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Critical non-Abelian vortex in four dimensions and little string theory

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As was shown recently, non-Abelian vortex strings supported in four-dimensional $\mathcal{N} = 2$ supersymmetric QCD with the U(2) gauge group and $N_f = 4$ quark multiplets (flavors) become critical superstrings. In addition to the translational moduli, non-Abelian strings under consideration carry six orientational and size moduli. Together, they form a ten-dimensional target space required for a superstring to be critical. The target space of the string sigma model is a product of the flat four-dimensional space and a Calabi-Yau noncompact threefold, namely, the conifold. We study closed string states which emerge in four dimensions and identify them with hadrons of four-dimensional $\mathcal{N} = 2$ QCD. One massless state was found previously; it emerges as a massless hypermultiplet associated with the deformation of the complex structure of the conifold. In this paper, we find a number of massive states. To this end, we exploit the approach used in LST little string theory, namely, the equivalence between the critical string on the conifold and noncritical $c = 1$ string with the Liouville field and a compact scalar at the self-dual radius. The states we find carry "baryonic" charge (its definition differs from standard). We interpret them as "monopole necklaces" formed (at strong coupling) by the closed string with confined monopoles attached.

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I. INTRODUCTION

It was recently shown [1] that the non-Abelian solitonic vortex string in a certain four-dimensional (4D) Yang-Mills theory becomes critical at strong coupling. This particular 4D theory in which the non-Abelian vortex is critical is $\mathcal{N} = 2$ supersymmetric QCD with the U(2) gauge group, four quark flavors and the Fayet-Iliopoulos (FI) [2] parameter ξ.

Non-Abelian vortices were first discovered in $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $\mathrm{U}(N)$ and $N_f \geq N$ flavors of quark hypermultiplets [3–6]. The non-Abelian vortex string is $1/2$ Bogomolny-Prasad-Sommerfeld (BPS) saturated and therefore has $\mathcal{N} = (2,2)$ supersymmetry on its world sheet. In addition to four translational moduli characteristic of the Abrikosov-Nielsen-Olesen (ANO) strings [7], the non-Abelian string carries orientational moduli as well as the size moduli if $N_f > N$ [3–6] (see Refs, [8–11] for reviews). Their dynamics are described by the effective two-dimensional sigma model on the string world sheet with the target space

$$\mathcal{O}(-1)^{\oplus(N_f-N)}_{\mathbb{CP}^1}, \tag{1.1}$$

to which we will refer as the weighted CP model [$\mathrm{WCP}(N, N_f - N)$]. For $N_f = 2N$, the model becomes conformal. Moreover, for $N = 2$, the dimension of the orientational/size moduli space is six, and these moduli can be combined with four translational moduli to form a ten-dimensional space required for the superstring to become critical.[1] In this case, the target space of the world sheet 2D theory on the non-Abelian vortex string is $\mathbb{R}^4 \times Y_6$, where Y_6 is a noncompact six-dimensional Calabi-Yau manifold, the resolved conifold [12,13].

The main obstacle in describing the solitonic vortex string as a critical string is that the solitonic strings are typically thick. Their transverse size is given by $1/m$, where m is a typical mass scale of the four-dimensional fields forming the string. This leads to the presence of a series of higher-derivative corrections to the low-energy sigma model action. The higher-derivative corrections run in powers of ∂/m. They make the string world sheet "crumpled" [14], and the string does not produce linear Regge trajectories at small spins [1].

The higher-derivative corrections on the noncritical string world sheet are needed to be properly accounted for in order to improve the UV behavior of the string theory [15]. Without them, the low-energy world sheet sigma model does not lead to UV complete string theory. In particular, this means that, say, the ANO string in four dimensions [7] never becomes thin.

On the other hand, the non-Abelian vortex string on the conifold is critical and has a perfectly good UV behavior. This opens the possibility that it can become thin in a certain regime. This cannot happen in weakly coupled bulk

[1] This setup corresponds to the Virasoro central charge in the $\mathrm{WCP}(N, N_f - N)$ sector $c_{\mathrm{WCP}} = 9$.

theory because at weak coupling $m \sim g\sqrt{T}$ and is always small in the units of $\sqrt{T}$. Here, g is the gauge coupling constant of the four-dimensional $\mathcal{N} = 2$ QCD, and T is the string tension.

A conjecture was put forward in Ref. [1] that at strong coupling in the vicinity of a critical value of $g_c^2 \sim 1$ the non-Abelian string on the conifold becomes thin, and higher-derivative corrections in the action can be ignored. It is expected that the thin string produces linear Regge trajectories for all spins. The above conjecture implies[2] that $m(g^2) \to \infty$ at $g^2 \to g_c^2$. Moreover, it was argued in Refs. [16,17] that it is natural to expect that the critical point g_c where the vortex string becomes thin is the self-dual point $g_c^2 = 4\pi$; see Refs. [18,19].

A version of the string-gauge duality for 4D QCD was proposed [1]; at weak coupling, this theory is in the Higgs phase and can be described in terms of (s)quarks and Higgsed gauge bosons, while at strong coupling, hadrons of this theory can be understood as string states formed by the non-Abelian vortex string. This hypothesis was further explored by studying string theory for the critical non-Abelian vortex in Refs. [16,17].

The vortices in the $\mathrm{U}(N)$ theories under consideration are topologically stable and cannot be broken. Therefore, the finite length strings are closed. Thus, we focus on the closed strings. The goal is to identify closed string states with hadrons of the 4D $\mathcal{N} = 2$ QCD.

The first step of this program, namely, identifying massless string states was carried out in Refs. [16,17] using supergravity formalism. In particular, a single matter hypermultiplet associated with the deformation of the complex structure of the conifold was found as the only 4D *massless* mode of the string. Other states arising from the massless ten-dimensional graviton are not dynamical in four dimensions. In particular, the 4D graviton and unwanted vector multiplet associated with deformations of the Kähler form of the conifold are absent. This is due to noncompactness of the Calabi-Yau manifold we deal with and non-normalizability of the corresponding modes over six-dimensional space Y_6.

