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Fluctuations in quantum mechanics and field theories from a new version of semiclassical theory. II.

M. A. Escobar-Ruiz,[1,2,*] E. Shuryak,[3,†] and A. V. Turbiner[3,4,‡]

[1]*School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA*
[2]*Centre de Recherches Mathématiques, Université de Montreal, C.P. 6128, succursale Centre-Ville, Montréal QC H3C 3J7, Canada*
[3]*Department of Physics and Astronomy, Stony Brook University, Stony Brook, New York 11794-3800, USA*
[4]*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, 04510 México, D.F., México*

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This is the second paper on the semiclassical approach based on the density matrix given by the Euclidean time path integral with fixed coinciding end points. The classical path, interpolating between this point and the classical vacuum (called a "flucton"), as well as systematic one- and two-loop corrections were calculated in the first paper [M. A. Escobar-Ruiz, E. Shuryak, and A. V. Turbiner, Phys. Rev. D **93,** 105039 (2016)] for a double-well potential. Here, we extend them for a number of quantum-mechanical problems, such as an anharmonic oscillator and the sine-Gordon potential. The method is based on a systematic expansion in Feynman diagrams and thus can be extended to quantum field theories (QFTs). We show that the loop expansion in quantum mechanics resembles the leading-log approximations in QFT. In this sequel, we present a complete set of results obtained using this method in a unified way. Alternatively, starting from the Schrödinger equation we derive a *generalized* Bloch equation whose semiclassical-like, iterative solution generates the loop expansion. We rederive the two-loop expansions for all three of the above potentials and extend them to three loops, which has not yet been done via Feynman diagrams. All results for both methods are fully consistent with each other. An asymmetric (tilted) double-well potential (nondegenerate minima) is also studied using the second method.

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I. INTRODUCTION

Semiclassical approximations are well known tools, both in quantum mechanics and quantum field theory. Standard textbooks of quantum mechanics usually start with the Bohr-Sommerfeld quantization conditions and the semiclassical WKB approximation for the wave function. Unfortunately, extending such methods beyond the first correction for the one-dimensional case or those with separable variables, or to a multidimensional case, proved to be difficult. In mathematical physics, general constructions for multidimensional complexified saddle points are related to their extrema and the network of the so-called Lefschetz thimbles (solutions of the gradient flow equations) connecting them. Applications of such a theory in the path-integral context have been investigated by Witten [1], which inspired a number of subsequent studies. However, most of the paths considered below are real, and we will only turn to complexified paths at the end of the paper in Sec. V B. We will not discuss thimble solutions.

Semiclassical methods that start with Feynman path integrals, on the other hand, are applicable not only for many dimensions or many-body applications, but even for quantum field theories. The early applications of these methods included, e.g., the paper by Rossi and Testa [2] who used an expansion around classical paths.

The so-called instanton calculus has been developed for gauge theories and other models. Instantons are indeed responsible for many important phenomena in quantum field theories (QFTs), which we cannot discuss here. From a technical point of view, however, so far the calculations have not progressed beyond one loop in any of these QFT applications.

In our previous paper [3] (which we hereafter refer to as I), we discussed a different version of the semiclassical approach, based on Feynman's path-integral representation of the density matrix [4,5] analytically continued to imaginary (Euclidean) time. It corresponds to a transition from quantum mechanics to statistical mechanics: if the time is defined as a periodic variable with period $\beta = \hbar/T$, the density matrix corresponds to a quantum ensemble at nonzero temperature T. Thus, the paths (both classical and quantum) to be considered in this setting need to have the same periodicity. However, in this paper we will only focus on the zero-temperature limit, in which the period becomes infinite.

At the classical level, the theory is based on a classical (minimal action) periodic path, which extends from some

*mauricio.escobar@nucleares.unam.mx
†edward.shuryak@stonybrook.edu
‡turbiner@nucleares.unam.mx; alexander.turbiner@stonybrook.edu

arbitrary point x_0 to the "classical vacuum" (the minimum of the potentia) and returns. This path was introduced in Ref. [6] and was named the "fluction." We only recently learned that essentially the same construction was proposed later on in Ref. [7].

A general advantage of this approach is that the path integrals lead to systematic perturbative series, in the form of Feynman diagrams, with clear rules for each order. Textbook perturbative approaches for the wave functions do not have this feature, and basically are never used beyond, say, the first and second orders.

Of course, the higher level of generality comes with a heavy price. While the classical part is relatively simple, for the quantum part at the one-loop level one already needs to calculate determinants of certain differential operators. At two and more loops, Feynman diagrams need to be evaluated on top of spacetime-dependent backgrounds; therefore, those should be evaluated in the spacetime representation rather than in the energy-momentum one used most often in QFT applications. Most of the content of the first part of this paper is the explicit demonstration of how one can accomplish all of this, in analytic form, for three classical examples—quartic anharmonic and double-well, and sine-Gordon (Mathieu) potentials. Their quantum Hamiltonian is of the standard form

$$\mathcal{H} = -\frac{1}{2m}\partial_x^2 + V(x), \qquad \partial_x = \frac{d}{dx}, \tag{1}$$

where we use the units $\hbar = 1$ and $m = 1$ without loss of generality.

