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Flow of the $\Box R$ Weyl anomaly

Vladimir Prochazka* and Roman Zwicky†

Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, Scotland

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An important aspect of Weyl anomalies is that they encode information on the irreversibility of the renormalization group flow. We consider, $\Delta\bar{b} = \bar{b}^{\rm UV} - \bar{b}^{\rm IR}$, the difference of the ultraviolet and infrared value of the $\Box R$-term of the Weyl anomaly. The quantity is related to the fourth moment of the trace of the energy momentum tensor correlator for theories which are conformal at both ends. Subtleties arise for nonconformal fixed points as might be the case for infrared fixed points with broken chiral symmetry. Provided that the moment converges, $\Delta\bar{b}$ is then automatically positive by unitarity. Written as an integral over the renormalization scale, flow-independence follows since its integrand is a total derivative. Furthermore, using a momentum subtraction scheme (MOM) the 4D Zamolodchikov-metric is shown to be strictly positive beyond perturbation theory and equivalent to the metric of a conformal manifold at both ends of the flow. In this scheme $\bar{b}(\mu)$ can be extended outside the fixed point to a monotonically decreasing function. The ultraviolet finiteness of the fourth moment enables us to define a scheme for the $\delta\mathcal{L} \sim b_0 R^2$-term, for which the R^2-anomaly vanishes along the flow. In the MOM- and the R^2-scheme, $\bar{b}(\mu)$ is shown to satisfy a gradient flow type equation. We verify our findings in free field theories, higher derivative theories and extend $\Delta\bar{b}$ and the Euler flow $\Delta\beta_a$ for a Caswell-Banks-Zaks fixed point for QCD-like theories to next-to-next-to leading order using a recent $\langle G^2 G^2\rangle$-correlator computation.

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I. INTRODUCTION

It is well known that moments of the correlator of the trace of the energy momentum tensor (TEMT) provide information on the flow of Weyl anomalies in theories with an ultraviolet (UV) and an infrared (IR) conformal fixed point (FP). For example the 2D Weyl anomaly $\langle T^\rho{}_\rho\rangle_{\rm CFT} = -(\beta_c/(24\pi))R$, where $T^\rho{}_\rho$ is the TEMT, R the Ricci scalar and $\beta_c = 1$ for a free scalar field, is probed by the second moment

$$\Delta\beta_c^{2D} = \beta_c^{\rm UV} - \beta_c^{\rm IR} = 3\pi \int d^2x x^2 \langle\Theta(x)\Theta(0)\rangle_c \geq 0 \qquad (1)$$

of the TEMT in flat space $T^\rho{}_\rho|_{\rm flat} \to \Theta$. This formula is Cardy's version [1] of the celebrated c-theorem [2] and $\langle\cdots\rangle_c$ stands for the connected component of the vacuum expectation value (VEV). Positivity follows reflection positivity [1] or the positivity of the spectral representation [3]. In 3D there are no Weyl anomalies on dimensional grounds but a relation analogous to (1) exists for the moment of two electromagnetic currents related to the flow of the parity anomaly [4].

In this work we exploit the finiteness conditions for 2-functions, worked out in a previous paper [5], to obtain results on 4D Weyl-anomalies. In 4D an analogous relation to (1) has been proposed in [3,6,7] and indirectly in [8],

$$\Delta\bar{b} = \bar{b}^{\rm UV} - \bar{b}^{\rm IR} = \frac{1}{2^9 3}\int d^4x x^4 \langle\Theta(x)\Theta(0)\rangle_c \geq 0, \qquad (2)$$

where $\bar{b}$ is the $\Box R$-term [7] of the Weyl or conformal anomaly [9].[1,2]

$$\begin{aligned}\langle T^\rho{}_\rho(x)\rangle &= \frac{1}{\sqrt{g}}(-\delta_{s(x)})\ln\mathcal{Z} \\ &= -(\beta_a^{\rm IR}E_4 + \beta_b^{\rm IR}H^2 + \beta_c^{\rm IR}W^2) \\ &\quad + 4\bar{b}^{\rm IR}\Box H + 4\Lambda^{\rm IR},\end{aligned} \qquad (3)$$

and H is the commonly used shorthand [10,11]

$$H \equiv \frac{1}{(d-1)}R.$$

Above $\delta_{s(x)} \equiv \frac{\delta}{\delta s(x)}$ under $g_{\alpha\beta} \to e^{-2s(x)}g_{\alpha\beta}$, and E_4, W^2 and R are the Euler, the Weyl squared and the Ricci scalar. The superscript IR indicates that all modes have been integrated

*v.prochazka@ed.ac.uk.
†roman.zwicky@ed.ac.uk.

[1] In this paper the coefficients in front of the geometric invariants are denoted by β-functions, in (dis)accordance with [10–12] ([13,14]). The association of the letters a, b and c with the geometric invariants is variable in the literature. Our notation follows Shore's review [12] in this respect.

[2] The cosmological constant $\Lambda^{\rm IR}$ may or may not be tuned to zero by an appropriate UV-counterterm. Note that the parametrization $\langle T_{\alpha\beta}\rangle = g_{\alpha\beta}\Lambda^{\rm IR} + \cdots$ is being used. In QCD-like theories, for example, the cosmological constant receives contributions from the gluon condensate $\Lambda^{\rm IR}(\mu) = \beta(\mu)/2\langle G^2\rangle_\mu$-term. This term is essential for $\frac{d}{d\ln\mu}\langle T^\rho{}_\rho\rangle = 0$ cf. Sec. III C.

out. The quantities $\beta^{\rm IR}_{a,b,c}$ are scheme-independent and determined by the IR-theory. In the case where the IR-theory is a CFT, $\beta_b^{\rm CFT}=0$ [15,16] implies that $\beta_{a,c}^{\rm CFT}\neq 0$ are to be regarded as the true Weyl anomalies. Turning our attention to the $\bar{b}^{\rm IR}$-term, the first thing to notice is that this term shifts linearly when adding local term to the UV-action (conventions as in [5])

$$\mathcal{L}^{\rm UV}\to\mathcal{L}^{\rm UV}+\frac{1}{8}\omega_0 H^2,\qquad \bar{b}^{\rm IR}\to\bar{b}^{\rm IR}-\frac{1}{8}\omega_0. \quad (4)$$

This is presumably related to regularization dependence found in explicit computations (e.g., [9,15,17–20]). The $\Box R$-term is therefore sometimes viewed as not being meaningful. One of the main points of our paper is that in physical meaningful quantities, such as $\Delta\bar{b}$ (2), this ambiguity has to cancel. For the flow, ω_0 is merely to be seen as the initial condition which does not affect the difference $\Delta\bar{b}$. A valuable result of this paper is that we show that the $\Box R$ flow is given by,

$$\Delta\bar{b}=\frac{1}{8}\int_{-\infty}^{\infty}\chi_{\theta\theta}^{\mathcal{R}_\chi}(\mu')d\ln\mu', \quad (5)$$

an integral over $\chi_{\theta\theta}^{\mathcal{R}_\chi}$, the scale derivative of the renormalized $\langle\Theta\Theta\rangle^{\mathcal{R}_\chi}$-correlator. In particular we identify a MOM-type scheme for which

$$\chi_{\theta\theta}^{\rm MOM}(\mu)=\chi_{AB}^{\rm MOM}(\mu)\beta^A(\mu)\beta^B(\mu),\qquad \chi_{AB}^{\rm MOM}(\mu)\geq 0, \quad (6)$$

with $\chi_{AB}^{\rm MOM}(\mu)$, the 4D analogue of the Zamolodchikov-metric, being a positive definite matrix along the flow. Since $\chi_{\theta\theta}^{\rm MOM}$ can be written as a $\ln\mu$-derivative it follows that $\Delta\bar{b}$ is flow-independent. The positivity of $\chi_{AB}^{\rm MOM}$ allows us to define a $\bar{b}(\mu)^{\rm MOM}$ outside the FPs as a monotonically decreasing function. Building on the observation that $\Delta\bar{b}$ is UV-finite for asymptotically-free and asymptotically-safe flows [5], $\Delta\bar{b}\geq 0$ follows from the spectral representation. Furthermore, finiteness allows us to define a scheme (R^2-scheme) for which the R^2-anomaly vanishes along the flow. In this scheme $\bar{\bar{b}}\equiv\bar{b}(\mu)^{\rm MOM}_{R^2}$ is shown to obey a gradient flow type equation. Furthermore we provide $\Delta\bar{b}$ to NNLO in QCD-like theories using a recent NNLO computation of the $\langle G^2G^2\rangle$-correlator. The latter also extends to the Euler flow $\Delta\beta_a$ since it is proportional to $\Delta\bar{b}$ up to NNLO around the Caswell-Banks-Zaks (CBZ) FP.

The paper is organized as follows. The flow properties of $\Box R$ are presented in Sec. II with the MOM- and R^2-scheme for the $\langle\Theta\Theta\rangle$-correlator and the b-coupling defined in Secs. II A and II B respectively. The main part consists of the description of the properties of $\Delta\bar{b}$, $\bar{b}^{\mathcal{R}}(\mu)$ and the Zamolodchikov-metric $\chi_{AB}^{\mathcal{R}}$ in Sec. II C followed by a discussion of the IR- and UV-convergence of the correlator indicating potential limitations. Section II, which can be considered as the main part of the paper, is summarized in Sec. II E. The explicit scheme change of the Zamolodchikov-metric from the MOM- to the MS-scheme, $\Delta\bar{b}$ at a CBZ-FP and the renormalisation of G^2 in the R^2-scheme are discussed in Secs. III A, III B and III C respectively. Free field theory computation of scalars and fermions are presented in Sec. III D and a free higher derivative computation is deferred to Appendix D. Three derivations of (2) using anomalous Ward identities (WI), an anomaly matching argument and an explicit derivation in QCD-like theories are provided in Appendices A 1, A 2, and A 3 respectively. The antisymmetric part of the gradient flow equation is elaborated on in Appendix B. Comments on different ways of handling the gravity counterterms are discussed in Appendix C followed by the computation of $\Box R$-flow in a higher derivative theory in Appendix D. Conventions of the QCD β-function and the CBZ-FP are specified in Appendix E.

II. THE FLOW OF $\Box R$ (OR $\Delta\bar{b}$)

Before writing the fourth moment (2) in terms of the 4D Zamolodchikov-metric and showing positivity, monotonicity, and the gradient flow type property, we specify some definitions, notations and assumptions of this paper. We work with the conventions of a Euclidean field theory and assume the operator-part of the TEMT to be of the form $\Theta=\beta^Q[O_Q]$ (summation over Q is implied). We refer the reader our previous work [5] regarding the terminology of the TEMT $T^\rho{}_\rho$ which splits into an operator, equation of motion and gravity part. At the exception of the free field theory examples in Sec. III D dimensionless couplings are assumed. The bare interaction Lagrangian is parametrized by $\mathcal{L}=g_0^Q O_Q$, g_0^Q are couplings and $[O_Q]=\mathcal{Z}_Q{}^P O_P$ denote renormalized (composite) operators defined by the local quantum action principle (QAP) $\langle[O_A(x)]\rangle=(-\delta_{A(x)})\ln\mathcal{Z}$ where $\mathcal{Z}$ and $\delta_{A(x)}\equiv\frac{\delta}{\delta g^A(x)}$ are the partition function and the variational derivative of the localized coupling respectively. Curved space is a tool to expose the Weyl anomalies (3) at lower-point functions and has no further physical meaning in this work.

The object of study is the $\langle\Theta\Theta\rangle$-correlator ($\Theta=[\Theta]$ since it is physical e.g., [21,22])[3];

$$\Gamma_{\theta\theta}(p)=\int d^4x e^{ix\cdot p}\langle\Theta(x)\Theta(0)\rangle_c$$
$$=\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p)p^4+c_T\langle\Theta\rangle \quad (7)$$

[3]The restrictive structure of (7) follows from the flat-space translational WI $\int d^4xe^{ip\cdot x}\langle\Theta_{\alpha\beta}(x)\Theta_{\gamma\delta}(0)\rangle_c=P^{(0)}_{\alpha\beta\gamma\delta}\Gamma^{(0)}+P^{(2)}_{\alpha\beta\gamma\delta}\Gamma^{(2)}+P^{(CT)}_{\alpha\beta\gamma\delta}\langle\Theta\rangle$. From the traces of the spin 0 and 2 structures, $P^{(0)}_{\alpha\gamma\alpha\gamma}\sim p^4$ and $P^{(2)}_{\alpha\gamma\alpha\gamma}=0$ (note $P^{(CT)}_{\alpha\gamma\alpha\gamma}=c_T$), one infers that $\Gamma^{(0)}(p)\sim\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p)$.

for which $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p)$ is UV-finite [5] and IR-finite for $p > 0$ with subtleties for $p \to 0$ for theories spontaneously broken chiral symmetry to be discussed in Sec. II D 1. The contact term (CT) c_T is of no relevance for the flow itself, the scalar product is defined as usual $a \cdot b \equiv a_\alpha b^\alpha$ and $p \equiv \sqrt{p \cdot p}$. Defining

$$M^{(2)}_{\theta\theta}(p) = \hat{P}_2 \Gamma_{\theta\theta}(p) = \frac{1}{2^6 3} \int d^4x e^{ix\cdot p} x^4 \langle \Theta(x)\Theta(0)\rangle_c, \tag{8}$$

with

$$\hat{P}_2 = \frac{1}{2^6 3}(\partial_{p_\alpha}\partial_{p^\alpha})^2, \qquad \hat{P}_2 p^4 = 1, \tag{9}$$

the fourth moment (2) is then proportional to $M^{(2)}_{\theta\theta}(0)$. The bare quantities $M^{(2)}_{\theta\theta}(p)$ and $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p)$ satisfy unsubtracted Källén-Lehmann spectral/dispersion representations of the form[4];

$$\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p) = \int_0^\infty \frac{ds}{s^2} \frac{\rho(s)}{(s+p^2)} + \mathbb{C}^{\mathbb{1}}_{\theta\theta}(\infty), \tag{10}$$

$$M^{(2)}_{\theta\theta}(p) = \int_0^\infty ds \frac{s(s-p^2)\rho(s)}{(s+p^2)^5} + M^{(2)}_{\theta\theta}(\infty), \tag{11}$$

where the spectral function $\rho(s)$ is of mass dimension four and defined as a formal sum over a complete set of spin 0 physical states,

$$\rho(s) = (2\pi)^3 \sum_n \theta((p_n)_0)\delta(p_n^2 - s)|\langle n(p_n)|\Theta|0\rangle|^2 \geq 0, \tag{12}$$

with p_n denoting momenta in Minkowski-space and $\theta(x)$ is the step-function.