Moreover, it was also discussed [16,17] how the states seen in 4D $\mathcal{N} = 2$ QCD at weak coupling are related to what we obtain from the string theory at strong coupling. In particular, the hypermultiplet associated with the deformation of the complex structure of the conifold was interpreted as a monopole-monopole baryon [16,17].

In this paper, we make the next step and find a number of massive states of the closed non-Abelian vortex string, which we interpret as hadrons of 4D $\mathcal{N} = 2$ QCD. However, to this end, we cannot use our formulation of the critical string theory on the conifold. The point is that the coupling constant $1/\beta$ of the world sheet WCP(2,2) is not small. Moreover, β tends to zero once the 4D coupling g^2 approaches the self-dual value we are interested in. At $\beta \to 0$, the resolved conifold develops a conical singularity. The supergravity approximation does not work for massive states.[3]

To analyze the massive states, we apply a different approach, which was used for little string theories (LSTs); see Ref. [20] for a review. Namely, we use the equivalence between the critical string on the conifold and noncritical $c = 1$ string that contains the Liouville field and a compact scalar at the self-dual radius [21,22]. The latter theory (in the Wess-Zumino-Novikov-Witten (WZNW) formulation) can be analyzed by virtue of algebraic methods. The spectrum can be computed exactly [23–27].

The states that we find carry a "baryonic" charge—we interpret them as "monopole necklaces" formed (at strong coupling) by the closed string with confined monopoles attached.

It is worth mentioning that the solitonic vortex describes only nonperturbative states. Perturbative states, in particular massless states associated with the Higgs branch in the original 4D Yang-Mills theory are present for all values of the gauge coupling and are not captured by the vortex string dynamics.

The paper is organized as follows. In Sec. II, we review the world sheet sigma model emerging on the critical non-Abelian vortex string. In Sec. III, we briefly review conifold geometry and the massless state associated with deformations of the conifold complex structure. In Sec. IV, we describe the equivalent formulation in terms of the noncritical $c = 1$ string, and in Sec. V, we calculate its spectrum. Section VI presents an interpretation of states we found in terms of the baryonic monopole necklaces. We summarize our conclusions in Sec. VII.

II. NON-ABELIAN VORTEX STRING

A. Four-dimensional $\mathcal{N} = 2$ QCD

As was already mentioned, non-Abelian vortex-strings were first found in 4D $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $\mathrm{U}(N)$ and $N_f \geq N$ flavors of the quark hypermultiplets supplemented by the FI D term ξ [3–6]; see, for example, Ref. [10] for a detailed review of this theory. Here, we just mention that at weak coupling $g^2 \ll 1$ this theory is in the Higgs phase in which scalar components of quark multiplets (squarks) develop vacuum

[2] At $N_f = 2N$, the beta function of the 4D $\mathcal{N} = 2$ QCD is zero, so the gauge coupling g^2 does not run. Note, however, that conformal invariance in the 4D theory is broken by the FI parameter ξ, which does not run either.

[3] This is in contradistinction to the massless states. For the latter, we can perform computations at large β where the supergravity approximation is valid and then extrapolate to strong coupling. In the sigma-model language, this procedure corresponds to chiral primary operators. They are protected by $\mathcal{N} = (2,2)$ world sheet supersymmetry, and their masses are not lifted by quantum corrections.

expectation values (VEVs). These VEVs break the U(N) gauge group Higgsing *all* gauge bosons, while the global flavor SU(N_f) is broken down to the so-called color-flavor locked group. The resulting global symmetry is

$$\mathrm{SU}(N)_{C+F}\times \mathrm{SU}(N_f-N)\times \mathrm{U}(1)_B; \tag{2.1}$$

see Ref. [10] for more details. The unbroken global $\mathrm{U}(1)_B$ factor above is identified with a baryonic symmetry. Note that what is usually identified as the baryonic U(1) charge is a part of our 4D theory gauge group. "Our" U(1) is a combination of two U(1) symmetries; the first is a subgroup of the flavor SU(N_f), and the second is the global U(1) subgroup of U(N) gauge symmetry.

The 4D theory has a Higgs branch formed by massless quarks which are in the bifundamental representation of the global group (2.1) and carry baryonic charge; see Ref. [17] for more details. In the case $N=2$, $N_f=2N=4$, we will deal with here, the dimension of this branch is

$$\dim\mathcal{H}=4N(N_f-N)=16. \tag{2.2}$$

The above Higgs branch is noncompact and is hyper-Kählerian [19,28], and therefore its metric cannot be modified by quantum corrections [19]. In particular, once the Higgs branch is present at weak coupling, we can continue it all the way into strong coupling.

B. World sheet sigma model

The presence of color-flavor locked group $\mathrm{SU}(N)_{C+F}$ is the reason for the formation of the non-Abelian vortex strings [3–6]. The most important feature of these vortices is the presence of the so-called orientational zero modes. In the $\mathcal{N}=2$ 4D theory, these strings are 1/2 BPS saturated; hence, their tension is determined exactly by the FI parameter,

$$T=2\pi\xi. \tag{2.3}$$

Let us briefly review the model emerging on the world sheet of the non-Abelian critical string [1,16,17].

The translational moduli fields (they decouple from all other moduli) in the Polyakov formulation [29] are given by the action

$$S_0=\frac{T}{2}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu+\text{fermions}, \tag{2.4}$$

where σ^α ($\alpha=1,\,2$) are the world sheet coordinates, x^μ ($\mu=1,\ldots,4$) describing the $\mathbb{R}^4$ part of the string target space and $h=\det(h_{\alpha\beta})$, where $h_{\alpha\beta}$ is the world sheet metric, which is understood as an independent variable.

If $N_f=N$, the dynamics of the orientational zero modes of the non-Abelian vortex, which become orientational moduli fields on the world sheet, is described by the two-dimensional $\mathcal{N}=(2,2)$ supersymmetric $CP(N-1)$ model.

If one adds extra quark flavors, non-Abelian vortices become semilocal. They acquire size moduli [30]. In particular, for the non-Abelian semilocal vortex at hand, in addition to the orientational zero modes n^P ($P=1,2$), there are the so-called size moduli ρ^K ($K=1,\,2$) [3,6,30–33].