Since our ultimate aim is a generalization of the semiclassical theory to QFTs, our quantum-mechanical examples should be represented in a certain specific form of an anharmonic perturbation of the harmonic oscillator,

$$V(x) = \frac{\tilde{V}(gx)}{g^2} = \frac{1}{2}x^2 + a_3 g x^3 + a_4 g^2 x^4 + \ldots, \tag{2}$$

where $\tilde{V}$ has a minimum at $x = 0$ and always starts from quadratic terms, the frequency of the small oscillation near the minimum is set equal to one ($\omega = 1$), and g is the coupling constant. The classical (vacuum) energy is always taken to be zero, $V(0) = 0$, and $a_{2,3,\ldots}$ are parameters. We call (gx) the *classical* coordinate (see below). Both the classical coordinate and the Hamiltonian (1) are invariant with respect to the simultaneous change

$$x \to -x, \qquad g \to -g.$$

This implies that the energy is a function of g^2,

$$E = E(g^2). \tag{3}$$

The semiclassical expansion is done in powers of the small coupling g, in a way that is similar to perturbation theory which is also expanded in powers of the small coupling g.

Let us indicate the potentials we are going to study in the form (2):

(i) Quartic anharmonic oscillator (AHO),

$$V = \frac{1}{2}x^2(1 + g^2x^4). \tag{4}$$

(ii) Quartic double-well potential (DWP),

$$V = \frac{1}{2}x^2(1 - gx)^2. \tag{5}$$

(iii) Sine-Gordon (Mathieu) potential (SGP),

$$V = \frac{(1 - \cos gx)}{g^2}. \tag{6}$$

(iv) Quartic asymmetric (tilted) double-well potential (ADWP),

$$V = \frac{1}{2}x^2(1 + 2tgx + g^2x^2). \tag{7}$$

Here the parameter t "measures" the asymmetry of wells; for $t = \pm 1$ the wells are symmetric and we arrive at the DWP, while for $t = 0$ we arrive at the AHO.

In our previous paper [3] we used Feynman diagrams to calculate one- and two-loop corrections to the classical fluction action for the DWP potential. For the other two famous quantum-mechanical problems—the AHO and SGP—we had only presented the derivation of the Green functions and the (one-loop) determinants. For completeness, here we also add the results for the two-loop corrections for those two problems, calculated from the same set of Feynman diagrams. Like for the DWP case, the complete set of results looks surprisingly simple and compact. Remarkably, it does not contain any transcendental functions, logs, or polylogs, which appear for individual diagrams. Furthermore, the classical (fluction) action is always the WKB action $\int pdq$, but evaluated at *zero* energy; this general observation was missed before. To clarify the meaning of the loop expansion, we will be able to derive a certain Bloch-type Riccati equation, the iteration solution of which exactly generates the loop expansion. This equation will be called the *generalized Bloch equation*.

II. FLUCTUATION CORRECTIONS FROM THE FEYNMAN DIAGRAMS

The method has been extensively described in I and there is no need to repeat it here in detail. Its main idea is that quantum fluctuations around the classical fluction path x_{fluction} can be described by a standard expansion of the action in powers of $(x - x_{\text{fluction}})$, with the quadratic term given by the Green function, while the higher-order terms

produce vertices of the Feynman diagrams. We briefly review the main definitions.

A. Generality

By definition, the Feynman path integral gives the density matrix in quantum mechanics [4],

$$\rho(x_i, x_f, t_{\rm tot}) = N \int_{x(0)=x_i}^{x(t_{\rm tot})=x_f} Dx(t) e^{iS[x(t)]/\hbar}. \quad (8)$$

Here N is a normalization factor and S is the usual classical action of the problem,

$$S = \int_0^{t_{\rm tot}} dt \left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right],$$

for a particle of mass m in a static potential $V(x)$; it provides the weight of the paths in Eq. (8).

As it is well known (see, e.g., Ref. [5]), one can also apply these expressions in statistical mechanics. For this, one needs to change time into its Euclidean version $\tau = it$ defined on a circle with circumference $\beta = \tau_{\rm tot}$. Such a periodic time is known as the Matsubara time, and the density matrix of a quantum system is related to the probability for a thermal system with temperature

$$T = \hbar/\beta. \quad (9)$$

At $T \to 0$ the ground state of the quantum system is naturally recovered. The periodicity of the path implies that there is only one end point: $x_i = x_f = x_0$.

The main object of our study is the diagonal matrix element of the density matrix, which gives the probability for the specific coordinate value x_0 [or a particular field configuration $\phi_0(\vec{x})$ in QFT] in this ensemble,

$$P(x_0, \beta) = N \int_{x(0)=x_0}^{x(\beta)=x_0} Dx(\tau) e^{-S_E[x(\tau)]/\hbar}. \quad (10)$$

So, we take into account all (closed) trajectories starting and ending at x_0. Here the weight is defined via the Euclidean action

$$S_E = \int_0^{\beta} d\tau \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right].$$

Using the standard definition of the density matrix in terms of stationary states $|n\rangle$ with energy E_n, the sum over states becomes a set of decreasing exponentials,

$$P(x_0, \beta) = \sum_n |\psi_n(x_0)|^2 e^{-E_n\beta}. \quad (11)$$

In the limit of large β or low temperature T, in Eq. (11) the dominant term

$$P(x_0, \beta \to \infty) \sim |\psi_0(x_0)|^2 e^{-E_0\beta} \quad (12)$$

describes the ground state, which is the main state we are interested in.

