

Phases of higher spin black holes: Hawking-Page, transitions between black holes, and a critical point

Máximo Bañados,^{1,*} Gustavo Düring,^{1,†} Alberto Faraggi,^{1,2,‡} and Ignacio A. Reyes^{1,3,§}

¹*Pontificia Universidad Católica de Chile Instituto de Física, Pontificia Universidad Católica de Chile, Casilla 306 Santiago, Chile*

²*Departamento de Ciencias Físicas, Facultad de Ciencias Exactas, Universidad Andres Bello, Sazie 2212, Piso 7, Santiago, Chile*

³*Institut für Theoretische Physik und Astrophysik, Julius-Maximilians-Universität Würzburg, Am Hubland 97074, Germany*

(Received 11 March 2017; published 29 August 2017)

We study the thermodynamic phase diagram of three-dimensional $sl(N; \mathbb{R})$ higher spin black holes. By analyzing the semiclassical partition function we uncover a rich structure that includes Hawking-Page transitions to the AdS_3 vacuum, first order phase transitions among black hole states, and a second order critical point. Our analysis is explicit for $N = 4$ but we extrapolate some of our conclusions to arbitrary N . In particular, we argue that even N is stable in the ensemble under consideration but odd N is not.

DOI: 10.1103/PhysRevD.96.046017

Soon after black hole entropy [1,2] and radiation [3] were discovered, Gibbons and Hawking [4] showed that these properties can be derived directly from the Euclidean gravitational action. Black holes are now understood as part of a thermodynamical system with an associated semiclassical partition function

$$Z(\beta) = \sum_M e^{-\beta M + S(M)}. \quad (1)$$

For Schwarzschild black holes the entropy takes the famous value $S = \text{Area}/(4G) = 4\pi GM^2$ [5].

This partition function can be extended to more general black holes in various dimensions. Of particular interest in recent years has been a relatively new family of configurations, namely, three-dimensional black holes carrying higher spin charges [6].

In this paper we study the thermodynamics of higher spin theories by emphasizing the role of the partition function (1). We uncover a rich structure with several interesting features: (i) the existence of Hawking-Page transitions from black holes to the AdS_3 background; (ii) discontinuous phase transitions among black hole states with different macroscopic properties (van der Waals-like); and (iii) a second order transition and a critical point. For related work see [7–10].

We start by reviewing some applications of (1). The partition function is dominated by the configuration that minimizes the action

$$\Gamma_\beta(M) = \beta M - S(M). \quad (2)$$

In a more general setup, Γ will depend on additional charges and chemical potentials. This function, related to the mean field free energy, encodes the thermodynamic structure of the gravitational system and will be our main tool to analyze its phases. Consider Schwarzschild black holes, for example. A quick look shows that $\Gamma_\beta(M)$ does not have a minimum at all; the value $M = \beta/(8\pi G)$ is a maximum, revealing the well-known instability of this system in the canonical ensemble.

The instability of Schwarzschild black holes can be cured either by putting the system in a box [11] or by adding a negative cosmological constant [12], the latter case leading to the celebrated Hawking-Page transition. It is instructive to understand this phenomenon from the point of view of the action (2).

In Fig. 1 (left) we plot $\Gamma_\beta(M)$ for 3 + 1 Schwarzschild-AdS black holes for different values of β [13]. At high

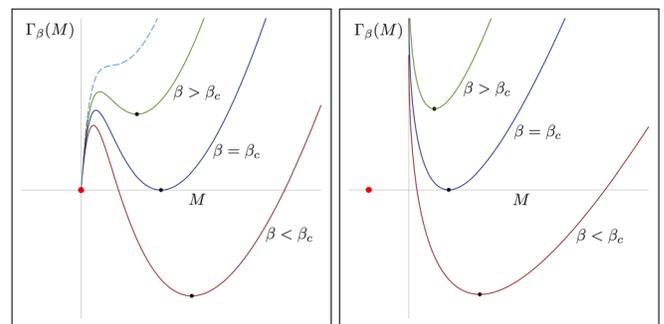


FIG. 1. Hawking-Page transition in 3 + 1 Schwarzschild-AdS black holes (left) and 2 + 1 BTZ black holes (right). The red dots represent the AdS_4 ($M = 0$) and AdS_3 ($M = -1/(8G)$) ground states, respectively. The dashed curve has no local minima but plays no role in the analysis.

