Assessing degrees of entanglement of phonon states in atomic Bose gases through the measurement of commuting observables

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We show that measuring commuting observables can be sufficient to assess that a bipartite state is entangled according to either nonseparability or the stronger criterion of "steerability." Indeed, the measurement of a single observable might reveal the strength of the interferences between the two subsystems, as if an interferometer were used. For definiteness, we focus on the two-point correlation function of density fluctuations obtained by *in situ* measurements in homogeneous one-dimensional cold atomic Bose gases. We then compare this situation to that found in transonic stationary flows mimicking a black hole geometry where correlated phonon pairs are emitted on either side of the sonic horizon by the analogue Hawking effect. We briefly apply our considerations to two recent experiments.

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I. INTRODUCTION

Quantum field theory allows the creation of pairs of correlated (quasi)particles via strong variations of the classical background [1]. When focusing on the correlations between the two particles, two cases are particularly clear. Firstly, when the background is homogeneous and time dependent, one obtains pairs of quanta with opposite wave numbers, as is the case in an expanding homogeneous universe; see Refs. [2–5] for works discussing these correlations. Secondly, when the background is stationary but inhomogeneous the pairs of created quanta carry opposite energies, as is the case for electro-production in a constant electric field and for the (Hawking) radiation emitted by a black hole [6].

In both cases, pair production can be stimulated by quasiparticles already present, or emerge via excitation of vacuum fluctuations. The latter contribution is of particular interest, as it gives rise to entangled states. Entanglement is a well-defined notion for pure states; for mixed states, on the other hand, some care is needed to properly define which subset is to be considered "entangled." Indeed, historically, several inequivalent criteria have been discussed and compared, see e.g. [7]. In this paper we only consider two of them: nonseparability [8,9], which is particularly simple, and the older and stronger criterion based on the possibility of steering the outcome of a measurement on a subsystem having already measured the state of its partner [10]. These notions are recalled in Appendix A for bosonic degrees of freedom, which is the case we shall consider in this paper. For each of these criteria, we also present some inequalities relating observable quantities that are sufficient for the criterion to be satisfied. This step is crucial as it translates the criterion, which is defined in rather abstract terms, at the level of observables. For instance, in homogeneous systems, the bipartite state of phonons of wave vectors $k, -k$ is necessarily nonseparable whenever the following inequality is satisfied [11,12]:

$$n_k n_{-k} - |c_k|^2 < 0, \tag{1}$$

where $n_{\pm k} = \langle \hat{b}^\dagger_{\pm k} \hat{b}_{\pm k} \rangle$ give the mean occupation number of particles with wave number $\pm k$, and the norm of $c_k = \langle \hat{b}_k \hat{b}_{-k} \rangle$ accounts for the strength of the correlation between the k and $-k$ quanta.

Having identified the relevant inequalities, one should then address the question of their observability, that is, identify possible sets of measurements which are sufficient to assess that the inequality is violated, and therefore that the state under consideration is necessarily entangled. At first sight, it seems natural to consider measurements of noncommuting observables. In fact, to be able to verify that some Bell inequality is violated, it is necessary to consider some set of noncommuting observables, see [13,14] for bosonic degrees of freedom. This is the line of thought that was adopted in [15] and further advocated in [16]. However, when considering pair creation of quasiparticles of opposite wave numbers $k, -k$ in homogeneous systems, it was noticed in [17,18] and further clarified in [4] that *in situ* measurements of the kth Fourier component of the connected part of the density-density correlation function at some time t, $G^{(2)}(k; t)$, can be sufficient to assess the nonseparability of the state.[1]

In the present paper we pursue the analysis undertaken in [4]. We clarify and extend it in several directions, first by

[1]It should be pointed out that the commuting measurements we propose could certainly be reproduced within a classical framework. But this does not concern us, for our aim is not to show the impossibility of a classical description of the results. Instead, it is to see what can be inferred from them when adopting a quantum mechanical framework from the outset.

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considering the stronger criterion of "steerability", then by distinguishing the isotropic and anisotropic homogeneous cases. In this paper we also consider globally *inhomogeneous* background flows which contain two homogeneous domains where the two-point function $G^{(2)}$ can be analyzed in k space. Our motivation there is to reconsider the entanglement of phonon pairs produced by the analogue Hawking effect in a stationary transonic flow, following the observability [19,20] and the experimental implementation [21] of the criterion studied in [22].

The paper is organized as follows. In Sec. II, we study density fluctuations in globally homogeneous backgrounds. We demonstrate that *in situ* measurements of density fluctuations performed at a given time can contain enough information to assess that the bipartite phonon state of wave numbers k, $-k$ is nonseparable, or even obeys the stronger criterion of steerability. In Sec. III, we study the statistical properties of the density fluctuations in the asymptotic homogeneous domains of a stationary transonic flow mimicking a black hole geometry. We conclude in Section IV. In Appendix A, we recall the two notions of entanglement described above, while in Appendix B we consider the extra information about the phonon state one could extract by measuring phase fluctuations (which do not commute with measurements of the density).

II. *IN SITU* MEASUREMENTS OF ATOMIC DENSITY FLUCTUATIONS

For simplicity and definiteness, we consider elongated (i.e., effectively one-dimensional) atomic condensates, with transverse dimensions much smaller than their length. Such condensates can be realized using a cylindrically symmetric harmonic potential where the transverse trapping frequency $\omega_\perp$ is much larger than the longitudinal one, see [4,23–25]. The transverse excitations then have high frequencies of order $\omega_\perp$, and therefore, low-frequency excitations are fully characterized by their longitudinal momentum k and their energy ω_k (we work in units where $\hbar \equiv 1$). These are well-defined when the effective one-dimensional background condensate is homogeneous and stationary in a sufficiently large domain, i.e., with extension in space (respectively in time) larger than $1/|k|$ (respectively $1/\omega_k$).

In this paper, when measuring density fluctuations, we shall assume that the above conditions are satisfied, though in general the background can be inhomogeneous and/or nonstationary on larger scales. In Sec. II A we introduce the relevant quantities applicable to stationary backgrounds, while in Secs. II B and II C we restrict our attention to systems that are also globally homogeneous. The extension to inhomogeneous stationary systems is delayed until Sec. III.

A. Generalities

We work in the standard second-quantized formalism and adopt the Bogoliubov approximation [26,27], where the field operator for the dilute Bose gas is written $\hat{\Phi}(t,x) = e^{-i\mu t + iKx}(\Phi_0 + \delta\hat{\phi}(t,x))$, where Φ_0 is a c number that describes the condensed fraction of the gas, μ is the chemical potential, K is the condensate momentum, and $\delta\hat{\phi}$ describes perturbations on top of the condensate. Since we assume (local) homogeneity and stationarity of the background, $|\Phi_0|^2 \equiv \rho_0$ is constant, and is equal to the one-dimensional number density of condensed atoms. We can also refer to these as atoms with zero momentum (relative to the condensate). Since we have explicitly factored out the spatial component of the condensate wave function, Φ_0 is independent of x, and can be taken to be real and positive so that $\Phi_0 = \sqrt{\rho_0}$. As an operator, the total atom number density is

$$\begin{aligned}\hat{\rho}(t,x) &= \hat{\Phi}^\dagger(t,x)\hat{\Phi}(t,x),\\ &\approx \rho_0 + \sqrt{\rho_0}(\delta\hat{\phi}(t,x) + \delta\hat{\phi}^\dagger(t,x)),\end{aligned} \tag{2}$$

where in the second line we have neglected the nonlinear contribution of the perturbations. Linear density fluctuations are thus described by the operator $\delta\hat{\rho} = \sqrt{\rho_0}(\delta\hat{\phi} + \delta\hat{\phi}^\dagger)$. In the body of this paper, we shall only use *in situ* measurements of $\delta\hat{\rho}(t,x)$ performed at some time t. Using the equal-time commutators $[\hat{\Phi}(t,x), \hat{\Phi}(t,x')] = 0$ and $[\hat{\Phi}(t,x), \hat{\Phi}^\dagger(t,x')] = \delta(x - x')$, one easily verifies that $\delta\hat{\rho}(t,x)$ and $\delta\hat{\rho}(t',x')$ commute with each other when $t = t'$. Hence only commuting measurements are considered in what follows.

At quadratic order, the statistical properties of $\delta\hat{\rho}(t,x)$ are encoded in the (equal-time) two-point function

$$\begin{aligned}G^{(2)}(t,x;t,x') &= \langle \delta\hat{\rho}(t,x)\delta\hat{\rho}(t,x')\rangle,\\ &= \langle \hat{\rho}(t,x)\hat{\rho}(t,x')\rangle - \langle \hat{\rho}(t,x)\rangle\langle \hat{\rho}(t,x')\rangle,\end{aligned} \tag{3}$$

where the expression on the second line makes clear that $G^{(2)}(t,x;t,x')$ is the connected part of the density-density correlation function. Hence the contributions of coherent states of phonons are removed by the subtraction. This two-point function has been experimentally studied by repeated measurements of $\rho(x)$ in Refs. [21,28,29], which motivated the present study.[2]

The density fluctuations include all atoms carrying a nonzero momentum (relative to the condensate), and thus all nonconstant Fourier components of the density profile. Indeed, it is useful to invoke homogeneity of the background to write explicitly the Fourier components of the density fluctuations and the density-density two-point function: for a region of length L and a wave vector $k \in 2\pi\mathbb{Z}/L$, we have

[2] To see what additional information can be extracted from phase measurements, in Appendix B we study other two-point functions involving the phase fluctuation $\delta\hat{\theta}$, which does not commute with $\delta\hat{\rho}$, even at equal time.

$$G^{(2)}(t,k;t,k') \equiv \int_0^L \mathrm{d}x\, e^{-ikx} \int_0^L \mathrm{d}x' e^{ik'x'} G^{(2)}(t,x;t,x') = \langle \hat{\rho}_k(t) \hat{\rho}_{k'}^\dagger(t) \rangle, \tag{4}$$

where

$$\hat{\rho}_k(t) \equiv \int_0^L e^{-ikx} \hat{\rho}(t,x) \mathrm{d}x. \tag{5}$$

Because $\hat{\rho}(t,x)$ is a Hermitian operator, it follows immediately from (5) that $\hat{\rho}_k^\dagger(t) = \hat{\rho}_{-k}(t)$, and therefore (since at equal time different Fourier components $\hat{\rho}_k$ always commute) that $\hat{\rho}_k(t)$ and $\hat{\rho}_k^\dagger(t)$ commute with each other; the ordering of the operators on the right-hand side of Eq. (4) is thus irrelevant. Returning to Eq. (2) for the expression for the density fluctuations, we get

$$\hat{\rho}_k(t) = \delta\hat{\rho}_k(t) = \sqrt{N}(\hat{\phi}_k(t) + \hat{\phi}_{-k}^\dagger(t)), \tag{6}$$

where $N = \rho_0 L$ is the number of condensed atoms in the region of length L, and where the normalization factor $\sqrt{N}$ has been chosen so that the operators $\hat{\phi}_k$ satisfy the standard equal-time commutation relation

$$[\hat{\phi}_k(t), \hat{\phi}_{k'}^\dagger(t)] = \delta_{k,k'}. \tag{7}$$

Note the two contributions to $\hat{\rho}_k$ in Eq. (6), the first of which destroys an atom of momentum k, and the second of which creates an atom of momentum $-k$. The appearance of these two operators is required for the identity $\hat{\rho}_k^\dagger(t) = \hat{\rho}_{-k}(t)$ to be satisfied, and is thus instrumental in ensuring that $\hat{\rho}_k(t)$ commutes with $\hat{\rho}_k^\dagger(t)$. Both have the effect of reducing the momentum by k, and are indistinguishable when measuring the atomic density at a given time. As a result they interfere when evaluating the Fourier components of the two-point function in Eq. (4). It is precisely these interferences that we shall exploit for assessing the nonseparability of the state.