B. The classical path: Flucton

We assume for simplicity that the potential (2) has a global minimum at $x = 0$,

$$V(x) \geq 0 \quad \text{and} \quad \frac{d}{dx} V|_{x=0} = 0.$$

Thus, the exponent in Eq. (11) is non-negative, $S_E \geq 0$.

In Euclidean time τ the kinetic energy changes sign, which is effectively equivalent to flipping the sign of the potential, $V(x) \to -V(x)$, turning a minimum into a maximum. Now, we ask if there exists a real path, starting at some arbitrary point x_0 at $\tau = 0$ and returning to it after the required time duration, at the Matsubara time $\tau = \beta$. The lowest-action path of this kind is the classical path, which we call the *flucton*. Its energy is defined by its period.

Since in this work we deal only with the quantum-mechanical limit of vanishing temperature $T \to 0$, the Matsubara time goes to infinity. It is clear then that the particle should spend a divergently long time near the turning point, which is the case when $x_t \to 0$, the location of the maximum of $-V$ (see Fig. 1). Evidently, such a classical path with an infinite period is the one with zero energy, $E = 0$. The basic idea is that such a classical path with zero energy $E = 0$ "climbs up the hill" to its maximum at $x = 0$.

Let us find the flucton paths explicitly. They of course satisfy the second-order classical equation of motion (EOM), but in the one-dimensional case it is much easier to use the energy conservation at $E = 0$,

$$\frac{d}{d\tau} x(\tau) = \sqrt{2V(x)}. \quad (13)$$

The circular trajectory emerging in Eq. (13)—which starts and ends at x_0 and passes through $x = 0$ at time β—is called a *flucton* [6],

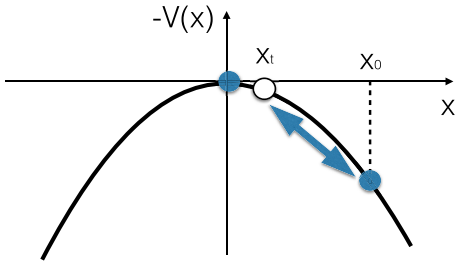

FIG. 1. The sketch of the inverted potential $-V$ versus the coordinate x. The flucton is the classical trajectory starting and ending at the same initial point x_0. At nonzero temperature it goes through the turning point x_t (see text). At zero temperature x_t coincides with the location of the maximum, $x_t = 0$.

$$x_{\text{flucton}}(0) = x_{\text{flucton}}(\beta) = x_0,$$

$$x_{\text{flucton}} = x_{\text{flucton}}(\tau; x_0, \beta).$$

It enables us to evaluate the transition amplitude $P(x_0, \beta)$ (10). Putting

$$\phi(x_0, \beta) \equiv -\log[P(x_0, \beta)]$$

in Eq. (12) and expanding the classical action around the flucton, we obtain

$$\phi(x_0,\beta) = S_{\text{flucton}} + \frac{1}{2}\log(N^{-2}\det(O_{\text{flucton}})) + \text{loops}, \quad (14)$$

where $S_{\text{flucton}} \equiv S_E[x_{\text{flucton}}]$ and

$$O_{\text{flucton}} \equiv -\frac{d^2}{d^2\tau} + \frac{\partial^2}{\partial x^2} V(x)|_{x=x_{\text{flucton}}(\tau;x_0,\beta)} \quad (15)$$

is a Schrödinger-type operator in the variable τ with x_0, β as parameters.

In order to construct the loop expansion (14), three building blocks are used.

(i) The action S_{flucton} of the flucton: In the limit $\beta \to \infty$, the action S_{flucton} provides the dominant term of the phase of the ground-state function [see Eq. (12)] and exactly reproduces the WKB result (obtained from the Riccati-Bloch equation for the phase).

(ii) The determinant of O_{flucton} [Eq. (15)] describes the quadratic quantum fluctuations.

(iii) The loop corrections: These are true quantum corrections decreasing at large distances, and in the present formalism they are given by explicit Feynman diagrams in the flucton background.

While the computation of S_{flucton} [Eq. (14)] is relatively simple, the evaluation of $\det O_{\text{flucton}}$ already involves the diagonalization of a certain nontrivial second-order differential operator. Usually, it is a highly nontrivial calculation which is enormously simplified by the generalized Riccati-Bloch equation (see below).

While in our paper I we discussed several quantum-mechanical problems to some extent, the calculations were done to two loops only for the DWP. It is clear now that the formalism can be applied for *any* perturbed-harmonic-oscillator-type potential [Eq. (2)]. Therefore, without explanations, we now simply list the full set of results for the AHO, DWP, and SGP and also for the ADWP. The units used assume a particle mass $m = 1$ and the Planck constant $\hbar = 1$.

1. Relating the determinant and the Green function

In this section we calculate the quadratic-order quantum oscillations around a classical (flucton) path, namely, the determinant (14). For the harmonic oscillator, the potential $\partial^2 V(x)/\partial x^2|_{x=x_{\text{flucton}}}$ in Eq. (15) is just a constant, so in this case the fluctuations do not depend on the classical path, and direct diagonalization of the operator (15) [8] shows that

$$N^{-2}\text{Det}(O_{\text{flucton}}) = 2\pi \sinh\beta.$$

In general, a direct diagonalization of Eq. (15) is highly nontrivial and analytical results for it are extremely rare.

