,a}^{\lambda_{a,0}}f_{M-1,a}^{\lambda_{a,M-1}},F_{\{r\}}^{a\dagger}\right\rangle_W^*\times\left\langle F_{\{qp\}}^a f_{0,a}^{\lambda_{a,0}}f_{M-1,a}^{\lambda_{a,M-1}},F_{\{r\}}^{a\dagger}\right\rangle_V.$$

Note that the sum over $\lambda_{i,j}$ can be performed for each a independently. So, we can move the sum over $\lambda_{i,j}$ inside the product over a to give

$$\Delta E_r=\frac{1}{N^2}\sum_{K=1}^{M-1}{\sum_{p,q}}'\frac{KLM|\det C|^s}{E_r-E_p-E_q}\prod_{a=1}^{s}\times\left(\sum_{i=1}^{4}\langle F_{\{qp\}}^aZ_i^a,F_{\{r\}}^{a\dagger}\rangle_W^*\langle F_{\{qp\}}^aZ_i^a,F_{\{r\}}^{a\dagger}\rangle_V\right),\tag{5.4}$$

where $Z^a=(1,f_0^a,f_{M-1}^a,f_0^af_{M-1}^a)$.

A. Ground energy correction

In principle, we can now calculate one-loop self-energy for any single trace energy state with Eq. (5.4). But in general, the calculation is tedious. Let us consider the simplest case that ψ_r is the ground state, i.e., $F_{\{r\}}=1$. For convenience, we denote $\langle O,1\rangle_{V,W}$ as $\langle O\rangle_{V,W}$. We only consider the $s=1$ case here, since $s>1$ cases are simply products of the $s=1$ case.

We need to calculate the vacuum expectation value $\langle\cdots h\exp(\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger)\rangle$. In terms of eigenoperators, h_{kl} contains quadratic terms of the form $A_{nm}^\dagger F_m^\dagger F_n^\dagger$, $A_{-n,m}F_m^\dagger F_n$, and $A_{nm}F_nF_m$ and a constant term μ, as Eq. (4.11) shows. Since $F_m\exp(\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger)\psi_G^{(K)}\psi_G^{(L)}=F_m\psi_G^{(M)}=0$, only the $F^\dagger F^\dagger$ and the constant terms make a nonzero contribution:

$$\left\langle\cdots h\exp\left(\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger\right)\right\rangle=\frac{2}{M}\left\langle\cdots(A_{nm}^\dagger F_n^\dagger F_m^\dagger+\mu)\exp\left(\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger\right)\right\rangle.$$

To calculate the result of $A_{nm}^\dagger F_n^\dagger F_m^\dagger$ term, we need to express F_m in terms of a linear combination of f_k and $f_k^\dagger$, as (2.27) shows, and commute f_k through the exponential. This is done in Appendix D. Using Eq. (D1), we have

$$\langle F_{\{qp\}}Z_i\rangle_{V,W}=\frac{2}{M}\left(\mu'_{V,W}+B_{mn}^{(V,W)}\frac{\partial}{\partial D_{mn}}\right)\times\left\langle F_{\{qp\}}Z_i\exp\left(\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger\right)\right\rangle,\tag{5.5}$$

where

$$\mu'_{V,W}=\mu_{V,W}-\mathrm{Tr}(S^*C^{-1}A_{V,W}^\dagger),\qquad B_{V,W}=C^{-1}A_{V,W}^\dagger(C^{-1})^T$$

with $\mu_{V,W}$ and $A_{V,W}$ defined in (4.12). Finally, the vacuum expectation values on the rhs of (5.5) can be calculated using

$$\left\langle f_{i_1}f_{i_2}\cdots f_{i_{2n-1}}f_{i_{2n}}\exp\left(\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger\right)\right\rangle={\sum_{P\in S_{2n}}}'(-)^PD_{i_{P(1)}i_{P(2)}}D_{i_{P(3)}i_{P(4)}}\cdots D_{i_{P(2n-1)}i_{P(2n)}},$$

where S_{2n} is the set of all permutations of $2n$ integers, $(-)^P$ is the signature of permutation P, and $\sum'$ indicates the sum over permutations satisfying

$$P(1)<P(2),\qquad P(3)<P(4),\ldots,P(2n-1)<P(2n),$$
$$P(1)<P(3)<P(5)<\cdots<P(2n-1).$$

Combining the above together, we can calculate the one-loop self-energy of the ground state. As the complete formula is very complicated, we do not bother writing it here. In Appendix E, we show examples of using formula (5.4) to calculate the one-loop self-energies of the $M=3$, $s=1$ and $M=3$, $s=2$ cases. For $M=3$, $s=1$, we have

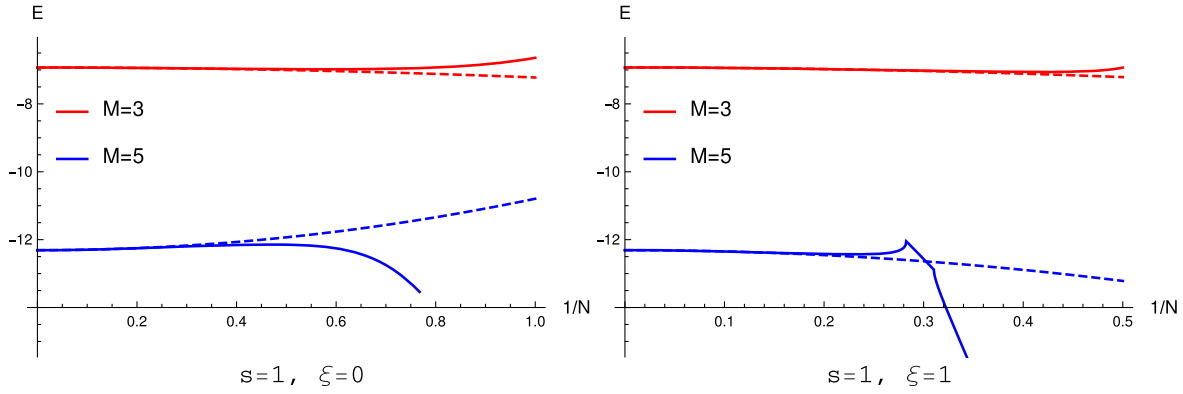

FIG. 2. Ground energy as a function of $1/N$ for the $s = 1, \xi = 0$ and $s = 1.\xi = 1$ cases. The solid lines plot the exact numerical results by the method of Ref. [10]. The dashed lines plot the $1/N^2$ order perturbation results.

$$\Delta E_G = \frac{1}{N^2}\left[-3\left(3\sqrt{3}-5\right)\xi^2 + 2\left(12-7\sqrt{3}\right)\xi - \frac{3}{2}\left(3\sqrt{3}-5\right)\right],$$

and for $M = 3$, $s = 2$, we have

$$\Delta E_G = \frac{1}{N^2}\left(-66\sqrt{3}\xi^4 + 360\xi^3 - 230\sqrt{3}\xi^2 + 180\xi - \frac{33\sqrt{3}}{2}\right).$$

In general, ΔE_G is a polynomial of ξ of degree $2s$.

B. Large M behavior

We conclude this section by considering the large M behavior of ΔE_G.[4] The vacuum expectation values in (5.4) only depends on the ratio K/M and therefore can be considered as $\mathcal{O}(1)$. So, when M is large,

$$\Delta E_G \sim \frac{1}{N^2}\sum_{K=1}^{M-1}{\sum_{p,q}}' \frac{KLM|\det C|^s}{E_G - E_p - E_q}. \tag{5.6}$$

In (5.6), the factor KLM scales as M^3, $|\det C|$ scales as $M^{-s/4}$ by Eq. (2.30), and the sum over K gives another factor of M. These three parts produce a factor scale as $M^{4-s/4}$.

We then consider the large M behavior of $1/(E_G - E_p - E_q)$. When s is even, both p and q can be ground states, and hence $1/(E_G - E_p - E_q) \sim \mathcal{O}(M)$ by Eq. (2.22). When s is odd, M has to be odd in order to have the physical M-bit ground state, and one of the small strings must have an even bit number. It implies that the ground state of one small chain is forbidden by the cyclic constraint (2.25). Therefore, $1/(E_G - E_p - E_q) \sim \mathcal{O}(1)$ for odd s.

Combining the above together, we have

$$\Delta E_G \sim \begin{cases} M^{5-s/4} & \text{for even } s \\ M^{4-s/4} & \text{for odd } s \end{cases}. \tag{5.7}$$

In analogy with the standard string theory, we can infer from Eq. (5.7) the critical Grassmann dimension of the model, where Lorentz invariance in $1+1$ dimensions is regained. In the light cone coordinates, P^+ is identified as mM, and P^- is identified as E. So, the Poincaré invariant dispersion relation $P^- \sim 1/P^+$ implies $E \sim 1/M$. Therefore, the Lorentz invariance requires $s = 24$. The model in the special $s = 24$ case is called the protostring model [11].

VI. NUMERICAL RESULTS

We have derived a formula for the one-loop correction to the ground energy. As Appendix E shows, however, the calculation is tedious even for the simplest case. We therefore turn to numerical computation.[5] As the complexity of the calculation grows dramatically, the highest M for which we performed numerical computation is 27 for $s = 1$ and 16 for $s = 2$ and continues decreasing as s increases. Since only the ground energy is considered, we will simply write the ground energy as E and its correction as ΔE and also suppress the $1/N^2$ factor.