* maxbanados@fis.puc.cl

† gduring@fis.puc.cl

‡ faraggi@fis.puc.cl

§ iareyes@uc.cl

temperatures (burgundy curve, small β), there is a clear minimum satisfying

$$\beta = \frac{\partial S(M)}{\partial M}. \quad (3)$$

As the temperature drops, however, the solution to this equation ceases to be the global minimum of $\Gamma_\beta(M)$. For $\beta > \beta_c$, the AdS₄ background with $M = 0$ is the preferred state. This transition from the black hole dominated phase to the vacuum is called Hawking-Page transition. Notice that the minimum of $\Gamma_\beta(M)$ is continuous at $\beta = \beta_c$ but its derivative is not.

The analysis for three-dimensional BTZ black holes is similar, except that the AdS vacuum is now a bound state separated from the black hole continuum by a mass gap. In Fig. 1 (right) we plot $\Gamma_\beta(M) = \beta(M - M_{\text{AdS}}) - S(M)$ for three values of β [14]. For convenience we have shifted the action such that $\Gamma_\beta(M_{\text{AdS}}) = 0$. We see that a local minimum satisfying (3) exists for all temperatures but only for $\beta < \beta_c$ does this state have less action than AdS₃ space.

We now move on to study the phase structure of three-dimensional higher spin gravity. This theory is topological in nature and does not have a known description in terms of metric fields ($g_{\mu\nu}$, $g_{\mu\nu\rho}$, etc.). Instead, it must be formulated as a Chern-Simons theory where the basic variables are $sl(N; \mathbb{R})$ ($N \times N$, real, traceless) matrices A_μ . In a radial gauge, and restricting to static and circularly symmetric configurations, one is left with two such matrices, A_t and A_φ , satisfying

$$[A_t, A_\varphi] = 0. \quad (4)$$

This is the remnant of the Chern-Simons equations of motion. We refer the reader to the extensive literature on this subject for more details.

The gauge invariant information carried by the fields is characterized by the $N - 1$ charges

$$Q_n = \frac{1}{n} \text{Tr}(A_\varphi^n), \quad n = 2, \dots, N. \quad (5)$$

The possible values these charges can take depend on the spatial topology of spacetime. We shall consider two classes of solutions. First, there is the AdS₃ vacuum, for which the spatial topology is a plane and the cycle $\varphi \sim \varphi + 2\pi$ is contractible. As a consequence, the eigenvalues of A_φ are imaginary and quantized, so as to render a smooth field [15]:

$$\text{Eigen}(A_\varphi^{\text{AdS}}) = \frac{i}{2}(N - 1, N - 3, \dots, 1 - N). \quad (6)$$

The corresponding charges can be computed from (5). Notice that $Q_n^{\text{AdS}} = 0$ for odd n . For $N = 4$, which will be our main example, we find

$$Q_2^{\text{AdS}} = -\frac{5}{2}, \quad Q_3^{\text{AdS}} = 0, \quad Q_4^{\text{AdS}} = \frac{41}{16}. \quad (7)$$

The time direction is noncompact so the matrix A_t^{AdS} is only restricted by (4) and not by regularity. This is important. The residual freedom in A_t^{AdS} is just what is needed in order to match the chemical potentials of AdS₃ to those of black holes (just like thermal AdS and Schwarzschild-AdS can both be put at the same temperature).

The second class of solutions we are interested in are black holes. The spatial topology in this sector is a semi-infinite cylinder, with the boundary at $r = 0$ corresponding to the horizon. This space is homeomorphic to the punctured disk. Since the cycle $\varphi \sim \varphi + 2\pi$ is not contractible, A_φ (and therefore Q_n) is left unrestricted. We take the eigenvalues of A_φ to be real for black hole configurations, in consistency with the definition given in [16] in a supersymmetric context. For $N = 4$ we parametrize the eigenvalues as

$$\text{Eigen}(A_\varphi) = \frac{1}{2}(2\lambda_1 + \lambda_2, 2\lambda_1 - \lambda_2, -2\lambda_1 + \lambda_3, -2\lambda_1 - \lambda_3). \quad (8)$$

It follows that the black hole charges read

$$\begin{aligned} Q_2 &= 2\lambda_1^2 + \frac{1}{4}\lambda_2^2 + \frac{1}{4}\lambda_3^2, \\ Q_3 &= \frac{1}{2}\lambda_1(\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3), \\ Q_4 &= \lambda_1^4 + \frac{1}{32}(\lambda_2^4 + \lambda_3^4) + \frac{3}{4}\lambda_1^2(\lambda_2^2 + \lambda_3^2). \end{aligned} \quad (9)$$