In 1978, Brown and Creamer [9] invented the a way to relate the determinant and the Green function, reducing it to a calculation of a symbolic one-loop Feynman diagram. One can apply their procedure to quantum mechanics. When the potential $V_{\text{flucton}} \equiv V(x_{\text{flucton}}(\tau; x_0, \beta))$ depends on some parameter, it can be varied. To this end, we rewrite the potential as

$$V_{\text{flucton}} = 1 + W(\tau; X, \beta),$$

where $X = X(x_0)$. Its variation, which results in an extra potential

$$\delta V_{\text{flucton}} = \frac{\partial W}{\partial X}\delta X, \quad (16)$$

is a perturbation: its effect can be evaluated by the Feynman diagram

$$\frac{\partial \log \text{Det}(O_{\text{flucton}})}{\partial X} = \int d\tau G(\tau,\tau)\frac{\partial V_{\text{flucton}}(\tau)}{\partial X} \quad (17)$$

which contains a derivative of the potential as a vertex and the "loop" Green function $G(\tau, \tau)$,

$$O_{\text{flucton}} G(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2), \quad (18)$$

at the same point $\tau_1 = \tau_2 = \tau$ (see Fig. 2). For simplicity, the dependence of $G(\tau_1, \tau_2)$ on x_0 and β is omitted. Equation (17) relates the determinant and the Green function: if the rhs can be calculated, the derivative over X can be integrated back. Hence, if the Green function is known, one can calculate the determinant.

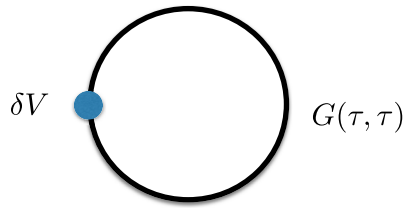

FIG. 2. Symbolic one-loop diagram, including variation of the fluctuation potential δV and the simplified "single-loop" Green function $G(\tau, \tau)$ (see Ref. [3]).

2. Two-loop-correction: Feynman diagrams

The loop expansion for $P(x_0,\beta)$ can be written in the form

$$\text{loops} = 1 + g^2 B_1 + g^4 B_2 + \cdots, \tag{19}$$

where $B_n = B_n(x_0)$ is the $(n+1)$-loop contribution. Equivalently, in Eq. (14) the loop expansion is of the form

$$\text{loops} = g^2 B_1 + g^4 \tilde{B}_2 + \cdots, \tag{20}$$

where $\tilde{B}_n = \tilde{B}_n(x_0), n > 1$ is made out of loop contributions B_n, e.g., $\tilde{B}_2 = -B_1^2/2 + B_2$ etc.

In general, for any potential of the form (2) the two-loop correction B_1 is given by the sum of three Feynman diagrams (see Fig. 3).

In Fig. 3, diagram a is given by a one-dimensional integral, while diagrams b_1 and b_2 correspond to two-dimensional integrals. Explicitly,

$$\begin{aligned}
a &\equiv -\frac{1}{8}\int_0^\infty [v_4(\tau)G^2(\tau,\tau) - v_{4,0}G_0^2(\tau,\tau)]d\tau,\\
b_1 &\equiv \frac{1}{12}\int_0^\infty \int_0^\infty [v_3(\tau_1)v_3(\tau_2)G^3(\tau_1,\tau_2)\\
&\quad - v_{3,0}v_{3,0}G_0^3(\tau_1,\tau_2)]d\tau_1 d\tau_2,\\
b_2 &\equiv \frac{1}{8}\int_0^\infty \int_0^\infty [v_3(\tau_1)v_3(\tau_2)G(\tau_1,\tau_1)G(\tau_1,\tau_2)G(\tau_2,\tau_2)\\
&\quad - v_{3,0}v_{3,0}G_0(\tau_1,\tau_1)G_0(\tau_1,\tau_2)G_0(\tau_2,\tau_2)]d\tau_1 d\tau_2,
\end{aligned} \tag{21}$$

where

$$v_k(\tau) = \frac{\partial^k}{\partial x^k}V(x)|_{x=x_{\text{flucton}}}, \qquad k = 3,4$$

are the vertices on the flucton background,

$$v_{k,0} = \frac{\partial^k}{\partial x^k}V(x)|_{x=0}, \qquad k = 3,4$$

denote the "vacuum vertices," the Green function $G(\tau_1,\tau_2)$ is defined in Eq. (18), and

$$G_0 = \frac{e^{-|\tau_1-\tau_2|}}{2} - \frac{e^{-\tau_1-\tau_2}}{2}, \tag{22}$$

is the "harmonic propagator." Its presence is related to the necessity to subtract (spacetime-divergent) anharmonic effects, and is unrelated to the fluctons.

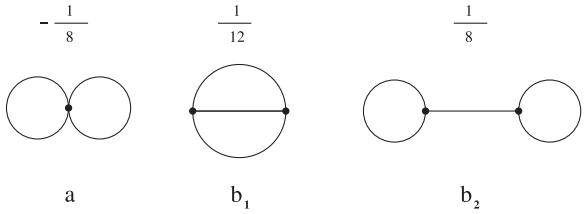

FIG. 3. Diagrams contributing to the two-loop correction $B_1 = a + b_1 + b_2$. The signs of the contributions and symmetry factors are indicated.