We first compare the perturbation results with the exact numerical results, which are obtained by the method of Ref. [10]. Figure 2 plots the change of ground energy with respect to the $1/N$ for $M = 3$ and 5 in the $s = 1$ case. The solid lines are exact numerical results, and the dashed lines are $\mathcal{O}(1/N^2)$ perturbation results. We see that the two types of results match very well for N large enough. One interesting observation is that, when N is small, the

[4]The large M discussion is mainly based on comments by Charles Thorn.

[5]The source code for the numerical computation can be found in Ref. [12].

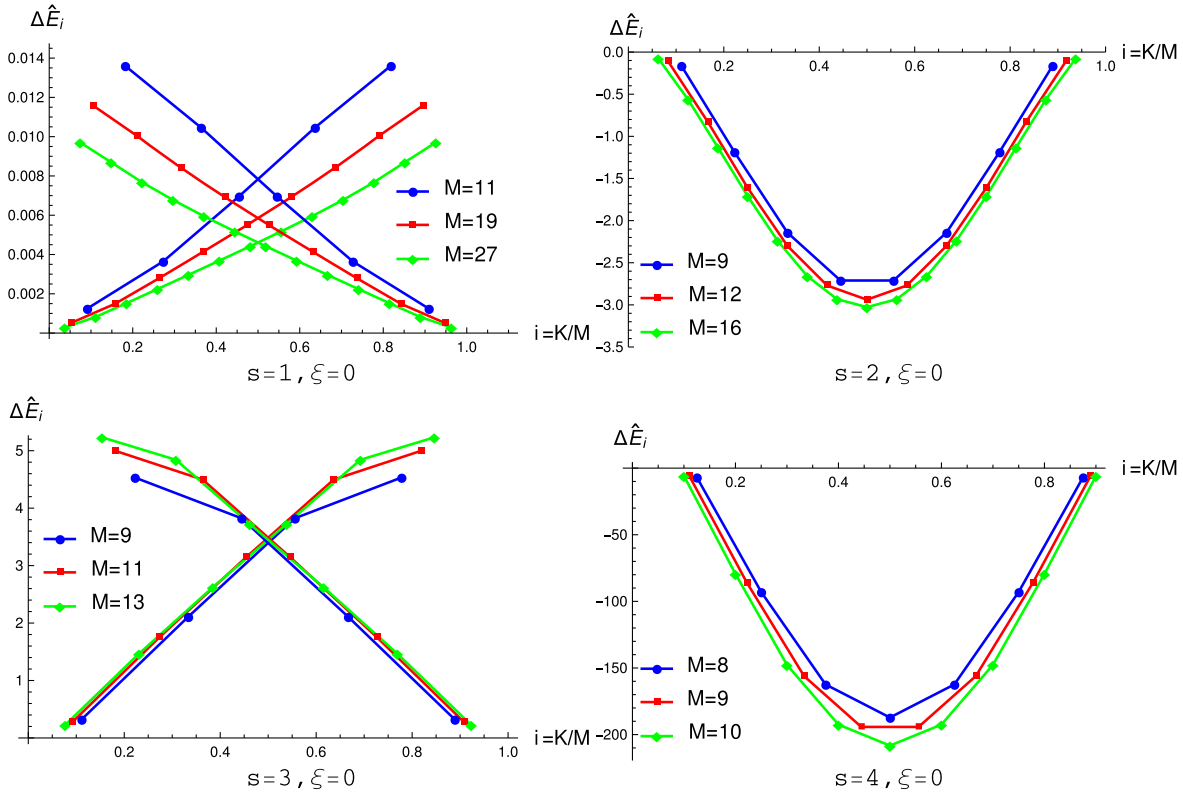

FIG. 3. $\Delta\hat{E}_i$ as a function of $i = K/M$. For odd s cases, the curves above the horizontal axis are for odd K points, and the curves below are for even K points.

perturbation results of $M = 3$ are lower than the exact results, while the perturbation results of $M = 5$ are above the exact results. It implies that the $\mathcal{O}(1/N^4)$ correction is positive for $M = 3$ and negative for $M = 5$.

We then verify the large M behavior of ΔE. Instead of plotting ΔE with respect to M, we study its "inner structure," that is the contribution of each K to ΔE, denoted by ΔE_i and defined as

$$\Delta E = \sum_{K=1}^{M-1} \Delta E_i, \qquad i = \frac{K}{M}.$$

Since the power of M in the large M behavior of ΔE_i is 1 lower than that of ΔE, we introduce the normalized ΔE_i to remove the M dependence:

$$\Delta\hat{E}_i = \begin{cases} \Delta E_i M^{-4+s/4} & \text{for even } s \\ \Delta E_i M^{-3+s/4} & \text{for odd } s \end{cases}.$$

We expect that, for fixed s and ξ, $\Delta\hat{E}_i$ only depends on the ratio K/M.

The plots of $\Delta\hat{E}_i$ as a function of $i = K/M$ are shown in Fig. 3, where $\xi = 0$ for all four plots. When s is odd, only odd values of M are allowed, and each M has two curves, one for odd K points and the other one for even K points, for a reason that will be clear shortly. For $s = 2, 3, 4$ cases, the curves of different M values are very close to each other, so the asymptotic behavior is evident. For the $s = 1$ case, the gaps between consecutive curves become smaller as M increases, which is consistent with the expected asymptotic behavior. It is therefore fair to conclude that the large M behavior is confirmed.

The fact that there are two curves for each M in odd s cases can be understood as follows. Let us consider the $s = 1$ case and take examples of $K = 1$ and $K = 2$, where the former has a much lower contribution to ΔE than the latter according to the plots. Assuming that M is large enough, we have the other small chains with bit number $L \gg K$. Since M is odd, L is even for $K = 1$ and odd for $K = 2$. The lowest energies of these two cases, which are equal to $-4\cot\frac{\pi}{2L} - 4\cos\frac{\pi}{2K} + 8$ according to (2.24b) and the cyclic constraint (2.25), differ only by $\mathcal{O}(1)$. Now, we compare these two cases in the low energy regime, in which the gap between energy levels and the lowest energies are at most of order $1/M$. Consider the numbers of states in the low energy regime. Because of the cyclic constraint, only chains with an even bit number have excited states with energy gaps of order $1/M$ above the lowest energy. For $K = 1$, the number of states in the low energy regime roughly equals $P(L/2)$, the partition number of $L/2$; for $K = 2$, it equals $P(2/2) = 1$. It implies that the low energy regime of $K = 1$ is much denser than the one of $K = 2$. Therefore, for large enough M, the $K = 1$ case has much

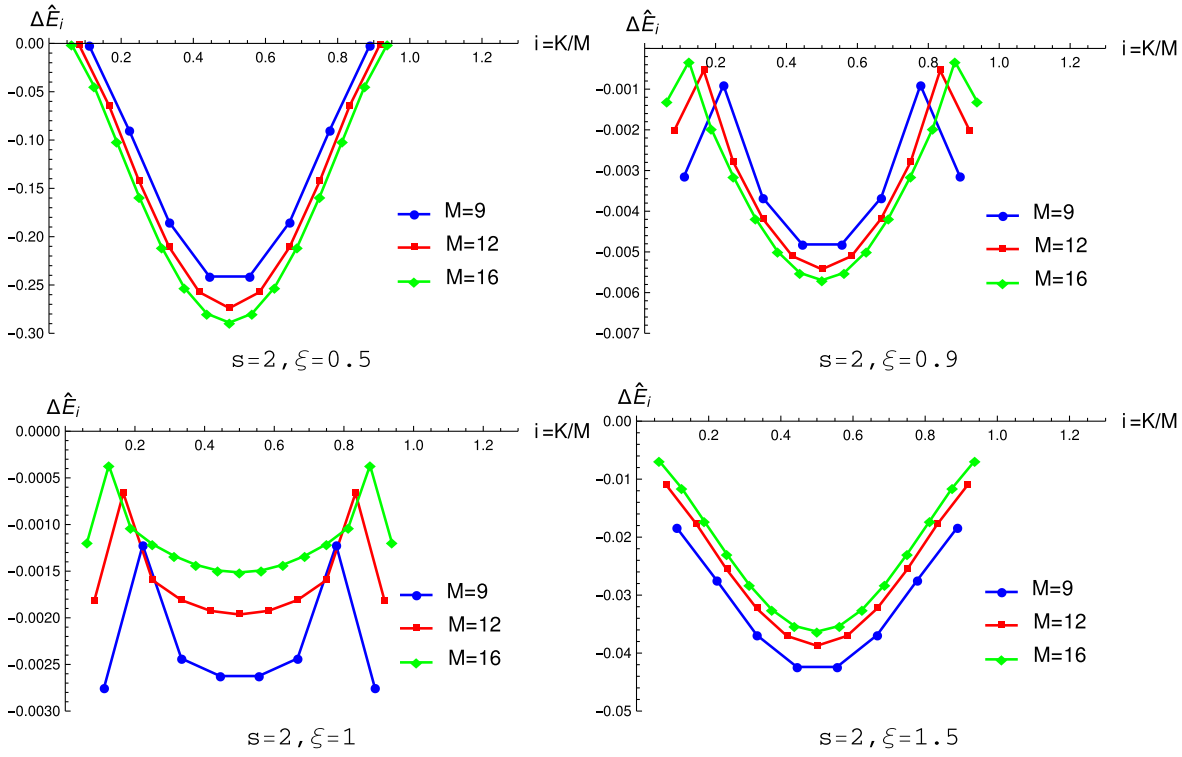

FIG. 4. $\Delta\hat{E}_i$ as a function of $i = K/M$ for the cases of $s = 2$.

lower average energy than the $K = 2$ case. This reasoning holds when K is small. Hence, small odd K cases have a lower contribution to ΔE than small even K.