From the representations Eqs. (10), (11) one deduces that $M^{(2)}_{\theta\theta}(p) - \mathbb{C}^{\mathbb{1}}_{\theta\theta}(p)$ is a finite p-dependent function for which $M^{(2)}_{\theta\theta}(0) - M^{(2)}_{\theta\theta}(\infty) = \mathbb{C}^{\mathbb{1}}_{\theta\theta}(0) - \mathbb{C}^{\mathbb{1}}_{\theta\theta}(\infty)$ holds. Furthermore, using and with Eqs. (7), (8) it follows that

$$M^{(2)}_{\theta\theta}(0) = \mathbb{C}^{\mathbb{1}}_{\theta\theta}(0), \qquad M^{(2)}_{\theta\theta}(\infty) = \mathbb{C}^{\mathbb{1}}_{\theta\theta}(\infty). \tag{13}$$

Together with (2) this implies

$$\Delta\bar{b} = \frac{1}{8} M^{(2)}_{\theta\theta}(0), \tag{14}$$

which modifies to $\Delta\bar{b} = \frac{1}{8}(M^{(2)}_{\theta\theta}(0) - M^{(2)}_{\theta\theta}(\infty))$ in the case where there is a finite contribution at infinity. This is for instance necessary when adding a term (4) to the UV Lagrangian as discussed towards the end of Sec. II A 1. The reason for introducing $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p)$ is that, contrary to $M^{(2)}_{\theta\theta}(p)$, it is monotonic in p allowing us to define a positive Zamolodchikov-metric in the MOM-scheme (cf. Sec. II A 1). We stress that $M^{(2)}_{\theta\theta}(0)$ is a bare, μ-independent, quantity and in the case where it is IR- and UV-finite (cf. Sec. II D) it therefore qualifies as a physical observable. Three different derivations of (2) are given in Appendices A 1, A 2 and A 3.

A. Generic scheme-definition for the ⟨ΘΘ⟩-Correlator

In order to perform a RG-analysis, the bare term in (8) is written as a sum of a renormalized term and a counterterm,

$$M^{(2)}_{\theta\theta}(g^{\mathcal{Q}}(p/\mu_0)) = M^{(2),\mathcal{R}}_{\theta\theta}(p/\mu, g^{\mathcal{Q}}(\mu/\mu_0)) + L^{\mathbb{1},\mathcal{R}}_{\theta\theta}(g^{\mathcal{Q}}(\mu/\mu_0)). \tag{15}$$

Above μ_0 is some reference scale and $M^{(2),\mathcal{R}}_{\theta\theta}(p/\mu, a_s(\mu)) = M^{(2),\mathcal{R}}_{\theta\theta}(p/\mu_0, \mu/\mu_0, a_s(\mu))$ but most of the time $M^{(2)}_{\theta\theta}(p)$, $M^{(2),\mathcal{R}}_{\theta\theta}(p,\mu)$ and $L^{\mathbb{1},\mathcal{R}}_{\theta\theta}(\mu)$ are used as shorthands. Since $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p) - M^{(2)}_{\theta\theta}(p)$ is finite one may use the same renormalization prescription for $\mathbb{C}^{\mathbb{1}}_{\theta\theta}$

$$\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p) = \mathbb{C}^{\mathbb{1},\mathcal{R}}_{\theta\theta}(p,\mu) + L^{\mathbb{1},\mathcal{R}}_{\theta\theta}(\mu), \tag{16}$$

connecting with the notation in our previous work [5]. Crucially, it is the choice (15) of splitting the bare correlation function into a nonlocal renormalized part $M^{(2),\mathcal{R}}_{\theta\theta}$ and a local part $L^{\mathbb{1},\mathcal{R}}_{\theta\theta}$ (counterterm) which defines a scheme $\mathcal{R}$ and introduces a renormalization scale μ.[5] The anomalous part of the equation above is

$$\begin{aligned}\chi^{\mathcal{R}}_{\theta\theta}(\mu) &= \left(\frac{d}{d\ln\mu} - 2\epsilon\right) M^{(2),\mathcal{R}}_{\theta\theta} \\ &= \left(\frac{d}{d\ln\mu} - 2\epsilon\right) \mathbb{C}^{\mathbb{1},\mathcal{R}}_{\theta\theta}(p,\mu) \\ &= -\left(\frac{d}{d\ln\mu} - 2\epsilon\right) L^{\mathbb{1},\mathcal{R}}_{\theta\theta}(\mu),\end{aligned} \tag{17}$$

[4]The dispersion relation for the correlation function (7) reads $\Gamma^{\mathcal{R}}_{\theta\theta}(p) = \int_0^\infty ds \frac{\rho(s)}{s+p^2} + \omega^{\mathcal{R}}_4(\mu) + \omega^{\mathcal{R}}_2(\mu)p^2 + \omega_0 p^4$. The constants $\omega^{\mathcal{R}}_{2,4}(\mu)$ take care of the quadratic and quartic divergences whereas the logarithmic part is convergent and ω_0 is therefore a true constant independent of μ. Eq. (10) and (11) are obtained from the ones above by using $p^4 \mathbb{C}^{\mathbb{1}}_{\theta\theta}(p) = \Gamma_{\theta\theta}(p) - \Gamma_{\theta\theta}(0) - p^2 \frac{d}{dp^2}\Gamma_{\theta\theta}(0)$ and $M^{(2)}_{\theta\theta}(p) = \hat{P}_2\Gamma_{\theta\theta}(p)$.

[5]In perturbation theory the counterterm is a Laurent series in ϵ and requires the scale μ. Nonperturbatively the scale p is identified with μ cf. next section. Moreover, in what follows $\mathcal{R}$ refers to the split (15) and we do not specify the renormalization of the couplings and operators, linked by the quantum action principle, other than assuming a mass-independent scheme.

the quantity entering (5) and related to the R^2-anomaly [5,10,11] [Eq. (48) of the 3rd reference]. The μ-dependence arising through the coupling $\chi^{\mathcal{R}}_{\theta\theta}(\mu) = \chi^{\mathcal{R}}_{\theta\theta}(g^Q(\mu))$. In both equations above the $\epsilon \to 0$ limit is smooth and we do therefore not distinguish between a four and d-dimensional $\chi^{\mathcal{R}}_{\theta\theta}$ and adapt the same attitude to other quantities.

1. Definition of a MOM-scheme for the 2-point function

Below we define a scheme which is most effectively imposed on $\mathbb{C}^{\mathbb{1}}_{\theta\theta}$ rather than $M^{(2),\mathcal{R}}_{\theta\theta}$. The renormalization condition is

$$\mathbb{C}^{\mathbb{1},\mathrm{MOM}}_{\theta\theta}(p,\mu)|_{p=\mu} = 0, \tag{18}$$

that the renormalized two-point function equals zero at $p = \mu$ (recall $p \equiv \sqrt{p^2}$) which is straightforwardly implemented by

$$\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p) = \underbrace{(\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p) - \mathbb{C}^{\mathbb{1}}_{\theta\theta}(\mu))}_{\mathbb{C}^{\mathbb{1},\mathrm{MOM}}_{\theta\theta}(p,\mu)} + \underbrace{\mathbb{C}^{\mathbb{1}}_{\theta\theta}(\mu)}_{L^{\mathbb{1},\mathrm{MOM}}_{\theta\theta}(\mu)}. \tag{19}$$

This is equivalent to the so-called MOM-scheme (and variations thereof), introduced for lattice Monte-Carlo simulations [23], where the renormalized momentum space correlation function is set to its tree-level value for some momentum configuration set to equal μ. A solution to Eqs. (17), (18) is given by

$$\mathbb{C}^{\mathbb{1},\mathrm{MOM}}_{\theta\theta}(p,\mu) = \int_{\ln p/\mu_0}^{\ln \mu/\mu_0} \chi^{\mathrm{MOM}}_{\theta\theta}(\mu') d\ln\mu', \tag{20}$$

and therefore

$$\begin{aligned}\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p) &= \underbrace{\int_{\ln p/\mu_0}^{\ln \mu/\mu_0} \chi^{\mathrm{MOM}}_{\theta\theta}(\mu') d\ln\mu'}_{\mathbb{C}^{\mathbb{1},\mathrm{MOM}}_{\theta\theta}(p,\mu)} \\ &\quad + \underbrace{\int_{\ln \mu/\mu_0}^{\infty} \chi^{\mathrm{MOM}}_{\theta\theta}(\mu') d\ln\mu' + \mathbb{C}^{\mathbb{1}}_{\theta\theta}(\infty)}_{L^{\mathbb{1},\mathrm{MOM}}_{\theta\theta}(\mu)} \\ &= \int_{\ln p/\mu_0}^{\infty} \chi^{\mathrm{MOM}}_{\theta\theta}(\mu') d\ln\mu' + \mathbb{C}^{\mathbb{1}}_{\theta\theta}(\infty).\end{aligned} \tag{21}$$

Together with (14) this implies Eq. (5) in the MOM-scheme and allows us to obtain $\chi^{\mathrm{MOM}}_{\theta\theta}(\mu)$ from $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p)$ as follows

$$\chi^{\mathrm{MOM}}_{\theta\theta}(\mu) = -\frac{d}{d\ln p}\bigg|_{p=\mu} \mathbb{C}^{\mathbb{1}}_{\theta\theta}(p). \tag{22}$$

Since the Lie derivative with respect to the β-function vector field commutes with the β-functions themselves (cf. Sec. II C 3 for more details)

$$\chi^{\mathcal{R}}_{\theta\theta} = \beta^A\beta^B\chi^{\mathcal{R}}_{AB} \tag{23}$$

holds. Together with p-independence of the β-functions this implies in the MOM-scheme

$$\chi^{\mathrm{MOM}}_{AB}(\mu) = -\frac{d}{d\ln p}\bigg|_{p=\mu} \mathbb{C}^{\mathbb{1}}_{AB}(p,\mu). \tag{24}$$

Above

$$\begin{aligned}\Gamma_{AB}(p,\mu) &= \int d^4x e^{ip\cdot x}\langle[O_A(x)][O_B(0)]\rangle_c \\ &= p^4\mathbb{C}^{\mathbb{1}}_{AB}(p,\mu) + \cdots\end{aligned} \tag{25}$$

in analogy with (8) where the μ-dependence comes from the renormalization of $[O_{A,B}]$. Eq. (24) is consistent with the representation of the Zamolodchikov-metric in conformal field theories (CFTs) $\mathbb{C}^{\mathbb{1}}_{AB}(p,\mu) = -\chi^{\mathrm{MOM}}_{AB}(\mu)\ln(p/\mu_0) +$ const (e.g., [24]) where the coupling space is referred to as a conformal manifold. The difference is that we consider the Zamolodchikov-metric flowing between two FPs rather than in a CFT only. Transformation under scheme changes for $\chi^{\mathrm{MOM}}_{\theta\theta}$ and χ^{MOM}_{AB} are discussed in Sec. II C 3. The formulas of this section allow us to clarify that (14) invariant under (4) is to be adapted to

$$\Delta\bar{b} = \frac{1}{8}(M^{(2)}_{\theta\theta}(0) - M^{(2)}_{\theta\theta}(\infty)). \tag{26}$$

In order to see this note that (13) still holds under (4), $M^{(2)}_{\theta\theta}(\mu) \to M^{(2)}_{\theta\theta}(\mu) + \omega_0$, and that in (26) the arbitrary ω_0 simply cancels in the difference on the right-hand side (RHS).

2. Positivity of the Zamolodchikov-metric in the MOM-scheme

From the positivity of the spectral function $\rho(s) \geq 0$ and (10) it follows that $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(p)$ strictly increasing when p decreases. This in turn with (21) implies that

$$\chi^{\mathrm{MOM}}_{\theta\theta}(\mu) > 0 \quad \text{for } \mu \geq 0. \tag{27}$$

From the spectral representation of $\mathbb{C}^{\mathbb{1}}_{AB}$ and (24) it follows that the Zamolodchikov-metric χ^{MOM}_{AB} itself,

$$\chi^{\mathrm{MOM}}_{AB}(\mu) > 0 \quad \text{for } \mu \geq 0, \tag{28}$$

is also a positive matrix along the flow. In both cases strict positivity is tied to nontrivial unitary theories. Note that even if the spectral representation of $\mathbb{C}^{\mathbb{1}}_{AB}$ had a logarithmic divergence then it would vanish under the p-derivative.

In 2D a positive definite Zamolodchikov-metric has been defined by Osborn [25] through the Weyl consistency relations and later in [26] via a derivative of a configuration space cutoff. Our definitions seem more closely related to the latter than the former. We are not aware of a direct extension of the definitions in [25,26] to 4D. However, such a question has been raised in the review [27] without any detailed analysis.

B. A scheme for which the R^2-anomaly (or β_b) vanishes along the flow

The general formalism allows us to define different schemes for different couplings by splitting the bare coupling into a renormalized and counterterm part. This applies in particular to gravity couplings, related to vacuum graphs,

$$\mathcal{L}_{\text{gravity}} = -(a_0 E_4 + b_0 H^2 + c_0 W^2). \tag{29}$$

Below we define scheme for b_0, named R^2-scheme, for which $\beta_b = 0$ outside the FP *and* for which $\bar{b}$ is governed by a gradient flow type equation. It is noted that this is *a priori* possible since $\beta_b = 0$ for CFTs [15,16] which define the endpoints of the flow. At the technical level $\beta_b = 0$ is established by the remarkable link between $\langle\Theta\ldots\Theta\rangle$-correlators and the gravity terms (29) by the QAP e.g., [10–12].

We find it helpful to think of b_0 as the coupling of the R^2-term similar to the role of the QCD-coupling and the field strength tensor squared G^2. Although the R^2-term is not quantized itself, $b(\mu)$ runs since it mixes with other dynamical operators, e.g., the G^2-term in QCD-like theories. The key observation is that the UV-finiteness of the fourth moment [or $\mathbb{C}^{\mathbb{1}}_{\theta\theta}(0)$] (8) then allows to absorb this finite part into the renormalization of G^2 in which case $\beta_b = 0$ along the flow.

In order to make this statement transparent it proves useful to briefly digress and clarify the effect of the choice of scheme for a coupling g^Q on the conjugate renormalized composite operator $[O_Q]$. A choice of scheme $\mathcal{R}_1$ is given, as usual, by a separation of the bare coupling into a renormalized coupling $g^{Q,\mathcal{R}_1}(\mu)$ and counterterm $L_Q^{\mathcal{R}_1}(\mu)$

$$g_0^Q = \mu^{d-4}(g^{Q,\mathcal{R}_1}(\mu) + L_Q^{\mathcal{R}_1}(\mu)). \tag{30}$$

For clarity let us mention that we have previously suppressed the $\mathcal{R}_1$-label when talking about dynamical couplings. The bare couplings are independent of the RG-scale, $\frac{d}{d\ln\mu} g_0^Q = 0$, and $L_Q^{\mathcal{R}_1}(\mu)$ therefore determines $g^{Q,\mathcal{R}_1}(\mu)$ up to a constant which has to be determined experimentally. The local QAP defines the renormalized composite operator by

$$\langle [O_Q(x)]^{\mathcal{R}_1}_{\mathcal{R}_2}\rangle = (-\delta_{g^Q(x)})|^{g^A = g^{A,\mathcal{R}_1}}_{v = v^{\mathcal{R}_2}} \ln \mathcal{Z}, \tag{31}$$

where $v = a, b, c$ from (29) and g^A are generic couplings. In principle one may choose different schemes for different couplings and parameters which leads to a proliferation of scheme dependences on the left-hand side (LHS).