The gauged formulation of the effective world sheet theory for the orientational and size moduli is as follows [34]. One introduces the U(1) charges ±1, namely, $+1$ for n's and -1 for ρ's,

$$S_1=\int d^2\sigma\sqrt{h}\Big\{h^{\alpha\beta}\Big(\tilde{\nabla}_\alpha\bar{n}_P\nabla_\beta n^P+\nabla_\alpha\bar{\rho}_K\tilde{\nabla}_\beta\rho^K\Big)+\frac{e^2}{2}(|n^P|^2-|\rho^K|^2-\beta)^2\Big\}+\text{fermions}, \tag{2.5}$$

where

$$\nabla_\alpha=\partial_\alpha-iA_\alpha,\qquad \tilde{\nabla}_\alpha=\partial_\alpha+iA_\alpha, \tag{2.6}$$

and A_α is an auxiliary gauge field. The limit $e^2\to\infty$ is implied. Equation (2.5) represents the WCP(2,2) model.[4]

The total number of real bosonic degrees of freedom in (2.5) is six, where we take into account the D-term constraint and the fact that one U(1) phase can be gauged away. As was already mentioned, these six internal degrees of freedom are combined with the four translational moduli from (2.4) to form a ten-dimensional space needed for the superstring to be critical.

In the semiclassical approximation, the coupling constant β in (2.5) is related to the 4D SU(2) gauge coupling g^2 via [10]

$$\beta\approx\frac{4\pi}{g^2}. \tag{2.7}$$

Note that the first (and the only) coefficient is the same for the 4D QCD and the world sheet beta functions. Both vanish at $N_f=2N$. This ensures that our world sheet theory is conformal.

The total bosonic world sheet action is

$$S=S_0+S_1. \tag{2.8}$$

Since the non-Abelian vortex string is 1/2 BPS, it preserves $\mathcal{N}=(2,2)$ in the world sheet sigma model, which is

[4]Both the orientational and the size moduli have logarithmically divergent norms; see, e.g., Ref. [31]. After an appropriate infrared regularization, logarithmically divergent norms can be absorbed into the definition of relevant two-dimensional fields [31]. In fact, the world-sheet theory on the semilocal non-Abelian string is not exactly the WCP($N,\tilde{N}$) model [33]; there are minor differences. The actual theory is called the zn model. Nevertheless, it has the same infrared physics as the model (2.5) [35].

necessary to have $\mathcal{N}=2$ space-time supersymmetry [36,37]. Moreover, in Ref. [17], it is shown that the string theory of the non-Abelian critical vortex is type IIA.

The global symmetry of the world sheet sigma model (2.5) is

$$SU(2)\times SU(2)\times U(1), \tag{2.9}$$

i.e., exactly the same as the unbroken global group in the 4D theory (2.1) at $N=2$ and $N_f=4$. The fields n and ρ transform in the following representations:

$$n:\ (\mathbf{2},0,0), \qquad \rho:\ (0,\mathbf{2},1). \tag{2.10}$$

C. Thin string regime

As is well known [18,19], the 4D Yang-Mills theory at hand possesses a strong-weak coupling duality, namely,

$$\tau\to\tau_D=-\frac{1}{\tau}, \qquad \tau=i\frac{4\pi}{g^2}+\frac{\theta_{4D}}{2\pi}, \tag{2.11}$$

where θ_{4D} is the four-dimensional θ angle.

The 2D coupling constant β can be naturally complexified, too, if we include the θ term in the action of the model (2.5),

$$\beta\to\beta+i\frac{\theta_{2D}}{2\pi}.$$

The exact relation between 4D and 2D couplings is

$$\exp(-2\pi\beta)=-h(\tau)[h(\tau)+2], \tag{2.12}$$

where the function $h(\tau)$ is a special modular function of τ defined in terms of the θ functions,

$$h(\tau)=\theta_1^4/(\theta_2^4-\theta_1^4).$$

This function enters the Seiberg-Witten curve for our 4D theory [18,19]. Equation (2.12) generalizes the quasiclassical relation (2.7). Derivation of the relation (2.12) will be presented elsewhere [38,39].

Note that the 4D self-dual point $g^2=4\pi$ is mapped onto the 2D self-dual point $\beta=0$.

According to the hypothesis formulated in Ref. [1], our critical non-Abelian string becomes thin in the strong coupling limit in the self-dual point $\tau_c=i$ or $g_c^2=4\pi$. This gives

$$m^2\to\xi\times\begin{cases} g^2, & g^2\ll 1\\ \infty, & g^2\to 4\pi\\ 16\pi^2/g^2, & g^2\gg 1\end{cases}, \tag{2.13}$$

where the dependence of m at small and large g^2 follows from the quasiclassical analysis [10] and duality (2.11), respectively.

Thus, we expect that the singularity of mass m lies at $\beta=0$. This is the point where the non-Abelian string becomes infinitely thin, higher-derivative terms can be neglected, and the theory of the non-Abelian string reduces to (2.8). The point $\beta=0$ is a natural choice because at this point we have a regime change in the 2D sigma model *per se*. This is the point where the resolved conifold defined by the D term in (2.5) develops a conical singularity [13].

III. MASSLESS 4D BARYON AS DEFORMATION OF THE CONIFOLD COMPLEX STRUCTURE

In this section, we briefly review the only 4D massless state associated with the deformation of the conifold complex structure. It was found in Ref. [17]. As was already mentioned, all other modes arising from massless ten-dimensional graviton have non-normalizable wave functions over the conifold. In particular, the 4D graviton is absent [17]. This result matches our expectations since we started with $\mathcal{N}=2$ QCD in the flat four-dimensional space without gravity.