We next consider the effect of the ξ parameter. Figure 4 shows the plots of $\Delta\hat{E}_i$ with respect to $i = K/M$ for $s = 2$ with different values of ξ. From the plots, the $\xi = 0.5$ and $\xi = 1.5$ cases show a smooth asymptotic behavior as the cases in Fig. 3. But when ξ is close to 1, curves are not smooth and intersect each other. When $\xi < 1$, the curve moves downward as M increases, which implies that ΔE decreases as M increases. So, ΔE is not bounded from below, and the system is not stable. In contrast, when $\xi > 1$, the curve moves upward as M increases, which implies a stable system. This is related to a special feature of the $\xi = 1$ case. Recall that the Hamiltonian has an H_1 part shown as (A3a). This part produces a term like $-s\mathrm{Tr}\bar{\phi}_{12\cdots s}\bar{\phi}_{12\cdots s}\phi_{12\cdots s}\phi_{12\cdots s}$. When s is even, $\phi_{12\cdots s}$ is a scalar, and this term behaves like a scalar potential with a negative coefficient, which leads to a dangerous instability. But when $\xi = 1$, this term is canceled exactly by $s\xi\Delta H$. That being said, for even s, $\xi = 1$ is the minimal value for the potential to be bounded from below. To build a physical string bit model for even s, we should require $\xi \geq 1$.

We next study the dependence of $\Delta\hat{E} = \sum_i \Delta\hat{E}_i$ on s. Figure 5 plots the change of $\ln|\Delta\hat{E}|$ with respect to s for

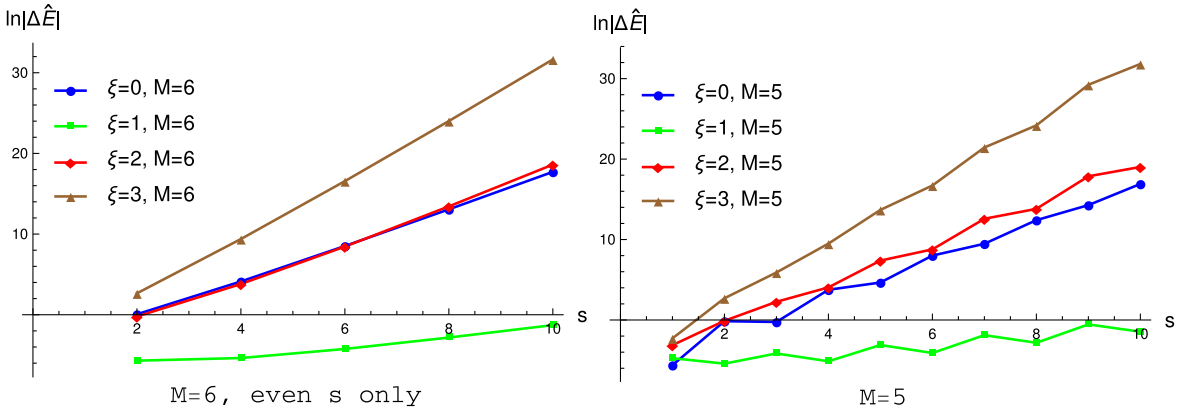

FIG. 5. $\ln|\Delta\hat{E}|$ as a function of s for $M = 6$ and $M = 5$ cases. Note that for $M = 6$ the $\xi = 0$ (blue) and $\xi = 2$ (red) curves almost coincide.

chains of $M=5$ and $M=6$. For $M=5$, we sampled s from 1 to 10; for $M=6$, only even s points are sampled as its ground states only survive in even s cases. For each M, we choose $\xi=0$, 1, 2, 3. For $M=6$, all the curves almost rise linearly. Of all four curves, $\xi=3$ is the steepest one, and $\xi=1$ is the flattest one. $\xi=0$ and $\xi=2$ almost coincide with each other. For the $M=5$ case, the overall trends of the curves are the same as $M=6$ except for slight oscillations between even and odd s points. For $\xi=0$, 1, the oscillation is relatively noticeable, and for $\xi=3$, it is negligible. Actually, if only even s points of $M=5$ are sampled, the plots are almost the same as $M=6$. The exponential dependence of $\Delta\hat{E}$ on s stems from the fact that each ground state has 2^s degeneracies. The fact that $\xi=1$ has a lower slope than others is also related to the fact that $\xi=1$ is the boundary for ΔE to be bounded from below.

VII. CONCLUSION

We have presented a formalism to calculate the cubic interaction vertices for the stable string bit model. With the vertices, we calculated the one-loop self-energies of the model in both analytical and numerical ways.

From the large M behavior of one-loop self-energies, we found that the Lorentz invariance requires the critical dimension of the model to be $s=24$, which then leads to the protostring model. One interesting interpretation of $s=24$ is as follows [13]. Out of the 24 dimensions, 16 of them are paired to form 8 compactified bosonic dimensions, and the rest 8 remain as a fermionic dimension. Thus, it has the same degrees of freedom as the superstring model. The large M behavior of ΔE_G is determined by the ground states contribution of the small chains. Notwithstanding that the number of excited states grows exponentially with respect to M [10], the excited states contributions are canceled out due to the fermionic nature of string bits. These results support the idea of formulating string theory by string bit models.

The future research of this work can be done in several ways. One can improve the numerical computation to study higher M or s cases. One can also apply the formalism to other calculations, e.g., four strings interaction, or to study higher-loop corrections and find the Feynman rules of the model.

ACKNOWLEDGMENTS

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APPENDIX A: HAMILTONIAN AND ITS ACTION ON COLOR SINGLETS

The (anti)communication relations among string-bit creation and annihilation operators is

$$\begin{aligned}\left[(\phi_{a_1\cdots a_n})^\beta_\alpha,(\bar{\phi}_{b_1\cdots b_m})^\delta_\gamma\right]_\pm &\equiv (\phi_{a_1\cdots a_n})^\beta_\alpha(\bar{\phi}_{b_1\cdots b_m})^\delta_\gamma-(-1)^{mn}(\bar{\phi}_{b_1\cdots b_m})^\delta_\gamma(\phi_{a_1\cdots a_n})^\beta_\alpha\\ &=\delta_{mn}\delta^\delta_\alpha\delta^\beta_\gamma\sum_P(-1)^P\delta_{a_1b_{P_1}}\cdots\delta_{a_nb_{P_n}},\end{aligned}\tag{A1}$$

where the sum runs over all permutations of $1,2,\ldots,n$.

The Hamiltonian of the model consists of $\mathcal{O}(1)$ terms and $\mathcal{O}(1/N)$ terms. The $\mathcal{O}(1)$ terms are the generalization of the Hamiltonian of the $s=1$ string bit model [8,10]

$$\begin{aligned}H^{s=1}=\frac{2}{N}\mathrm{Tr}[&(\bar{a}^2-i\bar{b}^2)a^2-(\bar{b}^2-i\bar{a}^2)b^2+(\bar{a}\,\bar{b}+\bar{b}\,\bar{a})ba\\&+(\bar{a}\,\bar{b}-\bar{b}\,\bar{a})ab],\end{aligned}\tag{A2}$$

where $\bar{a}=\bar{\phi}$ and $\bar{b}=\bar{\phi}_1$. $H^{s=1}$ produces the Green-Schwarz Hamiltonian [14,15] at $N=\infty$.

$H^{s=1}$ is generalized to $\sum_{i=1}^5 H_i$, where [9,11]

$$H_1=\frac{2}{N}\sum_{n=0}^{s}\sum_{k=0}^{s}\frac{s-2n}{n!k!}\mathrm{Tr}\bar{\phi}_{a_1\cdots a_n}\bar{\phi}_{b_1\cdots b_k}\phi_{b_1\cdots b_k}\phi_{a_1\cdots a_n},\tag{A3a}$$

$$H_2=\frac{2}{N}\sum_{n=0}^{s-1}\sum_{k=0}^{s-1}\frac{(-1)^k}{n!k!}\mathrm{Tr}\bar{\phi}_{a_1\cdots a_n}\bar{\phi}_{bb_1\cdots b_k}\phi_{b_1\cdots b_k}\phi_{ba_1\cdots a_n},\tag{A3b}$$

$$H_3=\frac{2}{N}\sum_{n=0}^{s-1}\sum_{k=0}^{s-1}\frac{(-1)^k}{n!k!}\mathrm{Tr}\bar{\phi}_{ba_1\cdots a_n}\bar{\phi}_{b_1\cdots b_k}\phi_{bb_1\cdots b_k}\phi_{a_1\cdots a_n},\tag{A3c}$$

$$H_4=\frac{2i}{N}\sum_{n=0}^{s-1}\sum_{k=0}^{s-1}\frac{(-1)^k}{n!k!}\mathrm{Tr}\bar{\phi}_{a_1\cdots a_n}\bar{\phi}_{b_1\cdots b_k}\phi_{bb_1\cdots b_k}\phi_{ba_1\cdots a_n},\tag{A3d}$$

$$H_5=-\frac{2i}{N}\sum_{n=0}^{s-1}\sum_{k=0}^{s-1}\frac{(-1)^k}{n!k!}\mathrm{Tr}\bar{\phi}_{ba_1\cdots a_n}\bar{\phi}_{bb_1\cdots b_k}\phi_{b_1\cdots b_k}\phi_{a_1\cdots a_n}.\tag{A3e}$$

One can check that for $s=1$ Eq. (A3) is reduced to Eq. (A2) if one identifies $\bar{\phi}$ as $\bar{a}$ and $\bar{\phi}_1$ as $\bar{b}$.