In the case of the three-loop contribution for the general potential (2), there exist 15 diagrams which contribute (see, e.g., Ref. [10]), while for the DWP the number of diagrams drops to 12 (see, e.g., Ref. [11]). Like in instanton calculus, we do not hope that all of these diagrams can be calculated analytically.

C. AHO

This section contains the expansion (14) up to two loops for the AHO.

The AHO is defined by the potential (4),

$$V(x) = \frac{1}{2}x^2(1+g^2x^2) = \frac{1}{2g^2}u^2(1+u^2), u = gx,$$

at zero temperature, $T = 0$. The classical flucton path solution with energy $E = 0$ of Eq. (13) is

$$x_{\text{flucton}}(\tau) = \frac{gx_0}{\cosh(|\tau|) + \sqrt{1+g^2x_0^2}\sinh(|\tau|)}.$$

The corresponding classical action is given by

$$S_E[x_{\text{flucton}}] = \frac{2}{3}\frac{(1+g^2x_0^2)^{\frac{3}{2}} - 1}{g^2}.$$

In the limit $g \to 0$ we recover the classical action for the harmonic oscillator, and at $x_0 \to \infty$ we obtain the expansion

$$S_E[x_{\text{flucton}}] = \frac{2}{3}gx_0^3 + \frac{1}{g}x_0 + \text{lower-order terms}. \tag{23}$$

It is convenient to introduce a new variable,

$$X_{\text{AHO}}(x) = (1+g^2x^2)^{\frac{1}{2}} = (1+u^2)^{\frac{1}{2}}, \qquad u = gx. \tag{24}$$

For the anharmonic oscillator, the Green function of the operator O_{flucton} [Eq. (15)] is given by

$$G(\tau_1,\tau_2;X_{\rm AHO}) = \frac{\mathrm{sech}[\frac{1}{2}(-|\tau_-|+\tau_+)](X_{\rm AHO}\cosh[\frac{1}{2}(|\tau_-|+\tau_+)]+\sinh[\frac{1}{2}(|\tau_-|+\tau_+)])}{4X_{\rm AHO}(\cosh[\frac{1}{2}(|\tau_-|+\tau_+)]+X_{\rm AHO}\sinh[\frac{1}{2}(|\tau_-|+\tau_+)])^2(1+X_{\rm AHO}\tanh[\frac{1}{2}(-|\tau_-|+\tau_+)])^2}$$
$$\times\left[X_{\rm AHO}\cosh[|\tau_-|-\tau_+]\left(1+3X_{\rm AHO}^2+X_{\rm AHO}(3+X_{\rm AHO}^2)\tanh\left[\frac{1}{2}(-|\tau_-|+\tau_+)\right]\right)\right.$$
$$+(1-X_{\rm AHO}^2)(4-5X_{\rm AHO}^2-3X_{\rm AHO}(|\tau_-|-\tau_+))\tanh\left[\frac{1}{2}(-|\tau_-|+\tau_+)\right]$$
$$\left.-X_{\rm AHO}(1+3X_{\rm AHO}(X_{\rm AHO}+(1-X_{\rm AHO}^2)(|\tau_-|-\tau_+)))\right], \tag{25}$$

where $\tau_- = \tau_2 - \tau_1$ and $\tau_+ = \tau_1 + \tau_2$.

Taking the above Green function and the "vertex"

$$\frac{\partial V_{\rm flucton}(\tau)}{\partial X_{\rm AHO}} = \frac{12(\sinh(\tau)+X_{\rm AHO}\cosh(\tau))}{(\cosh(\tau)+X_{\rm AHO}\sinh(\tau))^3}, \tag{26}$$

$(\tau > 0)$ to evaluate Eq. (17), we obtain analytically

$$\log \mathrm{Det}(O_{\rm flucton}) = 2\log[X_{\rm AHO}(1+X_{\rm AHO})]. \tag{27}$$

As for the next term—the two-loop correction B_1—the results for all three two-loop Feynman diagrams shown in Fig. 3 can be found analytically:

$$a = -\frac{3(32+96X+29X^2-74X^3-35X^4)}{560X^2(1+X)^2}$$
$$-\frac{9(2(2-3X^2)\log(2)-2(2-3X^2)\log[1+X]-3X(1-X^2)\mathrm{PolyLog}[2,\frac{X-1}{1+X}])}{70X(1-X^2)},$$
$$b_1 = -\frac{140+184X+272X^2+193X^3-478X^4-455X^5}{840X^3(1+X)^2}$$
$$+\frac{3(2(2-3X^2)\log(2)-2(2-3X^2)\log[1+X]-3X(1-X^2)\mathrm{PolyLog}[2,\frac{X-1}{1+X}])}{35X(1-X^2)},$$
$$b_2 = -\frac{140+528X+464X^2-169X^3-626X^4-385X^5}{560X^3(1+X)^2}$$
$$+\frac{3(2(2-3X^2)\log(2)-2(2-3X^2)\log[1+X]-3X(1-X^2)\mathrm{PolyLog}[2,\frac{X-1}{1+X}])}{70X(1-X^2)}, \tag{28}$$