In this sector, the matrix A_t is constrained by regularity to be [17]

$$\text{Eigen}(A_t) = \frac{1}{2}(N - 1, N - 3, \dots, 1 - N). \quad (10)$$

This is because the time cycle $t \sim t + 2\pi$ is contractible in the black hole topology. Finally, black holes have an entropy given by [18–21]

$$S(Q) = \text{Tr}(A_t A_\varphi). \quad (11)$$

For $N = 4$ this yields

$$S(\lambda) = \frac{1}{2}(8\lambda_1 + \lambda_2 + \lambda_3). \quad (12)$$

We are now ready to display the grand canonical partition function [22] we aim to calculate:

$$Z(\mu_2, \mu_3, \dots, \mu_n) = 1 + \sum_{\{\text{black holes}\}} e^{-k\Gamma_\mu(Q)}, \quad (13)$$

where the action is

$$\Gamma_\mu(Q) = \sum_{n=2}^N \mu_n(Q_n - Q_n^{\text{AdS}}) - S(Q). \quad (14)$$

The first term “1” in (13) corresponds to the AdS₃ bound state, whose contribution has been subtracted in (14) so that $\Gamma_\mu(Q^{\text{AdS}}) = 0$. The AdS charges have fixed values [for $N = 4$, see (7)]. The sum is then taken over the spectrum of black holes, that is, over all charges consistent with real eigenvalues of A_φ . Also, for convenience we have factored out the coupling constant “ k ” (Chern-Simons level). This parameter is related to the central charge by

$$k = \frac{2\pi c}{N(N^2 - 1)}, \quad c = \frac{3\ell}{2G},$$

and is a measure of the number of degrees of freedom in the system. The steepest descent approximation is justified in the limit $k \sim c \rightarrow \infty$. The real parameters μ_n are chemical potentials conjugate to Q_n . See [18] for details on this. The case of pure gravity is recovered for $N = 2$ after identifying $\mu_2 = \beta/(2\pi\ell)$ and $Q_2 = 2GM$.

We emphasize that the study of the partition function (13) represents a well-posed problem in its own right, independent from its relation to Chern-Simons theories and higher spin black holes. In fact, this problem has striking similarities with the mean field description of some condensed matter systems such as liquid crystals [23–25].

Notice that, for any N , the charges Q_{2n} are always semipositive while Q_{2n+1} have no definite sign. This explains why even N can yield a stable partition function but odd N cannot. Indeed, the action (14) is a polynomial of degree N in the eigenvalues of A_φ . Since the leading terms enter as $\Gamma_\mu(\lambda) = \mu_N Q_N(\lambda) + \dots$, the conditions $N \in 2\mathbb{N}$ and $\mu_N > 0$ guarantee that $\Gamma_\mu(\lambda)$ is bounded from below. The sum (13) then converges. It turns out that $N = 4$ is the simplest, nontrivial, stable theory; $N = 2$ exhibits a Hawking-Page transition but no transitions between black hole states, and $N = 3$ is unstable. From now on we concentrate on $N = 4$.

In principle, the computation of (13) involves a sum over the charges Q_n . This is inconvenient because the entropy (12) has a very simple form in terms of the eigenvalues but not as a function of the charges themselves. Expressing S as a function of Q_n would involve inverting (9). Happily, this is not necessary. We shall now argue that the sum over charges can be traded for a sum over eigenvalues, up to logarithmic corrections that we discard in large k limit.

From (9) we find that the Jacobian for the change of variables $Q_n \rightarrow \lambda_i$ is

$$\left| \frac{\partial Q}{\partial \lambda} \right| = \frac{1}{32} \lambda_2 \lambda_3 (4\lambda_1 + \lambda_2 + \lambda_3)(4\lambda_1 - \lambda_2 + \lambda_3) \times (4\lambda_1 + \lambda_2 - \lambda_3)(4\lambda_1 - \lambda_2 - \lambda_3). \quad (15)$$