The target space of the sigma model (2.5) is defined by the D-term condition

$$|n^P|^2-|\rho^K|^2=\beta. \tag{3.1}$$

The U(1) phase is assumed to be gauged away. We can construct the U(1) gauge-invariant "mesonic" variables

$$w^{PK}=n^P\rho^K. \tag{3.2}$$

These variables are subject to the constraint $\det w^{PK}=0$, or

$$\sum_{\alpha=1}^{4}w_\alpha^2=0, \tag{3.3}$$

where

$$w^{PK}=\sigma_\alpha^{PK}w_\alpha,$$

and the σ matrices above are $(1,-i\tau^a)$, $a=1,\ 2,\ 3$. Equation (3.3) defines the conifold Y_6. It has the Kähler Ricci-flat metric and represents a noncompact Calabi-Yau manifold [12,13,34]. It is a cone that can be parametrized by the noncompact radial coordinate

$$\tilde{r}^2=\sum_{\alpha=1}^{4}|w_\alpha|^2 \tag{3.4}$$

and five angles; see Ref. [12]. Its section at fixed $\tilde{r}$ is $S_2\times S_3$.

At $\beta = 0$, the conifold develops a conical singularity, so both S_2 and S_3 can shrink to zero. The conifold singularity can be smoothed out in two distinct ways: by deforming the Kähler form or by deforming the complex structure. The first option is called the resolved conifold and amounts to introducing a nonzero β in Eq. (3.1). This resolution preserves the Kähler structure and Ricci flatness of the metric. If we put $\rho^K = 0$ in (2.5), we get the $CP(1)$ model with the S_2 target space (with the radius $\sqrt{\beta}$). The resolved conifold has no normalizable zero modes. In particular, the modulus β, which becomes a scalar field in four dimensions has non-normalizable wave function over the Y_6 [17].

As explained in Refs. [17,40], non-normalizable 4D modes can be interpreted as (frozen) coupling constants in the 4D theory. The β field is the most straightforward example of this, since the 2D coupling β is related to the 4D coupling; see Eq. (2.12).

If $\beta = 0$, another option exists, namely, a deformation of the complex structure [13]. It preserves the Kähler structure and Ricci flatness of the conifold and is usually referred to as the *deformed conifold*. It is defined by deformation of Eq. (3.3), namely,

$$\sum_{\alpha=1}^{4} w_\alpha^2 = b, \tag{3.5}$$

where b is a complex number. Now, the S_3 cannot shrink to zero, and its minimal size is determined by b.

The modulus b becomes a 4D complex scalar field. The effective action for this field was calculated in Ref. [17] using the explicit metric on the deformed conifold [12,41,42],

$$S(b) = T \int d^4x |\partial_\mu b|^2 \log \frac{T^2 L^4}{|b|}, \tag{3.6}$$

where L is the size of $\mathbb{R}^4$ introduced as an infrared regularization of the logarithmically divergent b field norm.[5]

We see that the norm of the b modulus turns out to be logarithmically divergent in the infrared. The modes with the logarithmically divergent norm are at the borderline between normalizable and non-normalizable modes. Usually, such states are considered as "localized" on the string. We follow this rule. We can relate this logarithmic behavior to the marginal stability of the b state; see Ref. [17]. This scalar mode is localized on the string in the same sense in which the orientational and size zero modes are localized on the vortex-string solution.

The field b being massless can develop a VEV. Thus, we have a new Higgs branch in 4D $\mathcal{N} = 2$ QCD, which is developed only for the self-dual value of the coupling constant $g^2 = 4\pi$.

The logarithmic metric in Eq. (3.6) in principle can receive both perturbative and nonperturbative quantum corrections in the sigma model coupling $1/\beta$. However, for $\mathcal{N} = 2$ theory, the nonrenormalization theorem of Ref. [19] forbids the dependence of the Higgs branch metric on the 4D coupling constant g^2. Since the 2D coupling β is related to g^2, we expect that the logarithmic metric in (3.6) will stay intact. We confirm this expectation in the next section.

In Ref. [17], the massless state b was interpreted as a baryon of 4D $\mathcal{N} = 2$ QCD. Let us explain this. From Eq. (3.5), we see that the complex parameter b (which is promoted to a 4D scalar field) is singlet with respect to both SU(2) factors in (2.9), i.e., the global world sheet group.[6] What about its baryonic charge?

Since

$$w_\alpha = \frac{1}{2} \mathrm{Tr}[(\bar{\sigma}_\alpha)_{KP} n^P \rho^K], \tag{3.7}$$

we see that the b state transforms as

$$(1, 1, 2), \tag{3.8}$$

where we used Eqs. (2.10) and (3.5). In particular, it has the baryon charge $Q_B(b) = 2$.

To conclude this section, let us note that in the type IIA superstring the complex scalar associated with deformations of the complex structure of the Calabi-Yau space enters as a 4D $\mathcal{N} = 2$ hypermultiplet. Other components of this hypermultiplet can be restored by $\mathcal{N} = 2$ supersymmetry. In particular, the 4D $\mathcal{N} = 2$ hypermultiplet should contain another complex scalar $\tilde{b}$ with baryon charge $Q_B(\tilde{b}) = -2$. In the stringy description, this scalar comes from the ten-dimensional 3-form; see Ref. [43] for a review.

IV. NONCRITICAL $c = 1$ STRING

As was explained in Sec. I, the critical string theory on the conifold is hard to use for calculating the spectrum of massive string modes because the supergravity approximation does not work. In this paper, we take a different route and use the equivalent formulation of our theory as a noncritical $c = 1$ string theory with the Liouville field and a compact scalar at the self-dual radius [21,22].

Noncritical $c = 1$ string theory is formulated on the target space

$$\mathbb{R}^4 \times \mathbb{R}_\phi \times S^1, \tag{4.1}$$

[5] The infrared regularization on the conifold $\tilde{r}_{\max}$ translates into the size L of the 4D space because variables ρ in Eq. (3.4) have an interpretation of the vortex string sizes, $\tilde{r}_{\max} \sim TL^2$.

[6] Which is isomorphic to the 4D global group (2.1) at $N = 2$, $N_f = 4$.

where $\mathbb{R}_\phi$ is a real line associated with the Liouville field ϕ and the theory has a linear in ϕ dilaton, such that string coupling is given by

$$g_s = e^{-\frac{Q}{2}\phi}. \tag{4.2}$$

Generically, the above equivalence is formulated between the critical string on noncompact Calabi-Yau spaces with isolated singularity on the one hand and the noncritical $c = 1$ string with the additional Ginzburg-Landau $\mathcal{N} = 2$ superconformal system [21] on the other hand. In the conifold case, this extra Ginzburg-Landau factor in (4.1) is absent [44].