We now add $\mathcal{O}(1/N)$ terms to the Hamiltonian. As Refs. [8,10] show, the $N=\infty$ behavior is not affected by the $\mathcal{O}(1/N)$ terms

$$\Delta H^{s=1}=\frac{2}{N}\mathrm{Tr}\left[\bar{a}\,\bar{b}\,ba+\bar{b}\,\bar{a}\,ab+\bar{a}^2a^2+\bar{b}^2b^2-\tilde{M}^{s=1}\right],\tag{A4}$$

$$\tilde{M}^{s=1} = \mathrm{Tr}(\bar{a}a + \bar{b}b) - \frac{1}{N}(\mathrm{Tr}\bar{a}\mathrm{Tr}a + \mathrm{Tr}\bar{b}\mathrm{Tr}b). \quad \text{(A5)}$$

By analogy with $H^{s=1}$, $\Delta H^{s=1}$ can be generalized to the arbitrary s case as

$$\Delta H = \frac{2}{N}\left(\sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}\mathrm{Tr}\bar{\phi}_{b_1\cdots b_k}\bar{\phi}_{a_1\cdots a_n}\phi_{a_1\cdots a_n}\phi_{b_1\cdots b_k} - \tilde{M}\right), \quad \text{(A6a)}$$

$$\tilde{M} = \sum_{n=0}^{s}\frac{1}{n!}\mathrm{Tr}\bar{\phi}_{a_1\cdots a_n}\phi_{a_1\cdots a_n} - \frac{1}{N}\sum_{n=0}^{s}\frac{1}{n!}\mathrm{Tr}\bar{\phi}_{a_1\cdots a_n}\mathrm{Tr}\phi_{a_1\cdots a_n}. \quad \text{(A6b)}$$

Combining the two parts together, we have the complete form of the Hamiltonian for arbitrary s,

$$H = \sum_{i=1}^{5} H_i + s\xi\Delta H, \quad \text{(A7)}$$

where ξ is a real constant.

H commutes with the supersymmetry operators

$$Q^a = \sum_{n=0}^{s-1}\frac{(-1)^n}{n!}\mathrm{Tr}\Big[e^{i\pi/4}\bar{\phi}_{a_1\cdots a_n}\phi_{aa_1\cdots a_n} + e^{-i\pi/4}\bar{\phi}_{aa_1\cdots a_n}\phi_{a_1\cdots a_n}\Big], \quad \text{(A8)}$$

$$\{Q^a, Q^b\} = 2M\delta_{ab}, \quad \text{(A9)}$$

which will guarantee equal numbers of bosonic and fermionic eigenstates at each energy level.

Using the commutation relations (A1), we obtain the action of H_i on single trace states [11]

$$H_1 T(\theta_1,\ldots,\theta_M)|0\rangle = 2\sum_{k=1}^{M}\left(s - 2\theta_k^a\frac{d}{d\theta_k^a}\right)T(\theta_1,\ldots,\theta_M)|0\rangle + \frac{2}{N}\sum_{k=1}^{M}\left(s - 2\theta_k^a\frac{d}{d\theta_k^a}\right)\sum_{l\neq k,k+1}T(\theta_1,\ldots,\theta_k)T(\theta_{k+1},\ldots,\theta_{l-1})|0\rangle,$$

$$H_2 T(\theta_1,\ldots,\theta_M)|0\rangle = 2\sum_{k=1}^{M}\theta_k^a\frac{d}{d\theta_{k+1}^a}T(\theta_1,\ldots,\theta_M)|0\rangle + \frac{2}{N}\sum_{k=1}^{M}\sum_{l\neq k,k+1}\theta_k^a\frac{d}{d\theta_l^a}T(\theta_1,\ldots,\theta_k)T(\theta_{k+1},\ldots,\theta_{l-1})|0\rangle,$$

$$H_3 T(\theta_1,\ldots,\theta_M)|0\rangle = 2\sum_{k=1}^{M}\theta_{k+1}^a\frac{d}{d\theta_k^a}T(\theta_1,\ldots,\theta_M)|0\rangle + \frac{2}{N}\sum_{k=1}^{M}\sum_{l\neq k,k+1}\theta_l^a\frac{d}{d\theta_k^a}T(\theta_1,\ldots,\theta_k)T(\theta_{k+1},\ldots,\theta_{l-1})|0\rangle,$$

$$H_4 T(\theta_1,\ldots,\theta_M)|0\rangle = -2i\sum_{k=1}^{M}\theta_k^a\theta_{k+1}^a T(\theta_1,\ldots,\theta_M)|0\rangle - \frac{2i}{N}\sum_{k=1}^{M}\sum_{l\neq k,k+1}\theta_k^a\theta_l^a T(\theta_1,\ldots,\theta_k)T(\theta_{k+1},\ldots,\theta_{l-1})|0\rangle,$$

$$H_5 T(\theta_1,\ldots,\theta_M)|0\rangle = -2i\sum_{k=1}^{M}\frac{d}{d\theta_k^a}\frac{d}{d\theta_{k+1}^a}T(\theta_1,\ldots,\theta_M)|0\rangle - \frac{2i}{N}\sum_{k=1}^{M}\sum_{l\neq k,k+1}\frac{d}{d\theta_k^a}\frac{d}{d\theta_l^a}T(\theta_1,\ldots,\theta_k)T(\theta_{k+1},\ldots,\theta_{l-1})|0\rangle.$$

Similarly, the action of ΔH on a single trace state is

$$\Delta H T(\theta_1,\ldots,\theta_M)|0\rangle = \frac{2}{N}\sum_{i=1}^{M}\sum_{j\neq i+1}^{M}T(\theta_j,\ldots\theta_i)T(\theta_{i+1},\ldots\theta_{j-1})|0\rangle.$$

The actions of H_i on double traces are [11]

$$H_1 T(\theta_1\cdots\theta_K)T(\eta_1\cdots\eta_L)|0\rangle_{\text{Fussion}} = \frac{2}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\left(s - 2\theta_k^a\frac{d}{d\theta_k^a}\right)T(\theta_{k+1}\cdots\theta_k\eta_l\cdots\eta_{l-1})|0\rangle + \frac{2}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\left(s - 2\eta_l^a\frac{d}{d\eta_l^a}\right)T(\theta_k\cdots\theta_{k-1}\eta_{l+1}\cdots\eta_l)|0\rangle,$$

$$H_2 T(\theta_1\cdots\theta_K)T(\eta_1\cdots\eta_L)|0\rangle_{\text{Fussion}} = \frac{2}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\theta_k^a\frac{d}{d\eta_l}T(\theta_{k+1}\cdots\theta_k\eta_l\cdots\eta_{l-1})|0\rangle + \frac{2}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\eta_l^a\frac{d}{d\theta_k^a}T(\theta_k\cdots\theta_{k-1}\eta_{l+1}\cdots\eta_l)|0\rangle.$$

Similarly, the action of ΔH on the double trace states is

$$\begin{aligned}\Delta H T(\theta_1\cdots\theta_K)T(\eta_1\cdots\eta_L)|0\rangle_{\text{Fussion}} &= \frac{2}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}T(\theta_{k+1}\cdots\theta_k\eta_l\cdots\eta_{l-1})|0\rangle \\ &+\frac{2}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}T(\theta_k\cdots\theta_{k-1}\eta_{l+1}\cdots\eta_l)|0\rangle.\end{aligned}$$

1. Derivation of ΔH

It is not obvious how to generalize $\Delta H^{s=1}$ to arbitrary s cases. We actually obtain the generalization from the relation

$$\mathrm{Tr}G^2 = N(\Delta H - H'),$$

which has been proven in Appendix E of Ref. [10] for $s=1$. Here, the color operator G is defined as [7]

$$G^\beta_\alpha = \sum_{n=0}^{s}\frac{1}{n!}(\bar\phi_{a_1\cdots a_n}\phi_{a_1\cdots a_n} - :\phi_{a_1\cdots a_n}\bar\phi_{a_1\cdots a_n}:)^\beta_\alpha,$$

and both ΔH and H' are supersymmetric and of $\mathcal{O}(1/N)$. The notation $:\phi_{a_1\cdots a_n}\bar\phi_{a_1\cdots a_n}:$ indicates the normal ordering of $\phi_{a_1\cdots a_n}\bar\phi_{a_1\cdots a_n}$. In $s=1$, we have [10]

$$H'^{s=1} = \frac{2}{N}\mathrm{Tr}(\bar a:a\bar a:a + \bar b:b\bar a:a - \bar a:b\bar b:a).$$

One can verify that the action of G^β_α on any color singlet vanishes: $G^\beta_\alpha|\text{any color singlet}\rangle = 0$. We therefore have $(\Delta H - H') = 0$ in the color singlet space.