We see that $|\frac{\partial Q}{\partial \lambda}| = 0$ happens precisely when two or more eigenvalues of A_φ coincide. In that case, A_φ is non-diagonalizable and the solution becomes extremal [16]. Of course, we restrict our attention to solutions bounded by extremal black holes; the points where the map $Q_n \rightarrow \lambda_i$ is not invertible are never touched. For orientation, recall that in the spectrum of $2 + 1$ black holes, the angular momentum J is bounded by $-M < J < M$, with $J = \pm M$ corresponding to extremal solutions. The map $Q_n \rightarrow \lambda_i$ fails to be invertible at the extreme points $J = \pm M$. Going back to the general case, the domain of λ_i should lie within any of the “wedges” defined by the planes $|\frac{\partial Q}{\partial \lambda}| = 0$. Different wedges correspond to different branches of the inverse map $Q \rightarrow \lambda(Q)$ and have different entropy functions $S(Q)$. We will work in the wedge that includes the BTZ black hole ($\lambda_1 = \lambda_2 = \lambda_3 > 0$), for which all the factors in (15) are positive.

So, given a set of values for the chemical potentials μ_2 , μ_3 and μ_4 , we want to compute the values of λ_1 , λ_2 and λ_3 that minimize the action $\Gamma_\mu(\lambda)$. This configuration will dominate the partition function (13). In particular, we would like to study the continuity of the λ_i obtained in this way as one varies μ_2 , μ_3 and μ_4 .

First, we have checked explicitly, within a wide range of chemical potentials, that the minimum of $\Gamma_\mu(\lambda)$ is never achieved by extremal black holes. The minimum either occurs for the AdS₃ ground state, with $\Gamma_\mu(Q^{\text{AdS}}) = 0$, or it lies in the interior of the BTZ wedge [26]. We do not need to worry about extremal solutions.

Next, we separate the region in the space of chemical potentials where black hole states dominate from the region where the AdS ground state is preferred. The interface between these two regions is defined by the equation

$$\min_{\substack{\{\text{black}\} \\ \{\text{holes}\}}} \Gamma_\mu(Q) = \Gamma_\mu(Q^{\text{AdS}}), \quad (16)$$

and its graphical representation is shown in Fig. 2. The interior of the surface corresponds to the AdS dominated phase. Outside black holes dominate. Crossing this surface in any direction gives a first order Hawking-Page transition.

Let us now concentrate on the region of black hole dominance and look for the global minimum satisfying

$$\frac{\partial \Gamma_\mu(\lambda)}{\partial \lambda_i} = 0. \quad (17)$$

[Recall that $\Gamma_\mu(\lambda)$ is built from (14) after using (9) to write the charges in terms of the eigenvalues, together with the expressions (12) for the entropy and (7) for the background charges]. These equations can be simplified by setting $\mu_2 = 5\mu$ and rescaling $\mu_3 \rightarrow \mu_3\mu^2$, $\mu_4 \rightarrow \mu_4\mu^3$ and $\lambda_i \rightarrow \lambda_i/\mu$. The μ dependence then drops out from (17), reducing the black hole thermodynamics to a two dimensional phase diagram.

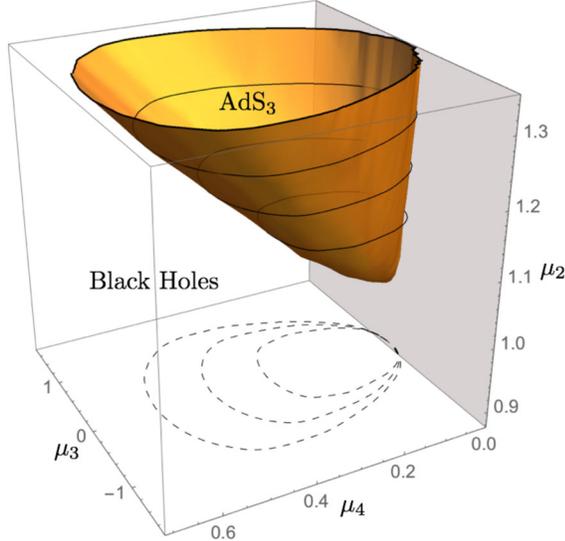


FIG. 2. Hawking-Page surface for the $sl(4; \mathbb{R})$ theory. The minimum is located at $\mu_2 = 1$, $\mu_3 = 0$, $\mu_4 = 0$. This coincides with the critical temperature of $2 + 1$ BTZ black holes after identifying $\mu_2 = \beta/(2\pi\ell)$.

Figure 3 shows the phase diagram for the charge Q_2 . Similar results are obtained for the other two charges. One observes that from the origin stems a critical line across which the system exhibits a first order phase transition between two macroscopically different black hole states. The line ends at a critical point (in red). We now show that at the critical point a second order phase transition takes place in which the minima of $\Gamma_\mu(\lambda)$ become degenerate.