In Refs. [21,44,45], it was argued that noncritical string theories with the string coupling exponentially falling off at $\phi \to \infty$ are holographic. The string coupling goes to zero in the bulk of the space-time, and nontrivial dynamics (LST)[7]; is localized on the "boundary." In our case, the boundary is the four-dimensional space in which $\mathcal{N} = 2$ QCD is defined.

The holography for our non-Abelian vortex string theory is most welcome and expected. We start with $\mathcal{N} = 2$ QCD in 4D space and study the solitonic vortex string. In our approach, ten-dimensional space formed by 4D "real" space and six internal moduli of the string is an artificial construction needed to formulate the string theory of a special non-Abelian vortex. Clearly, we expect that all nontrivial real physics should be localized exclusively on the 4D boundary. In other words, we expect that LST in our case is nothing other than 4D $\mathcal{N} = 2$ supersymmetric QCD at the self-dual value of the gauge coupling $g^2 = 4\pi$ (in the hadronic description).

The linear dilaton in (4.2) implies that the bosonic stress tensor of $c = 1$ matter coupled to 2D gravity is given by

$$T_{--} = -\frac{1}{2}[(\partial_z\phi)^2 + Q\partial_z^2\phi + (\partial_z Y)^2]. \tag{4.3}$$

The compact scalar Y represents $c = 1$ matter and satisfies the following condition:

$$Y \sim Y + 2\pi Q. \tag{4.4}$$

Here, we normalize the scalar fields such that their propagators are

$$\langle\phi(z), \phi(0)\rangle = -\log z\bar{z}, \qquad \langle Y(z), Y(0)\rangle = -\log z\bar{z}. \tag{4.5}$$

The central charge of the supersymmetrized $c = 1$ theory above is

$$c_{\phi+Y}^{\mathrm{SUSY}} = 3 + 3Q^2. \tag{4.6}$$

The criticality condition for the string on (4.1) implies that this central charge should be equal to 9. This gives

$$Q = \sqrt{2}. \tag{4.7}$$

Deformation of the conifold (3.5) translates into adding the Liouville interaction to the world sheet sigma model [21]

$$\delta L = b\int d^2\theta e^{-\frac{\phi+iY}{Q}}. \tag{4.8}$$

The conifold singularity at $b = 0$ corresponds to the string coupling constant becoming infinitely large at $\phi \to -\infty$; see Eq. (4.2). At $b \neq 0$, the Liouville interaction regularizes the behavior of the string coupling, preventing the string from propagating to the region of large negative ϕ.

In fact, the $c = 1$ noncritical string theory can also be described in terms of a two-dimensional black hole [46], which is the $\mathrm{SL}(2,R)/\mathrm{U}(1)$ coset WZNW theory [21,22,26,47] at the level

$$k = \frac{2}{Q^2}. \tag{4.9}$$

This relation implies in the case of the conifold ($Q = \sqrt{2}$) that

$$k = 1, \tag{4.10}$$

where k is the total level of the Kac-Moody algebra in the supersymmetric version (the level of the bosonic part of the algebra is then $k_b = k + 2 = 3$). The target space of this theory has the form of a semi-infinite cigar; the field ϕ associated with the motion along the cigar cannot take large negative values due to semi-infinite geometry. In this description, the string coupling constant at the tip of the cigar is $g_s \sim 1/b$.

V. VERTEX OPERATORS AND THE SPECTRUM

In this section, we consider vertex operators for the noncritical string theory on (4.1) and calculate the string spectrum.

A. Vertex operators

Vertex operators for the string theory on (4.1) are constructed in Ref. [21]; see also Refs. [26,44]. Primaries of the $c = 1$ part for large positive ϕ (where the target space becomes a cylinder $\mathbb{R}_\phi \times S^1$) take the form

$$V_{j,m} \approx \exp(\sqrt{2}j\phi + i\sqrt{2}mY). \tag{5.1}$$

[7]The main example of this behavior is nongravitational LST in the flat six-dimensional space formed by the world volume of parallel NS5 branes.

For the self-dual radius (4.7) (or $k = 1$), the parameter $2m$ in Eq. (5.1) is an integer. In the left-moving sector, $2m$ is the total momentum plus the winding number along the compact dimension Y. For the right-moving sector, we introduce $2\bar{m}$, which is the winding number minus momentum; then, we consider operators with $\bar{m} = \pm m$.

The primary operator (5.1) is related to the wave function over "extra dimensions" as follows:

$$V_{j,m} = g_s \Psi_{j,m}(\phi, Y).$$

The string coupling (4.2) depends on ϕ. Thus,

$$\Psi_{j,m}(\phi, Y) \sim e^{\sqrt{2}(j+\frac{1}{2})\phi + i\sqrt{2}mY}. \qquad (5.2)$$

We look for string states with normalizable wave functions over the extra dimensions which we will interpret as hadrons of 4D $\mathcal{N} = 2$ QCD. The condition for the string states to have normalizable wave functions reduces to [21]

$$j \le -\frac{1}{2}. \qquad (5.3)$$

The scaling dimension of the primary operator (5.1) is

$$\Delta_{j,m} = m^2 - j(j+1). \qquad (5.4)$$

Unitarity implies that it should be positive,

$$\Delta_{j,m} > 0. \qquad (5.5)$$

The spectrum of the allowed values of j and m in (5.1) was determined exactly by using the Kac-Moody algebra for the coset $\mathrm{SL}(2,R)/\mathrm{U}(1)$ in Refs. [23–27]; see Ref. [48] for a review. Both discrete and continuous representations were found. Parameters j and m determine the global quadratic Casimir operator and the projection of the spin on the 3-axis,

$$J^2|j,m\rangle = -j(j+1)|j,m\rangle, \quad J^3|j,m\rangle = m|j,m\rangle, \qquad (5.6)$$

where J^a $(a = 1,\ 2,\ 3)$ are global $\mathrm{SL}(2,R)$ currents. We have

(i) *Discrete representations* with

$$j = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots, \quad m = \pm\{j, j-1, j-2, \ldots\}. \qquad (5.7)$$

(ii) *Principal* continuous representations with

$$j = -\frac{1}{2} + is, \quad m = \text{integer} \ \text{or} \ m = \text{half-integer}, \qquad (5.8)$$

where s is a real parameter.