It should also be noticed that measurements of $\hat{\rho}_k(t)$ performed at different times do not generally commute. When we refer to such measurements, we do so in a "weakly noncommuting" sense: the measurements would be noncommuting if performed on the same experimental realization, but (since the condensate is generally destroyed when the density profile $\rho(x)$ is measured[3]) the measurements are actually performed on different realizations of the same system.

As explained in textbooks [27,30], the Hamiltonian of linear perturbations is diagonalized by writing $\hat{\phi}_k = u_k \hat{\varphi}_k + v_k \hat{\varphi}_{-k}^\dagger$, where $u_k^2 - v_k^2 = 1$[4] in order to preserve the bosonic commutation relation (7) with $\hat{\phi}_k$ replaced by $\hat{\varphi}_k$. Then Eq. (6) is equivalent to

$$\hat{\rho}_k(t) = \sqrt{N}(u_k + v_k)(\hat{\varphi}_k(t) + \hat{\varphi}_{-k}^\dagger(t)). \tag{8}$$

The operators $\hat{\varphi}_k$ correspond to collective excitations (phonons), with $\hat{\varphi}_k$ ($\hat{\varphi}_k^\dagger$) destroying (creating) a phonon of momentum k relative to the condensate. The fact that they diagonalize the Hamiltonian means that (so long as the background is stationary) the k and $-k$ sectors decouple, so that we can write

$$\hat{\varphi}_k(t) = \hat{b}_k e^{-i\omega_k t}, \qquad \hat{\varphi}_{-k}^\dagger(t) = \hat{b}_{-k}^\dagger e^{i\omega_{-k} t}, \tag{9}$$

where the operators $\hat{b}_k$ and $\hat{b}_{-k}^\dagger$ do not depend on time. The lab frame frequency ω_k is related to the condensate rest frame frequency Ω_k by a Doppler shift:

$$\omega_k - vk = \Omega_k \equiv c|k|\sqrt{1 + k^2\xi^2/4}, \tag{10}$$

where v is the flow velocity of the condensate, c is the speed of low-frequency phonons, and $\xi = 1/mc$ is the healing length (m being the atomic mass). Here Ω_k is taken to be positive, since the negative-frequency solutions are automatically accounted for by the Hermitian conjugate operators in the second of Eqs. (9).

B. Homogeneous systems

Let us first analyze density fluctuations on condensates which are stationary and globally homogeneous. We also assume that the phonon states are statistically homogeneous. Then, the two-point function of Eq. (3) depends only on the spatial interval $|x - x'|$, and not on x and x' individually. The expectation value of $\hat{\rho}_k(t)\hat{\rho}_{k'}^\dagger(t)$ can be nonzero only when $k = k'$; the nontrivial part of the two-point function is therefore simply $\langle \hat{\rho}_k(t) \hat{\rho}_k^\dagger(t) \rangle$, and can be expressed in terms of only k and t. In fact, the Fourier-transformed two-point function necessarily has the form

$$G^{(2)}(k,t) = N(u_k + v_k)^2 (1 + n_k + n_{-k} + 2\mathrm{Re}[c_k e^{-2i\Omega_k t}]). \tag{11}$$

It is thus fully governed by the following expectation values:

$$n_{\pm k} = \langle \hat{b}_{\pm k}^\dagger \hat{b}_{\pm k} \rangle, \qquad c_k = \langle \hat{b}_k \hat{b}_{-k} \rangle. \tag{12}$$

The mean occupation numbers $n_{\pm k}$ are real and positive, while c_k is in general a complex number. The time-dependence of $G^{(2)}(k,t)$ comes only from the last term of Eq. (11), since $n_{\pm k}$ and c_k are constant in time whenever the

[3]This is not necessarily the case if one follows the rather sophisticated method proposed in [15].

[4]u_k and v_k are then uniquely defined by the additional relation $u_k/v_k = -(1 + k^2\xi^2/2 + k\xi\sqrt{1 + k^2\xi^2/4})$, where ξ is the healing length defined later in the paragraph.

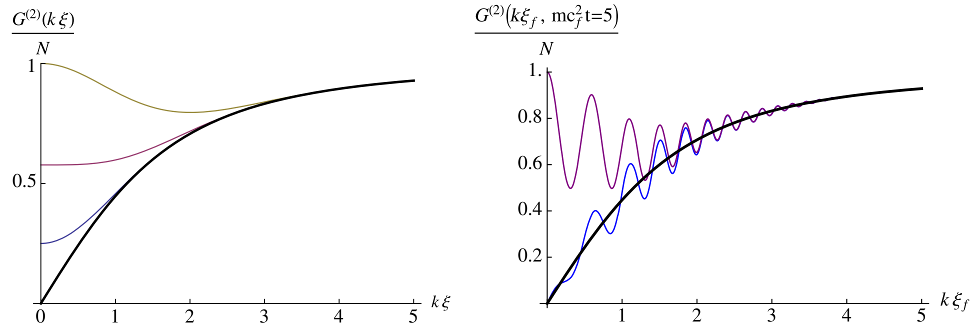

FIG. 1. Fourier transform of equal-time density-density correlation function. On the left is shown $G^{(2)}(k)/N$ of Eq. (11) when the phonon state itself is stationary and thermal, i.e. $c_k = 0$ and $2n_k + 1 = \coth(\Omega_k/2T)$. The various curves correspond to different temperatures: $T/mc^2 = 0$ (black), 1/4 (blue), $1/\sqrt{3}$ (purple) and 1 (yellow). On the right is shown $G^{(2)}(k)/N$ after a lapse of time t following an increase in c^2 by a factor of 2 (the same situation as in Fig. 5 of [4]). The two curves correspond to different initial temperatures: $T = 0$ (blue) and $T = mc_{\rm in}^2$ (purple), where $c_{\rm in}$ is the initial value of the phonon speed, and where c_f and ξ_f are the final values of the phonon speed and the healing length, respectively. As the lapse of time t increases, at fixed k, $G^{(2)}(k)$ oscillates with the final frequency $2\Omega_k$, and the number of oscillations visible in the plot of $G^{(2)}(k)/N$ increases.

background is stationary.[5] Note that, being a function of k only, Eq. (11) is manifestly Galilean invariant.

Measurements of $G^{(2)}(k,t)$ thus allow us to extract a certain amount of information about the phonon state. The first thing to notice is that, even in the absence of any phonons (i.e. $n_{\pm k} = c_k = 0$), $G^{(2)}(k)$ does not vanish but takes the value

$$G^{(2)}_{\rm vac}(k) = N(u_k + v_k)^2, \tag{13}$$

which is due to vacuum fluctuations, see the thick black curve in the left plot of Fig. 1. The presence of uncorrelated phonons (i.e. $n_{\pm k} \neq 0$, $c_k = 0$) increases this value by a relative amount of $n_k + n_{-k}$, see the colored curves in the left plot which correspond to thermal states with temperatures respectively equal to $T/mc^2 = 1/4$, $1/\sqrt{3}$ and 1. In fact, in a thermal state, one has $n_k = n_{-k} = (e^{\Omega_k/T} - 1)^{-1}$ where T is the temperature (in the condensate rest frame), and $G^{(2)}(k)$ thus becomes

$$G^{(2)}(k) = N(u_k + v_k)^2 \coth\left(\frac{\Omega_k}{2T}\right). \tag{14}$$

The ratio $G^{(2)}(k)/N$ is precisely the "static structure factor" shown (with $T/mc^2 = 0$ and 1/4) in Fig. 7.4 of [27] [for the sake of comparison, their healing length is defined as $1/(\sqrt{2}mc)$, which is a factor of $1/\sqrt{2}$ smaller than ours given after Eq. (10)]. In the high-k limit, it tends to 1, whereas in the low-k limit it approaches T/mc^2 (where we emphasize that, because of Galilean invariance, the $k \to 0$ limit gives the temperature in the rest frame of the condensate). By careful measurements of $\rho(x)$, it is thus possible to extract the physical quantities N, ξ and T, as reported in [28].

It should be noticed that it is the total number of phonons $n_k + n_{-k}$ which is extracted, as we are unable to distinguish between the left- and right-moving sectors. Note that this is closely related to the fact that $\hat{\rho}_k$ and $\hat{\rho}_k^\dagger$ commute. In effect, the indistinguishable character of n_k and n_{-k} is the price we pay in restricting ourselves to commuting measurements. Moreover, if we allow for anisotropic states where n_k and n_{-k} are characterized by different temperatures in the limit $k \to 0$, say $T_{\rm rm}$ and $T_{\rm lm}$, then the low-k limit of $G^{(2)}(k)/N$ would yield the arithmetic mean of these two, i.e. it would approach

$$\frac{G^{(2)}(k)}{N} \to \frac{T_{\rm rm} + T_{\rm lm}}{2mc^2}. \tag{15}$$

Finally, the presence of correlations (i.e. $c_k \neq 0$) causes the two-point function to vary sinusoidally with a frequency $2\Omega_k$, a relative amplitude $2|c_k|$ and a phase equal to the phase of c_k, see the right plot of Fig. 1 for two examples. We refer to our former work [4] where these two examples are obtained after having modified the trapping frequency in the perpendicular directions, $\omega_\perp$, in such a way that the square of the phonon speed c^2 increases by a factor of 2, while starting with different initial temperatures. (Notice that in that work, we effectively used a different normalization convention for the Fourier transform of $\hat{\rho}$, so that there was no factor of N out front in the expression for $G^{(2)}(k)$.) The rate of change of $c^2(t)$ is chosen in such a way that it is slow with respect to $\omega_\perp$, so as not to cause any oscillations of the condensate itself; see Ref. [4]. As explained in that work, longitudinal phonon modes with frequencies much lower than the rate of change of $\omega_\perp$ respond to that change as if it were sudden, inducing a significant mode amplification. It should also be recalled that, when starting from vacuum, this dynamical Casimir effect (DCE) leads to the spontaneous production of

[5] In this paper, we shall not describe the preceding time-dependent processes which have engendered these expectation values. The interested reader is referred to [2–4] where several scenarios are studied in detail. One of these scenarios is briefly discussed after Eq. (15).