where $X \equiv X_{\rm AHO}$ and $\mathrm{PolyLog}[n,z] = \sum_{k=1}^{\infty} z^k/k^n$ is the polylogarithm. Each diagram provides a contribution in the form of a sum of rational (meromorphic) functions and a transcendental function. Functionally, both functions are similar in each diagram. Eventually, after summing all three diagrams a, b_1, and b_2, the two-loop correction B_1 takes an amazingly simple form:

$$B_1^{(\rm AHO)} = \frac{(1-X_{\rm AHO})(5+16X_{\rm AHO}+25X_{\rm AHO}^2+17X_{\rm AHO}^3)}{12X_{\rm AHO}^3(1+X_{\rm AHO})}, \tag{29}$$

where all transcendental contributions are *canceled out* and the answer turns out to be the meromorphic function of $X_{\rm AHO}$ *only*. We will observe similar cancellations for the DWP and SGP below.

D. DWP

The *double-well potential* is defined as in Eq. (5),

$$V=\frac{1}{2}x^2(1-gx)^2=\frac{1}{2g^2}u^2(1-u)^2,\qquad u=gx.$$

Its two degenerate minima are situated at $x=0$ and $x=1/g$, respectively. In the zero-temperature limit $T=1/\beta=0$, the fluctuon trajectory is given by [3]

$$x_{\text{fluctuon}}(\tau)=\frac{x_0(\cosh(|\tau|)-\sinh(|\tau|))}{1+gx_0(1-\cosh(|\tau|)+\sinh(|\tau|))},$$

and the corresponding classical action reads

$$S_E[x_{\text{fluctuon}}]=x_0^2\left(1+\frac{2gx_0}{3}\right).$$

For this case it is convenient to introduce the variable

$$X_{\text{DWP}}(x)\equiv u=gx, \tag{30}$$

cf. Eq. (24). In this variable the corresponding Green function takes the form

$$\begin{aligned}G(\tau_1,\tau_2)=&\frac{e^{-|\tau_1-\tau_2|}}{2(e^{\tau_1}(1+X)-X)^2(e^{\tau_2}(1+X)-X)^2}[8e^{\frac{1}{2}(\tau_1+\tau_2+3|\tau_1-\tau_2|)}X^3(1+X)-8e^{\frac{1}{2}(3\tau_1+3\tau_2+|\tau_1-\tau_2|)}X(1+X)^3\\&+e^{2(\tau_1+\tau_2)}(1+X)^4-6e^{(\tau_1+\tau_2+|\tau_1-\tau_2|)}X^2(1+X)^2|\tau_1-\tau_2|+e^{(\tau_1+\tau_2+|\tau_1-\tau_2|)}(6X^4(\tau_1+\tau_2)\\&+12X^3(1+\tau_1+\tau_2)+6X^2(3+\tau_1+\tau_2)+4X-1)-e^{2|\tau_1-\tau_2|}X^4],\end{aligned} \tag{31}$$

for $\tau_1,\tau_2>0$, where $X=X_{\text{DWP}}$.

Substituting Eq. (31) and the "vertex"

$$\frac{\partial V_{\text{fluctuon}}(\tau)}{\partial X_{\text{DWP}}}=\frac{6e^{\tau}(X_{\text{DWP}}+e^{\tau}(1+X_{\text{DWP}}))}{(e^{\tau}(1+X_{\text{DWP}})-X_{\text{DWP}})^3} \tag{32}$$

into the rhs Eq. (17) gives

$$\frac{\partial\log\text{Det}(O_{\text{fluctuon}})}{\partial X_{\text{DWP}}}=\frac{4}{1+X_{\text{DWP}}}, \tag{33}$$

cf. Eq. (27).

The results for all three two-loop Feynman diagrams shown in Fig. 3 can be found analytically:

$$\begin{aligned}a=&\frac{3}{560X^2(1+X)^4}\times\left(24X-60X^2-520X^3-1024X^4-832X^5-245X^6\right.\\&\left.-24(1+X)^2(1-4X-18X^2-12X^3)\log[1+X]+288X^2(1+X)^4\text{PolyLog}\left[2,\frac{X}{1+X}\right]\right),\\b_1=&-\frac{1}{280X^2(1+X)^4}\times\left(24X-60X^2-520X^3-1024X^4-832X^5-245X^6\right.\\&\left.-24(1+X)^2(1-4X-18X^2-12X^3)\log[1+X]+288X^2(1+X)^4\text{PolyLog}\left[2,\frac{X}{1+X}\right]\right),\\b_2=&-\frac{1}{560X^2(1+X)^4}\times\left(24X-60X^2+1720X^3+5136X^4+4768X^5+1435X^6\right.\\&\left.-24(1+X)^2(1-4X-18X^2-12X^3)\log[1+X]+288X^2(1+X)^4\text{PolyLog}\left[2,\frac{X}{1+X}\right]\right),\end{aligned} \tag{34}$$

where $X=X_{\text{DWP}}$.

Again, the full two-loop correction takes an amazingly simple form:

$$B_1^{(\text{DWP})}=-X_{\text{DWP}}\frac{(4+3X_{\text{DWP}})}{(1+X_{\text{DWP}})^2}, \tag{35}$$

cf. Eq. (29). All transcendental contributions are again canceled out, and the answer is the meromorphic function of X_{DWP} *only* [3].