Returning to our task we define the coupling

$$b_0 = \mu^{d-4}(b^{\mathcal{R}_b} + L_b^{\mathcal{R}_b}), \tag{32}$$

in analogy with (30) and assume a renormalisation scheme $\mathcal{R}_\chi$ for the $\langle\Theta\Theta\rangle$-correlator.[6] A double variation of the metric ($g_{\mu\nu} \to e^{-2s(x)} g_{\mu\nu}$) is finite since both the partition function and the metric are finite. When Fourier transformed and projected on the p^4-structure one obtains

$$\begin{aligned} &\int d^d x e^{ip\cdot x}((-\delta_{s(x)})(-\delta_{s(0)}) \ln \mathcal{Z})|_{p^4} \\ &= \int d^d x e^{ip\cdot x}\langle\Theta(x)\Theta(0)\rangle|_{p^4} + 8 b_0 = [\text{finite}]. \end{aligned} \tag{33}$$

This implies the nontrivial, known, relation

$$L_b^{\mathcal{R}_b} = -\frac{1}{8} L_{\theta\theta}^{\mathbb{1},\mathcal{R}_\chi} + [\text{finite}], \tag{34}$$

quoted for in the MS-scheme in [28]. The difference in signs in (34) is somewhat unfortunate but imposes itself in this sector cf. [5] for more detailed remarks.

The observation that the finiteness of $L_{\theta\theta}^{\mathbb{1},\mathcal{R}_\chi}$ implies the finiteness of $L_b^{\mathcal{R}_b}$ can be used to define a scheme, which we call R^2-scheme, $b_0 = \mu^{d-4}(b^{R^2} + L_b^{R^2})$ with

$$b^{R^2} = b + L_b, \qquad L_b^{R^2} = 0. \tag{35}$$

This is equivalent to saying that it is not necessary to renormalize since there are no divergences. In the R^2-scheme we therefore have that

$$\beta_b^{R^2}(\mu) = -\left(\frac{d}{d\ln\mu} - 2\epsilon\right) L_b^{R^2} = 0, \qquad \mu \geq 0. \tag{36}$$

This means that the b^{R^2}-coupling does not receive RG-running by other dynamical operators.[7] All that remains is to determine the previously mentioned unknown constant by experiment. The VEV of the TEMT, $\langle T^\rho{}_\rho\rangle$, is of course invariant under scheme-changes as illustrated in Sec. III C for QCD-like theories.

[6] We comment on other ways of handling the R^2-term in the literature in Appendix C.

[7] Where the characterization "other" refers to the fact that the R^2-gravity term is not quantized and therefore does not contribute to the running of the b^{R^2}-coupling. Whether or not in such a case a scheme exist where the R^2-coupling does not run is beyond the scope of investigations of this work. This question can be posed in a well-defined framework, modulo ghosts due to higher derivatives, since R^2-gravity has been shown to be renormalizable [29].

Before continuing towards the flow of $\Box R$-term we digress in discussing whether or not schemes could exist for which the other Weyl-anomaly (3) vanish along the flow. An *a priori* no-go argument is that, unlike the R^2-anomaly, the other anomalies have generically a non-zero flow difference. We consider two types of gravitational trace anomalies (cf. [30] for a more refined discussion without inclusion of $\Box R$ though):

(1) β-functions terms. For the $\beta_{a,c}$-function terms, the analogous argument as above would require L_a and L_c to be finite.
   (a) $\beta_a E_4$-term: The counterterm of E_4 has been shown to be finite only when multiplied by ϵ [31]. This is typical for topological terms since their nontotal derivative parts are necessarily evanescent. The local QAP then implies finiteness constraints on ϵL_x where L_x is the counterterm associated with the topological invariant. Since L_a is not finite we conclude that there does not exist a scheme where β_a can be set to zero along the flow.
   (b) $\beta_c W^2$-term: The W^2 term is associated with the spin 2 part of the $\langle\Theta_{\rho\sigma}\Theta_{\lambda\nu}\rangle$-correlator. The latter is generically divergent in the relevant structure contrary to the $\langle\Theta\Theta\rangle$-correlator. The essential point is that the TEMT is protected in the UV by the additional couplings originating from the dynamical β-functions. For example in QCD-like theories $\Theta \sim \beta G^2 + \cdots$ whereas $\Theta_{\rho\sigma} = \frac{1}{4} g_{\rho\sigma} G^2 - G_{\rho\alpha} G^{\alpha}_{\sigma} + \cdots$. In the convergence criterium for asymptotically free theories in [5], this means that $n_{\Theta\Theta} = 2$ and $n_{\Theta_{\rho\sigma}\Theta_{\lambda\nu}} = 0$ which satisfies and violates the convergence criteria in Sec. III.1 of this reference. Hence we conclude that L_c is not finite when the regulator is removed and β_c can therefore not be set to zero.

(2) $\bar{b}\Box R$-term: Is not a β-function term and therefore does not derive from (36). Thus the same trick is not applicable.

C. Properties of $\Delta\bar{b}$, $\bar{b}^{\mathcal{R}_\chi}_{\mathcal{R}_b}(\mu)$ and the Zamolodchikov-metric $\chi^{\mathcal{R}_\chi}_{AB}$

Clarifying the properties of the quantities $\Delta\bar{b}$, $\bar{b}^{\mathcal{R}_\chi}_{\mathcal{R}_b}(\mu)$ and $\chi^{\mathcal{R}_\chi}_{AB}$ is linked to understanding their scheme dependences. The following hierarchy or degree of complication emerges. The global flow $\Delta\bar{b}$ (Sec. II C 1) is scheme-independent. The local flow properties, discussed in Sec. II C 2, are scheme-dependent. The infinitesimal change along the flow $\frac{d}{d\ln\mu}\bar{b}^{\mathcal{R}_\chi} = \frac{1}{8}\chi^{\mathcal{R}_\chi}_{AB}\beta^A\beta^B$ (A3) is dependent on the $\mathcal{R}_\chi$-scheme and the local value $\bar{b}^{\mathcal{R}_\chi}_{\mathcal{R}_b}(\mu)$ is dependent on both the $\mathcal{R}_\chi$- and $\mathcal{R}_b$-scheme.

1. Properties of $\Delta\bar{b}$ (global flow)

Let us summarize the various ways in which $\Delta\bar{b}$ (14) can be expressed as an integral using (2), (10) and (21)[8]

$$\Delta\bar{b} = \frac{1}{8}(M^{(2)}_{\theta\theta}(0) - M^{(2)}_{\theta\theta}(\infty))$$
$$= \frac{1}{2^9 3}\int d^4x x^4 \langle\Theta(x)\Theta(0)\rangle_c^{\omega_0} \quad (37)$$
$$= \frac{1}{8}\int_{-\infty}^{\infty}\chi^{\mathcal{R}}_{\theta\theta}(\mu')d\ln\mu' \quad (38)$$
$$= \frac{1}{8}\int_0^{\infty} ds\frac{\rho(s)}{s^3} > 0. \quad (39)$$

The following properties are immediate

(i) *Positivity:* $\Delta\bar{b} > 0$ follows from the positivity of the spectral function $\rho(s) \geq 0$ as well as the positivity of the Zamolodchikov-metric in the MOM-scheme (28). Since $\Theta_{\rm CFT} \to 0$ and therefore $\Delta\bar{b}|_{\rm CFT} = 0$, a nonzero value measures the departure from conformality. Note that the $\Theta(x)\Theta(0)$-correlator can be interpreted as a probe that records a response of a theory with couplings $g^A(\mu = x^{-1})$.

(ii) *Scheme-independence* of $\Delta\bar{b}$ follows from the scheme-independence of the spectral function $\rho(s)$ and the fact that the spectral representation does not require subtractions. Similarly since $\Delta\bar{b}$ can be expressed in terms of a bare correlation function (37) the scheme-independence of the latter implies scheme-independence of $\Delta\bar{b}$. Further remarks on scheme dependence and independence can be found in Sec. II C 3.

(iii) *Flow-independence* follows from combing Eqs. (17) and (A3) into

$$\frac{d}{d\ln\mu}\bar{b}^{\mathcal{R}_\chi} = \frac{1}{8}\frac{d}{d\ln\mu}\mathbb{C}^{\mathbb{1},\mathcal{R}_\chi}_{\theta\theta}(p,\mu), \quad (40)$$

which shows that the flow of $\bar{b}^{\mathcal{R}_\chi}$ derives from a potential and is therefore independent of the flow itself. More explicitly this equation, when integrated over $d\ln\mu$ and particularised to the MOM-scheme, gives

$$\Delta\bar{b} = \frac{1}{8}(\mathbb{C}^{\mathbb{1},\rm MOM}_{\theta\theta}(p,\infty) - \mathbb{C}^{\mathbb{1},\rm MOM}_{\theta\theta}(p,0))$$
$$\overset{(19)}{=} \frac{1}{8}(\mathbb{C}^{\mathbb{1}}_{\theta\theta}(0) - \mathbb{C}^{\mathbb{1}}_{\theta\theta}(\infty))$$
$$\overset{(13)}{=} \frac{1}{8}(M^{(2)}_{\theta\theta}(0) - M^{(2)}_{\theta\theta}(\infty)), \quad (41)$$

[8]Formally $\langle\Theta(x)\Theta(0)\rangle_c^{\omega_0} = \langle\Theta(x)\Theta(0)\rangle_c - \omega_0\Box^2\delta(x)$ where $\langle\Theta(x)\Theta(0)\rangle_c$ is evaluated by any regulator respecting the symmetries and $\omega_0 = M^{(2)}_{\theta\theta}(\infty)$ is assumed for definiteness. The regulator $\mathcal{R}$ can be removed smoothly since the moment is UV-finite.

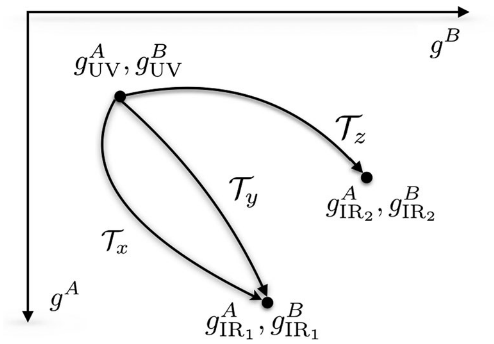

FIG. 1. Possible RG-flow trajectories from an UV-FP $g^A_{\rm UV}, g^B_{\rm UV}$ to an IR-FPs. The trajectories $\mathcal{T}_{x,y}$ and $\mathcal{T}_z$ flow into the IR$_1$- and IR$_2$-FPs respectively. Hence $\Delta\bar{b}_{\mathcal{T}_x} = \Delta\bar{b}_{\mathcal{T}_y} \neq \Delta\bar{b}_{\mathcal{T}_z}$ with the last statement being the generic case.

Eq. (26). Hence this derivation provides an alternative to the one presented in Sec. II A 1. Flow-independence only poses itself for two or more couplings, as illustrated in Fig. 1, and translates in our case to the question whether (the difference of) 2-point functions can depend on the approach in coupling space. Local reversibility of RG-flows implies that this cannot be the case. If one assumes for example that the RG-flow can be linearised around a FP then the limit is automatically uniform and the flow therefore independent of the path.

Equivalently flow-independence can be obtained by rewriting (38) as line integral of a vector $V^{\mathcal{R}}_B$ over coupling space

$$\Delta\bar{b} = \frac{1}{8}\int_{-\infty}^{\infty} \beta^A\beta^B\chi^{\mathcal{R}}_{AB} d\ln\mu' = \frac{1}{8}\int_{\vec{g}_{\rm IR}}^{\vec{g}_{\rm UV}} V^{\mathcal{R}}_B dg^B. \quad (42)$$

Path-independence follows from $V^{\mathcal{R}}_B$ being curl-free which is true if and only if $V^{\mathcal{R}}_B$ derives from a potential $V^{\mathcal{R}}_B = -\partial_B f^{\mathcal{R}}$. Contracted by β^B gives $\chi^{\mathcal{R}}_{\theta\theta} = \beta^B V^{\mathcal{R}}_B = -\beta^B\partial_B f^{\mathcal{R}} = -\frac{d}{d\ln\mu} f^{\mathcal{R}}$ for which $f^{\mathcal{R}} = L^{\mathbb{1},\mathcal{R}}_{\theta\theta}$ is a solution (17). We refer the reader to Appendix B for related and refined discussion of these quantities. Note that we have used that $L^{\mathbb{1},\mathcal{R}}_{\theta\theta}$ is independent of b in writing $\beta^B\partial_B L^{\mathbb{1},\mathcal{R}}_{\theta\theta}$ as a total $\ln\mu$-derivative of $L^{\mathbb{1},\mathcal{R}}_{\theta\theta}$.

It should be added that flow-independence is not straightforward in the case where the coupling manifold is topologically nontrivial e.g., not simply connected. In this case the Stokes like argumentation (42) breaks down and the correlation functions in (41) are multivalued. This topic certainly deserves further study but is beyond the scope of this paper and we refer the reader to Ref. [32] for recent discussion on how to count RG-flows.

2. Properties of $\bar{b}(\mu) = \bar{b}^{\mathcal{R}_\chi}_{\mathcal{R}_b}(\mu)$ outside the fixed points (local flow)

The extension of $\bar{b}$ outside the FP is scheme-dependent. It is dependent on the scheme for the $\langle\Theta\Theta\rangle$-correlator and the b-coupling which were discussed in Secs. II A and II B respectively. Hence generically $\bar{b}(\mu) = \bar{b}^{\mathcal{R}_\chi}_{\mathcal{R}_b}(\mu)$. For extending the flow integral the preferred scheme is the MOM-scheme where the Zamolodchikov metric is positive and properties of monotonicity and gradient flow follow.

(i) *Monotonicity:* From (21) we may define,

$$\bar{b}^{\rm MOM}_{\mathcal{R}_b}(\mu) = \bar{b}^{\rm UV}_{\mathcal{R}_b} - \frac{1}{8}\int_{\ln\mu/\mu_0}^{\infty} \chi^{\rm MOM}_{\theta\theta}(\mu') d\ln\mu', \quad (43)$$

a flow dependent extension satisfying the boundary conditions $\bar{b}^{\rm MOM}_{\mathcal{R}_b}(\infty) = \bar{b}^{\rm UV}_{\mathcal{R}_b}$ and $\bar{b}^{\rm MOM}_{\mathcal{R}_b}(0) = \bar{b}^{\rm IR}_{\mathcal{R}_b}$. Due to the positivity of $\chi^{\rm MOM}_{\theta\theta}$ (27) the function $\bar{b}^{\rm MOM}_{\mathcal{R}_b}(\mu)$ is monotonically decreasing along the flow (with decreasing μ).