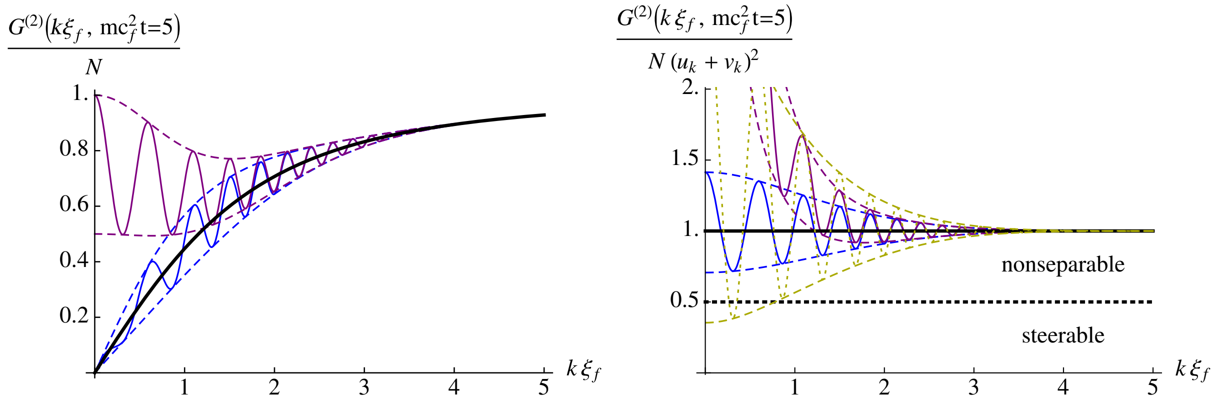

FIG. 2. Assessing the entanglement of the phonon state. On the left are plotted the same density-density correlation functions as in the right panel of Fig. 1, with their upper and lower envelopes shown in dashed lines. Nonseparability is guaranteed when the minimum value of $G^{(2)}(k,t)$ is less than the correlation function associated to vacuum fluctuations, i.e. to the dipping of the lower envelope below the thick black curve. For the blue curve (at $T=0$) the threshold is crossed for all k, whereas for the purple curve (at $T=mc_{\text{in}}^2$) it is crossed only for $k\xi \gtrsim 1.2$. On the right are shown the same correlation functions, normalized by $(u_k+v_k)^2$ so that the nonseparability threshold occurs at exactly 1. Also included there in yellow is the result having increased c^2 by a factor of 8 (at $T=0$). For $k\xi \lesssim 0.8$, this curve satisfies the sufficient condition for steerability given in Eq. (17) and indicated by the thick horizontal dotted line.

maximally entangled pairs with opposite wave vectors $k, -k$, by which we mean that the maximal value of $|c_k|$ (for a given n_k) allowed by quantum mechanics is reached [see Eq. (A7)].

C. Assessing entanglement

We now turn to the extraction of the degree of entanglement of the phonon state by studying the behavior of $G^{(2)}(k,t)$ of Eq. (3). To this end, we shall use the notions of nonseparability and steerability, which are briefly recalled in Appendix A.

As mentioned in the introduction, inequality (1) is a sufficient condition for nonseparability of the $k, -k$ bipartite state (and, in fact, is also necessary whenever the state is Gaussian) [11,12,20]. In Appendix A, we further demonstrate that a sufficient condition for (1) to be satisfied is

$$G^{(2)}(k,t) < G^{(2)}_{\text{vac}}(k) = N(u_k+v_k)^2, \qquad (16)$$

for some time t. This is one of our key results, and was previously reported (for isotropic states with $n_k = n_{-k}$) in [4]. The indistinguishability of the k and $-k$ sectors in the expression $\hat{\rho}_k(t)\hat{\rho}_k^\dagger(t)$ entering $G^{(2)}(k,t)$ causes the density-density measurements to act as an effective interferometer for these two channels. With varying t, it gives us access to the two-mode phase space spanned by the modes k and $-k$, see Eq. (11). Inequality (16), which expresses the (periodic) reduction of the noise below its vacuum value, implies the existence of a subfluctuant direction in this phase space [11], thus directly revealing the entanglement of the state. This dipping below vacuum noise is a key feature of several practical measurements of entanglement, see also e.g. [15,31,32]. Crucially, it is directly revealed by observations performed at a single time only; in particular, it does not yield any of the individual expectation values $n_{\pm k}$ or c_k, but only that they stand in a certain relation to one another. Indeed, the extraction of $n_{\pm k}$ and c_k separately (which in the present settings are equivalent to the knowledge of the full covariance matrix, see Appendix A) *would* require the performance of noncommuting measurements, given that the number operators $\hat{n}_{\pm k} = \hat{b}^\dagger_{\pm k}\hat{b}_{\pm k}$ do not commute with $\hat{c}_k = \hat{b}_k\hat{b}_{-k}$, nor even the Hermitian part of $\hat{c}_k$ with its anti-Hermitian part.

In practical terms, to verify if inequality (16) is satisfied, it suffices to look for the lower enveloping curve of $G^{(2)}(k,t)$, i.e., the minimum value reached by $G^{(2)}(k,t)$ when varying time. In the left plot of Fig. 2, in dotted lines with the corresponding colors, we have added the upper and lower envelopes for the two examples considered in the right plot of Fig. 1. To facilitate the reading, on the right plot of Fig. 2 we represent the ratio $G^{(2)}(k,t)/G^{(2)}_{\text{vac}}(k)$; the state is then nonseparable whenever the curve drops below 1. One clearly sees that the lower envelope of the blue curve (which corresponds to the case with a vanishing temperature) is below the threshold for all values of k, as can be understood from the fact that the final phonon state in that case is a pure two-mode squeezed state. By contrast, the lower envelope of the purple curve (which corresponds to an initial temperature equal to mc_{in}^2, where c_{in} is the initial phonon speed) dips below the nonseparability threshold only for $k\xi \gtrsim 1.2$.

On the right plot of Fig. 2, we have also added a thick dotted black line showing the threshold

$$G^{(2)}(k,t) < \frac{G^{(2)}_{\text{vac}}(k)}{2}, \qquad (17)$$

which is a sufficient condition for steerability, see Appendix A. It is not crossed by either of the two cases shown on the left plot, although, for the case with a vanishing temperature (the blue curve), the phonon state is in fact steerable for all values of k. The reason is that there is a minimal value of n_k below which the threshold (17) cannot be reached, even for maximally entangled states; see Eq. (A13) in Appendix A. To show that there is no problem of principle, we have added a third case (shown in yellow) which is obtained (for a vanishing initial temperature) when varying the trapping frequency $\omega_\perp$ in the perpendicular direction in such a way that the effective speed of sound c^2 appearing in Eq. (10) changes by a factor of 8, and not by a factor of 2 as for the blue curve. This greater change induces enough mode amplification for certain modes (those with $k\xi \lesssim 0.8$) to dip below the threshold (17), even though, as for the blue curve, the state is pure and all two-mode systems $(k, -k)$ are in fact steerable.

III. INHOMOGENEOUS STATIONARY BACKGROUND

We now turn to the complementary case of a flow profile corresponding to an analogue black hole [33,34], by which we mean that the one-dimensional flow is transonic and stationary [35] and that the velocity increases in the direction of the flow. Hence the background is necessarily inhomogeneous but we shall also assume that it possesses (sufficiently long) homogeneous regions on both sides of the sonic horizon so that $\hat{\rho}_k$, the Fourier components of the density $\hat{\rho}(x)$, can be extracted on either side.[6] An example of such a flow, known as the "waterfall" solution [38], is shown in the left panel of Fig. 3, with the flow to the right so that $v > 0$. It has been chosen because it is close to that used in the experimental work [21], see also Figs. 1 and 2 of [39]. It is now well-established that the transonic character of the background flow gives rise to a steady pair production of outgoing phonons carrying opposite frequencies $\pm\omega$ (in similar fashion to the above described DCE which produces phonon pairs of opposite wave vectors $\pm k$).

When the dispersive effects [governed by the healing length ξ, see Eq. (10)] can be neglected, it can be shown [33–35] that phase fluctuations propagating in a transonic background flow obey a (massless) d'Alembertian equation in an effective black hole metric. The analogue event horizon is located where the flow velocity v crosses the sound speed c. This is captured by the analogue metric, which reads

$$ds^2 = -c^2(x)dt^2 + (dx - v(x)dt)^2. \tag{18}$$

In this case, the steady phonon emission is thermal, with temperature (measured in the "stationary" frame in which the flow profile $v(x)$ is at rest) proportional to the rate of change of $v - c$ at the sonic horizon:

$$T_H = \frac{1}{2\pi}\partial_x(v - c)|_{\mathrm{hor}}. \tag{19}$$

This follows immediately from the analogy with the Hawking radiation emitted by black holes [33]. When dispersive effects are no longer neglected, this result appears at leading order in $T_H\xi/c$, as can be shown by solving the stationary Bogoliubov-de Gennes equation on transonic flows [35].

If stationarity is achieved in an experiment, then the density-density correlation function (3) depends only on the time difference and not on t and t' individually. Given the form of $\hat{\rho}_k(t)$ in Eqs. (8) and (9), this ensures that the expectation value of $\hat{\rho}_k(t)\hat{\rho}^\dagger_{k'}(t)$ in the asymptotic flat regions can be nonzero only when the corresponding frequencies ω_k and $\omega_{k'}$ (measured in the stationary frame) are equal in magnitude.[7]

The following developments have a lot in common with Refs. [19,20], which are also based on *in situ* measurements of density fluctuations. Our first aim is to contrast the rather simple procedure of Sec. II to the rather complicated procedure adopted in these references. We also wish to clarify the origin and consequences of the commuting character of the measurements.