E. SGP

In this section we consider the *sine-Gordon potential* (6),

$$V=\frac{1}{g^2}(1-\cos(gx))=\frac{1}{g^2}(1-\cos u),\qquad u=gx,$$

with an infinite number of degenerate vacua situated periodically in x.

For this potential, the flucton at $T = 0$ takes the form

$$x_{\text{flucton}}(\tau) = \frac{4\text{arccot}[(\cosh\tau - \sinh\tau)\cot(\frac{gx_0}{4})]}{g}.$$

The classical flucton action gives

$$S_E[x_{\text{flucton}}] = \frac{16\sin^2(\frac{gx_0}{4})}{g^2}.$$

For the SGP, we introduce the variable

$$X_{\text{SGP}}(x) = \frac{gx}{4} = \frac{u}{4}, \qquad u = gx, \tag{36}$$

cf. Eqs. (24) and (30).

The standard construction yields the following Green function:

$$\begin{aligned} G(\tau_1,\tau_2) = {} & \frac{1}{8(\cosh(\tau_1)+\cos(2X)\sinh(\tau_1))} \times \frac{1}{(\cosh(\tau_2)+\cos(2X_3)\sinh(\tau_2))} \\ & \times \left[2(\tau_1+\tau_2-|\tau_2-\tau_1|)\sin^2(2X) + 8\cos(2X)\sinh^2\left(\frac{1}{2}(\tau_1+\tau_2-|\tau_2-\tau_1|)\right)\right. \\ & \left. + (3+\cos(4X))\sinh(\tau_1+\tau_2-|\tau_2-\tau_1|)\right], \end{aligned} \tag{37}$$

for $\tau_1, \tau_2 > 0$, where $X = X_{\text{SGP}}$.

In this case, the "vertex"

$$\frac{\partial V_{\text{flucton}}(\tau)}{\partial X_{\text{SGP}}} = -\frac{16e^{2\tau}\sec(X_{\text{SGP}})^2\tan(X_{\text{SGP}})(e^{2\tau}-\tan(X_{\text{SGP}})^2)}{(e^{2\tau}+\tan(X_{\text{SGP}})^2)^3}$$

and direct evaluation of Eq. (17) leads to the amazingly simply expression

$$\frac{\partial \log\text{Det}(O_{\text{flucton}})}{\partial X_{\text{SGP}}} = 4\tan[X_{\text{SGP}}], \tag{38}$$

cf. Eqs. (27) and (33).

As for the three two-loop Feynman diagrams shown in Fig. 3, we are only able to analytically calculate diagram a,

$$\begin{aligned} a = -\text{Re}\Big[\frac{1}{640}(5 - 2\sec^2(X_{\text{SGP}}) + \sec^4(X_{\text{SGP}}) \\ - 8\text{PolyLog}[2, -\tan^2(X_{\text{SGP}})] \\ + 8[\csc^2(X_{\text{SGP}}) - \sec^2(X_{\text{SGP}})]\log[\cos(X_{\text{SGP}})])\Big], \end{aligned} \tag{39}$$

where $\text{Re}[x]$ denotes the real part of x. Irrational contributions appear again [see the second line of Eq. (39)] as in the AHO [Eq. (28)] and DWP cases [Eq. (34)]. We were not able to analytically calculate $b_{1,2}$ contributions. However, we can calculate these two-dimensional Feynman integrals numerically. We make two assumptions: (i) all irrational contributions cancel in the sum $B_1 = a + b_1 + b_2$ [see the second line of Eq. (39)], and (ii) the sum $B_1 = a + b_1 + b_2$ is given by a polynomial in $\sec^2(X)$ of degree two,

$$A - D\sec^2(X) + C\sec^4(X),$$

[cf. the first line of Eq. (39)] with the coefficients A, D, and C. Based on these assumptions, we fit the numerical data of B_1 and find that the coefficients A, D, and C can be calculated explicitly. It leads to a very simple expression for the two-loop correction:

$$B_1^{(\text{SGP})} = -\frac{g^2}{16}\tan^2(X_{\text{SGP}}). \tag{40}$$

Eventually, this expression will be verified numerically. Later on, this result will be derived in quantum mechanics using the generalized Bloch equation.

F. The results summarized

The combined results from Secs. III C, III D, and III E show that the expansion (14) of the phase of the ground-state function for the AHO, DWP, and SGP is given explicitly by

$$
\begin{aligned}
2\phi^{(\mathrm{AHO})}(g,x) &= \frac{2}{3}\frac{(1+g^2x^2)^{\frac{3}{2}}-1}{g^2} + \log\left[\frac{(1+g^2x^2+\sqrt{1+g^2x^2})}{2}\right] \\
&\quad + g^2\frac{(1-\sqrt{1+g^2x^2})(5+16\sqrt{1+g^2x^2}+25(1+g^2x^2)+17(1+g^2x^2)^{\frac{3}{2}})}{12(1+g^2x^2)^{\frac{3}{2}}(1+\sqrt{1+g^2x^2})}+\cdots, \\
2\phi^{(\mathrm{DWP})}(g,x) &= \frac{1}{g^2}\left((gx)^2+\frac{2(gx)^3}{3}\right)+2\log(1+gx)-g^2\frac{(gx)(4+3gx)}{(1+gx)^2}+\cdots, \\
2\phi^{(\mathrm{SGP})}(g,x) &= \frac{16}{g^2}\sin^2\left(\frac{gx}{4}\right)+2\log\left[\cos\left(\frac{gx}{4}\right)\right]-\frac{g^2}{16}\tan^2\left(\frac{gx}{4}\right)+\ldots,
\end{aligned}
\tag{41}
$$

which is the Laurent expansion in powers of g^2 if the variable $u = gx$ is introduced.