For systems with a single order parameter a critical point occurs when the first, second and third derivatives of the free energy vanish. The simplest generalization to the case with multiple order parameters is to demand that the Hessian matrix has one null eigenvalue, with the rest being strictly positive. Calling v^i the corresponding normalized eigenvector, we further require that the third derivative of $\Gamma_\mu(\lambda)$ along v^i vanishes. Thus, in addition to (17), the critical point must satisfy

$$v^j \frac{\partial^2 \Gamma_\mu(\lambda)}{\partial \lambda_i \partial \lambda_j} = 0, \quad v^i v^j v^k \frac{\partial^3 \Gamma_\mu(\lambda)}{\partial \lambda_i \partial \lambda_j \partial \lambda_k} = 0. \quad (18)$$

To ensure that we still have a minimum, the fourth derivative along v^i should be positive. Following this approach we find that the $N = 4$ higher spin theory exhibits two mirror critical points, the first of which is located at

$$\begin{aligned} \mu_2 &= 5\mu, & \lambda_1 &= \frac{0.3299}{\mu}, & v^1 &= -0.2061, \\ \mu_3 &= 8.7184\mu^2, & \lambda_2 &= \frac{0.0854}{\mu}, & v^2 &= 0.0349, \\ \mu_4 &= 2.9299\mu^3, & \lambda_3 &= \frac{1.0259}{\mu}, & v^3 &= -0.9779, \end{aligned} \quad (19)$$

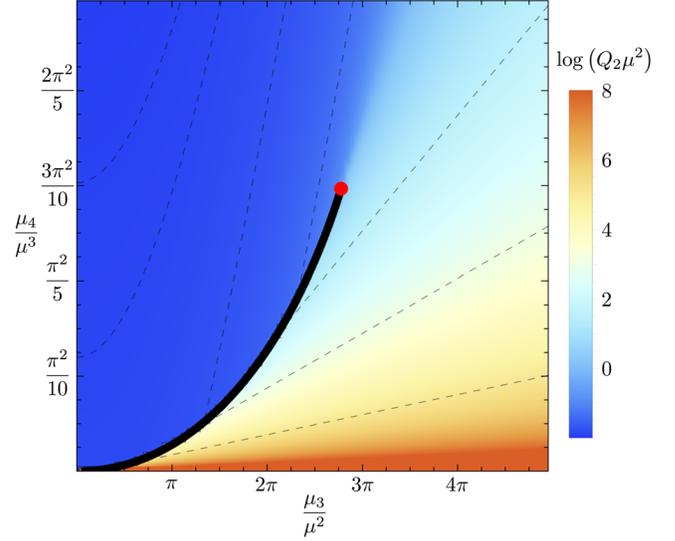


FIG. 3. Phase diagram of the $N = 4$ system, with the second order critical point in red and the first order critical line in black. The complete diagram is symmetric with respect to the vertical axis. $\mu = \mu_2/5$.

and corresponds to the one displayed in Fig. 3. The second point (not shown in Fig. 3) is obtained by exchanging $\lambda_2 \leftrightarrow \lambda_3$ ($Q_3 \rightarrow -Q_3$) and $\mu_3 \rightarrow -\mu_3$, which is a symmetry of the action (14). The values of the charges and the entropy at the critical points can be computed directly from (9) and (12). One can check that in the range $1.0886 < \mu < 6.3591$ the second order phase transition takes place inside the Hawking-Page surface and is therefore not relevant. The endpoints of this interval correspond to the intersection of the critical point with Hawking-Page surface.