A. Practical and conceptual difficulties

Let us briefly discuss the main properties characterizing density fluctuations in transonic flows, with particular emphasis on their differences with respect to the globally homogeneous case, and the complications thereby induced.

Firstly, at fixed $|\omega|$, three stationary modes are mixed by the scattering on a transonic flow [35,39], rather than two as on a globally homogeneous background. One of these phonons is copropagating with the flow, while the other two are counterpropagating; adopting a common notation, these shall be labeled by the superscripts v and u, respectively. Taking the rest frame frequency $\Omega > 0$ (and given that the flow velocity $v > 0$), the v phonon has positive wave number k while the u phonons have negative k. More important are their (conserved) frequencies $\pm\omega$ in the stationary frame, which for $\Omega > 0$ give the signs of their energies: the v phonon has positive energy while the two u phonons have opposite energies.[8] Two types of phonon pair carrying zero total energy are thus spontaneously produced: (u, u) pairs involving the two

[6]The assumption of homogeneity far from the horizon is rather mild as it has been shown that transonic flows analogous to black holes, i.e., with the flow velocity $v(x, t)$ increasing along the direction of the flow, obey "no-hair" theorems [36,37]: they expel perturbations away from the sonic horizon where v crosses the sound speed c. Therefore stationary asymptotically uniform transonic flows act (in a finite interval of x containing the sonic horizon) as attractors for neighboring flows.

[7]As a result, the correlation pattern projected onto the (k, k')-plane is more complicated than the strict $k = k'$ condition characterizing pair production in globally homogeneous flows; see [40] for the structure of these correlations after time-of-flight measurements, and [41] for the corresponding curves in a stationary inhomogeneous water wave system.

[8]Whereas the two positive-energy phonons can propagate throughout the entire space, the negative-energy phonon exists only in the region where the flow is supersonic; see Fig. 1 in Ref. [22] for the spacetime trajectories followed by the three types of phonon.

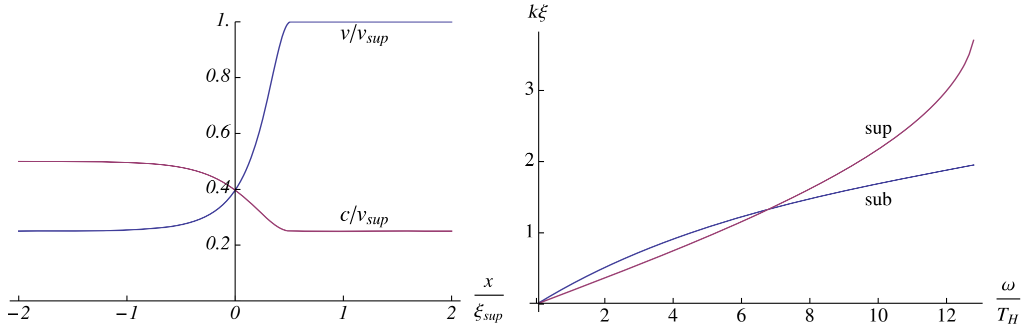

FIG. 3. Flow and dispersion profiles for a "waterfall" configuration. On the left are shown the flow velocity v and the low-frequency phonon speed c, both normalized with respect to the supersonic (downstream) flow velocity v_{sup}. In the supersonic region, the Mach number $M_{\mathrm{sup}} = 4$ fixes that in the subsonic (upstream) region at $M_{\mathrm{sub}} = 1/2$, which is close to that reported in [21]; see also [39]. The position is labelled such that v and c cross at $x = 0$, which corresponds to the analogue event horizon. On the right are shown the corresponding behaviors of the wave vectors of counterpropagating phonons (both in the subsonic and supersonic regions and adimensionalized by the local value of the healing length) as functions of ω/T_H, where T_H is given in Eq. (19). It can be seen that, when ω/T_H is small enough ($\lesssim 4$), dispersive effects are small, which means that Eqs. (18) and (19) provide reliable approximations.

counterpropagating modes, and (u, v) pairs involving the copropagating v mode.

Despite this 3-mode mixing, it can be shown that the inequality,

$$n^u_\omega n^u_{-\omega} - |c^{uu}_\omega|^2 < 0, \tag{20}$$

guarantees that the bipartite state characterizing the two u modes (and obtained by tracing over the v mode) is nonseparable [22]. In strict analogy with the quantities entering Eq. (1), we have $n^u_{\pm\omega} = \langle \hat{b}^{u\dagger}_{\pm\omega} \hat{b}^u_{\pm\omega} \rangle$ and $c^{uu}_\omega = \langle \hat{b}^u_\omega \hat{b}^u_{-\omega} \rangle$, where $\hat{b}^u_{\pm\omega}$ ($\hat{b}^{u\dagger}_{\pm\omega}$) destroys (creates) a u phonon of frequency $\pm\omega$. It can also be shown that the couplings involving the v mode are generally smaller than those relating the two u modes [22]. Therefore, the v mode is essentially a spectator, and a fair understanding of the physics can be reached by focusing on the (u, u) coupling terms. That said, from an experimental point of view, the indistinguishability of n_k and n_{-k} when measuring the k-th Fourier transform of density fluctuations means that one cannot simply discard the v modes when one attempts to assess the nonseparability of the state, for the mean occupation numbers will be polluted by their presence.

Secondly, and more crucially, is the fact that the u phonons of opposite energy appear on opposite sides of the sonic horizon and then propagate away from each other in separate regions of space. As a result, one is now necessarily dealing with two distinct atomic densities: $\hat{\rho}^{\mathrm{sub}}(x)$ in the subsonic (upstream) region, and $\hat{\rho}^{\mathrm{sup}}(x')$ in the supersonic (downstream) region. Explicitly, on Fourier transforming the density operator (and arbitrarily setting $t = 0$ since the state is stationary and all measurements are made at equal time), we have (see Eq. (8) and [19])

$$\begin{aligned} \hat{\rho}^{\mathrm{sub}}_k &= \sqrt{N^{\mathrm{sub}}}(u^{\mathrm{sub}}_k + v^{\mathrm{sub}}_k)(\hat{b}^{\mathrm{sub}}_k + (\hat{b}^{\mathrm{sub}}_{-k})^\dagger), \\ \hat{\rho}^{\mathrm{sup}}_{k'} &= \sqrt{N^{\mathrm{sup}}}(u^{\mathrm{sup}}_{k'} + v^{\mathrm{sup}}_{k'})(\hat{b}^{\mathrm{sup}}_{k'} + (\hat{b}^{\mathrm{sup}}_{-k'})^\dagger). \end{aligned} \tag{21}$$

We have added superscripts 'sub' and 'sup' to the Bogoliubov coefficients u_k and v_k to indicate that they depend on the healing lengths ξ^{sub} and ξ^{sup} defined on either side. We have also added superscripts "sub" and "sup" to the phonon operators $\hat{b}_k$ because they encode different modes with support in nonoverlapping regions of space; for instance, the commutator $[\hat{b}^{\mathrm{sub}}_k, (\hat{b}^{\mathrm{sup}}_{k'})^\dagger]$ vanishes even when $k = k'$. Finally, to facilitate the reading, we use an unprimed k to refer to a measurement made in the subsonic region, and a primed k' to refer to a corresponding measurement in the supersonic region. It will prove convenient to employ this notation in the remainder of this section.

The fact that the entangled outgoing u phonons propagate in different regions has two important consequences. On the one hand, they produce a nonlocal long-distance correlation pattern in the (x, x')-plane which signals the production of correlated phonon pairs of type (u, u) [42,43]. This nonlocal correlation pattern has been reported in the experimental work [21] and agrees with theoretical predictions to a large extent [39]. On the other hand, as far as entanglement is concerned, their associated density fluctuations no longer (locally) interfere. This implies that one can no longer assess the entanglement of the state by simply comparing, as we did in Eq. (16), the measured value of $G^{(2)}(k)$ for some k with the corresponding vacuum expression $G^{(2)}_{\mathrm{vac}}(k)$.

Instead, one is forced to follow the more indirect procedure proposed in [19], which involves combining three different measurements of the density-density correlation function. Indeed, each of the three quantities entering Eq. (20) should be estimated by pairing differently $\hat{\rho}^{\mathrm{sub}}_k$ and $\hat{\rho}^{\mathrm{sup}}_{k'}$ of the first and second lines of Eq. (21), where k and k' are both negative (i.e. they correspond to u modes) and are related by $\omega^{\mathrm{sub}}_k = -\omega^{\mathrm{sup}}_{k'}$; see Eq. (10) for the expression for ω_k (recalling that c and ξ are different in the two asymptotic regions), and the right panel of Fig. 3 for the behavior of these solutions in the flow shown in the left panel. The wave numbers should thus be considered as functions of the frequency, which can be shown explicitly by writing $k = k^u_\omega$ and $k' = k'^u_\omega$. In short, the mean occupation numbers n^u_ω and $n^u_{-\omega}$ should be extracted, respectively, from the expectation values of $\hat{\rho}^{\rm sub}_k \hat{\rho}^{\rm sub\dagger}_k$ and $\hat{\rho}^{\rm sup}_{k'} \hat{\rho}^{\rm sup\,\dagger}_{k'}$, while the correlation term c^{uu}_ω should be extracted from the expectation value of $\hat{\rho}^{\rm sub}_k \hat{\rho}^{\rm sup}_{k'}$.[9]

B. Explicit expressions

We first consider the autocorrelation $G^{(2)\rm sub}(k) = \langle \hat{\rho}^{\rm sub}_k \hat{\rho}^{\rm sub\dagger}_k \rangle$ in the subsonic region, noting that the autocorrelation in the supersonic region is entirely analogous. Assuming the stationarity of the phonon state, we have

$$
\begin{aligned}
G^{(2)\rm sub}(k,k) &= N^{\rm sub}(u^{\rm sub}_k + v^{\rm sub}_k)^2(1 + n^{\rm sub}_k + n^{\rm sub}_{-k}) \\
&= G^{(2)\rm sub}_{\rm vac}(k)(1 + n^{\rm sub}_k + n^{\rm sub}_{-k}). \qquad (22)
\end{aligned}
$$