To find ΔH, we expand $\mathrm{Tr}G^2$ and match its terms with $H'^{s=1}$ and $\Delta H^{s=1}$. By direct calculation, we have

$$\begin{aligned}\mathrm{Tr}G^2 = &\sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}\phi_{a_1\cdots a_n}\bar\phi_{b_1\cdots b_k}\phi_{b_1\cdots b_k} + \sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}\mathrm{Tr}:\phi_{a_1\cdots a_n}\bar\phi_{a_1\cdots a_n}::\phi_{b_1\cdots a_k}\bar\phi_{b_1\cdots b_k}: \\ &-\sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}\phi_{a_1\cdots a_n}:\phi_{b_1\cdots b_k}\bar\phi_{b_1\cdots b_k}: - \sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}\mathrm{Tr}:\phi_{a_1\cdots a_n}\bar\phi_{a_1\cdots a_n}:\bar\phi_{b_1\cdots b_k}\phi_{b_1\cdots b_k}.\end{aligned}$$

We calculate each term on the rhs of $\mathrm{Tr}G^2$ and obtain

$$\begin{aligned}\text{first term} &= N\sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}\phi_{a_1\cdots a_n} + \sum_{n=0}^{s}\sum_{k=0}^{s}\frac{(-1)^{nk}}{n!k!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}:\phi_{a_1\cdots a_n}\bar\phi_{b_1\cdots b_k}:\phi_{b_1\cdots b_k}, \\ \text{second term} &= N\sum_{n=0}^{s}\frac{1}{n!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}\phi_{a_1\cdots a_n} + \sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}:\phi_{b_1\cdots b_k}\bar\phi_{b_1\cdots b_k}:\phi_{a_1\cdots a_n}, \\ \text{third term} &= -\sum_{n=0}^{s}\frac{1}{n!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}\mathrm{Tr}\phi_{a_1\cdots a_n} - \sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}\mathrm{Tr}\bar\phi_{b_1\cdots b_k}\bar\phi_{a_1\cdots a_n}\phi_{a_1\cdots a_n}\phi_{b_1\cdots b_k}, \\ \text{fourth term} &= -\sum_{n=0}^{s}\frac{1}{n!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}\mathrm{Tr}\phi_{a_1\cdots a_n} - \sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}\bar\phi_{b_1\cdots b_k}\phi_{b_1\cdots b_k}\phi_{a_1\cdots a_n}.\end{aligned} \tag{A10}$$

Combining the above together, we have

$$\begin{aligned}\mathrm{Tr}G^2 = &\sum_{n=0}^{s}\sum_{k=0}^{s}\frac{(-1)^{nk}}{n!k!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}:\phi_{a_1\cdots a_n}\bar\phi_{b_1\cdots b_k}:\phi_{b_1\cdots b_k} + \sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}:\phi_{b_1\cdots b_k}\bar\phi_{b_1\cdots b_k}:\phi_{a_1\cdots a_n} \\ &-\sum_{n=0}^{s}\sum_{k=0}^{s}\frac{2}{n!k!}\mathrm{Tr}\bar\phi_{b_1\cdots b_k}\bar\phi_{a_1\cdots a_n}\phi_{a_1\cdots a_n}\phi_{b_1\cdots b_k} + N\sum_{n=0}^{s}\frac{2}{n!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}\phi_{a_1\cdots a_n} - \sum_{n=0}^{s}\frac{2}{n!}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}\mathrm{Tr}\phi_{a_1\cdots a_n}.\end{aligned}$$

Comparing the terms of $\mathrm{Tr}G^2$ with $H'^{s=1}$ and $\Delta H^{s=1}$, we can identify

$$H' = \frac{1}{N}\sum_{n=0}^{s}\sum_{k=0}^{s}\frac{1}{n!k!}(\mathrm{Tr}\bar\phi_{a_1\cdots a_n}:\phi_{b_1\cdots b_k}\bar\phi_{b_1\cdots b_k}:\phi_{a_1\cdots a_n} + (-)^{nk}\mathrm{Tr}\bar\phi_{a_1\cdots a_n}:\phi_{a_1\cdots a_n}\bar\phi_{b_1\cdots b_k}:\phi_{b_1\cdots b_k})$$

and ΔH as (A6). One can verify that both H' and ΔH commute with the supersymmetry operators Q^a (A8).

APPENDIX B: VERIFYING THE NORMALIZATION CONDITION FOR $\bar{\psi}_G$

In this Appendix, we show that the conjugate eigenfunction of $s = 1$, defined as Eq. (3.6), satisfies the normalization condition (3.1). We first show that $\int d^M\theta\bar{\psi}_G\psi_G = 1$. For odd M,

$$\begin{aligned}\int d^M\theta\bar{\psi}_G\psi_G &= (-i)^{\lfloor M/2\rfloor}\int d^M\theta\alpha_0\prod_{i=1}^{\lfloor M/2\rfloor}(-s_i + c_i\alpha_{M-i}\alpha_i)(c_i - s_i\alpha_{M-i}\alpha_i)\\ &= (-i)^{\lfloor M/2\rfloor}\int d^M\theta\alpha_0\prod_{i=1}^{\lfloor M/2\rfloor}(s_i^2 + c_i^2)\alpha_{M-i}\alpha_i\\ &= (-i)^{\lfloor M/2\rfloor}\int d^M\theta\prod_{i=1}^{M}\alpha_{M-i}\\ &= 1,\end{aligned}$$

where in the last step we used[6]

$$\int d^M\theta\prod_{i=1}^{M}\alpha_{M-i} \equiv \int d^M\theta\alpha_{M-1}\alpha_{M-2}\cdots\alpha_0 = i^{\lfloor (M-1)/2\rfloor}. \tag{B1}$$

Similarly, we can show $\int d^M\theta\bar{\psi}_G\psi_G = 1$ for even M.

To show $\int d^M\theta\bar{\psi}_G\psi_r = 0$ for $r \neq G$, it suffices to show that $\int d^M\theta\bar{\psi}_G F_k^\dagger\psi'$ vanishes for all $0 \leq k \leq M-1$ and any eigenfunction ψ'. If $k = 0$, it clearly vanishes because both $F_0^\dagger$ and $\bar{\psi}_G$ contain the Grassmann odd operator α_0. If $0 < k < M/2$,

$$\int d^M\theta\bar{\psi}_G F_k^\dagger\psi' = \int d^M\theta(\tilde{F}_k^{\dagger\pm}\bar{\psi}_G)\psi'. \tag{B2}$$

The rhs of (B2) vanishes because of

$$\tilde{F}_k^{\dagger\pm}\bar{\psi}_G = 0, \tag{B3}$$

which can be verified by checking that

$$\tilde{F}_k^{\dagger\pm}(-s_k + c_k\alpha_{M-k}\alpha_k) = \tilde{F}_{M-k}^{\dagger\pm}(-s_k + c_k\alpha_{M-k}\alpha_k) = 0, \qquad [\tilde{F}_k^{\dagger\pm}, -s_l + c_l\alpha_{M-l}\alpha_l] = 0, \qquad k \neq l, k \neq M-l.$$

Similarly, we can show that $\tilde{F}_k^{\dagger\pm}\bar{\psi}_G = 0$ for $M/2 \leq k \leq M-1$. Therefore, the normalization condition $\int d^M\theta\bar{\psi}_G\psi_r = \delta_{Gr}$ is proved.

APPENDIX C: CALCULATION OF h_{kl}

In this Appendix, we will find the expression of h_{kl}^a in terms of lowering and raising operators. The h_{kl}^a in the language of θ_k is

$$h_{kl} = -2\left(1 - 2\theta_k\frac{d}{d\theta_k}\right) - 2\theta_k\frac{d}{d\theta_l} - 2\theta_l\frac{d}{d\theta_k} - 2i\theta_k\theta_l - 2i\frac{d}{d\theta_k}\frac{d}{d\theta_l} + 2\xi + 2\delta_{k,l}.$$

We now temporarily drop the last two constant terms and will add them back in the end of the calculation.

[6]We do not prove the formula (B1) here. But we have verified it by the *Mathematica* program.