(iii) *Exceptional* continuous representations with

$$-\frac{1}{2} \le j < 0, \qquad m = \text{integer}. \qquad (5.9)$$

We see that discrete representations include the normalizable states localized near the tip of the cigar, while the continuous representations contain non-normalizable states[8] see Eq. (5.3). This nicely matches our qualitative expectations.

Discrete representations contain states with a negative norm. To exclude the ghost states, a restriction for spin j is imposed [23–25,27,48]:

$$-\frac{k+2}{2} < j < 0. \qquad (5.10)$$

Thus, for our value $k = 1$, we are left with only two allowed values of j,

$$j = -\frac{1}{2}, \qquad m = \pm\left\{\frac{1}{2}, \frac{3}{2}, \ldots\right\} \qquad (5.11)$$

and

$$j = -1, \qquad m = \pm\{1, 2, \ldots\}. \qquad (5.12)$$

Below, in this section, we will first consider normalizable string states from discrete representations, and finally, in Sec. V E, we discuss the physical interpretation of the continuous representations.

B. Massless baryon

Our first task now is to rederive the massless baryon b associated with deformations of the conifold complex structure (see Ref. [17] and Sec. III) within the framework of the noncritical Liouville string theory described above. To this end, we consider vertex operators for 4D scalars. The 4D scalar vertices V^S in the $(-1,-1)$ picture have the form [21]

$$V^S_{j,m}(p_\mu) = e^{-\varphi} e^{ip_\mu x^\mu} V_{j,m}, \qquad (5.13)$$

where φ represents bosonized ghosts and p_μ is the 4D momentum of the string state. These states are the lowest components of $\mathcal{N} = 2$ multiplets in four dimensions. Also, the Gliozzi-Scherk-Olive (GSO) projection restricts the integer $2m$ for the operator in Eq. (5.13) to be odd [21,49],

$$m = l + \frac{1}{2}, \qquad |l| = 0, 1, 2, \ldots \qquad (5.14)$$

[8]We will discuss the case $j = -\frac{1}{2}$, which is on the borderline between normalizable and non-normalizable states, in the next subsection.

The condition for the state (5.13) to be physical is

$$\frac{p_\mu p^\mu}{8\pi T} + \frac{(2l+1)^2}{4} - j(j+1) = \frac{1}{2}, \qquad (5.15)$$

where we used Eqs. (5.4) and (5.14). This determines the masses of the 4D scalars,

$$\frac{(M^S)^2_{j,l}}{8\pi T} = -\frac{p_\mu p^\mu}{8\pi T} = \frac{(2l+1)^2}{4} - \frac{1}{2} - j(j+1), \qquad (5.16)$$

where the Minkowski 4D metric with the diagonal entries $(-1, 1, 1, 1)$ is used.

Consider the states that are on the borderline between normalizable and non-normalizable, namely, the states with

$$j = -\frac{1}{2}. \qquad (5.17)$$

For $l = 0$ ($m = \frac{1}{2}$) and $l = -1$ ($m = -\frac{1}{2}$), Eq. (5.16) gives the lightest states with

$$M^S_{j=-\frac{1}{2},l=0} = M^S_{j=-\frac{1}{2},l=-1} = 0. \qquad (5.18)$$

This is our massless baryon b associated with deformations of the complex structure of the conifold (plus antibaryon $\tilde{b}$). To confirm this, let us show that it has a logarithmically normalizable wave function in terms of the conifold radial coordinate $\tilde{r}$; see Eq. (3.6).

For $j = -\frac{1}{2}$, all states (5.13) have constant wave functions with respect to the Liouville coordinate ϕ. Thus, the norm of these states is $(\phi_{\max} - \phi_{\min})$. To relate ϕ to $\tilde{r}$, we note that $\phi_{\min}$ is determined by the Liouville interaction term (4.8), which becomes of the order of unity at this point. This gives

$$\phi \sim \log \tilde{r}^2, \qquad (5.19)$$

where we used that $\tilde{r}^2_{\min} = |b|$. In particular, this gives $\log \tilde{r}^2_{\max}/|b|$ for the norm of the massless state with $l = 0$, as expected [see Eq. (3.6)].

C. Massive 4D scalars

Now, consider the states (5.13) with arbitrary values of l in (5.14), still assuming $j = -\frac{1}{2}$. From Eq. (5.16), we obtain their masses

$$(M^S)^2_{j=-\frac{1}{2},l} = 8\pi T l(l+1), \qquad |l| = 0, 1, 2, \ldots \qquad (5.20)$$

All these states are logarithmically normalizable with respect to the conifold radial coordinate.

What are their quantum numbers with respect to the 4D global group (2.1)? They are all invariant with respect to SU(2) factors. To determine their baryon charge, note that the $U(1)_B$ transformation of b in the Liouville interaction (4.8) is compensated by a shift of Y. Therefore, m in Eq. (5.1) is proportional to the baryon charge. Normalization is fixed by the massless baryon b, which has $Q_B(b) = 2$ at $m = \frac{1}{2}$, $l = 0$. This implies

$$Q_B(V_{j,m}) = 4m. \qquad (5.21)$$

We see that the momentum m in the compact Y direction is in fact the baryon charge of a string state. In particular, 4D scalar states (5.20) are all baryons for positive l and antibaryons for negative l with

$$Q_B = 4l + 2.$$

The masses of 4D scalars as a function of the baryonic charge are shown in Fig. 1.