As in (16), we see the key role played by the vacuum two-point function $G^{(2)\rm sub}_{\rm vac}(k)$ in extracting the observable quantity, here $n^{\rm sub}_k + n^{\rm sub}_{-k}$, characterizing the phonon state. As could have been expected from the analysis of the former section, it is the total occupation number $n^{\rm sub}_k + n^{\rm sub}_{-k}$ that is extracted from the density measurements. Hence the v modes do (positively) contribute to the measurements of $\hat{\rho}^{\rm sub}_k \hat{\rho}^{\rm sub\dagger}_k$ which therefore only gives an upper bound for the u-mode occupation number $n^u_\omega = n^{\rm sub}_k$, where $k = k^u_\omega$. Note also that, if the state is stationary, then since $|\omega^u_k| \neq |\omega^v_k|$ whenever the flow velocity is nonzero, we necessarily have on each side $c_k = 0$, i.e. u and v modes of wave numbers k and $-k$ are completely uncorrelated. Whereas, in the homogeneous case, the interference between phonon modes of opposite wave numbers is useful because it combines precisely those two modes that are entangled by the time-varying background, here a possible interference (for instance due to some lack of stationarity of the background flow) would be a hindrance in that it combines modes which are not related by the stationary analogue Hawking effect, and thus would pollute the measurements.[10]

It turns out that the extraction of the correlation term c^{uu}_ω is not polluted by v modes when the stationarity of the phonon state is assumed and when the sub- and supersonic regions are sufficiently well-separated that any residual contribution due to the finite width of the autocorrelation can be safely ignored. Indeed, using again $k = k^u_\omega$ and $k' = k'^u_\omega$, one easily verifies that the only term in the cross-correlation which is nonvanishing has the following form:

$$
\begin{aligned}
&G^{(2)\rm sub/\,sup}(k,-k') \\
&= \langle \hat{\rho}^{\rm sub}_k \hat{\rho}^{\rm sup}_{k'} \rangle \\
&= \sqrt{N^{\rm sub} N^{\rm sup}}(u^{\rm sub}_k + v^{\rm sub}_k)(u^{\rm sup}_{k'} + v^{\rm sup}_{k'}) c_{k,k'}, \qquad (23)
\end{aligned}
$$

where we have defined $c_{k,k'} \equiv \langle \hat{b}^{\rm sub}_k \hat{b}^{\rm sup}_{k'} \rangle$. Importantly, in order to extract $|c^{uu}_\omega|$ from $|c_{k,k'}|$, it is not sufficient to use k and k' with opposite values of ω; one should also carefully choose the relative location and extension of the spatial windows used in the sub- and supersonic regions so as to maximize the overlap of the phonon modes with opposite ω, see the function $F(\omega)$ entering Eq. (32) in Ref. [20].[11] Assuming that the optimum overlap is reached (i.e. $|F(\omega)| = 1$), one gets the sought-after relation $|c^{uu}_\omega| = |c_{k,k'}|$; more generally, we have $|c^{uu}_\omega| \geq |c_{k,k'}|$, i.e., the measured value $|c_{k,k'}|$ gives an underestimation of the strength of the correlations present in the phonon state.

In brief, under the assumption of stationarity, noticing that the populations $n^u_{\pm\omega}$ are both being overestimated by the $G^{(2)}$ measurements if one ignores the population of the v modes, while the correlation strength $|c^{uu}_\omega|$ is being underestimated if the optimum overlap between the two windows is not reached, Eq. (20) combined with the above equations tells us that a sufficient criterion for nonseparability is [19,20]

$$
\begin{aligned}
\Delta^{(2)}(k,k') \equiv\ & [G^{(2)\rm sub}(k,k) - G^{(2)\rm sub}_{\rm vac}(k)] \\
& \times [G^{(2)\,\rm sup}(k',k') - G^{(2)\,\rm sup}_{\rm vac}(k')] \\
& - |G^{(2)\rm sub/\,sup}(k,-k')|^2 < 0. \qquad (24)
\end{aligned}
$$

[9]It thus appears that one needs to measure separately the three ingredients entering condition (20). This raises a puzzling question, since on the one hand, the occupation numbers $\hat{n}^u_{\pm\omega} = \hat{b}^{u\dagger}_{\pm\omega} \hat{b}^u_{\pm\omega}$, considered as operators, do *not* commute with the operator $\hat{c}^{uu}_\omega = \hat{b}^u_\omega \hat{b}^u_{-\omega}$, while on the other hand these three quantities are extracted from measurements (performed at a given time) of $\hat{\rho}(x)$ for various values of x which do commute. The resolution of this paradox comes from the operator content of $\hat{\rho}^{\rm sub}_k$ and $\hat{\rho}^{\rm sup}_{k'}$ of Eq. (21). Each of them contains *with equal weight* a destruction operator and a creation operator corresponding to modes with opposite values of k. This guarantees that the three relevant combinations $\hat{\rho}^{\rm sub}_k \hat{\rho}^{\rm sub\dagger}_k$, $\hat{\rho}^{\rm sup}_{k'} \hat{\rho}^{\rm sup\,\dagger}_{k'}$ and $\hat{\rho}^{\rm sub}_k \hat{\rho}^{\rm sup}_{k'}$ commute with each other. Although they do not contribute to the three uu quantities entering Eq. (20), the v operators $(\hat{b}^v_{-k})^\dagger, (\hat{b}^v_{-k'})^\dagger$ are necessary to ensure the commutation of the density measurements.

[10]For the sake of clarity, we wish to emphasize that the v modes which are coupled to the u modes by the density measurements, and which are related to them through having the same magnitude of wave number $|k|$, are to be distinguished from the v modes which are coupled to the u modes by the analogue Hawking effect, related by having the same magnitude of frequency $|\omega|$, and which only appear in the downstream (supersonic) region; see Eq. (58) of [35]. In the homogeneous case, by contrast, the modes coupled by the DCE are *precisely* those which are also coupled by the density measurements.

[11]This function $F(\omega)$ plays a similar role as the factor multiplying $2RT$ in Eq. (9) of the Hong-Ou-Mandel paper [31]. As mentioned there, one immediately sees that the strongest interference occurs when this factor is 1, which is its maximum value. The occurrence of such a function limiting the relative intensity of the correlations is very general, and when its maximum value is less than 1, it encodes decoherence effects; see e.g. Ref. [44].

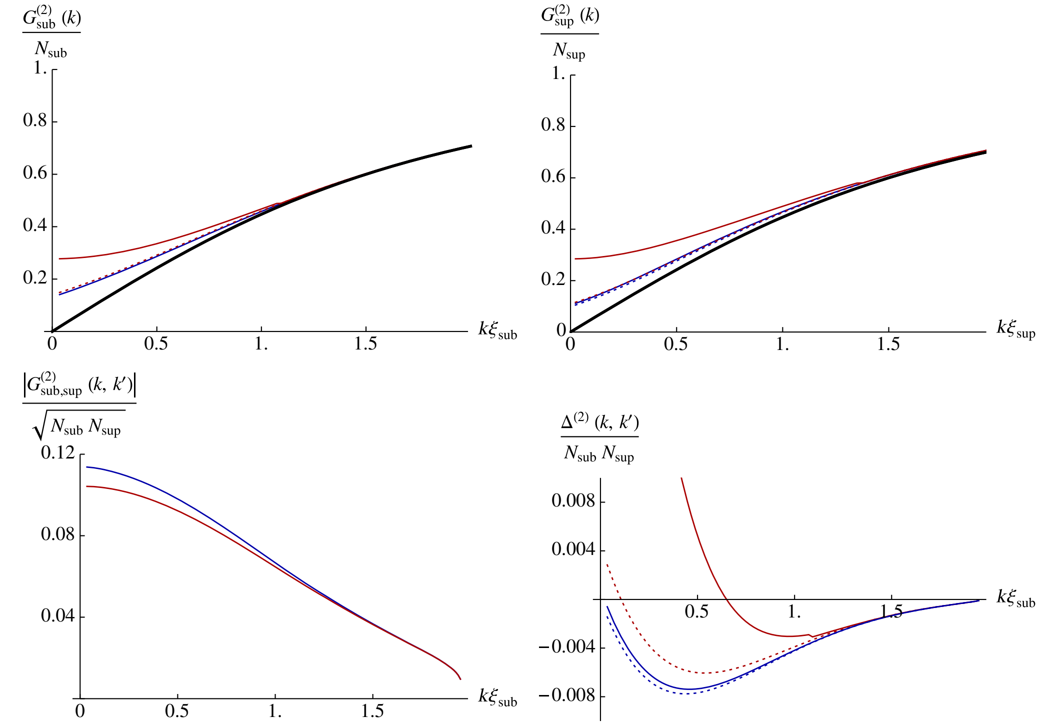

FIG. 4. Density-density correlation function for a waterfall in BEC, with Mach number in the supersonic region $M_{\rm sup} = 4$ (close to that reported in [21]). The plots in the upper row show the autocorrelation $G^{(2)}(k)$ in the subsonic (left panel) and supersonic (right panel) regions, while the lower left panel plots the cross-correlation $G^{(2)}(k,k')$. The lower right panel shows instead the difference appearing in inequality (24). The variously colored curves correspond to different initial temperatures (in the frame of the condensate) of the incident v modes: $T^v_{\Omega,\rm in} = 0$ (blue curves) and $T^v_{\Omega,\rm in} = 2T_H$ (red curves—the kinks appearing in these are numerical artifacts due to assuming that the scattering is trivial above $\omega_{\rm max}$, the maximum frequency at which (u,u) phonon pairs are produced). In the upper plots, the thick black curves show the autocorrelation in vacuum, which must be subtracted before insertion into inequality (24). The solid curves are those that are actually observed; by contrast the dotted curves are those that would result if the measurements were able to distinguish u and v modes, and only the occupation numbers of u modes were to appear in all expressions. When the initial temperature $T^v_{\Omega,\rm in} = 2T_H$, the pollution of the density measurements by the presence of v modes dramatically affects the observations of $G^{(2)}(k)$ in the subsonic and supersonic regions (and, therefore, also the computed value of $\Delta^{(2)}$), as can be seen by comparing the solid and dotted red curves in the upper plots and in the lower right plot. Indeed, since the u-v coupling is small, the occupation numbers of u modes remain essentially the same whether or not there is an initial thermal state, as indicated by the fact that the dotted red curves are almost the same as the blue curves in the upper plots.