(ii) *Gradient flow type equation:* From the anomalous WI (A2) the following Eq. (A3) was derived[9];

$$\frac{1}{8}\chi^{\mathcal{R}_\chi}_{AB}\beta^A\beta^B = \frac{d}{d\ln\mu}\bar{b}^{\mathcal{R}_\chi} = (\beta^A\partial_A + \beta^{\mathcal{R}_b}_b\partial_b)\bar{b}^{\mathcal{R}_\chi}_{\mathcal{R}_b}. \quad (44)$$

For $\mathcal{R}_\chi =$ MOM-scheme, (44) would be a gradient flow type equation if it were not for the $\beta^{\mathcal{R}_b}_b$-term. Since the latter vanishes in the $\mathcal{R}_b = R^2$-scheme one can then obtain a gradient flow equation. For a compact presentation of the gradient flow formulas the following notation is introduced

$$\bar{\bar{b}}(\mu) \equiv \bar{b}^{\rm MOM}_{R^2}(\mu), \qquad \mathcal{G}_{AB}(\mu) \equiv \frac{1}{8}\chi^{\rm MOM}_{AB}(\mu), \quad (45)$$

and $T = -\ln\mu$, increasing towards the IR, and shorthand $\dot{} = \frac{d}{dT}$. The equation then assumes the familiar form

$$\dot{\bar{\bar{b}}} = -\beta^A\partial_A\bar{\bar{b}} = -\mathcal{G}_{AB}\beta^A\beta^B < 0. \quad (46)$$

One then obtains the gradient flow type equation of the form

$$\partial_A\bar{\bar{b}} = (\mathcal{G}_{AB} + \tilde{\mathcal{G}}_{AB})\beta^B, \quad (47)$$

[9]Note that the $\mathcal{R}_\chi$ scheme-dependence of the Zamolodchikov-metric and $\bar{b}$ ought to cancel on the RHS of the second equation. Equation (44) is equivalent to one of Osborn's Weyl consistency relations cf. Eq. (3.10c) in [25] upon identifying $\chi^{\rm MS}_{AB} \to -\chi^a_{AB}$ and $4\bar{b}^{\rm MS} \to d + \frac{1}{2}U_I\beta^I$.

where $\tilde{\mathcal{G}}_{AB} = -\tilde{\mathcal{G}}_{BA}$ is an antisymmetric part whose form is discussed in Appendix B. In the case where the antisymmetric part vanishes, (47) becomes a proper gradient flow equation and can be inverted to give $\beta^B = \mathcal{G}^{AB}\partial_A \bar{\bar{b}}$ where $\mathcal{G}^{AB} \equiv (\mathcal{G}_{AB})^{-1}$ is the inverse matrix which exists since the eigenvalues of $\mathcal{G}^{AB}$ are strictly positive (28). Covariance of Eq. (47) under couplings scheme change is shown in the next section. Note that Eq. (47) is though not covariant under $\mathcal{R}_b$-scheme changes.

3. Transformation of the Zamolodchikov-metric under scheme changes

The Zamolodchikov-metric has been implicitly defined through (17) and (23) in an arbitrary scheme $\mathcal{R}_\chi$ and explicitly for the MOM-scheme (24). The expression of $\Delta\bar{b}$ (37) is obviously scheme independent and so the question of how the scheme dependence of $\chi^{\mathrm{MOM}}_{\theta\theta}$ cancels in the representation (38) is of interest which we aim to clarify in this section. It is appropriate to distinguish between the scheme dependence due to renormalization condition (18), denoted by $\mathcal{R}_\chi$, and a redefinition of the g^Q-scheme of the dynamical couplings (30) which we have ignored for most part of the paper. It is worthwhile to emphasise that the transformations have a geometric interpretation in the space of couplings in that the $\mathcal{R}_\chi$-transformation is governed by a Lie derivative on a 2-tensor (infinitesimal change of a tensor along a flow) and the g^Q-transformation corresponds to a coordinate change (generalized rotation).

(1) Changing the renormalization from $\mathcal{R}_{\chi 1}$ to $\mathcal{R}_{\chi 2}$ corresponds to

$$M^{\mathcal{R}_{\chi 2}}_{AB} = M^{\mathcal{R}_{\chi 1}}_{AB} + \omega_{AB}, \qquad L^{\mathbb{1},\mathcal{R}_{\chi 2}}_{AB} = L^{\mathbb{1},\mathcal{R}_{\chi 1}}_{AB} - \omega_{AB}, \tag{48}$$

where ω_{AB} is finite, local and μ-dependent. The split $M_{AB}(p,\mu) = M^{\mathcal{R}}_{AB}(p,\mu) + L^{\mathbb{1},\mathcal{R}}_{AB}(\mu)$ is defined in analogy to (15) with regards to the $\langle O_A O_B\rangle$-correlator (25). With (17) and (23) this results in

$$\delta\chi_{\theta\theta} = \chi^{\mathcal{R}_{\chi 2}}_{\theta\theta} - \chi^{\mathcal{R}_{\chi 1}}_{\theta\theta} = \mathcal{L}_\beta \omega = \beta^Q \partial_Q \omega = \frac{d}{d\ln\mu}\omega,$$
$$\delta\chi_{AB} = \chi^{\mathcal{R}_{\chi 2}}_{AB} - \chi^{\mathcal{R}_{\chi 1}}_{AB} = \mathcal{L}_\beta \omega_{AB} = \beta^Q \partial_Q \omega_{AB} + \{(\partial_B \beta^Q)\omega_{AQ} + A \leftrightarrow B\}, \tag{49}$$

where the abbreviation $\omega = \beta^A\beta^B\omega_{AB}$ was used and $\mathcal{L}_\beta$ denotes the Lie derivative with respect to the vector field β^A. Hence (38) is manifestly invariant under the scheme change (48) provided ω vanishes at both the UV and IR boundary. Under such circumstances a scheme change might be regarded as being cohomologically trivial. Incidentally (49) also clarifies that the Zamolodchikov metric for a scheme, other than MOM-scheme, is defined as follows[10]

$$\chi^{\mathcal{R}}_{AB} = -\mathcal{L}_\beta L^{\mathbb{1},\mathcal{R}}_{AB}(\mu)$$
$$= -(\beta^Q \partial_Q L^{\mathbb{1},\mathcal{R}}_{AB}(\mu) + \{(\partial_A \beta^Q) L^{\mathbb{1},\mathcal{R}}_{QB}(\mu) + A \leftrightarrow B\}). \tag{50}$$

It is noteworthy that this does not correspond to a total derivative with respect to $\ln\mu$.

(2) Independence under a change in the coupling constant scheme follows from the β-function as well as χ^{MOM}_{AB} transforming as tensors. Going from the scheme $g^P \to g'^P$ results in

$$\beta'^P = \frac{\delta g'^P}{\delta g^A}\beta^A, \qquad \chi'_{PQ} = \frac{\delta g^A}{\delta g'^P}\frac{\delta g^B}{\delta g'^Q}\chi^{\mathrm{MOM}}_{AB}, \tag{51}$$

where the first equation results from the chain rule and so does the second since χ^{MOM}_{AB} is derived from

$$\langle [O_A(x)][O_B(0)]\rangle'_c = -\delta_{\prime A(x)})(-\delta_{\prime B(0)}) \ln\mathcal{Z}$$
$$= \frac{\delta g^P}{\delta g'^A}\frac{\delta g^Q}{\delta g'^B}(-\delta_{P(x)})(-\delta_{Q(0)}) \ln\mathcal{Z}$$
$$= \frac{\delta g^P}{\delta g'^A}\frac{\delta g^Q}{\delta g'^B}\langle [O_P(x)][O_Q(0)]\rangle_c, \tag{52}$$

where the prime denotes the change of the coupling scheme and $\delta_{\prime A(x)} = \delta/\delta g'^A(x)$. Clearly $\beta'^P\beta'^Q\chi'_{PQ} = \beta^A\beta^B\chi_{AB}$ which shows the scheme independence.

D. UV and IR convergence the $\Delta\bar{b}$-integral representation

For Eqs. (2), (39), (38) being a valid way to compute $\Delta\bar{b}$ the integrals need to be finite. We shall see that (39) is not finite in the spontaneously broken phase which implies that either $\Delta\bar{b}$ diverges or that the formalism needs to be

[10] Equation (50) can either be derived by straightforward computation or one may use that on a scalar, with no explicit μ-dependence, $\frac{d}{d\ln\mu} = \beta^C\partial_C = \mathcal{L}_\beta$ and that the Lie derivative along a vector field acts trivially on itself. The reason that the general definition is more involved, than the MOM-scheme, (24) is that the μ-, unlike the p-, derivative does not commute with the β-functions.

adapted. Before investigating the representation (39) it is instructive to consider $\Delta\bar{b} \sim \int d^4x x^4 \langle\Theta(x)\Theta(0)\rangle_c$ (2). First, $\langle\Theta\rangle$ is well-defined since $\langle T^\rho{}_\rho\rangle$ is scale independent and differs from $\langle\Theta\rangle$ by the finite Weyl-anomalies vanishing in flat space. Hence it is the correlation of the two Θ-operators which is subject to potential divergences in the UV ($x \to 0$) as well as the IR (for $x \to \infty$).

The technical discussion parallels the one in [33] with a slightly more refined discussion on the subtle case of the chirally broken phase in Sec. II D 1. In order to analyse the UV- and IR-convergence one needs to investigate the behavior of the spectral function close to the FP. In the case where the scaling dimension (i.e., classical plus anomalous dimension) of the most relevant operator is Δ the spectral density (12) behaves like $\rho(s) \sim s^{\Delta-2}$ and from (39)

$$\Delta\bar{b} \sim \int_0^\infty \frac{ds}{s} s^{\Delta-4}. \tag{53}$$

It is understood that the identity operator (i.e., the cosmological constant term), for which $\Delta = 0$, is subtracted by an appropriate UV-counterterm as otherwise $\rho(s) \sim s^{-2}$.

It is useful to distinguish the cases of a nontrivial and a trivial FP. [i.e., asymptotically safe (AS) and asymptotically free (AF)]. The case where there is spontaneous breaking of chiral symmetry is subtle cf. Sec. II D 1. For the AS-case $\Delta_{\rm UV} > 4$ and $\Delta_{\rm IR} < 4$ in which case the dispersion representation (53) converges both in the UV and the IR. For the AF-case $\Delta = 4$ (53) is potentially both divergent in the IR and UV requiring a refined discussion taking into account the logarithmic behavior. In our previous paper [5] it was shown that AF-free theories, including the multiple coupling case, converges in the UV. In perturbation theory this can be seen by resumming the logarithms order by order. An IR-AF theory behaves in the same way with $s \to s^{-1}$ which leads to the same integral as in the UV [33] and is therefore convergent.

In conclusion in all cases where the theory is a CFT in the IR and UV the integral representations (39), (38), and (2) are finite and do hold. Potential problems with the formulas occur when the theory is not a CFT in either the UV or IR. This is not surprising since for the IR effective action derivation of (2) (cf. Appendix A 2), conformality at the FPs is an assumption. The cases where the FPs are not conformal include the free massive nonconformally coupled scalar and the free massive vector boson (cf. Sec. III D), as well as the chirally broken case which might belong to the former type in the IR. A few short comments on extending the framework to include dimensionful couplings. Generally dimensionful couplings should not worsen the UV-convergence. For example applying the fourth moment projector $\hat{P}_2$ to the fermion correlator $m^2\langle\bar{q}q\bar{q}q\rangle$, in Appendix B in [5], the $p \to \infty$ limit exists ensuring UV-finiteness. The convergence in the IR is less obvious but if the dimensionful parameter is a mass the latter can act as an IR cutoff and is therefore expected to improve the IR-behavior.

1. Spontaneous broken chiral symmetry in the infrared

The case of spontaneously broken chiral symmetry (e.g., QCD) is more cumbersome when viewed from standard chiral perturbation theory. The π Goldstone bosons are free scalars in the far IR and the operator-part of the TEMT contains a term $\Theta = -\frac{1}{2}\Box\pi^2 + \cdots$ at the classical level (e.g., [34]). This EMT cannot undergo the improvement proposed in [35] which removes the term above, since the improvement term is incompatible with chiral symmetry [33,34,36,37]. This is reflected in the generally accepted view that chiral symmetry and conformal symmetry are incompatible with each other.

Adapting the view that chiral symmetry is not compatible with conformal symmetries may lead to problems since in this case $\beta_b^{\rm IR} \neq 0$ and the $\Delta\bar{b}$ formulas might need to be reconsidered. The most concrete way is to approach the problem by computation. In the limit of free pions the $\langle\Theta(x)\Theta(0)\rangle \to \frac{1}{4}\langle\Box\pi^2(x)\Box\pi^2(0)\rangle$ correlator corresponds to a bubble graph which contributes a term of the form $\Gamma_{\theta\theta}(p) \sim p^4 \ln(4m_\pi^2 + p^2) + \cdots$ to the TEMT-correlator where a quark mass m_q ($m_\pi^2 \sim m_q\Lambda_{\rm QCD}$) was introduced as an IR-regulator. (cf., the closely related discussion in and around Eq. (2.26) in [33]). This leads to $M_{\theta\theta}^{(2)}(0) \sim \ln(4m_\pi^2) + \cdots$ which diverges in the chiral limit $m_q \to 0$. Unlike in the UV-case it does not seem possible that this behavior is improved by resumming interactions since corrections necessarily come with additional powers of p^2/f_π^2 where f_π is the pion decay constant. A series of the form $\ln(4m_\pi^2 + p^2)\sum_{n\geq 0} x_n (p^2/f_\pi^2)^n (\ln(4m_\pi^2 + p^2))^{a_n}$ with $a_n \leq n$ does not resume to an expression which is finite in the limit p^2, $m_\pi^2 \to 0$. This is the case since each coefficient $n \geq 1$ vanishes in this limit and the non-zero x_0-term gives rise to a divergence.[11] Hence if $\Theta \to -\frac{1}{2}\Box\pi^2$ is the correct prescription for a chirally broken theory then

[11] In principle $\Gamma_{\theta\theta}(p) \sim p^4 \ln(4m_\pi^2 + p^2) + \cdots$. might also affect the formula for $\Delta\beta_a$ when expressed as a dispersion relation of the four-dilaton scattering amplitude [13,38]. Note that the four dilaton scattering amplitude contains a term proportional to $\Gamma_{\theta\theta}(p)$, where two dilatons couple to the same TEMT on each side, e.g., [33,39] Eq. (3.7) in the first reference. This term does not vanish when the individual dilatons are put on shell since the p^2 variable corresponds to the sum of two dilaton momenta $p^2 = (p_1 + p_2)^2$. Whether or not such a divergence is cancelled by other terms deserves some further study. Clearly it is at most the formula and not the a-theorem itself which is troubled by the chiral phase. Due to the topological nature of the Euler term β_a is well-defined at each end. Therefore one may introduce a mass for the quarks and compute $\Delta\beta_a$ via a two-step process $\Delta\beta_a = \Delta\beta_a|_{m_q\neq 0} - \Delta\beta_a|_{N_f^2-1\ \text{free scalars}}$ in order to take into account the $N_f^2 - 1$ free massless Goldstone bosons.

this implies that $\Delta\bar{b}$ diverges or that the formula (39) has to be amended. Whether or not this prescription is really correct is not known to our knowledge in the sense of being verified by experiment.