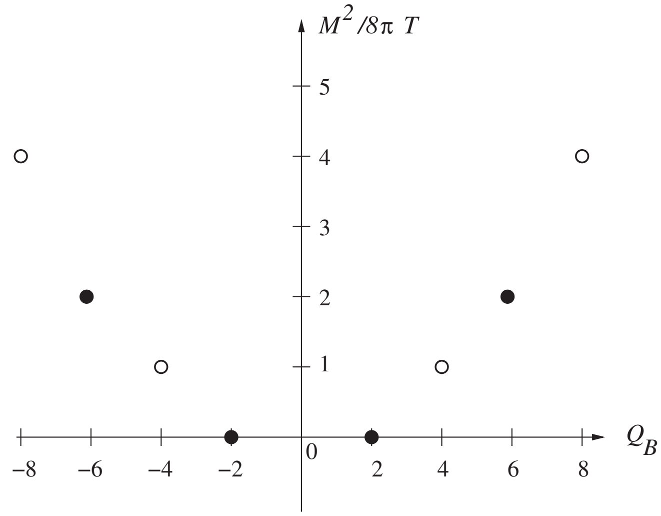

FIG. 1. Spectrum of spin-0 and spin-2 states as a function of the baryonic charge. Closed and open circles denote spin-0 and spin-2 states, respectively.

To conclude this subsection, let us note that the second allowed value of j, $j = -1$ in Eq. (5.12) is excluded by the GSO projection, which selects only half-integer values of m for states (5.13); see Eq. (5.14).

Note also that the 4D scalar states found above are the lowest components of $\mathcal{N} = 2$ multiplets. Other components can be restored by virtue of 4D $\mathcal{N} = 2$ supersymmetry.

D. Spin-2 states

At the next level, we consider 4D spin-2 states coming from space-time "gravitons." The corresponding vertex operators are given by

$$V^G_{j,m}(p_\mu) = \xi_{\mu\nu}\psi^\mu_L \psi^\nu_R e^{-\varphi} e^{ip_\mu x^\mu} V_{j,m}, \qquad (5.22)$$

where $\psi^{\mu}_{L,R}$ are the world sheet superpartners to 4D coordinates x^{μ}; moreover, $\xi_{\mu\nu}$ is the polarization tensor.

The condition for these states to be physical takes the form

$$\frac{p_{\mu}p^{\mu}}{8\pi T} + m^2 - j(j+1) = 0. \quad (5.23)$$

The GSO projection selects now $2m$ to be even, $m = l$, $|l| = 0, 1, 2, \ldots$ [21], and thus we are left with only one allowed value of j, $j = -1$ in Eq. (5.12). Moreover, the value $m = l = 0$ is excluded. This leads to the following expression for the masses of spin-2 states:

$$(M^G)^2_{j,l} = 8\pi T l^2, \qquad |l| = 1, 2, \ldots. \quad (5.24)$$

We see that all spin-2 states are massive. This confirms the result in Ref. [17] that no massless 4D graviton appears in our theory. It also matches the fact that our boundary theory, 4D $\mathcal{N} = 2$ QCD, is defined in flat space without gravity.

All states with masses (5.24) are baryons for $l > 0$ and antibaryons for $l < 0$, with the baryon charge $Q_B = 4m = 4l$. The masses of 4D spin-2 states as a function of the baryonic charge are shown in Fig. 1.

We expect that all 4D states with a given baryon charge considered in this and previous subsections are the lowest states of the Regge trajectories linear in 4D spin.

E. Non-normalizable states

In Ref. [17], the continuum spectrum of non-normalizable states was interpreted as unstable string states. Since the Liouville coordinate ϕ is related to the radial coordinate $\tilde{r}$ on the conifold [see (5.19)], the modes with $j > -\frac{1}{2}$ are power non-normalizable on the conifold. The conifold radial coordinate $\tilde{r}$ has the physical interpretation of a distance from the string axis in 4D space; see Eqs. (3.4) and (3.2). Therefore, the wave functions of the non-normalizable states are saturated at large distances from the vortex-string axis in 4D.

These states are *not* localized on the string. The infinite norm of these states should be interpreted as an instability. Namely, these states decay into massless bifundamental quarks inherent to the Higgs branch of our four-dimensional $\mathcal{N} = 2$ QCD. This instability is present already at the perturbative level; see Sec. II and Ref. [17].

Our vortex string has a conceptual difference compared to the fundamental string. In compactifications of the fundamental string, all states present in four dimensions are string states. The string theory for the vortex strings of Refs. [1,17] is different. The string states describe only nonperturbative physics at strong coupling, such as mesons and baryons. The perturbative massless moduli states seen at week coupling are not described by this theory. In particular, the Higgs branch (and associated massless bifundamental quarks) found at weak coupling can be continued to the strong coupling; they persist there. It can intersect other branches but cannot disappear (for quarks with the vanishing mass terms) [19].

One class of non-normalizable unstable modes is given by the exceptional continuous representation (5.9). For $|m| = 1, 2, \ldots,$ the continuous spectra parametrized by j in Eq. (5.9) start from the thresholds given by masses (5.24).

Another class of unstable string states corresponds to the principal representation given by complex values of j,

$$j = -\frac{1}{2} + is; \quad (5.25)$$

see Eq. (5.8). These states have continuous spectra parametrized by s. The parameter s has a clear-cut interpretation of a momentum along the Liouville direction. Therefore, we interpret these states as decaying modes of the string states interacting with the perturbative bifundamental quarks rather than the hadronic states of $\mathcal{N} = 2$ QCD.

Much in the same way as for the exceptional representation, the principal continuous spectra for half-integer m start from thresholds given by masses (5.20). Using this picture, we are led to conclude that the continuous spectra contain multiparticle states formed by a given baryon and a number of emitted bifundamental quarks with zero total baryonic charge. This issue needs future clarification.

VI. PHYSICAL INTERPRETATION OF STRING STATES

In this section, we reveal a physical interpretation of all baryonic states found in the previous section as monopole necklaces.