To illustrate what this procedure entails, we have plotted in Fig. 4 the three relevant Fourier transforms of $G^{(2)}(x,x')$, as well as the difference of (24), that are theoretically obtained by solving the BdG equation on a stationary transonic flow described by an exact solution (called a "waterfall" or "half-soliton") of the GPE on a step function potential [39]. These stationary asymptotically homogeneous flows form a one-parameter family of solutions that can be labeled by the asymptotic Mach number $M_{\rm sup} = v_{\rm sup}/c_{\rm sup}$ in the downstream supersonic region; the corresponding Mach number in the upstream subsonic region is $M_{\rm sub} = M_{\rm sup}^{-1/2}$. We worked with $M_{\rm sup} = 4$ as it matches what has been observed in [21]. We also considered two initial temperatures (in the condensate rest frame) for the incident v modes: $T^v_{\Omega,\rm in} = 0$ and $T^v_{\Omega,\rm in} = 2T_H$, where T_H [of Eq. (19)] is the effective low-frequency temperature of the emitted u phonons.

In terms of the u and v modes that combine to give the autocorrelation of Eq. (22), Hawking radiation is inherently anisotropic: the u- and v-mode populations on any one side are generally very different. If the ingoing v-mode state is close to vacuum, the $k \to 0$ limit of $G^{(2)}(k)$ [governed by Eq. (15)] will then be $T^u_\Omega/2mc^2$. Notice that the extracted temperature of the u modes is that measured in the condensate rest frame; i.e., it is associated to the rest frame frequency Ω rather than to the conserved frequency ω, these being related by a Doppler shift, see Eq. (10).[12] In the limit $k \to 0$, they become proportional, namely $\Omega = \omega/(1-M)$ where $M = v/c$ is the Mach

[12]In fact, the inverse transformation should be considered to express the initial temperature of the condensate $T_{\rm in}$, naturally expressed in terms of Ω, in terms of the conserved frequency ω governing the Hawking effect; see Sec. II C of [22].

number. Therefore, the associated temperatures are related in the same manner: we have, for the counterpropagating u modes,

$$T^u_\Omega = \frac{T^u_\omega}{|1 - M|}. \tag{25}$$

Taking into account the above facts that, when the ingoing state is vacuum, in the subsonic region the v modes are in their vacuum state and the outgoing u modes have a ω temperature close to T_H of Eq. (19), we arrive at the conclusion that the $k \to 0$ limit of $G^{(2)}(k)$ is

$$\frac{G^{(2)}_{\mathrm{sub}}(k)}{N_{\mathrm{sub}}} \to \frac{1}{2}\frac{T_H}{mc^2_{\mathrm{sub}}}\frac{1}{1 - M_{\mathrm{sub}}}. \tag{26}$$

As far as we know, this combination of effects has not yet been discussed in the literature, although it can be shown to agree with Eq. (35) of [39]. We should also point out that, for the waterfall flow (close to the flow realized in [21]) with $M_{\mathrm{sub}} = 1/2$, the product $2(1 - M_{\mathrm{sub}}) = 1$. In that case, we recover from Eq. (26) the standard expression $G^{(2)}(k)/N \to T_H/mc^2$ [see below Eq. (14)], as if neither of the above effects were modifying $G^{(2)}(k)$. Notice finally that a more complicated expression than (26) governs the low-k behavior of $G^{(2)}(k)$ in the supersonic region due to the production of v modes; see the first term in Eq. (53) of [35].

IV. CONCLUSION

In this paper, we have considered the properties of the equal-time density-density correlation function in homogeneous domains of elongated effectively one-dimensional atomic cold gases. We began by studying globally homogeneous systems, examining the time-dependence of the Fourier transform of the density-density correlation function at fixed wave number k. In this case, $G^{(2)}(k,t)$, the kth component of the two-point function, displays periodic oscillations whose amplitude depends on the strength of the correlations between the phonon modes of opposite wave numbers. These oscillations arise from the interference between the contributions of these modes when measuring the atomic density. We demonstrated that if the minimal value periodically reached by $G^{(2)}(k,t)$ goes below $G^{(2)}_{\mathrm{vac}}(k)$, the constant value $G^{(2)}(k)$ has in the vacuum state, then the bipartite phonon state of wave numbers $k, -k$ is necessarily nonseparable. Remarkably, repeated *in situ* measurements of the atomic density $\rho(x)$ at a given time (one for which $G^{(2)}(k,t) < G^{(2)}_{\mathrm{vac}}(k)$) are sufficient to assess the nonseparability of the bipartite phonon state. In this, we see that the commuting character of these measurements does not prevent one from having access to the entanglement of the state, in contrast to the fact that the knowledge of all entries of the covariance matrix does require the performance of noncommuting measurements. In fact, a closer analysis shows that *in situ* measurements of $\rho(x)$ performed at an appropriate time give us access to the particular combination of the elements of the covariance matrix which governs the degree of entanglement between the two subsystems. In mathematical terms, the combination appearing in (1) gives the lowest eigenvalue of the determinant used in the function $\mathcal{P}$ which appears in the generalized Peres-Horodecki criterion. Hence the sign of the difference entering (1) is equal to the sign of $\mathcal{P}$ (see Appendix A). By a similar analysis (performed at the end of Sec. II C), we showed that the stronger criterion of steerability can also be experimentally verified by studying the behavior of $G^{(2)}(k)$ at a given time, i.e. by commuting measurements of density fluctuations. We hope that forthcoming experiments could exploit the simple analysis of Sec. II C to assess entanglement of bipartite phonon states $(k, -k)$.

In the second part of the paper, we studied the density fluctuations measured asymptotically on each side of a transonic stationary flow whose velocity increases along the direction of the flow, mimicking a black hole metric and giving rise to the steady production of phonon pairs of zero total energy by a process analogous to the Hawking effect. The relevant phonons, i.e. those which are counterpropagating with respect to the flow, are emitted on either side of the horizon, so that (in stark contrast to the homogeneous case) they live in nonoverlapping regions of space and do not (directly) interfere. It is thus no longer possible to extract information about the entanglement of the relevant phonon modes by directly observing the oscillations of the density-density correlations. Rather, one should combine three different measurements of these correlations. Two of them are autocorrelations performed in each asymptotic region, and give upper bounds for the mean occupations numbers $n^u_{\pm\omega}$ of the relevant phonons. The reason they give only upper bounds is our inability to subtract the (weak but unknown) contributions of the copropagating modes, which act as spectator modes in the present process in that they are only weakly involved in the mode mixing taking place near the sonic horizon. The third measurement instead is a cross-correlation, coming from Fourier components of the two-point function evaluated on either side of the horizon. It gives a lower bound for the norm of c^{uu}_ω which governs the strength of the correlations between the two relevant phonon modes. Although the spectator copropagating modes do not contribute to the mean value of this measurement (because we assumed stationarity), their contribution to the operators being measured guarantees that the third measurement commutes with the former two. To us this is a remarkable illustration of the counterintuitive nature of quantum mechanics.

ACKNOWLEDGMENTS

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APPENDIX A: DEGREES OF ENTANGLEMENT IN BIPARTITE SYSTEMS

Entanglement is one of the most telling signs of the quantum nature of physics. Moreover it is a rich and often subtle subject. Whereas for pure states, "entanglement" is a fairly well-defined property, with several equivalent formulations, it was pointed out by Werner [8] and subsequent authors that for mixed states these formulations are no longer equivalent. This leads to a hierarchy of different degrees of entanglement for general quantum states. Here we consider two such notions: *nonseparability* and *steerability* (defined below). In preparation for the study of density perturbations in atomic Bose gases presented in the main text, we restrict our attention to two-mode bosonic states, each single-mode subsystem having its own quantum amplitude operators $\hat{b}_j$ and $\hat{b}_j^\dagger$ ($j = 1, 2$) subject to the usual bosonic commutation relation $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{i,j}$.

1. The criteria

a. Nonseparability

Nonseparable states are best defined as the complement of the set of separable states, for which an explicit definition can be given. For a fixed pair of subsystems denoted 1 and 2, such that the Hilbert space of the whole system is the tensor product $H_1 \otimes H_2$ of the Hilbert spaces for each subsystem, the bipartite state $\hat{\rho}_{1,2}$ is said to be *separable* with respect to subsystems 1 and 2 whenever it can be written in the form[13]

$$\hat{\rho}_{1,2} = \sum_a P_a \hat{\rho}_1^a \otimes \hat{\rho}_2^a, \tag{A1}$$

where the $\hat{\rho}_j^a$ are states pertaining to each of the subsystems separately and the $P_a \geq 0$ are real numbers. Then $\sum_a P_a = 1$, and the state $\hat{\rho}_{1,2}$ has the properties of a probability distribution. When more than one $P_a \neq 0$, $\hat{\rho}_{1,2}$ entails correlations between the subsystems 1 and 2, but only in a classical sense: the overall state can be obtained by using a random number generator to pick the factorized $a^{\rm th}$-state $\hat{\rho}_1^a \otimes \hat{\rho}_2^a$ distributed according to the probability distribution P_a, and placing each subsystem separately in the states $\hat{\rho}_1^a$ and $\hat{\rho}_2^a$. Note that uncorrelated states are necessarily separable. Conversely, *nonseparable* states are those which cannot be written in the form (A1). Such states are necessarily correlated, but their correlations cannot be accounted for by the above classical means.

b. Steerability

The notion of steerability was originally formulated along the lines of thinking present in the original EPR paper [45]. The idea is to consider making measurements on one subsystem, say 1, and using the results of such measurements to infer the values of correlated quantities for the second subsystem 2.[14] As formulated more recently by Reid [47], when considering a pair of noncommuting operators $\hat{A}_2$ and $\hat{B}_2$ with a c-number commutator,[15] the mathematical description of steering can be written as

$$\Delta_{\rm inf} A_2 \cdot \Delta_{\rm inf} B_2 < \frac{1}{2} |[\hat{A}_2, \hat{B}_2]|. \tag{A2}$$

Here, $\Delta_{\rm inf} A_2$ refers to the inferred standard deviation of A_2 on subsystem 2 having measured A_1 on subsystem 1, that is,

$$\Delta_{\rm inf} A_2 = \sqrt{\langle (\hat{A}_2 - \bar{A}_2(A_1))^2 \rangle}, \tag{A3}$$

where $\bar{A}_2(A_1)$ is the conditional (mean) value of $\hat{A}_2$ given that a (projective) measurement of $\hat{A}_1$ on subsystem 1 yields the eigenvalue A_1, and given that the state of the system is $\hat{\rho}_{1,2}$. Several comments are needed in order to appreciate the content of inequality (A2). Firstly, a measurement of $\hat{B}_1$, which does not commute with $\hat{A}_1$, is performed on subsystem 1 when computing $\Delta_{\rm inf} B_2$. Secondly, the right-hand side of (A2) gives the usual lower bound for the product of the standard deviations of A_2 and B_2 according to the Heisenberg uncertainty principle applied to subsystem 2. The latter is steerable by subsystem 1, then, when correlations between the two subsystems are so strong that the inferred standard deviations of subsystem 2 are able to violate the Heisenberg uncertainty relation. Thirdly, in general the steerability criterion is asymmetric: it is different depending on which of the two subsystems is measured and which is being

[13]The fixedness of the subsystems is crucial in this definition, in that the allowed canonical transformations belong to the set of *local linear transformations* $\mathrm{Sp}(2,\mathbb{R})_1 \otimes \mathrm{Sp}(2,\mathbb{R})_2$, where $\mathrm{Sp}(2,\mathbb{R})_j$ is the real symplectic group restricted to subsystem j; see Eq. (15) in Ref. [9].