Hence the caveat to the reasoning above is that we do not know for sure whether the chirally broken phase is a CFT in the IR or not. The degrees of freedom of an effective theory are not always necessarily clear *a priori* or simply a working assumption justified *a posteriori* by their success. Low energy QCD is described by an effective theory of pions, known as chiral perturbation theory (χPT), which is extremely successful in many domains but whether or not IR conformality *per se* has been tested is unclear. For example it has been advocated [39] that to describe three-flavor $\chi\mathrm{PT}_3$ it might be advantageous to supplement χPT with an additional pseudo-Goldstone (dilaton) resulting from the spontaneous (anomalous) breaking of scale invariance. The effective theory is known as $\chi\mathrm{PT}_\sigma$ [39] and it is currently unclear whether or not this is a valid description in the sense of improved convergence over $\chi\mathrm{PT}_3$. The EMT undergoes an improvement in the dilaton field, which is not constrained by chiral symmetry breaking, and seems to eliminate some of the dangerous kinetic terms (cf. Eq. (3.7) in [40]) discussed above. The remaining kinetic terms are absent in the case where the low energy constants $c_{1,2}(\mu) \to 1$ for m_q, $\mu \to 0$ which is the chiral-scale limit advocated in [40]. In summary in $\chi\mathrm{PT}_\sigma$ the EMT can be improved in the dilaton sector which in principle allows for the elimination of the previously discussed and dangerous $\Box\pi^2$-term. It would be interesting to compute (2) nonperturbatively on the lattice and to check whether or not a chiral logarithm of the form $\ln m_\pi^2 \sim \ln m_q$ is present.

E. Section summary

Since this section is the heart-piece of this work we summarize before continuing the paper. The integral representations Eqs. (2), (39), (38) are well-defined when the theory is conformal in the IR and UV. The latter might not be the case for the chirially broken IR-phase (cf., Sec. II D 1) and the free field theories of the nonconformally coupled scalar and vector particle (cf., Sec. III D). For the latter two cases the operator-part of TEMT, which excludes equation of motion terms, reads $\Theta = -\frac{1}{2}\Box\phi^2$ and $\Theta = -\frac{1}{2}\Box A_\nu^2$ which are only scale but not conformal invariant and the $\Delta\bar{b}$-integral (2) diverges in the IR and UV respectively. The IR and UV divergences of the free field correlators also seem to be the underlying reason why these cases are found to be regularization dependent in actual calculation [15,17–20] as documented in the classic textbook of Birrell and Davies [9]. In summary if (37) is well-defined then positivity and scheme-independence of the spectral function imply the global properties $\Delta\bar{b} \geq 0$ and $\Delta\bar{b}$ scheme-independence. Flow-independence follows from the fact that the integrand of (38) can be written as a total $\ln\mu$ derivative (17) (with $\epsilon \to 0$-limit implied). The extension of $\bar{b}(\mu)$ outside the FP is scheme-dependent. In the MOM-scheme (cf. Sec. II A 1) for the $\langle\Theta\Theta\rangle$-correlator, positivity of the Zamolodchikov-metric, as derived in Sec. II A 2, allows us to extend the $\bar{b}(\mu)$ as a monotonically decreasing function (43) and in the R^2-scheme (cf. Sec. II B) for the b_0R^2-term, $\bar{b}(\mu)$ is shown to satisfy a gradient flow type Eq. (46).

III. EXAMPLES IN QCD-LIKE AND FREE FIELD THEORIES

Below details on renormalization are illustrated in Secs. III A and III C for QCD-like theories and examples are given for a CBZ-FP and free field theories in Secs. III B and III D respectively. Other examples, such as the $\mathcal{O}(N)$ sigma model in the large N limit, can be found in the earlier work [6]. This reference also discusses examples in $d = 4 - \epsilon$ and $d = 3$ dimensions which are not directly related to our work since we strictly adapt $d = 4$ in association with the $\Box R$-flow.

A. Zamolodchikov-Metric in the MOM- and MS-Scheme

In this section we exemplify the Zamolodchikov-metric in QCD-like theories in the MOM-scheme and the MS. The result can be extracted to NNLO using a recent computation of field-strength correlator in [41]. The convention for the QCD coupling and the logarithmic β-function are given in Appendix A 3. With these definitions the operator-part of the TEMT reads $\Theta = \frac{\beta}{2}[G^2]$ and therefore $\chi_{\theta\theta} = \frac{1}{4}\beta^2\chi_{gg}$.

The MOM-metric is obtained by using (22) and identifying $\mathbb{C}^{\mathbb{1}}_{gg}(a_s(p)) = 16C_0^{GG}$ in [Eq 4.18] [41]

$$\chi_{gg}^{\mathrm{MOM}} = -\frac{d}{d\ln p}\bigg|_{p=\mu} \mathbb{C}^{\mathbb{1}}_{gg}(a_s(p)) = \frac{n_g}{2\pi^2}\left(1 + a_s\left(\frac{73}{3}C_A - \frac{28}{3}N_F T_F\right)\right) + \mathcal{O}(a_s^2), \quad (54)$$

with $a_s \equiv g^2/(4\pi)^2$ and the standard group theoretic symbols are specified in Appendix A 3. In principle we could quote $\mathcal{O}(a_s^2)$ but refrain from doing so since we believe that there is no further insight to be gained from it. The MS-metric is obtained by using (17) and identifying $L_{gg}^{\mathbb{1},\mathrm{MS}} = 16Z_0$ in [Eq 4.18] [41] ($L_{gg}^{\mathbb{1},\mathrm{MS}} = r_{gg}^{\mathbb{1}(1)}\epsilon^{-1} + \mathcal{O}(\epsilon^{-2})$)

$$\chi_{gg}^{\mathrm{MS}} = -\left(\frac{d}{d\ln\mu} - 2\epsilon\right)L_{gg}^{\mathbb{1},\mathrm{MS}} = 2\partial_{a_s}(a_s r_{gg}^{\mathbb{1}(1)}) = \frac{n_g}{2\pi^2}\left(1 + 2a_s\left(\frac{17}{2}C_A - \frac{10}{3}N_F T_F\right)\right) + \mathcal{O}(a_s^2). \quad (55)$$

A few remarks are in order. First, the LO expression is the same in both schemes and positive in accordance with positivity in CFTs. The $\mathcal{O}(a_s)$ coefficient differs but in the absence of the knowledge of the higher terms no firm conclusions can be drawn on positivity. Nevertheless it is instructive to see for what number of flavors the sign of the second term changes. If we fix $N_c = 3$ then the critical number is $N_F^c|_{\rm MOM} \simeq 15.6$ and $N_F^c|_{\rm MS} \simeq 5.1$ in the MOM- and MS-scheme respectively. This indicates that the MOM-scheme is more likely to be positive than the MS-scheme. In fact $N_F^c|_{\rm MOM} \simeq 15.6$ is very close to a CBZ-FP where the critical coupling is very small and positivity can be expected to hold for the first few coefficients of $\chi_{gg}^{\rm MOM}$. The difference between the MOM- and MS-metric at $\mathcal{O}(a_s)$ is due to the $\mathcal{O}(a_s)/\epsilon^2$-term in the bare term. Hence the single logarithm $\epsilon \ln(p^2)$, relevant to the definition of the metric, needs to be complemented with an additional $\mathcal{O}(\epsilon)$-term which cannot be deduced without further computation in order to obtain a finite result. Yet since the $\mathcal{O}(a_s)/\epsilon^2$-term equals $-\beta_0\mathcal{O}(a_s^0)/\epsilon$-term, the difference between the two metrics has to be proportional to β_0 which is easily verified

$$\chi_{gg}^{\rm MOM} - \chi_{gg}^{\rm MS} = \frac{n_g}{2\pi^2} 2\beta_0 a_s + \mathcal{O}(a_s^2). \tag{56}$$

Note, the β_0-coefficient is consistent with the generic formula for a scheme change (49).

B. Caswell-Banks-Zaks fixed point

The CBZ-FP [42,43] is a perturbative IR-FP which is analytically tractable and therefore often serves to illustrate conformal window studies explicitly. The CBZ-FP in QCD-like theories (cf. Apendix A 3 for the conventions) is found by tuning N_c and N_f in some quark representation such that $\beta(a_s^{\rm IR}) = 0$ with β approximated by some low order in perturbation theory and crucially $a_s^{\rm IR}$ being small. This amounts to keeping the parameter $\kappa = -\frac{3}{2}\frac{\beta_0}{N_c} \ll 1$ small and introducing the following power counting $a_s \sim \mathcal{O}(\kappa)$ and $\beta \sim \mathcal{O}(\kappa^2)$.

Since $\Delta\bar{b}$ is determined from the 2-point function we may use the recent NNLO computation of the $\langle G^2 G^2\rangle$-correlator [41] ($\Theta = \beta/2[G^2]$ in QCD-like theories) to obtain $\Delta\bar{b}$ and β_a to NNLO which is $\mathcal{O}(\kappa^4)$. Concretely $\Delta\bar{b}$ is obtained from (38)

$$\begin{aligned}\Delta\bar{b} &= \frac{1}{8}\int_{-\infty}^{\infty} \chi_{gg}^{\rm MS}\left(\frac{\beta}{2}\right)^2 d\ln\mu' \\ &= \frac{1}{32}\int_0^{a_s^{\rm IR}} \partial_u\left(\frac{\beta}{u}\right) u r_{gg}^{\mathbb{1}(1)}(u) du,\end{aligned} \tag{57}$$

where to deduce the second equality, (55) and integration by parts were used. The first pole residue $r_{gg}^{\mathbb{1}(1)}$, known from [44], is quoted in [5] [Sec. III.4.2.] in the notation used here. Using the formula above we get

$$\begin{aligned}\Delta\bar{b} &= \frac{-\beta_1 r_{gg}^{\mathbb{1}(1,0)}}{64}(a_s^{\rm IR})^2 \\ &\quad - \frac{1}{96}(2\beta_2 r_{gg}^{\mathbb{1}(1,0)} + \beta_1 r_{gg}^{\mathbb{1}(1,1)})(a_s^{\rm IR})^3 \\ &\quad - \frac{1}{64}\left(\frac{3}{2}\beta_3 r_{gg}^{\mathbb{1}(1,0)} + \beta_2 r_{gg}^{\mathbb{1}(1,1)} + \frac{1}{2}\beta_1 r_{gg}^{\mathbb{1}(1,2)}\right)(a_s^{\rm IR})^4 \\ &\quad + \mathcal{O}(a_s^5).\end{aligned} \tag{58}$$

Solving $\beta(a_s{}^{\rm IR}) = 0$ up to the fourth order gives

$$a_s^{\rm IR} = -\frac{\beta_0}{\beta_1}\left(1 + \frac{\beta_0\beta_2}{\beta_1^2} + \beta_0^2\frac{(2\beta_2^2 - \beta_1\beta_3)}{\beta_1^4}\right) + O(\beta_0^4). \tag{59}$$

Inserting this expression into (58) and using (E4) the final result of this section reads

$$\begin{aligned}\Delta\bar{b} &= \frac{1}{7200\pi^2} N_c^2\kappa^2\left(1 + 2\left(\frac{7}{25}\right)^2\kappa + \frac{53\times 4231}{3^3\times 25^4}\kappa^2\right) \\ &\quad + O(\kappa^5).\end{aligned} \tag{60}$$

Note that LO and NLO expression agrees with Ref. [10]. The $O(\kappa^4)$ term is new and it is observed that the factor of ζ_3 has dropped from the final expression. With the knowledge of the four loop expression $r_{gg}^{\mathbb{1}(1,3)}$ one could easily extend this expression to $O(\kappa^5)$ by using the evaluation of the β-function to five loops [45]. It is noted that since $\kappa = -3/2\beta_0/N_c > 0$ in the conformal window the above expression is manifestly positive in accordance with (2). Effectively (60) corresponds to Euler flow difference $\Delta\beta_a/2$ since it can be shown that in QCD-like theories $\Delta\beta_a = 2\Delta\bar{b} + \mathcal{O}(\kappa^6)$ [31].[12]

C. The R^2-scheme in QCD-like theories and the renormalization of G^2

It is instructive to consider the case of a QCD-like theory to understand what happens in this R^2-scheme. From the work of Hathrell [11], related to QED but sufficient for our purposes, the relevant part of the TEMT reads

$$\langle T^\rho{}_\rho\rangle = \frac{1}{4}(d-4)\langle G^2\rangle - (d-4)b_0 H^2 + \cdots, \tag{61}$$

in terms of bare quantities. The relation of the latter to the renormalized finite quantities is as follows

[12]It is presumably possible to obtain the Zamolodchikov-metric for the β_a-flow, $\chi_{gg}^g \sim G_{gg}$ (notation as in [10,38] on the LHS and RHS respectively) in QCD-like theories from Eq. (2.20) in [38]. For a one coupling theory the antisymmetric $S_{gg} = 0$, $\chi_{gg}^{\rm MS} = -\chi_{gg}^a \sim \mathcal{A}_{gg}$ is known to NNLO and the knowledge of $\chi_{ggg}^b \sim B_{ggg}$ to NLO seems sufficient to get χ_{gg}^g at NNLO.

$$\frac{1}{4}(d-4)\langle G^2\rangle = \frac{\hat{\beta}}{2}\langle [G^2]\rangle^{\rm MS} + (d-4)\mu^{d-4}\left(L_b^{\rm MS} - \frac{\beta_b^{\rm MS}}{d-4}\right)H^2 + \cdots, \tag{62}$$

where $b_0 = \mu^{d-4}(b^{\rm MS} + L_b^{\rm MS})$ and the MS-scheme dependence has been labeled. In both equations the dots stand for terms which are not essential for our discussion. Note that when $\langle T^\rho{}_\rho\rangle$ is expressed in terms of renormalized quantities the L_b-term cancels and the $(d-4)bH^2$ vanishes in the $\epsilon \to 0$ limit and $\langle T^\rho{}_\rho\rangle = \beta/2\langle [G^2]\rangle^{\rm MS} - \beta_b^{\rm MS} H^2 + \cdots$.