Using (2.13b), we express θ_k and $\frac{d}{d\theta_k}$ in terms of α_n and β_n:

$$\theta_k \frac{d}{d\theta_k} = \frac{1}{M}\sum_{n,m=0}^{M-1} \alpha_n\beta_m \exp\left(2\pi i k\frac{m+n}{M}\right)$$
$$\theta_k \frac{d}{d\theta_l} + \theta_l \frac{d}{d\theta_k} = \frac{1}{M}\sum_{n,m=0}^{M-1} (\alpha_n\beta_m + \alpha_m\beta_n)\left[\exp\left(2\pi i\frac{kn+lm}{M}\right)\right]$$
$$\theta_k\theta_l = \frac{1}{M}\sum_{n,m=0}^{M-1} \alpha_n\alpha_m \exp\left(2\pi i\frac{kn+lm}{M}\right)$$
$$\frac{d}{d\theta_k}\frac{d}{d\theta_l} = \frac{1}{M}\sum_{n,m=0}^{M-1} \beta_n\beta_m \exp\left(2\pi i\frac{kn+lm}{M}\right).$$

Substituting the above into h_{kl} and rearranging, we obtain

$$h_{kl} = h^{(1)}_{kl} + h^{(0)}_{kl},$$

where $h^{(0)}_{kl}$ are the terms with zero modes and $h^{(1)}_{kl}$ are the terms without,

$$h^{(1)}_{kl} = -2 + \frac{4}{M}\sum_{n,m=1}^{M-1} \alpha_n\beta_m \exp\left(2\pi i k\frac{m+n}{M}\right) - \frac{2}{M}\sum_{n,m=1}^{M-1} (\alpha_n\beta_m + \alpha_m\beta_n + i\alpha_n\alpha_m + i\beta_n\beta_m)\exp\left(2\pi i\frac{kn+lm}{M}\right),$$
$$h^{(0)}_{kl} = \frac{2}{M}\sum_{n=1}^{M-1} (\alpha_n\beta_0 + \alpha_0\beta_n - i\alpha_n\alpha_0 - i\beta_n\beta_0)\left[\exp\left(2\pi i\frac{kn}{M}\right) - \exp\left(2\pi i\frac{ln}{M}\right)\right].$$

Let us first consider $h^{(1)}_{kl}$. We express nonzero modes α_m and β_m in terms of raising and lowering operators. Using

$$\alpha_k = c_k F^\dagger_{M-k} + s_k F_k, \qquad \beta_k = -s_k F^\dagger_{M-k} + c_k F_k, \qquad k = 1,\ldots,M-1,$$

we have

$$\alpha_n\alpha_m = c_n c_m F^\dagger_{M-n}F^\dagger_{M-m} + s_n s_m F_n F_m + c_n s_m F^\dagger_{M-n}F_m - c_m s_n F^\dagger_{M-m}F_n + c_n s_n \delta_{m+n,M}, \tag{C1a}$$
$$\alpha_n\beta_m = -c_n s_m F^\dagger_{M-n}F^\dagger_{M-m} + c_m s_n F_n F_m + c_n c_m F^\dagger_{M-n}F_m + s_n s_m F^\dagger_{M-m}F_n + s_n^2\delta_{m+n,M}, \tag{C1b}$$
$$\beta_n\beta_m = s_n s_m F^\dagger_{M-n}F^\dagger_{M-m} + c_n c_m F_n F_m - c_m s_n F^\dagger_{M-n}F_m + c_n s_m F^\dagger_{M-m}F_n + c_n s_n \delta_{m+n,M}. \tag{C1c}$$

We then apply Eqs. (C1) to $h^{(1)}_{kl}$, collect like terms, and antisymmetrize $F^\dagger F^\dagger$ and FF terms to give

$$h^{(1)}_{kl} = \frac{2}{M}\sum_{n,m=1}^{M-1} (A^\dagger_{nm}F^\dagger_n F^\dagger_m + A_{nm}F_n F_m + 2A_{-n,m}F^\dagger_n F_m) - \frac{2}{M}\cot\frac{\pi}{2M} + \frac{2}{M}\cot\frac{2(k-l)+1}{2M}\pi, \tag{C2}$$

where

$$A_{nm} = \exp\left(2\pi i k\frac{m+n}{M}\right)\sin\frac{m-n}{2M}\pi + \frac{i}{2}\left[\exp\left(2\pi i\frac{km+ln}{M}\right)\exp\left(\pi i\frac{m-n}{2M}\right) - m\leftrightarrow n\right]. \tag{C3}$$

Similarly, applying

$$\alpha_n\beta_0 = \exp\left(-\frac{i\pi}{4}\right)c_nF^\dagger_{M-n}F_0 + s_nF_nF_0 \qquad \alpha_0\beta_n = \exp\left(\frac{i\pi}{4}\right)(s_nF^\dagger_{M-n}F^\dagger_0 + c_nF^\dagger_0F_n)$$
$$\alpha_n\alpha_0 = \exp\left(\frac{i\pi}{4}\right)(c_nF^\dagger_{M-n}F^\dagger_0 - s_nF^\dagger_0F_n) \qquad \beta_n\beta_0 = \exp\left(-\frac{i\pi}{4}\right)(-s_nF^\dagger_{M-n}F_0 + c_nF_nF_0)$$

to $h^{(0)}_{kl}$ yields

$$h^{(0)}_{kl} = \frac{2}{M}\sum_{n=1}^{M-1}(\alpha_n\beta_0 + \alpha_0\beta_n - i\alpha_n\alpha_0 - i\beta_n\beta_0)\left[\exp\left(2\pi i\frac{kn}{M}\right) - \exp\left(2\pi i\frac{ln}{M}\right)\right] = \frac{2}{M}\sum_{n=1}^{M-1}X^\dagger_n(F^\dagger_0F^\dagger_n - F^\dagger_nF_0) + \text{H.c.},$$

where

$$X_n = i\left[\exp\left(2\pi i\frac{ln}{M}\right) - \exp\left(2\pi i\frac{kn}{M}\right)\right]\exp\left(-\frac{in\pi}{2M}\right).$$

Now from Eq. (C3), we see that

$$\begin{aligned}A_{n0} &= -\exp\left(2\pi i\frac{kn}{M}\right)\sin\frac{n\pi}{2M} + \frac{i}{2}\left[\exp\left(2\pi i\frac{ln}{M}\right)\exp\left(-\frac{n\pi i}{2M}\right) - \exp\left(2\pi i\frac{kn}{M}\right)\exp\left(\frac{n\pi i}{2M}\right)\right]\\ &= -\frac{i}{2}\exp\left(2\pi i\frac{kn}{M}\right)\exp\left(-\frac{in\pi}{2M}\right) + \frac{i}{2}\exp\left(2\pi i\frac{ln}{M}\right)\exp\left(-\frac{in\pi}{2M}\right)\\ &= \frac{1}{2}X_n.\end{aligned}$$

Hence, to add $h^{(0)}_{kl}$ terms to $h^{(1)}_{kl}$, we can simply change the m, n index of (C2) to start from 0. Finally, adding back the constant terms, we have

$$h_{kl} = \frac{2}{M}\sum_{n,m=0}^{M-1}(A^\dagger_{nm}F^\dagger_nF^\dagger_m + A_{nm}F_nF_m + 2A_{-n,m}F^\dagger_nF_m) + \frac{2}{M}\mu, \tag{C4}$$

where A_{mn} is given by (C3) and

$$\mu = -\cot\frac{\pi}{2M} + \cot\frac{2(k-l)+1}{2M}\pi + M\xi. \tag{C5}$$

APPENDIX D: CALCULATION OF $\langle\cdots\bar{A}_{nm}F^\dagger_nF^\dagger_m\exp(\frac{1}{2}f^\dagger_kD_{kl}f^\dagger_l)\rangle$

In this Appendix, we will derive the formula

$$\left\langle\cdots\bar{A}_{nm}F^\dagger_nF^\dagger_m\exp\left(\frac{1}{2}f^\dagger_kD_{kl}f^\dagger_l\right)\right\rangle = -\frac{2}{M}\left[\mathrm{Tr}(S^*C^{-1}\bar{A}) + B_{mn}\frac{\partial}{\partial D_{mn}}\right]\left\langle\cdots\exp\left(\frac{1}{2}f^\dagger_kD_{kl}f^\dagger_l\right)\right\rangle, \tag{D1}$$

where $\bar{A}\equiv A^\dagger$ and $B = C^{-1}\bar{A}(C^{-1})^T$, and the relations among $F^\dagger_n$, f_k, and $f^\dagger_k$ are given by

$$F_m = \sum_{n=0}^{M-1}(f_nC_{mn} + f^\dagger_nS_{mn}), \qquad 0\le m\le M-1, \tag{D2}$$

with $C_{mn}D_{nl} + S_{ml} = 0$.