[14]Schrödinger [46] coined the term *"steering"* to describe the ability to "steer" the subsystem 2 into an eigenstate of either of two noncommuting observables $\hat{A}_2, \hat{B}_2$, by choosing the measurement made on the subsystem 1.

[15]We refer the reader to the more recent work [48] for a discussion when the commutator is a q number, a scenario that is beyond the scope of the present paper. We are grateful to Iacopo Carusotto for pointing out some imprecision in our former presentation of the criterion.

"steered." Finally, steerability is a stronger entanglement criterion than nonseparability; see the forthcoming analysis and Refs. [7,10].

2. Sufficient inequalities

The mathematical definitions of nonseparability and steerability given above are too general to be straightforwardly applied to experimental data. It is then useful to identify inequalities relating observable quantities which, when violated, are sufficient to assess that the state is entangled according to one of the above criteria.

Before doing so, we remind the reader (see Eq. (B3) of [12]) that quantum mechanical settings imply that the following inequalities cannot be violated:

$$|c_{12}|^2 \le n_1(n_2+1) \le n_2(n_1+1), \tag{A4}$$

where $n_i = \langle \hat{b}_i^\dagger \hat{b}_i \rangle$ and $c_{12} = \langle \hat{b}_1 \hat{b}_2 \rangle$. We emphasize that these inequalities apply to any bipartite state, and therefore, the following entanglement criteria will be limited by them. On the other hand, measurements which violate them would imply that the usual bosonic commutation relations do not apply, or that quantum mechanics altogether is brought into question.

a. Nonseparability

A commonly used sufficient criterion for nonseparability is the generalized Peres-Horodecki (gPH) criterion, an algebraic condition on the covariance matrix of a two-state system. It is expressed by Simon [9] using the two-mode operators $\hat{X} = [\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2]$ and the covariance matrix $V_{\alpha\beta} = \langle \{ \Delta\hat{X}_\alpha, \Delta\hat{X}_\beta \} \rangle /2$, where, as usual, $\hat{q}_j = (\hat{b}_j + \hat{b}_j^\dagger)/\sqrt{2}$ and $\hat{p}_j = (\hat{b}_j - \hat{b}_j^\dagger)/(\sqrt{2}i)$ for each subsystem. A necessary condition for separability is

$$V + \frac{i}{2}\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \ge 0, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A5}$$

That is, if the state is separable, the operator on the left-hand side of (A5) must be positive-semidefinite. Violation of the inequality is therefore a sufficient condition for the state to be nonseparable.[16] It can be written in the equivalent form $\mathcal{P} \ge 0$, where $\mathcal{P}$ is a scalar function of the elements of the covariance matrix [9]. Much of the literature works directly with the function $\mathcal{P}$, see e.g. Figure 4 of [15]. This has the advantage of generality, but at the expense of obtuseness of the expressions.

If we restrict ourselves to states for which the two-point function $G^{(2)}(t, x; t', x')$ is either homogeneous (when the background is homogeneous) or stationary (when the background is stationary), the gPH criterion can be simplified. Expectation values such as $\langle \hat{b}_i^2 \rangle$ or $\langle \hat{b}_i^\dagger \hat{b}_j \rangle$ (where $i \ne j$) must then vanish, and the gPH criterion for nonseparability takes the form (see Appendix B of [12])

$$\mathcal{P} = ((n_1+1)(n_2+1) - |c_{12}|^2)(n_1 n_2 - |c_{12}|^2) < 0, \tag{A6}$$

where $n_i = \langle \hat{b}_i^\dagger \hat{b}_i \rangle$ and $c_{12} = \langle \hat{b}_1 \hat{b}_2 \rangle$. Using the inequalities of Eq. (A4), one verifies that the first term in brackets in (A6) is necessarily positive. For the class of states here considered, the gPH criterion is thus equivalent to

$$n_1 n_2 - |c_{12}|^2 < 0, \tag{A7}$$

which is exactly condition (1). Interestingly, for *all* two-mode states, (A7) is sufficient for the gPH criterion to be satisfied, and hence for the state to be nonseparable (see Table 1 of [20], and Appendix B of [12]).

When assessing nonseparability in the homogeneous case (see Sec. II C), we use another sufficient criterion, namely inequality (16). Given the form taken by $G^{(2)}(k, t)$ in a homogeneous state [see Eq. (11)], this is equivalent to

$$\frac{n_1 + n_2}{2} - |c_{12}| < 0. \tag{A8}$$

That is, the "product" condition (A7) is replaced by the "sum" condition (A8). In the isotropic case $n_1 = n_2$, the proof of the sufficiency of the sum condition is immediate, for it is then equivalent to the product condition; whereas in the anisotropic case $n_1 \ne n_2$, the inequality

$$\left(\frac{n_1 + n_2}{2}\right)^2 - n_1 n_2 = \left(\frac{n_1 - n_2}{2}\right)^2 > 0 \tag{A9}$$

guarantees that (A8) implies (A7). The sum condition (A8) is thus a sufficient condition for nonseparability that can be accessed directly via measurements of $G^{(2)}(k, t)$ of Eq. (11).

b. Steerability

For the subclass of homogeneous or stationary states, using Ref. [7] a sufficient condition for steerability can be shown to be

$$\Delta_{\rm steer}^{1\to 2} \equiv n_2\left(n_1 + \frac{1}{2}\right) - |c_{12}|^2 < 0. \tag{A10}$$

Note that it is asymmetric with respect to the exchange of n_1 and n_2, which reflects the fact that it depends on which subsystem is being steered by the other. In the present case, it is subsystem 2 which is steered by subsystem 1, hence the arrow in the superscript of $\Delta_{\rm steer}^{1\to 2}$. Note that for Gaussian states, (A10) is also a necessary criterion.

As for nonseparability, there is a sum condition for steerability that is sufficient for inequality (A10) to be satisfied. To derive it, we use the inequality

[16] In fact, if the state $\hat{\rho}_{1,2}$ is Gaussian, violation of inequality (A5) is also necessary for the nonseparability of the state.

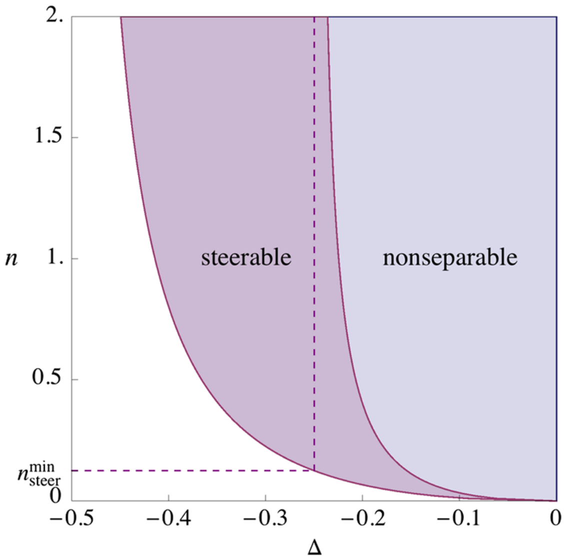

FIG. 5. Entangled states. Assuming either homogeneity or stationarity of the two-mode state, along with isotropy $n_1 = n_2 \equiv n$, the color in the (n, Δ)-plane (where $\Delta \equiv n - |c|$) determines the degree of entanglement of a Gaussian state (see footnote 16). The white region is physically inaccessible, since it does not conform to (A4). The entire shaded region corresponds to physical states for which $\Delta < 0$, and thus to nonseparable states; since we have assumed isotropy, the product and sum conditions (A7) and (A8) are equivalent. Within this region, the darker shaded region corresponds to physical states for which (A10) is satisfied. The dashed line corresponds to the strong steerability condition (A12), and it crosses the outermost limit of physical states at $n = n_{\rm steer}^{\rm min}$ of Eq. (A13).

$$\left(\frac{n_1+n_2}{2}+\frac{1}{4}\right)^2 - n_2\left(n_1+\frac{1}{2}\right) = \left(\frac{n_1-n_2}{2}+\frac{1}{4}\right)^2 \geq 0, \tag{A11}$$

which guarantees that (A10) is satisfied whenever

$$\frac{n_1+n_2}{2} - |c_{12}| < -\frac{1}{4}. \tag{A12}$$

Unlike the product condition, the sum condition is symmetric under the exchange of n_1 and n_2; thus, if it is satisfied, each of the two subsystems is steerable by the other. Notice that it has the same structure as the symmetric inequality (A8). Therefore, using Eq. (11), it can also be expressed directly in terms of $G^{(2)}(k,t)$ and $G^{(2)}_{\rm vac}(k)$, giving rise to inequality (17). This condition is represented by the dashed vertical line in Fig. 5 and by the horizontal dotted line in Fig. 2.

It should be noticed that it crosses the outermost limit of physical states with $n_1 = n_2 \equiv n$ for finite values of the mean occupation number n and the correlation strength $|c|$, i.e.

$$n_{\rm steer}^{\rm min} = \frac{1}{8}, \qquad |c|_{\rm steer}^{\rm min} = \sqrt{n_{\rm steer}^{\rm min}(n_{\rm steer}^{\rm min}+1)} = \frac{3}{8}. \tag{A13}$$

This can be understood from the fact that the realization of inequality (17) requires sufficiently large oscillations of the $G^{(2)}(k,t)$ with respect to $G^{(2)}_{\rm vac}(k)$. It is interesting to notice that it is the symmetric condition (A12) that is used in the recent work [49], where the threshold is correctly pointed out. Indeed, they measure the variances of linear combinations of operators pertaining to subsystems 1 and 2 [as in Eq. (6)], which in the language of Sec. II amounts to measuring $G^{(2)}(k,t)$ of Eq. (11).