Consider first the weak coupling domain $g^2 \ll 1$ in four-dimensional $\mathcal{N} = 2$ QCD. It is in the Higgs phase; N squarks condense. Therefore, non-Abelian vortex strings confine monopoles. However, the monopoles cannot be attached to the string endpoints. In fact, in the $U(N)$ theories, confined monopoles are junctions of two distinct elementary non-Abelian strings [5,6,50] (see Ref. [10] for a review). As a result, in four-dimensional $\mathcal{N} = 2$ QCD, we have monopole-antimonopole mesons in which the monopole and antimonopole are connected by two confining strings. In addition, in the $U(N)$ gauge theory, we can have baryons appearing as a closed necklace configuration of $N \times (\text{integer})$ monopoles [10]. For the $U(2)$ gauge group, the lightest baryon presented by such a necklace configuration consists of two monopoles; see Fig. 2.

Moreover, the monopoles acquire quantum numbers with respect to the global symmetry group (2.1). To see that this is the case, note that in the world sheet theory on the vortex string the confined monopole is seen as a kink

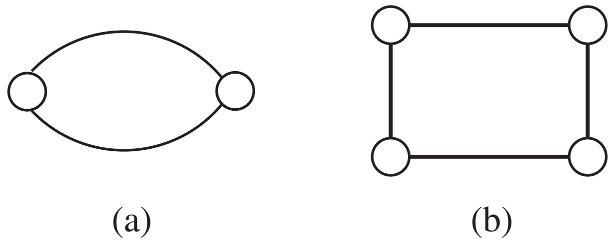

FIG. 2. Examples of the monopole necklace baryons: a) Massless b baryon with $Q_B = 2$; b) Spin-2 baryon with $Q_B = 4$. Open circles denote monopoles.

interpolating between two distinct vacua (i.e., distinct elementary non-Abelian strings) in the corresponding 2D sigma model [5,6,50]. At the same time, we know that the sigma model kinks at strong coupling are described by the n^P and ρ^K fields [51,52] [for the sigma model described by Eq. (2.5), it was shown in Ref. [53]) and therefore transform in the fundamental representations[9] of two non-Abelian factors in Eq. (2.1).

As a result, the monopole baryons in our case can be singlets, triplets, or bifundamentals or form higher representations of both SU(2) global groups in Eq. (2.1). With respect to the baryonic $\mathrm{U}(1)_B$ symmetry in Eq. (2.1), the monopole baryons can have charges

$$|Q_B(\text{baryon})| = 0, 1, 2...; \tag{6.1}$$

see Eq. (2.10). In particular, nonzero baryonic charge is associated with the ρ kinks. In the U(2) gauge theory, the monopole necklace can be formed by an even number of monopoles.

All these nonperturbative stringy states are heavy at weak coupling, with mass of the order of $\sqrt{\xi}$, and therefore can decay into screened quarks that are lighter and, eventually, into massless bifundamental screened quarks.

Now, we pass to the self-dual point $\beta = 0$ in the strong coupling region. As was already discussed, all string states found in Sec. V have nonzero baryon charge $Q_B = 4m$; see Eq. (5.21). The lightest state (the massless b state) has $Q_B = 2$. It can be formed by minimum two monopoles; see Fig. 2. The spin-2 massive state (5.22) with $m = 1$ can be formed by the monopole necklace with a minimum of four monopoles.

All stringy monopole necklace baryons found in Sec. V are singlets with respect to two SU(2) factors in Eq. (2.1). They are metastable and can decay into pairs of massless bifundamental quarks in the singlet channel with the same baryon charge. The metastability of stringy baryons on the string side is reflected in the logarithmic divergence of their norm and the presence of continuous spectra. Detailed studies of the nonperturbative Higgs branch formed by VEVs of massless b and interactions of stringy baryons with massless bifundamental quarks are left for future work.

[9]Strictly speaking, to make both bulk monopoles and world sheet kinks well defined as localized objects, one should introduce an infrared regularization, say, a small quark mass term. When we take the limit of the zero quark masses, the kinks become massless and smeared all over the closed string. However, their global quantum numbers stay intact.

VII. CONCLUSIONS

Previously, we observed that non-Abelian vortex strings supported in four-dimensional $\mathcal{N} = 2$ supersymmetric QCD with the U(2) gauge group and $N_f = 4$ flavors of quark hypermultiplets can represent critical superstrings in ten-dimensional target space $\mathbb{R}^4 \times Y_6$ where Y_6 is a noncompact Calabi-Yau manifold. This can be called "reverse holography." Indeed, we start from a well-defined four-dimensional Yang-Mills theory and, analyzing the vortex strings it supports, add six extra dimensions—the moduli of the string world sheet theory—which relates our construction (at the critical value of the coupling constant $g^2 = 4\pi$ corresponding to $\beta = 0$) to a critical string theory on a six-dimensional conifold.

In this paper, the mass spectrum of the string states is determined using equivalent formulation in terms of $c = 1$ noncritical string theory with the Liouville field. The string states *per se* are identified with the hadronic states in the four-dimensional theory. Since the extradimensional space is not compact, the above identification becomes possible because our string theory is holographic; nontrivial physics is "projected" to the 4D boundary. This behavior is typical for LSTs. The reason for holography is that the string coupling constant exponentially falls off at large values of the noncompact Liouville coordinate [see (4.2)], and the bulk physics becomes trivial and decouples.

Of course, the holography of our string theory is expected since our starting point was 4D $\mathcal{N} = 2$ QCD. Holography ensures the presence of normalizable string states localized in 4D space, which we identified as hadrons of $\mathcal{N} = 2$ QCD. Also, we qualitatively interpret nonnormalizable states as decay modes of the hadronic states.

The Higgs branch existing in our four-dimensional $\mathcal{N} = 2$ QCD leads to strictly massless quark hypermultiplets, which are evident at weak coupling and survive the transition to strong coupling. This implies a peculiar behavior in the infrared. In particular, the masses of the string states derived in Secs. V C and V D are in fact the end points of the branch cuts. Note, however, that exactly the same situation would take place in QCD with massless quarks giving rise to massless pions. Say, every lowest-lying state with the given baryon number would represent the beginning of a branch cut. The pion mass could be lifted by an arbitrarily small perturbation. Disentangling infrared effects of the Higgs branch from physics of the critical string under consideration will be the subject of a subsequent work.

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