Thus the question is what happens to this picture in the R^2-scheme. Taking the definition into account (35) we see that the equations above change to

$$\frac{1}{4}(d-4)\langle G^2\rangle = \frac{\hat{\beta}}{2}\langle [G^2]\rangle^{R^2} + \cdots, \tag{63}$$

with $b_0 = \mu^{d-4} b^{R^2}$. When inserted in (61) this gives the same scheme-independent VEV of the TEMT

$$\langle T^\rho{}_\rho\rangle = \frac{\beta}{2}\langle [G^2]\rangle^{\rm MS} - \beta_b^{\rm MS} H^2 + \cdots = \frac{\beta}{2}\langle [G^2]\rangle^{R^2} + \cdots, \tag{64}$$

when expressed in terms of renormalized quantities in the $\epsilon \to 0$ limit. The above reasoning can be restated as $\hat{\beta}(\langle [G^2]\rangle^{\rm MS} - \langle [G^2]\rangle^{R^2}) = 2(\beta_b^{\rm MS} - \beta_b^{R^2})H^2 = 2\beta_b^{\rm MS} H^2$ valid up to terms previously denoted by dots.

D. $\Delta\bar{b}$ in free field theory

Free field theory flows are instructive and relevant since they describe the transition from an asymptotically free theory to the chiral broken phase of free massless goldstone bosons [46]. A higher derivative massive free field theory computation is deferred to Appendix D. Concretely we think of a massive free field of spin s consisting of $(2s+1)$ degrees of freedom in the UV which decouple in the IR. Within this setup (2), or the adaption

$$\Delta\bar{b} = \frac{1}{8}\hat{P}_2|_{p=0}\int d^4x e^{ix\cdot p}\langle\Theta(x)\Theta(0)\rangle_c, \tag{65}$$

with $\hat{P}_2$ defined in (9), can be considered as an efficient $\Box R$-anomaly calculator provided (cf. Sec. II E) that the integral is convergent in the IR and the UV. For this to be the case conformality ought to be broken by soft terms only. This is the case for the free massive conformally coupled scalar and fermion for which the operator-part of the TEMT are $\Theta = m^2\phi^2$ and $\Theta = m\bar{q}q$ (Dirac fermion) respectively.

Using the formula (65) we get

$$\Delta\bar{b}_{(0,0)} = \frac{1}{8}m^4\mathcal{B}_0''(0,m^2) = 1[\text{unit}],$$
$$\Delta\bar{b}_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})} = -\frac{1}{8}m^2(2m^2\mathcal{B}_0''(0,m^2) + \mathcal{B}_0'(0,m^2)) = 6[\text{unit}], \tag{66}$$

where [unit] is a normalization factor

$$[\text{unit}] = \frac{1}{3840\pi^2}, \tag{67}$$

$(2880 = 3/4\cdot 3840$ converting to the conventions of [9]) and

$$\mathcal{B}_0(p^2,m^2) = \frac{\Gamma(\epsilon)}{(4\pi)^2}\int_0^1 dx(m^2 + x(1-x)p^2)^{-\epsilon}, \tag{68}$$

is the bubble-integral for equal mass scalars with primes denoting derivatives with respect to p^2 and Γ is the Euler function. It is readily seen that (66) agrees with the results in the literature [9] (cf., Table 1 of chapter 6.3) by taking into account the conversion $c|_{[9]} = 4/3\Delta\bar{b}$ and the factor two for Dirac versus Weyl fermions. The convergence of the integral presumably is in 1-to-1 correspondence with scheme-independence of direct computation using a regularization method to derive (3). For example the ζ- [47] and dimensional-regularization [17] yield the same result. This contrasts the case of the free nonconformally coupled scalar and the vector particle for which those methods yield different results. This is reflected here in that the formula (65) is IR and UV divergent for the nonconformally coupled scalar $\Theta = -\frac{1}{2}\Box\phi^2 + m^2\phi^2$ and the vector particle. This issue clearly deserves further study in view of the remarks at the beginning of this section. An interesting aspect is that the scalar to Dirac fermion ration is 6 for the $\Delta\bar{b}$ but 11 for $\Delta\beta_a$ and might therefore give rise to tighter bounds.

IV. SUMMARY AND OUTLOOK

Amongst the Weyl-anomaly contributions (3) the $\bar{b}\Box R$-term has received considerably less attention as compared to the Weyl and the Euler term, presumably because it is ambiguous $\bar{b} \to \bar{b} - \frac{1}{8}\omega_0$ under $\mathcal{L} \to \mathcal{L} + (\omega_0/72)R^2$ (4). Our starting point was the observation that whereas such an ambiguity is present in each theory it disappears in the flow, $\Delta\bar{b} \equiv \bar{b}^{\rm UV} - \bar{b}^{\rm IR}$, since the IR and UV ambiguity are identical. On the technical level the crucial ingredient is the UV-finiteness property of the $\langle\Theta\Theta\rangle$-correlator, discussed in our previous work [5], allowing us to identify $\Delta\bar{b}$ with a bare and therefore RG-scale invariant correlator (37). The quantity $\Delta\bar{b}$ describes the global flow properties, cf. Sec. II C 1, which include scheme-independence and positivity $\Delta\bar{b} > 0$ which are most clearly seen from the spectral representation (39) as previously observed [7]. The integral representation of $\Delta\bar{b}$ follows from an anomalous Ward identity (A2)

$$\Delta\bar{b} = \frac{1}{8}\int_{-\infty}^{\infty}(\chi^{\mathcal{R}}_{AB}\beta^A\beta^B)(\mu')d\ln\mu' \\ = \frac{1}{8}\int_{-\infty}^{\infty}\frac{d}{d\ln\mu'}\mathbb{C}^{1,\mathcal{R}}_{\theta\theta}(p,\mu')d\ln\mu'. \tag{69}$$

The integrand being a total derivative implies flow-independence of $\Delta\bar{b}$ which is one of the main results of this work. The quantity $\chi^{\mathcal{R}}_{AB}$ is the 4D analogue of the Zamolodchikov-metric and independence with respect to the $\langle O_A O_B\rangle^{\mathcal{R}}$-scheme is ensured by the local quantum action principle cf. Sec. II C 3.

The key point in discussing the local flow properties (cf. Sec. II C 2) is the discussion of scheme-dependences since flows, in general, are known to be scheme-dependent outside fixed points. The definition of the Zamolodchikov-metric $\chi^{\mathcal{R}_\chi}_{AB}$ (2-form) in the MOM-scheme (24) is considerably simpler than the generic Lie derivative definition (50). For the former positivity $\chi^{\rm MOM}_{AB}(\mu) \geq 0$ is shown to hold nonperturbatively using a spectral representation. This suffices to define a quantity ($\dot{} = -\frac{d}{d\ln\mu}$)

$$\bar{b}^{\rm MOM}_{\mathcal{R}_b}(\mu) = \bar{b}^{\rm UV}_{\mathcal{R}_b} - \frac{1}{8}\int_{\ln\mu/\mu_0}^{\infty}(\chi^{\rm MOM}_{AB}\beta^A\beta^B)(\mu')d\ln\mu', \qquad \dot{\bar{b}}^{\rm MOM}_{\mathcal{R}_b} < 0, \tag{70}$$

which is monotonically decreasing along the flow (43) where $\mathcal{R}_b$ is the scheme-prescription of the b_0R^2-term (32). Moreover the UV-finiteness [5] allows us to define a scheme, referred to as the R^2-scheme, for which the R^2-anomaly vanishes along the entire flow $\beta^{R^2}_b = 0$. In these particular schemes, $\bar{b}^{\rm MOM}_{\mathcal{R}_b}(\mu)$ obeys a gradient flow type Eq. (46), (47) which in the notation here reads

$$\dot{\bar{b}}^{\rm MOM}_{R^2}(\mu) = -\frac{1}{8}\chi^{\rm MOM}_{AB}\beta^A\beta^B < 0. \tag{71}$$

Furthermore in Sec. III B we extend $\Delta\bar{b}$ for Caswell-Banks-Zaks fixed point to NNLO using a recent computation of the $\langle G^2G^2\rangle$-correlator. This corresponds to fourth order in the Caswell-Banks-Zaks coupling and constitutes also an extension of the Euler flow $\Delta\beta_a$ (a-theorem) to the same order since $\Delta\beta_a = 2\Delta\bar{b}$ up to the sixth order [31].

It is noteworthy that, due to topological protection, β_a is well-defined at both the UV- and IR-CFT. As discussed above such a term is also irrelevant for $\Delta\bar{b}$ but requires an adaptation of the moment formula (2) to (37).[13]

Generally the $\Delta\bar{b}$-integral representations (37)–(38) are correct when conformality is broken by soft terms only, e.g., $\Theta = m^2\phi^2$ and $\Theta = m\bar{q}q$, in which case the integrals converge in the IR and UV and (37) can be regarded as a $\Box R$-anomaly calculator. UV-convergence is ensured for asymptotically safe and asymptotically free theories [5]. Free field theories are a class on their own, coherent with our finding that convergent correlation functions diverge at fixed order in perturbation theory. Since propagators of massive fields $\Phi^{(s)}$ of spin s contain terms scaling like $(k^2)^{s-1}$, the representation in (37) diverges in the UV for conformally coupled fields of spin 1 and higher.[14] UV-convergent cases include the previously quoted free spin 0 (conformally coupled) and spin 1/2 particles for which we find results (cf. Sec. III D) in accordance with direct $\Box R$-computations [9]. Nonconformal couplings of the type $\Theta = -\frac{1}{2}\Box\phi^2 + m\phi^2$ worsen the situation and already lead to UV-divergences in (37) for spin 0 fields. IR-divergences occur for nonconformally coupled spin 0 fields $\Delta\bar{b} \sim \ln(m_\phi)$ (cf., the discussion in Sec. II D 1).

The problems of a free spin 1 particle might be cured by using a gauge invariant formulation, e.g., providing mass to the spin 1 field via a Higgs-mechanism as mentioned elsewhere [6]. The nonconformally coupled scalar is relevant since it is associated with the Goldstone boson of a spontaneously broken chiral symmetry. The IR-divergence does not appear to resume to a finite expression cf., Sec. II D 1. Since chiral symmetry and conformal symmetry are regarded as excluding each other, removing the $\Box\pi^2$-term, with π denoting the Goldstone bosons, by the usual improvement [35] seems prohibited. If the prescription $\Theta \to -\frac{1}{2}\Box\pi^2$ is correct then $\Delta\beta_a$, the flow of the Euler term, still seems well-defined since its topological nature permits to bypass the problem in an efficient manner cf. footnote 11 What happens for the flow of $\Box R$ is less clear. It might either indicate that the flow $\Delta\bar{b}$ diverges or that the formulas need to be amended. It is possible that this situation may change should there exist a phase where scale symmetry is spontaneously broken

[13]One may distinguish a total of four scheme choices: the dynamical couplings g^Q, the b-coupling ($\mathcal{R}_b$-scheme), the choice of the 2-point function for the dynamical operators ($\mathcal{R}_\chi$-scheme) and ω_0R^2-term (4). Other than in Sec. II C 3 the scheme of the dynamical couplings have not been considered. The $\mathcal{R}_b$-scheme and the ω_0R^2-term are related in that $b_0 = \mu^{(d-4)}(b^{\mathcal{R}_b}(\mu) + L^{\mathcal{R}_b}_b(\mu) + \omega_0)$ where ω_0 is μ-independent cf. Appendix C for further remarks.

[14]This seems linked to the scheme-dependence found for direct evaluation of the spin 1 term via (3) cf. [9].

(Goldstone-Nambu realization) and the pion degrees of freedom are supplemented by a dilaton in which case improvement might be possible. Clearly the question of IR-divergencies of the chirally broken phase deserves further study.[15] The resolution for the $\Box R$-flow has the potential to render it more predictive for theories with broken chiral symmetry, e.g., a bound on the conformal window which differs from the one of the a-theorem.

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APPENDIX A: DERIVIATIONS OF $\Delta\bar{b} \sim \int d^4x x^4 \langle\Theta(x)\Theta(0)\rangle_c$

In this Appendix we derive the fourth moment formula for $\Delta\bar{b}$ (2) using anomalous WIs, the (Weyl) anomaly matching procedure by Komargodski and Schwimmer [13,48] and indirectly by veryfing (5) for QCD-like theories using result by Hathrell [11] on the renormalization of the field strength tensor in curved space in Secs. A 1, A 2, and A 3 respectively. We stress that the derivations of in Sec. A 1 and A 2 are general and do not rely on the specific interplay of σ and b in QCD-like theories.

1. The fourth moment and $\Delta\bar{b}$ from an anomalous Ward identity

Anomalous WIs can be obtained by applying operator combinations of the form $\mathcal{D}_\pm(x,\mu) \equiv -(\delta_{s(x)} \pm \beta^A \delta_{A(x)}(\mu))$ to the partition function. A single application gives

$$\begin{aligned}\mathcal{D}_-(x,\mu)\ln Z &= \sqrt{\mathrm{g}}(\langle T^\rho{}_\rho(x)\rangle^{\mathcal{R}} - \beta^A\langle[O_A(x)]\rangle^{\mathcal{R}}) \\ &= 4\bar{b}^{\mathcal{R}}\sqrt{\mathrm{g}}\Box H + \cdots,\end{aligned} \tag{A1}$$

where the dots stand for terms which cancel from the final expression. The quantity g denotes the determinant of the metric $\mathrm{g}_{\alpha\beta}$. Note, the μ-dependence of $\bar{b}$ is balanced on the LHS by the second term. The WIs are anomalous in the sense that they display the Weyl anomaly on the RHS of (A1). Applying a second $\mathcal{D}$-operator leads to

$$\begin{aligned}\mathcal{D}_+(0,\mu)\mathcal{D}_-(x,\mu)\ln Z|_{\mathrm{g}_{\alpha\beta}\to\delta_{\alpha\delta}} &= (\langle T^\rho{}_\rho(x)T^\lambda{}_\lambda(0)\rangle_c^{\mathcal{R}} - \beta^A\beta^B\langle[O_A(x)][O_B(0)]\rangle_c^{\mathcal{R}}) \\ &\quad + (2\langle T^\rho{}_\rho(x)\rangle^{\mathcal{R}} - \beta^B(\partial_B\beta^A)\langle[O_A(x)]\rangle^{\mathcal{R}})\delta(x) \\ &= -8\bar{b}^{\mathcal{R}}\Box^2\delta(x),\end{aligned} \tag{A2}$$

where the vanishing of the commutator, $[\delta_{s(x)}, \beta^A\delta_{A(0)}] = 0$, was used. The anomalous WI (A2) corresponds to Eq. (5.21) in [12] (in Minkowski space). With regard to the notation [12,25], the identification $4\bar{b}^{\mathcal{R}}(\mu) \equiv 4(\sigma^{\mathcal{R}}(\mu) - b^{\mathcal{R}}(\mu)) = \tilde{d}(\mu) \equiv d + \frac{1}{2}\beta^Q U_Q(\mu)$ and $4\bar{b}^{\mathrm{IR}} = d$, gives a consistent picture. Note that by combining different anomalous WIs some Weyl consistency conditions arise [12]. This is of little surprise since the commutator above encodes the essence of the Weyl consistency relations.