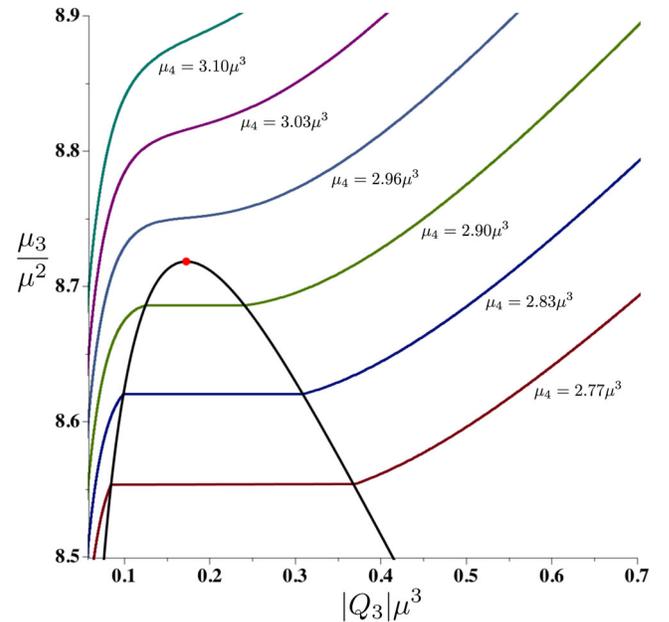


FIG. 4. μ_3 — Q_3 diagram with iso— μ_4 curves.

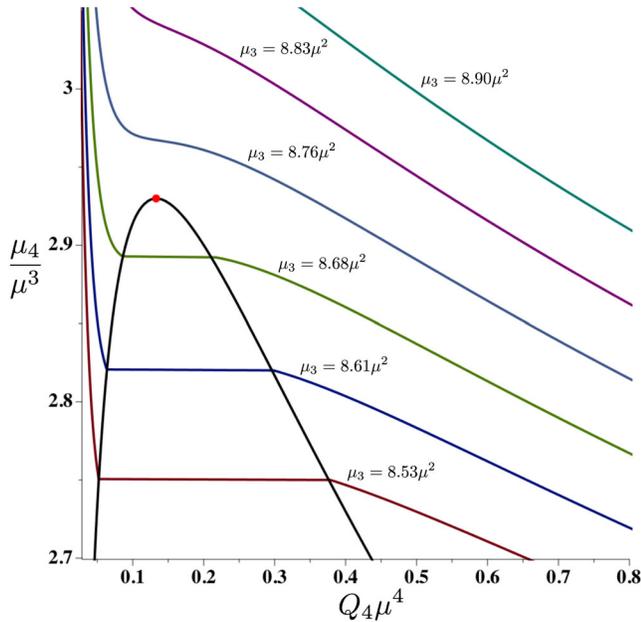


FIG. 5. μ_4 — Q_4 diagram with iso— μ_3 curves.

The phase diagram “ $\mu_3 - \mu_4$ ” is qualitatively similar to a “P-T” diagram for a liquid-gas transition. Therefore, a Van der Waals-like equation of state is expected to describe the different phases. Figures 4 and 5 are the analog of plotting “isotherms” in a “P-V” diagram for a liquid-gas system. A drastic change in the derivatives $\partial\mu/\partial Q$ is observed when crossing the critical line. Thus, we could identify the different regions as a “liquid” phase, in which the black holes are highly sensitive to any change in the charges, and a “gaseous” phase, characterized by considerably smaller values of $\partial\mu/\partial Q$.

It is important to notice that the phase transitions always occur between black hole solutions with $Q_4 \neq 0$, as seen explicitly in the “ $\mu_4 - Q_4$ ” diagram (Fig. 5). This fact guarantees that the states between which the system is transitioning have the same asymptotic (UV, far from the horizon) structure [27]; it is the spin of the highest spin charge that sets the value of the AdS radius and the central charge. As argued in [10], the corresponding asymptotic symmetry algebra for $N = 4$ is a \mathcal{W} -algebra associated to the (2,1,1) nonprincipal embedding of $sl(2)$ in $sl(4)$.

To conclude, in this paper we have studied the thermodynamic phase space properties of $sl(N; \mathbb{R})$ higher spin black holes. We have identified the even- N theories as those having well-defined (finite and stable) partition functions in the ensemble under consideration. These theories exhibit Hawking-Page transitions, just like any black hole coupled to a cosmological constant. Moreover, we find first order phase transitions between different higher spin black holes, as well as a second order transition and a critical point. According to the AdS/CFT correspondence, the phenomena observed in the bulk should have a counterpart at the boundary CFT. We leave the investigation of this issue for future work.

M. B. was partially funded by Fondecyt No. 1141221. M. B. would also like thank F. Quevedo for hospitality at ICTP, where part of this work was done. A. F. acknowledges support from Fondecyt No. 1160282. G. D. acknowledges support from Fondecyt No. 1150463. The work of I. R. was partially funded by CONICYT PCHA/Doctorado Nacional No. 2015149744.

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- $S(0) = 0$; the AdS_4 vacuum has no entropy. The critical temperature is $\beta_c = \pi\ell$.
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- [27] We thank one of the referees for pointing out this issue.