Let $X=\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger$, $|G\rangle=\exp(X)|0\rangle$; then,

$$F_n^\dagger \bar{A}_{nm}F_m^\dagger|G\rangle=\exp(X)\left(\bar{A}_{nm}F_n^\dagger F_m^\dagger-[X,\bar{A}_{nm}F_n^\dagger F_m^\dagger]+\frac{1}{2}[X,[X,\bar{A}_{nm}F_n^\dagger F_m^\dagger]]+\cdots\right)|0\rangle. \tag{D3}$$

Now, let us calculate each term in the parentheses of the rhs of Eq. (D3). For the first term,

$$\begin{aligned}\bar{A}_{nm}F_n^\dagger F_m^\dagger|0\rangle&=\left(\bar{A}_{nm}f_i^\dagger f_j^\dagger C_{ni}^* C_{mj}^*+\bar{A}_{nm}f_i S_{ni}^* f_j^\dagger C_{mj}^*\right)|0\rangle\\&=\left(\bar{A}_{nm}f_i^\dagger f_j^\dagger C_{ni}^* C_{mj}^*+\bar{A}_{nm}S_{ni}^* C_{mj}^*\left\{f_i,f_j^\dagger\right\}\right)|0\rangle\\&=\left[f_i^\dagger\left(C^\dagger\bar{A}C^*\right)_{ij}f_j^\dagger+\mathrm{Tr}\left(\bar{A}C^*S^\dagger\right)\right]|0\rangle.\end{aligned} \tag{D4a}$$

For the second term of the rhs of Eq. (D3), we first find

$$\left[\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger,F_m^\dagger\right]=\frac{1}{2}D_{kl}\left[f_k^\dagger f_l^\dagger,F_m^\dagger\right]=\frac{1}{2}D_{kl}\left(f_k^\dagger\left\{f_l^\dagger,F_m^\dagger\right\}-\left\{f_k^\dagger,F_m^\dagger\right\}f_l^\dagger\right)=\frac{1}{2}D_{kl}\left(f_k^\dagger\delta_{ln}S_{mn}^*-\delta_{kn}S_{mn}^*f_l^\dagger\right)=-S_{ml}^*D_{lk}f_k^\dagger,$$

where in the second step we used the identity $[AB,C]=A\{B,C\}-\{A,C\}B$ and in the last step we used the property that D_{kl} is antisymmetric. We then have

$$\begin{aligned}\left[\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger,F_n^\dagger\bar{A}_{nm}F_m^\dagger\right]&=F_n^\dagger\bar{A}_{nm}\left[\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger,F_m^\dagger\right]+\left[\frac{1}{2}f_k^\dagger D_{kl}f_l^\dagger,F_n^\dagger\right]\bar{A}_{nm}F_m^\dagger\\&=-F_n^\dagger\bar{A}_{nm}S_{ml}^*D_{lk}f_k^\dagger-S_{nl}^*D_{lk}f_k^\dagger\bar{A}_{nm}F_m^\dagger\\&=\bar{A}_{nm}S_{ml}^*D_{lk}\left(f_k^\dagger F_n^\dagger-\left\{F_n^\dagger,f_k^\dagger\right\}\right)-S_{nl}^*D_{lk}f_k^\dagger\bar{A}_{nm}F_m^\dagger\\&=\bar{A}_{nm}S_{ml}^*D_{lk}\left(f_k^\dagger F_n^\dagger-S_{nk}^*\right)-S_{nl}^*D_{lk}f_k^\dagger\bar{A}_{nm}F_m^\dagger\\&=\bar{A}_{nm}S_{ml}^*D_{lk}f_k^\dagger F_n^\dagger-S_{nl}^*D_{lk}f_k^\dagger\bar{A}_{nm}F_m^\dagger-\bar{A}_{nm}S_{ml}^*D_{lk}S_{nk}^*\\&=f_k^\dagger D_{kl}(S^\dagger)_{lm}\bar{A}_{mn}F_n^\dagger+f_k^\dagger D_{kl}(S^\dagger)_{ln}\bar{A}_{nm}F_m^\dagger-\bar{A}_{nm}S_{ml}^*D_{lk}(S^\dagger)_{kn}\\&=2f_k^\dagger(DS^\dagger\bar{A})_{km}F_m^\dagger-\mathrm{Tr}(\bar{A}S^*DS^\dagger).\end{aligned}$$

It then follows that

$$[X,F_n^\dagger\bar{A}_{nm}F_m^\dagger]|0\rangle=[2f_k^\dagger(DS^\dagger\bar{A}C^*)_{kl}f_l^\dagger-\mathrm{Tr}(\bar{A}S^*DS^\dagger)]|0\rangle. \tag{D4b}$$

For the third term rhs of Eq. (D3),

$$\begin{aligned}\left[X,\left[X,\bar{A}_{nm}F_n^\dagger F_m^\dagger\right]\right]&=\left[X,2f_k^\dagger\left(DS^\dagger\bar{A}\right)_{km}F_m^\dagger\right]-\cancelto{0}{\left[X,\mathrm{Tr}\left(\bar{A}S^*DS^\dagger\right)\right]}\\&=2f_i^\dagger\left(DS^\dagger\bar{A}\right)_{im}\left[\frac{1}{2}\sum_{k,l}f_k^\dagger D_{kl}f_l^\dagger,F_m^\dagger\right]\\&=-2f_k^\dagger\left(DS^\dagger\bar{A}S^*D\right)_{kl}f_l^\dagger.\end{aligned} \tag{D4c}$$

It follows that the higher order commutations all vanish. Substituting Eqs. (D4) into Eq. (D3), we have

$$\begin{aligned}F_n^\dagger \bar{A}_{nm} F_m^\dagger |G\rangle &= [f_k^\dagger (C^\dagger \bar{A} C^*)_{kl} f_l^\dagger + \mathrm{Tr}(\bar{A} C^* S^\dagger)]|G\rangle - [2 f_k^\dagger (D S^\dagger \bar{A} C^*)_{kl} f_l^\dagger - \mathrm{Tr}(\bar{A} S^* D S^\dagger)]|G\rangle - f_k^\dagger (D S^\dagger \bar{A} S^* D)_{kl} f_l^\dagger |G\rangle \\ &= \mathrm{Tr}[\bar{A}(C^* + S^* D) S^\dagger]|G\rangle + f_k^\dagger (C^\dagger \bar{A} C^* - 2 D S^\dagger \bar{A} C^* - D S^\dagger \bar{A} S^* D)_{kl} f_l^\dagger |G\rangle \\ &= \mathrm{Tr}[\bar{A}(C^* + S^* D) S^\dagger]|G\rangle + f_k^\dagger (C^\dagger \bar{A} C^* - D S^\dagger \bar{A} C^* + C^\dagger \bar{A} S^* D - D S^\dagger \bar{A} S^* D)_{kl} f_l^\dagger |G\rangle \\ &= \mathrm{Tr}[\bar{A}(C^* + S^* D) S^\dagger]|G\rangle + f_k^\dagger [C^\dagger \bar{A}(C^* + S^* D) - D S^\dagger \bar{A}(S^* D + C^*)]_{kl} f_l^\dagger |G\rangle \\ &= \mathrm{Tr}[\bar{A}(C^* + S^* D) S^\dagger]|G\rangle + f_k^\dagger [(C^* + S^* D)^T \bar{A}(C^* + S^* D)]_{kl} f_l^\dagger |G\rangle,\end{aligned}$$

where in the third equality we antisymmetrized the $2DS^\dagger \bar{A} C^*$ term to be $DS^\dagger \bar{A} C^* - (DS^\dagger \bar{A} C^*)^T$ and then used the fact that $\bar{A}$ and D are antisymmetric matrices. Now,

$$C^* + S^* D = C^* - C^* D^* D = C^*(I - D^* D) = C^*(I + DD^\dagger)^T = C^*(C^{-1} C^{-1\dagger})^T = (C^{-1})^T,$$

where in the second-to-last equality $I + DD^\dagger = C^{-1} C^{-1\dagger}$ follows from Eqs. (2.28) and (2.32). We therefore have

$$F_n^\dagger \bar{A}_{nm} F_m^\dagger |G\rangle = -\mathrm{Tr}(S^* C^{-1} \bar{A})|G\rangle + f_k^\dagger [C^{-1} \bar{A} (C^{-1})^T]_{kl} f_l^\dagger |G\rangle,$$

which implies (D1).

APPENDIX E: EXAMPLES OF $M = 3$

In this Appendix, as a demo of using (5.4) to calculate one-loop self-energy, let us consider the one-loop self-energy for the ground state of the $M = 3$, $s = 1$ and $M = 3$, $s = 2$ cases. For $M = 3$, we only need to calculate the $K = 1$ case since the contribution of $K = 2$ is the same as $K = 1$.

The C, S, and D matrices are

$$C = (1) \oplus \frac{1+\sqrt{3}}{4} \begin{pmatrix} e^{i\frac{\pi}{6}} & -e^{i\frac{2\pi}{3}} \\ e^{-i\frac{\pi}{6}} & e^{i\frac{2\pi}{3}} \end{pmatrix}, \qquad |\det C| = \frac{2+\sqrt{3}}{4}$$

$$S = (0) \oplus \frac{\sqrt{3}-1}{4} \begin{pmatrix} e^{-i\frac{\pi}{6}} & e^{-i\frac{2\pi}{3}} \\ e^{-i\frac{5\pi}{6}} & e^{-i\frac{\pi}{3}} \end{pmatrix}, \qquad D = (0) \oplus (2-\sqrt{3}) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

and matrices A, B, and constant μ' are

$$A^{(V)} = \begin{pmatrix} 0 & i\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ -i\frac{\sqrt{3}}{4} & 0 & \frac{3}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 \end{pmatrix}, \qquad A^{(W)} = \begin{pmatrix} 0 & \frac{i\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ -\frac{i\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}$$

$$B^{(V)} = \begin{pmatrix} 0 & \frac{1}{2}\sqrt{\frac{3}{2}} e^{3i\pi/4} & \sqrt{\frac{3}{2}}\left(1 - \frac{\sqrt{3}}{2}\right) e^{3i\pi/4} \\ \frac{1}{2}\sqrt{\frac{3}{2}} e^{-i\pi/4} & 0 & 3i(2-\sqrt{3}) \\ \sqrt{\frac{3}{2}}\left(1 - \frac{\sqrt{3}}{2}\right) e^{-i\pi/4} & -3i(2-\sqrt{3}) & 0 \end{pmatrix},$$

$$B^{(W)} = \begin{pmatrix} 0 & \sqrt{\frac{3}{2}} e^{3i\pi/4} & \sqrt{\frac{3}{2}}(2-\sqrt{3}) e^{3i\pi/4} \\ \sqrt{\frac{3}{2}} e^{-i\pi/4} & 0 & 0 \\ \sqrt{\frac{3}{2}}(2-\sqrt{3}) e^{-i\pi/4} & 0 & 0 \end{pmatrix},$$

$$\mu'^{(V)} = -3 + \sqrt{3} + 3\xi, \qquad \mu'^{(W)} = -4\sqrt{3} + 6\xi.$$

The operators f_n are

$$f_0 = F_0, f_1 = F_1^{(2)}, \qquad f_2 = e^{-i\pi/4}\left(\sqrt{\frac{2}{3}}F_0^{(1)} - F_0^{(2)}\sqrt{\frac{1}{3}}\right).$$