Applying $\int d^4x x^4$ to (A2) and differentiating with respect to the scale $\frac{d}{d\ln\mu}$ one obtains

$$\frac{d}{d\ln\mu}\bar{b}^{\mathcal{R}} = \frac{1}{8}\chi^{\mathcal{R}}_{AB}\beta^A\beta^B, \tag{A3}$$

upon using (17), (23) and $\Theta = \beta^A[O_A]$. Above we have directly assumed the $\epsilon \to 0$ limit and crucially used the fact that $\frac{d}{d\ln\mu}M^{(2),\mathcal{R}}_{ss}(p,\mu) = 0$, the renormalized counterpart of the $\langle T^\rho{}_\rho(x)T^\lambda{}_\lambda(0)\rangle_c^{\mathcal{R}}$-correlation function, is scale independent. This is the case because the counterterm b_0 in (29) is scale independent. Combining Eqs. (17) and (A3) one obtains Eq. (26), $\Delta\bar{b} = \frac{1}{8}(M^{(2)}_{\theta\theta}(0) - M^{(2)}_{\theta\theta}(\infty))$, with more detail shown in Sec. II C 1, which is equivalent to (2) and completes the task of this Appendix.

2. The fourth moment and $\Delta\bar{b}$ *à la* Komargodski and Schwimmer

The fourth moment formula for $\Delta\bar{b}$ (2) is derived here in close analogy to the second moment formula for β_c^{2D} in [48] building on the anomaly matching procedure in [13]. The derivation proceeds by matching the term $\bar{b}^{\mathrm{IR}}$ in the IR effective action

$$\ln\mathcal{Z} = -\bar{b}^{\mathrm{IR}}\int d^4x\sqrt{g}H^2 + \cdots, \tag{A4}$$

with the path integral expression. Above the dots stand for nonlocal and Weyl-invariant contributions. The local part of (A4) is dictated by the IR trace anomaly (3). The

[15] So does a systematic study of dimensionful couplings, e.g., [38] for local RG-formulations, beyond the remarks in Sec. II D.

correctness of (A4) follows from a Weyl-variation $g_{\mu\nu} \to e^{-2s(x)} g_{\mu\nu}$ for which $\langle T^\rho{}_\rho \rangle = (-\delta_{s(x)}) \ln \mathcal{Z}$ and $(-\delta_{s(x)}) H^2 = 4\Box H$. In what follows it is convenient to assume a conformally flat background $g_{\mu\nu} = e^{-2s(x)} \delta_{\mu\nu}$ for which

$$\ln \mathcal{Z} = -4\bar{b}^{\rm IR} \int d^4x (\Box s)^2 + \mathcal{O}(s^3). \tag{A5}$$

One might wonder whether the presence of W^2 and E_4 would interfere in this picture. This is not the case since for conformally flat background W^2 vanishes and E_4 does not contain a quadratic term in $s(x)$. In passing we remark that this fact is at the heart of the difficulty of establishing the 4D a-theorem ($\Delta\beta_a \geq 0$).

On the other hand $\ln \mathcal{Z}$ written as the Euclidean path integral over dynamical fields ϕ_i reads

$$\begin{aligned} \mathcal{Z} &= \left(\int \mathcal{D}\phi_i e^{-S_{dyn}(\phi_i, g_{\mu\nu}) + b_0 \int d^4x \sqrt{g} H^2} \right) \\ &= \left(\int \mathcal{D}\phi_i e^{-S_{dyn}(\phi_i, s) + 4 b_0 \int d^4x (\Box s)^2 + \mathcal{O}(s^3)} \right), \end{aligned} \tag{A6}$$

where the conformally flat metric was assumed in the second equality and b_0 is the bare gravitational counterterm with conventions specified in (29). Note that these conventions imply a somewhat unfortunate sign of the initial condition $b_0 = -\bar{b}^{\rm UV}$.[16] The quantity $\bar{b}^{\rm IR}$ is found by performing a derivative expansion of the quantum part of the path integral in order to match the $(\Box s)^2$-term in (A5). Concretely

$$\begin{aligned} \ln \int \mathcal{D}\phi_i e^{-S_{\rm dyn}(\phi_i, s)} = \ln \mathcal{Z}_0 - \int d^4x s(x) \langle \Theta(x) \rangle \\ + \frac{1}{2} \iint d^4x d^4y s(x) s(y) \langle \Theta(x) \Theta(y) \rangle \\ + \mathcal{O}(s^3), \end{aligned} \tag{A7}$$

where here $\langle \ldots \rangle$ refers to the flat-space VEV. The TEMT correlators appear in the expansion since $s(x)$ is the source term for the latter. The four derivative term (A5) is matched by Taylor expanding the double integral term in (A7) to fourth order

$$s(y) = s(x) + \cdots + \frac{1}{4!} (x-y)^\mu (x-y)^\nu (x-y)^\rho (x-y)^\sigma \partial_\mu \partial_\nu \partial_\rho \partial_\sigma s(x) + \mathcal{O}(\partial^5). \tag{A8}$$

Using the Euclidean rotational symmetry the following replacement

$$(x-y)^\mu (x-y)^\nu (x-y)^\rho (x-y)^\sigma \to \frac{1}{24} (x-y)^4 (\delta^{\mu\nu} \delta^{\rho\sigma} + \delta^{\mu\rho} \delta^{\nu\sigma} + \delta^{\mu\sigma} \delta^{\rho\nu}), \tag{A9}$$

is valid under the integral. Changing the integration variable to $y = z + x$ one gets

$$\begin{aligned} \frac{1}{2} \iint d^4x d^4y s(x) s(y) \langle \Theta(x) \Theta(y) \rangle_c = \frac{1}{2} \int d^4x s(x)^2 \int d^4z \langle \Theta(z) \Theta(0) \rangle_c + \cdots \\ + \frac{1}{32^7} \int d^4x (\Box s(x))^2 \int d^4z z^4 \langle \Theta(z) \Theta(0) \rangle_c. \end{aligned} \tag{A10}$$

Substituting (A7) in (A6) and using the derivative expansion (A10) leads to

$$-\bar{b}^{\rm IR} = \ln \mathcal{Z}|_{4 \int (\Box s)^2} = b_0 + \frac{1}{32^9} \int d^4z z^4 \langle \Theta(z) \Theta(0) \rangle_c, \tag{A11}$$

and

$$\Delta \bar{b} = \bar{b}^{\rm UV} - \bar{b}^{\rm IR} = \frac{1}{2^9 3} \int d^4x x^4 \langle \Theta(x) \Theta(0) \rangle_c \geq 0, \tag{A12}$$

then follows by using the initial condition $b_0 = -\bar{b}^{\rm UV}$ in the above equation. It is important to note that this derivation implicitly relies on the theories being conformal in the UV and IR since $\beta_b^{\rm CFT} = 0$ and so the $\bar{b}$ and β_b do not interfere in the Weyl anomaly (3) when reduced to a conformally flat background.

[16] It is instructive to underlay this statement in the language of the QCD-like example of Sec. A 3. Using $d = 4$, the following lengthy chain applies of equations $b_0 = b^{\rm UV} + L_b^{\rm UV} = b^{\rm UV} = -(\sigma^{\rm UV} - b^{\rm UV}) = -\bar{b}^{\rm UV}$, when taking into account that $L_b^{\rm UV} = 0$ and $\sigma^{\rm UV} = 0$.

Adding a term $\delta\mathcal{L} \sim \omega_0 R^2$ (4), resulting in $b_0 \to b_0 + \frac{1}{8}\omega_0$ does not affect (A12) since it is present in both the UV and IR term $\bar{b}^{\rm UV} = -b_0$ of $\bar{b}^{\rm IR} = -b_0 + \frac{1}{8}\mathbb{C}^{\mathbb{1}}_{\theta\theta}(0)$. Stated more simply b_0 is only an initial value which does not affect the difference accumulated in the flow. A more serious issue is the question as to whether the fourth moment converges in the UV and IR which is discussed in Sec. II D.

3. The fourth moment and $\Delta\bar{b}$ *à la* Hathrell in QCD-like theories

In this section we rederive the formula (5) in QCD-like theories by direct use of the expressions for $\beta_b^{\rm MS}$ & $L_{\theta\theta}^{\mathbb{1},{\rm MS}}$, the local QAP and results on the renormalization of G^2 in the external gravitational field. The link between the gravity counterterms (29) and $\langle\Theta\ldots\Theta\rangle$-correlators is given by the QAP and establishes $L_b^{\rm MS} = -\frac{1}{8}L_{\theta\theta}^{\mathbb{1},{\rm MS}}$ (34) which consists in our first step. The relation between b and $\bar{b}$ is as follows

$$\bar{b}(\mu) = \sigma(\mu) - b(\mu), \qquad \sigma^{\rm UV} = 0 \tag{A13}$$

where $\sigma(\mu) = \sigma(a_s(\mu))$ is a quantity related to the renormalization of G^2 in a curved background [11].[17] In some more detail the bare b_0 in the Lagrangian (29) [with $\epsilon \to 0$ allowed by finiteness of L_b (34)] is

$$b_0 \equiv b^{\rm UV} = b(\mu) + L_b(\mu), \tag{A14}$$

where we remind the reader that the μ-dependence arises from $a_s(\mu)$. From the explicit expression of $L_{\theta\theta}^{\mathbb{1},{\rm MS}}$ given in Sec. III. 1 of [5], it is observed that ($\epsilon \to 0$ implied)

$$\begin{aligned} L_b^{\rm MS}(\mu) &= -\frac{1}{32}\int_0^{a_s}\partial_u\left(\frac{\beta}{u}\right)u\left(1-\frac{u}{a_s}\right)r_{gg}^{\mathbb{1}(1)}(u)du \\ &= \frac{\beta_b^{\rm MS}}{2\beta} - \frac{1}{32}\int_0^{a_s}\partial_u\left(\frac{\beta}{u}\right)u r_{gg}^{\mathbb{1}(1)}(u)du \\ &= -\sigma^{\rm MS} - \frac{1}{32}\int_0^{a_s}\partial_u\left(\frac{\beta}{u}\right)u r_{gg}^{\mathbb{1}(1)}(u)du, \end{aligned} \tag{A15}$$

where in the last line the formula $\sigma = -\beta_b/(2\beta)$ [11] was used along with the formula for β_b[18]

$$\begin{aligned} \beta_b^{\rm MS} &= -\left(\frac{d}{d\ln\mu} - 2\epsilon\right)L_b^{\rm MS} \\ &= \frac{1}{16}\frac{\beta(a_s)}{a_s}\int_0^{a_s}\partial_u\left(\frac{\beta(u)}{u}\right)u^2 r_{gg}^{\mathbb{1}(1)}(u)du. \end{aligned} \tag{A16}$$

Taking the IR limit ($a_s \to a_s^{\rm IR}$) in (A15) we get

$$L_b^{\rm MS}(a_s^{\rm IR}) = -\sigma^{\rm IR,MS} - \frac{1}{32}\int_0^{a_s^{\rm IR}}\partial_u\left(\frac{\beta}{u}\right)u r_{gg}^{\mathbb{1}(1)}(u)du. \tag{A17}$$

Further using $L_b(a_s^{\rm IR}) = b^{\rm UV} - b^{\rm IR}$ (A14) and taking into account $\sigma^{\rm UV} = 0$ one arrives at

$$\Delta\bar{b} = \frac{1}{32}\int_0^{a_s^{\rm IR}}\partial_u\left(\frac{\beta}{u}\right)u r_{gg}^{\mathbb{1}(1)}(u)du, \tag{A18}$$

in agreement with (57). Since the latter is equivalent to the fourth moment (37) the task of this section is completed.

APPENDIX B: ON THE ASYMMETRIC PART TO THE GRADIENT FLOW EQUATION (47)

The goal of this Appendix is to discuss the antisymmetric part $\tilde{\mathcal{G}}_{AB}$ in (47). Clearly such a term does not affect global results since it vanishes when contracted by $\beta^A\beta^B$ in (46).

The possibility of such a term can be inferred directly from the definition $\chi_{AB}^{\mathcal{R}} = -\mathcal{L}_\beta L_{AB}^{\mathbb{1},\mathcal{R}}$ (50), related to $\mathcal{G}_{AB}$ as in (45). It is straightforward to obtain

$$\beta^A\chi_{AB}^{\mathcal{R}} = -\partial_B f^{\mathcal{R}} - \beta^A\tilde{\chi}_{AB}^{\mathcal{R}}, \tag{B1}$$

where

$$f^{\mathcal{R}} = L_{\theta\theta}^{\mathbb{1},\mathcal{R}} = \beta^A\beta^B L_{AB}^{\mathbb{1},\mathcal{R}}, \qquad \tilde{\chi}_{AB}^{\mathcal{R}} = \partial_{[A}F_{B]}^{\mathcal{R}}, \tag{B2}$$

with $F_B^{\mathcal{R}} = \beta^C L_{CB}^{\mathbb{1},\mathcal{R}}$ and the square bracket denoting antisymmetrization in the indices A and B as usual. Now

$$\begin{aligned} \partial_A F_B^{\mathcal{R}} &= \partial_A\beta^C L_{BC}^{\mathbb{1},\mathcal{R}} + \beta^C\partial_A L_{BC}^{\mathbb{1},\mathcal{R}} \\ &= \gamma_A^C L_{BC}^{\mathbb{1},\mathcal{R}} + \beta^C\partial_A L_{BC}^{\mathbb{1},\mathcal{R}}, \end{aligned} \tag{B3}$$

whose antisymmetric part is not obviously vanishing. Hence at this formal level the vanishing of $\tilde{\mathcal{G}}_{AB} = \frac{1}{8}\tilde{\chi}_{AB}^{\mathcal{R}} = \frac{1}{8}\partial_{[A}F_{B]}^{\mathcal{R}}$ cannot be concluded and $\tilde{\mathcal{G}}_{AB}$ has therefore to be included in (47). The antisymmetric part is the reason why Eq. (47) is referred to as gradient flow type rather than gradient flow only.