To summarize, although steerability is originally formulated in terms of inferred variances [see Eq. (A3)], a sufficient criterion can be expressed in terms of variances of linear combinations of operators pertaining to the two subsystems. Using the law of total variance [50], this possibility follows from the inequality

$$(\Delta_{\rm inf} A_2)^2 \leq \langle(\hat{A}_1 \pm \hat{A}_2 - \langle\hat{A}_1 \pm \hat{A}_2\rangle)^2\rangle, \tag{A14}$$

which guarantees the sufficiency of Eq. (A12) and therefore of Eq. (17).

For the interested reader, we add a simple illustration of the role of the inferred variance using the Wigner function of a Gaussian isotropic ($n_1 = n_2 \equiv n$) state (for a similar analysis based on the Husimi Q distribution, see Appendix D of [11]). The state is thus completely characterized by the expectation values n and c_{12}. We further assume that the correlation term c_{12} is real, so that the quadrature operators $\hat{q}_j$ and $\hat{p}_j$ introduced above can be used in the steerability criterion (A2). Using as variables the coherent state amplitudes $u_j = (q_j + ip_j)/\sqrt{2}$, the Wigner function of the bipartite state $\hat{\rho}_{1,2}$ is given by

$$W_{1,2}(u_1,u_2) = \mathcal{N} \exp\left(-\frac{|u_1|^2}{n+1/2}\right) \times \exp\left(-\frac{|u_2-\bar{u}_2(u_1)|^2}{(\Delta_{\rm inf} q_2)^2}\right), \tag{A15}$$

where $\mathcal{N}$ is a normalization prefactor, and where we have defined

$$\bar{u}_2(u_1) = \frac{c_{12}}{n+1/2} u_1^\star, \qquad (\Delta_{\rm inf} q_2)^2 = \frac{1}{2} + \frac{\Delta_{\rm steer}^{1\to 2}}{n+1/2}. \tag{A16}$$

Straightforward symmetry arguments show that $(\Delta_{\rm inf} p_2)^2 = (\Delta_{\rm inf} q_2)^2$. The first exponential factor in Eq. (A15) is (up to a normalization prefactor) the reduced Wigner function of subsystem 1 having traced over the degrees of freedom pertaining to subsystem 2. It is characterized by a width which is the total standard deviation of $\hat{q}_1$ (or $\hat{p}_1$). The second factor is the conditional Wigner function of subsystem 2 given that a measurement of $\hat{b}_1$ yields the value u_1. It is characterized both by a conditional mean $\bar{u}_2(u_1)$ and by a width which is the inferred standard deviation of $\hat{q}_2$ (or $\hat{p}_2$) given a measurement of $\hat{q}_1$ (or $\hat{p}_1$). Subsystem 2 is steerable by subsystem 1 whenever the inferred variance is smaller than its vacuum value of $1/2$, which occurs precisely when condition (A10) is satisfied.

APPENDIX B: ADDITIONAL INFORMATION FROM OTHER TYPES OF MEASUREMENT

1. Phase fluctuations and noncommuting measurements

In addition to density fluctuations, it is interesting to study the two-point correlation functions involving the phase fluctuations $\delta\hat{\theta}$ in order to see what is the extra information about n_k and c_k that could be extracted. Using again $\hat{\Phi}(t,x) = e^{-i\mu t+iKx}(\Phi_0 + \delta\hat{\phi}(t,x))$, one has

$$\delta\hat{\phi}(t,x) = \frac{\delta\hat{\rho}}{2\sqrt{\rho_0}} + i\sqrt{\rho_0}\delta\hat{\theta}. \tag{B1}$$

Then, as in (8), it is useful to work with the spatial Fourier transform and to express it using the phonon operators. One finds

$$\delta\hat{\theta}_k = \frac{\sqrt{N}}{2i\rho_0}(u_k - v_k)(\hat{\varphi}_k - \hat{\varphi}^\dagger_{-k}). \tag{B2}$$

Much as for the $\hat{\rho}_k$, we here have $\hat{\theta}^\dagger_k = \hat{\theta}_{-k}$ and $[\hat{\theta}_k, \hat{\theta}^\dagger_k] = 0$. Therefore, the correlation $\langle|\hat{\theta}_k|^2\rangle$ is well-defined, and we have

$$\begin{aligned}\langle|\hat{\theta}_k|^2\rangle &= \langle\hat{\theta}_k\hat{\theta}_{-k}\rangle \\ &= \frac{N}{4\rho_0^2}(u_k - v_k)^2(1 + n_k + n_{-k} - 2\mathrm{Re}[c_k e^{-2i\omega_k t}]).\end{aligned} \tag{B3}$$

Comparing with Eq. (11), we see that we gain little additional information from phase measurements: indeed, apart from the minus sign in the prefactor $(u_k - v_k)^2$, the essential difference occurs in the minus sign in front of the oscillating term. Thus, the measurement is just as if we had examined the density-density correlation shifted in time by half a period.

To conclude, let us now consider the information encoded in density-phase correlations:

$$\langle\{\hat{\rho}_k, \hat{\theta}_{-k}\}\rangle = 2L(i\delta n_k + \mathrm{Im}[c_k e^{-2i\omega_k t}]), \tag{B4}$$

where curly brackets signify the anti-commutator (used here because $\hat{\rho}_k$ and $\hat{\theta}_{-k}$ do *not* commute, since they obey $[\hat{\rho}_k, \hat{\theta}_{-k'}] = iL\delta_{k,k'}$). This measurement *does* give us access to new information: the degree of anisotropy $\delta n_k = (n_k - n_{-k})/2$.

2. Anisotropy

It is useful to investigate the effects of anisotropy further. Let us define $\delta n_k \equiv (n_k - n_{-k})/2$. Then the observable accessible from measurements of $\langle\hat{\rho}_k\hat{\rho}_{-k}\rangle$ is

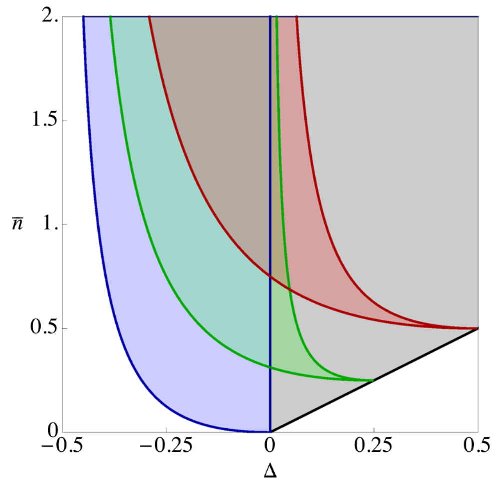

FIG. 6. Nonseparable states. Assuming either homogeneity or stationarity of the two-mode state, and if the degree of anistropy $\delta n \equiv |n_1 - n_2|/2$ is known, the point in the $(\bar{n}, \Delta)$-plane (where $\bar{n} \equiv (n_1 + n_2)/2$ and $\Delta \equiv \bar{n} - |c|$) determines the degree of entanglement of a Gaussian state (see footnote 16). The white region is inaccessible to quantum mechanical states. Within the shaded region, we have used different colors to illustrate the set of all nonseparable states when $\delta n = 0$ (blue), $1/4$ (green) and $1/2$ (red). The gray regions correspond to separable states. The left-most colored curve always represents the boundary of quantum mechanical states given the value of δn.

$$\frac{1}{2}(n_k + n_{-k}) - |c_k| = n_k - |c_k| - \delta n_k. \tag{B5}$$

On the other hand, the criterion that determines separability is $n_k n_{-k} - |c_k|^2 > 0$ or, taking the square root for ease of comparison with measurements, $\sqrt{n_k n_{-k}} - |c_k| > 0$. Substituting δn_k and Taylor expanding the square root, we find

$$\begin{aligned}\sqrt{n_k n_{-k}} - |c_k| &= n_k\sqrt{1 - 2\delta n_k/n_k} - |c_k| \\ &= n_k - |c_k| - \delta n_k - \frac{(\delta n_k)^2}{2n_k} + O((\delta n_k)^3).\end{aligned} \tag{B6}$$

Therefore, the theoretical and measurable criteria first differ at quadratic order in δn_k. That is, if there is a small degree of anisotropy, the theoretical separability threshold occurs slightly above that directly accessible to measurement: there is a small band just above the vacuum value of $\langle\hat{\rho}_k\hat{\rho}_{-k}\rangle$ where the corresponding state is nonseparable, and its thickness varies as $(\delta n_k)^2$.

Continuing in this vein, let us assume that we can place a lower bound on the degree of anisotropy, i.e. we have $|\delta n_k| > M_k > 0$ for some M_k. From what has been said so far, it is clear that there will exist states which do not satisfy $\frac{1}{2}(n_k + n_{-k}) - |c_k| < 0$, but which are nonetheless non-separable. Can we improve the sufficiency criterion in order to be able to recognize some of these states? We shall allow ourselves to collect measurements of $G^{(2)}(k,t)$ at different times, but even then, it is clear from Eq. (11) that we have experimental access only to $n_k + n_{-k}$ and $|c_k|$. The improved criterion must therefore contain only these values (in addition to the bound M_k). Let us first assume separability, i.e. $n_k n_{-k} - |c_k|^2 \geq 0$, and derive a necessary condition for this to be true; any violation of this condition will then constitute a sufficient condition for nonseparability. Firstly, we note that an equivalent separability criterion (from straightforward algebraic rearrangement) is

$$2(\delta n_k)^2 \leq \frac{1}{2}(n_k^2 + n_{-k}^2) - |c_k|^2. \tag{B7}$$

We then have

$$\begin{aligned} 4(\delta n_k)^2 &\leq 4(\delta n_k)^2 + 2(n_k n_{-k} - |c_k|^2) \\ &\leq (n_k + n_{-k})^2 - 4|c_k|^2 \\ &= (n_k + n_{-k} - 2|c_k|)(n_k + n_{-k} + 2|c_k|), \end{aligned} \tag{B8}$$

where in the first line we have used the standard criterion for separability and in the second line we have used the equivalent condition (B7). The result can be rewritten in the form

$$\frac{1}{2}(n_k + n_{-k}) - |c_k| \geq \frac{2(\delta n_k)^2}{n_k + n_{-k} + 2|c_k|}. \tag{B9}$$

Violation of this inequality is therefore sufficient to be able to assert the nonseparability of the state, and if we assume a known lower bound for $|\delta n_k| > M_k$, we can shift the nonseparability criterion to

$$\frac{1}{2}(n_k + n_{-k}) - |c_k| < \frac{2M_k^2}{n_k + n_{-k} + 2|c_k|}. \tag{B10}$$

The knowledge of δn on nonseparability is clearly illustrated in Fig. 6.

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