For $s = 1$, the eigenfunctions and their conjugates of 1-bit and 2-bit chains are shown in Table II. The contribution of $K = 1$ to the energy correction is

$$\Delta E_G^{K=1} = \frac{1}{N^2}\frac{KLM|\det C|}{E_G - E_G^{(1)} - E_1^{(2)}}\left(\left\langle F_1^{(2)} f_0\right\rangle_W^* \left\langle F_1^{(2)} f_0\right\rangle_V + \left\langle F_1^{(2)} f_2\right\rangle_W^* \left\langle F_1^{(2)} f_2\right\rangle_V\right). \tag{E1}$$

So, we need to calculate $\langle F_1^{(2)} f_0\rangle_{V,W}$ and $\langle F_1^{(2)} f_1\rangle_{V,W}$:

$$\begin{aligned}\left\langle F_1^{(2)} f_0\right\rangle_V &= \frac{2}{M}\mu'^{(V)}\cancelto{0}{\left\langle f_1 f_0 \exp\left(\frac{1}{2}f_k^\dagger D_{kl} f_l^\dagger\right)\right\rangle} + \frac{2}{M}B_{mn}^{(V)}\left\langle f_1 f_0 f_m^\dagger f_n^\dagger \exp\left(\frac{1}{2}f_k^\dagger D_{kl} f_l^\dagger\right)\right\rangle \\ &= \frac{4}{M}B_{0n}^{(V)}\left\langle f_1 f_n^\dagger \exp\left(\frac{1}{2}f_k^\dagger D_{kl} f_l^\dagger\right)\right\rangle \\ &= \frac{4}{M}B_{01}^{(V)} = \sqrt{\frac{2}{3}}e^{3i\pi/4},\end{aligned}$$

$$\begin{aligned}\left\langle F_1^{(2)} f_2\right\rangle_V &= \frac{2}{M}\mu'^{(V)}\left\langle f_1 f_2 \exp\left(\frac{1}{2}f_k^\dagger D_{kl} f_l^\dagger\right)\right\rangle + \frac{2}{M}B_{mn}^{(V)}\left\langle f_1 f_2 f_m^\dagger f_n^\dagger \exp\left(\frac{1}{2}f_k^\dagger D_{kl} f_l^\dagger\right)\right\rangle \\ &= -\frac{2}{M}\mu'^{(V)}D_{12} - \frac{4}{M}B_{12}^{(V)} = -2i\left(1 - \frac{1}{\sqrt{3}} + 2\xi - \sqrt{3}\xi\right).\end{aligned}$$

Likewise,

$$\left\langle F_1^{(2)} f_0\right\rangle_W = \frac{4}{M}B_{01}^{(W)} = 2\sqrt{\frac{2}{3}}e^{3i\pi/4}, \qquad \left\langle F_1^{(2)} f_2\right\rangle_W = -\frac{2}{M}\mu'^{(W)}D_{12} - \frac{4}{M}B_{12}^{(W)} = -8i\left(1 - \frac{2}{\sqrt{3}} + \xi - \frac{\sqrt{3}}{2}\xi\right).$$

Substituting above results and $E_G = -4\sqrt{3}$ into (E1) yields $\Delta E_G^{K=1} = -\frac{3}{2}(3\sqrt{3}-5)\xi^2 + (12 - 7\sqrt{3})\xi - \frac{3}{4}(3\sqrt{3}-5)$. We then have

$$\Delta E_G = 2\Delta E_G^{L=1} = -3\left(3\sqrt{3}-5\right)\xi^2 + 2\left(12 - 7\sqrt{3}\right)\xi - \frac{3}{2}\left(3\sqrt{3}-5\right).$$

For $s = 2$, the matrices and constants are the same as the $s = 1$ case. But as Table III shows, the energy eigenstates of small chains are different. The energy correction now is given by

$$\begin{aligned}\Delta E_G^{K=1} = &\frac{1}{N^2}\frac{KLM|\det C|^2}{E_G - E_G^{(2)} - 0}\left(\langle 1\rangle_W^*\langle 1\rangle_V + \langle f_2 f_0\rangle_W^*\langle f_2 f_0\rangle_V\right)^2 \\ &+ \frac{1}{N^2}\frac{KLM|\det C|^2}{E_G - E_1^{(2)} - 0}\left(\left\langle F_1^{(2)} f_0\right\rangle_W^* \left\langle F_1^{(2)} f_0\right\rangle_V + \left\langle F_1^{(2)} f_2\right\rangle_W^* \left\langle F_1^{(2)} f_2\right\rangle_V\right)^2.\end{aligned} \tag{E2}$$

Since we have calculated the $\langle F_1^{(2)} f_0\rangle_{V,W}$ and $\langle F_1^{(2)} f_2\rangle_{V,W}$ in the $s = 1$ case, we only need to find $\langle 1\rangle_{V,W}$ and $\langle f_2 f_0\rangle_{V,W}$.

For $K = 1$,

$$\begin{aligned}\langle 1\rangle_V &= \frac{2}{M}\mu'^{(V)}\left\langle \exp\left(\frac{1}{2}f_k^\dagger D_{kl} f_l^\dagger\right)\right\rangle + \frac{2}{M}B_{mn}^{(V)}\cancelto{0}{\left\langle f_m^\dagger f_n^\dagger \exp\left(\frac{1}{2}f_k^\dagger D_{kl} f_l^\dagger\right)\right\rangle} \\ &= \frac{2}{M}\mu'^{(V)} = \frac{2}{\sqrt{3}} - 2 + 2\xi,\end{aligned}$$

TABLE II. 1-bit and 2-bit energy eigenstates of $s = 1$ that do not contain zero modes.

M	ψ	Conjugate	Energy	Grading of ψ
1	$\psi_G^{(1)} = 1$	$\bar{\psi}_G^{(1)} = \theta_1$	$E_G^{(1)} = 0$	Even
2	$\psi_1^{(2)} = F_1^{(2)\dagger}\psi_G^{(2)}$	$\bar{\psi}_1^{(2)} = \tilde{F}_1^{(2)}\bar{\psi}_G^{(2)}$	$E_1^{(2)} = 4$	Odd

TABLE III. 1-bit and 2-bit energy eigenstates of $s = 2$ that do not contain zero modes.

M	ψ	Conjugate	Energy	Grading of ψ
1	$\psi_G^{(1)} = 1$	$\bar{\psi}_G^{(1)} = \theta_1$	$E_G^{(1)} = 0$	Even
2	$\psi_G^{(2)}$	$\bar{\psi}_G^{(2)}$	$E_G^{(2)} = -8$	Even
2	$\psi_1^{(2)} = F_{1,a=1}^{(2)\dagger}F_{1,a=2}^{(2)\dagger}\psi_G^{(2)}$	$\bar{\psi}_1^{(2)} = \tilde{F}_{1,a=2}^{(2)}\tilde{F}_{1,a=1}^{(2)}\bar{\psi}_G^{(2)}$	$E_1^{(2)} = 8$	Even

$$\langle f_2 f_0\rangle_V = \frac{2}{M}\mu'^{(V)}\left\langle f_2 f_0 \exp\left(\frac{1}{2}f_k^\dagger D_{kl} f_l^\dagger\right)\right\rangle^{0} + \frac{2}{M}B_{mn}^{(V)}\left\langle f_2 f_0 f_m^\dagger f_n^\dagger \exp\left(\frac{1}{2}f_k^\dagger D_{kl} f_l^\dagger\right)\right\rangle$$
$$= \frac{4}{M}B_{02}^{(V)} = \sqrt{2}\left(\frac{2}{\sqrt{3}} - 1\right)e^{3i\pi/4},$$

Likewise,

$$\langle 1\rangle_W = \frac{2}{M}\gamma'^{(W)} = -\frac{8}{\sqrt{3}} + 4\xi, \qquad \langle f_2 f_0\rangle_W = \frac{4}{M}B_{02}^{(W)} = 2\sqrt{2}\left(\frac{2}{\sqrt{3}} - 1\right)e^{3i\pi/4}.$$

Substituting the above into Eq. (E2), we obtain

$$\Delta E_G = 2\Delta E_G^{K=1} = \frac{1}{N^2}\left(-66\sqrt{3}\xi^4 + 360\xi^3 - 230\sqrt{3}\xi^2 + 180\xi - \frac{33\sqrt{3}}{2}\right).$$

From the results and the formula (5.4), we see that ΔE_G is a polynomial of ξ of degree $2s$.

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