An interesting question is as to whether $\tilde{\chi}_{AB}^{\mathcal{R}}$ is finite or not. Equation (B1) implies so since $\beta^A\chi_{AB}^{\mathcal{R}}$ and $\partial_B f^{\mathcal{R}}$ are both finite. The former is finite since $\chi_{AB}^{\mathcal{R}}$ is an anomalous dimension of the $\langle O_A O_B\rangle$-correlator and $\partial_B f^{\mathcal{R}}$ is the derivative of the finite quantity $f^{\mathcal{R}} = L_{\theta\theta}^{\mathbb{1},\mathcal{R}}$ [5]. In Eq. (B1) an evanescent term proportional to $2\epsilon\beta^A L_{AB}$ was omitted which comes from the d-dimensional relation $\chi_{AB}^{\mathcal{R}} = (2\epsilon - \mathcal{L}_\beta)L_{AB}^{\mathbb{1},\mathcal{R}}$ e.g., (17). Such a term can though safely be neglected since it is finite even in the free field theory limit. On a final note, the relation to Osborn's

[17] The quarks and gluons that are integrated out in an external gravitational field lead to a curvature term $\Box R$ which when divergent needs to be subtracted.

[18] From (A16) one infers that $\beta_b^{\rm MS} = \mathcal{O}(a_s^3)$ since $r_{gg}^{\mathbb{1}(1)} = \mathcal{O}(a_s^0)$ and that the R^2-anomaly-term is absent for theories with $\beta = -\beta_0 a_s$ which is the case for $\mathcal{N} = 1$ supersymmetric Yang-Mills theory.

formalism is that $\partial_{[A} F_{B]} \sim \partial_{[A} U_{B]}$ in the notation used in the Weyl consistency paper [25] and the formula $F_B^{\mathcal{R}} = \beta^C L_{CB}^{\mathbb{1},\mathcal{R}}$ resembles the one given in Eq. (2.17) in [49] in the 2D case.

APPENDIX C: DIFFERENT WAYS OF HANDLING THE GRAVITY COUNTERTERMS

The gravity counterterms $\mathcal{L}_{\text{gravity}} = -(a_0 E_4 + b_0 H^2 + c_0 W^2)$ (29) are not always treated uniformly in the literature. We first describe the two different ways and then show that they give rise to equivalent RG equation for the VEV of the TEMT.[19]

(1) The authors of Refs. [11,28,50] and ourselves (cf. Sec. II B) impose $\frac{d}{d\ln\mu} v_0 = 0$ for $v = a,\ b,\ c$ therefore treating the coefficients of the gravity terms like regular couplings. This leads to $\frac{d}{d\ln\mu}\langle T^\rho{}_\rho\rangle = 0$ for the generally accepted definition of $\langle T^\rho{}_\rho\rangle$ (3).

(2) Jack and Osborn decide not to treat v_0 as couplings but as pure counterterms (choosing the MS-scheme in particular), which translates in our notation to $v_0 = \mu^{(d-4)} L_v$. This then obviously leads to $\frac{d}{d\ln\mu}\langle T^\rho{}_\rho\rangle \neq 0$.

Hence one might wonder whether these two ways of dealing with the gravity counterterms are reconcilable. In fact, as Jack and Osborn remark, below Eq. (2.8) [10], these two ways are equivalent. Let us see how this works, assuming that the a_0 and the c_0 terms are not present which simplifies the presentation. In our way (item 1) the RG equation for the VEV of the TEMT is homogeneous and reads

$$\frac{d}{d\ln\mu}\langle T^\rho{}_\rho\rangle = \left(\frac{\partial}{\partial\ln\mu} + \beta^A\partial_A + \beta_b\partial_b\right)\langle T^\rho{}_\rho\rangle|_{b_0=\mu^{(d-4)}(b+L_b)} = 0. \tag{C1}$$

If treated *à la* Jack and Osborn (item 2) the RG equation is inhomogeneous

$$\frac{d}{d\ln\mu}\langle T^\rho{}_\rho\rangle = \left(\frac{\partial}{\partial\ln\mu} + \beta^A\partial_A\right)\langle T^\rho{}_\rho\rangle|_{b_0=\mu^{(d-4)}L_b} = 4\beta_b\Box H + \cdots, \tag{C2}$$

where $\beta_b = \frac{d}{d\ln\mu} L_b$ was used. Note that the $\partial_{\ln\mu}$-terms vanish in mass-independent schemes as assumed in this work. Now (C1) is seen to be equivalent upon noticing that $\langle T^\rho{}_\rho\rangle = 4\bar{b}\Box H + \cdots$ and using that $\partial_b\bar{b} = -1$. At last we would like to state that it is our understanding that in both formalisms one can add an arbitrary (μ-independent) constant to $b_0 \to b_0 + \mu^{d-4}\frac{1}{8}\omega_0$. This constant term is related to the famous $\Box R$-ambiguity in the trace anomaly [9,15,17–20] which arises in tree-level computations in form of scheme-ambiguities. Note that if ω_0 was μ-dependent then one would deduce different conclusions from the RG-equations. Let us note at last that the μ-independence of $\langle T^\rho{}_\rho\rangle$ might be of importance for the possibility of defining the gluon condensate as the derivative of the cosmological constant term with respect to the renormalized coupling $\langle [G^2]\rangle^{\mathcal{R}} = -2\partial_{\ln g^{\mathcal{R}}}\Lambda^{\text{IR}}$ [51,52].

APPENDIX D: FLOW-INDEPENDENCE OF A HIGHER DERIVATIVE FREE THEORY

In Ref. [53] the higher derivative theory, of the Lee-Wick type [54], was considered

$$\mathcal{L}_{\text{hd}} = \frac{1}{2}\left((\partial\phi)^2 + m^2\phi^2 + \frac{(\Box\phi)^2}{M^2}\right). \tag{D1}$$

It was found that the $\Box R$-flow is dependent on the ratio m/M and therefore not flow-independent [53]. The ratio of masses defines different trajectories in the coupling space, e.g., Fig. 1 for an illustration. Below we present a conformally coupled extension of this model which leads to a flow-independent result in accordance with our findings in Sec. II C 1 (for dimensionless couplings). In summary (D1) can be written in terms of two free massive fields one of them with negative norm. This is of no major concern since Lee-Wick field theories are known to be unitary in all examples at least at the one-loop level. The standard conformal $R\phi^2$-improvement is applied to each field separately. The $\Box R$-flow is then given by just twice the value for the free scalar field (66) which is in particular mass-independent.

The solution of the eom of (D1) shows that the 2-point function propagates two degrees of freedom [$m_{1,2}^2 = (M^2/2)(1 \mp \sqrt{1 - 4m^2/M^2})$]

$$\int d^4x e^{ix\cdot p}\langle\phi(x)\phi(0)\rangle = \frac{M^2}{(p^2+m_1^2)(p^2+m_2^2)} = \frac{M^2}{m_2^2 - m_1^2}\left(\frac{1}{p^2+m_1^2} - \frac{1}{p^2+m_2^2}\right). \tag{D2}$$

[19] This is our interpretation on the topic which emerged from illuminating exchange with Hugh Osborn.

These two degrees of freedom can be made explicit at the Lagrangian level by introducing an auxiliary field χ_2' [55]

$$\mathcal{L}_{\rm aux} = \frac{1}{2}((\partial\phi)^2 + m^2\phi^2 - M^2(\chi_2')^2 + 2\chi_2'\Box\phi). \quad \text{(D3)}$$

Upon using the eom $\chi_2' = (\Box/M^2)\phi$ of (D3), one recovers (D1). An even more convenient form is obtained by substituting $\phi = \chi_1' + \chi_2'$

$$\begin{aligned}\mathcal{L}_{12} &= \frac{1}{2}((\partial\chi_1')^2 - (\partial\chi_2')^2 + m^2(\chi_1' + \chi_2')^2 - M^2(\chi_2')^2) \\ &= \frac{1}{2}((\partial\chi_1)^2 - (\partial\chi_2)^2 + m_1^2\chi_1^2 - m_2^2\chi_2^2)). \quad \text{(D4)}\end{aligned}$$

In the second line we have passed to the mass eigenstates, $m_{1,2}^2$ quoted above, by a hyperbolic rotation conserving the kinematic structure. It is apparent that χ_1 and χ_2 correspond to free massive positive and a negative normed states respectively. The two scalar fields can be conformally coupled by the standard technique ($\eta = \frac{1}{6}$) [35]

$$\mathcal{L}_{12}^{\rm conf} = \frac{1}{2}((\partial\chi_1)^2 - (\partial\chi_2)^2 + m_1^2\chi_1^2 - m_2^2\chi_2^2 + \eta R(\chi_1^2 - \chi_2^2)). \quad \text{(D5)}$$

Conformality can be made manifest for a conformally flat metric $g_{\alpha\beta} = e^{-2s(x)}\delta_{\alpha\beta}$ introducing the Weyl-invariant fields $\bar{\chi}_{1,2} = e^{-s}\chi_{1,2}$. The function $s(x)$ conveniently act as a source for the TEMT. The action $S_{12}^{\rm conf}[s] = \int d^4x\sqrt{g}\mathcal{L}_{12}^{\rm conf}$ assumes the form ($\Delta\eta \equiv (\eta - \frac{1}{6})$)

$$\begin{aligned}S_{12}^{\rm conf}[s] = \frac{1}{2}\int d^4x((\partial\bar{\chi}_1)^2 - (\partial\bar{\chi}_2)^2 \\ + \bar{m}_1^2\bar{\chi}_1^2 - \bar{m}_2^2\bar{\chi}_2^2 + \Delta\eta\bar{R}(\bar{\chi}_1^2 - \bar{\chi}_2^2)), \quad \text{(D6)}\end{aligned}$$

where $\sqrt{g} = e^{-4s}$ has been absorbed into $\bar{m}_{1,2} = e^{-s}m_{1,2}$, $\bar{R} = 6(\Box s - (\partial s)^2)$ and here and below $(\partial\chi)^2 = \delta^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi$ is understood to be contracted with the flat metric. Crucially, the action (D6) is manifestly conformally invariant for $\eta = \frac{1}{6}$ up to the mass terms which break the symmetry softly. The TEMT then follows from

$$\langle\Theta(x)\rangle = (-\bar{\delta}_{s(x)})|_{s=0}\ln\mathcal{Z} = m_1^2\chi_1^2 - m_2^2\chi_2^2 + \mathcal{O}(\Delta\eta), \quad \text{(D7)}$$

where $\bar{\delta}_{s(x)}$ indicates that $\bar{\chi}_{1,2}$ but not $\bar{m}_{1,2}$ are kept fixed. This is the TEMT of two free massive fields for which $\Delta\bar{b}$ is then simply twice the result of a free field (66)

$$\Delta\bar{b}|_{\mathcal{L}_{\rm hd}} = 2\Delta\bar{b}_{(0,0)} = 2[\text{unit}]. \quad \text{(D8)}$$

It is interesting to note that the negative norm state gives a positive contribution to the $\Box R$-flow. This is intimately tied to the fact that Lee-Wick theories are unitary (at least at one-loop). Most importantly we find, contrary to [53], that this model is independent of the mass ratio and therefore flow-independent.

At last it might be instructive to give the conformally coupled higher derivative version of the action (D6) by performing the previous steps backwards

$$\begin{aligned}S_{\rm hd}^{\rm conf}[s] = \frac{1}{2}\int d^4x\bigg((\partial\bar{\phi})^2 + \bar{m}^2\bar{\phi}^2 + \frac{(\Box\bar{\phi})^2}{\bar{M}^2} \\ + \Delta\eta\bar{R}\bigg(\bar{\phi}^2\bigg(1 + \frac{\Delta\eta\bar{R}}{\bar{M}^2}\bigg) - \frac{2\bar{\phi}\Box\bar{\phi}}{\bar{M}^2}\bigg)\bigg), \quad \text{(D9)}\end{aligned}$$

where $\bar{M}^2 = \bar{m}_1^2 + \bar{m}_2^2$ was used. The corresponding higher derivative TEMT assumes the form

$$\langle\Theta(x)\rangle = (-\bar{\delta}_{s(x)})|_{s=0}\ln\mathcal{Z}_{\rm hd} = m^2\phi^2 - \frac{(\Box\phi)^2}{M^2} + \mathcal{O}(\Delta\eta), \quad \text{(D10)}$$

which one would naively expect from an improved version of (D1). Equation (D10) differs from the expression given in [53]. We have checked by explicit computation that (D10) [or (D9)] with (2) give the same result as in (D8).

APPENDIX E: CONVENTION FOR THE QCD-LIKE β-FUNCTION

In this work the bare β-function $\hat{\beta}$ of DR are defined by

$$\hat{\beta} = \frac{d\ln g}{d\ln\mu} = \frac{(d-4)}{2} + \beta = -\epsilon + \beta. \quad \text{(E1)}$$

The logarithmic β-function (E1) is convenient for QCD and is to do with the unusual appearance in the Langrangian $\mathcal{L} = \frac{1}{4g_0^2}G^2$. For multiple couplings $\mathcal{L} = g_0^Q O_Q$ the linear β-function guarantees that $\beta^A = \frac{d}{d\ln\mu}g^A$ transforms like a vector in coupling space. We parametrize

$$\begin{aligned}\beta &= -\beta_0 a_s - \beta_1 {a_s}^2 - \beta_2 {a_s}^3 - \beta_3 {a_s}^4 \ldots, \\ a_s &= \frac{\alpha_s}{4\pi} = \frac{g^2}{(4\pi)^2} \quad \text{(E2)}\end{aligned}$$

where β_{0-3} in $\overline{\rm MS}$-scheme can be found in Ref. [56]. The first two coefficients, which are universal in mass-independent scheme, read

$$\beta_0 = \left(\frac{11}{3}C_A - \frac{4}{3}N_F T_F\right),$$
$$\beta_1 = \left(\frac{34}{3}C_A^2 - \frac{20}{3}N_c N_F T_F - 4C_F T_F N_F\right),$$

where C_F, C_A are quadratic Casimir operators of the fundamental (quark) and adjoint (gluons) representations, N_F the number of quarks and $\mathrm{tr}[T^a T^b] = T_F \delta^{ab}$ is a Lie algebra normalization factor of the fundamental representation. These factors are given by

$$C_A = N_c, \qquad C_F = \frac{N_c^2 - 1}{2N_c}, \qquad T_F = \frac{1}{2}, \tag{E3}$$

for an $SU(N_c)$ gauge group.

1. The Caswell-Banks-Zaks fixed point

The CBZ-FP [42,43] corresponds to a large N_c, N_f limit with $N_f = \frac{11}{2}N_c - \kappa N_c$ and $\kappa \ll 1$. The $O(\kappa^4)$ calculation in Sec. III B corresponds to

$$\beta_0 = -\frac{2}{3}\kappa N_c; \qquad \beta_1 = -\left(\frac{25}{2} - \frac{13}{3}\kappa\right)N_c^2;$$
$$\beta_2 = -\left(\frac{701}{12} - \frac{53}{6}\kappa\right)N_c^3; \qquad \beta_3 = \left(\frac{14731}{144} + 275\zeta_3\right)N_c^4; \tag{E4}$$

where β_3 was given in [56].

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