III. REDERIVATION OF THE LOOP EXPANSION FROM THE SCHRÖDINGER EQUATION

Our ultimate aim remains a generalization of the semiclassical theory to QFTs; thus, all of our quantum-mechanical examples should be written as anharmonic perturbations of the harmonic oscillator,

$$V = \frac{\tilde{V}(gx)}{g^2}$$

[see Eq. (2)], where $\tilde{V}$ has a minimum at $x = 0$ and it always starts from quadratic terms. The classical vacuum energy is always taken to be zero, $V(0) = 0$. Note that $\hbar$ was put to one, as is traditionally done in QFTs. The semiclassical expansion is done in powers of the small coupling g instead of powers of $\hbar$. It should be in agreement with the so-called nonperturbative normalization of the non-Abelian gauge theory, where the coupling appears *only* in front of the action.

However, in a quantum-mechanical setting traditional units are different, and in the next subsection we show how one can reformulate these results as an expansion in powers of the Planck constant $\hbar$.

A. Quantum-mechanical meaning of the loop expansion: Generalized Bloch equation

In order to clarify the meaning of the semiclassical loop expansion in quantum mechanics, it is convenient to change notation as follows. Taking the AHO potential (4) as an example, we recall the units used. The mass of the particle $m = 1$, the frequency of the near-minimum oscillations $\omega = 1$, and the Planck constant $\hbar = 1$ are all put to unity. Now we want to restore $\hbar$ in the exponent of the quantum (statistical) weight $\exp(-S_E/\hbar)$, and make a shift to the "classical coordinate" $u = gx$. Now the Euclidean action is

$$S_E = \frac{1}{\hbar g^2}\int d\tau\left(\frac{\dot{u}^2}{2}+\frac{u^2(1+u^2)}{2}\right). \tag{42}$$

In other words, we have selected different unit of length, thereby eliminating the parameter in the nonlinear, quartic term. The coupling constant now appears only together with the Planck constant. So, one can put it to one, $g = 1$, using only $\hbar$ as a parameter of the loop semiclassical expansion. The classical equation of motion does *not* depend on it, and thus neither does the flucton solution itself. Furthermore, the Green function—which inverts the operator of quantum fluctuations around the flucton—depends on the classical coordinate $u = gx$ but does *not* depend on $\hbar$. A similar consideration can be made for a general potential (2) and we arrive at the Euclidean action

$$S_E = \frac{1}{\hbar g^2}\int d\tau\left(\frac{\dot{u}^2}{2}+\tilde{V}(u)\right). \tag{43}$$

The parameters of the problem can thus be defined as (i) the quantum parameter $\hbar g^2$ (or just $\hbar$ for the $g = 1$ choice) and (ii) the classical coordinate location $u_0 = gx_0$ under consideration. The loop expansion of the semiclassical theory we discuss is therefore redefined as just the Laurent expansion in powers of $\hbar g^2$, starting from the classical term $O(1/\hbar g^2)$, the determinant $O((\hbar g^2)^0)$, the two-loop diagrams $O((\hbar g^2)^1)$, and so on. For $g = 1$ the loop expansion appears as a Laurent expansion in $\hbar$. Naturally, its validity is expected when $S_{\mathrm{flucton}}/(\hbar g^2) \gg 1$, and thus at small $\hbar \ll 1$ and/or sufficiently large u_0. Below we quantify the accuracy of this expansion in detail.

B. Iterative solution of the Schrödinger equation

In this section we rederive first three terms of the loop expansion (14), and derive one more term, based on quantum mechanics, employing the usual Schrödinger equation for the wave function. Let us stress that this is the only part of our program which cannot be straightforwardly generalized to QFT, at least so far. It allows us to cross-check the results obtained in the loop expansion.

The first step is standard; we proceed from the Schrödinger equation of the wave function to that of its logarithmic derivative, which eliminates the overall normalization constant. In the second step we extract one power of the coordinate from the function,

$$xz(gx) = -\frac{\psi'(x)}{\psi(x)}. \tag{44}$$

This reflects the fact that since it is assumed that the original potential (2) has a minimum at $x = 0$, the logarithmic derivative of the wave function (the derivative of the phase) has to vanish at $x = 0$. By substituting it into the Schrödinger equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x),$$

where the Planck constant $\hbar = 1$, one gets the following equation for $z(gx)$:

$$gxz'(gx) + z(gx) - x^2 z(gx)^2 = 2E - \frac{2}{g^2}\tilde{V}(gx). \tag{45}$$

Now we redefine the coordinate $u = gx$ and obtain the form of the equation we will be solving,

$$g^2 uz'(u) + g^2 z(u) - u^2 z(u)^2 = 2g^2 E - 2\tilde{V}(u). \tag{46}$$

We call this equation *the generalized Bloch equation*. Here $z(u)$ represents the *reduced* logarithmic derivative. Now we proceed to solve Eq. (46).

C. Weak coupling expansions

1. Riccati-type equation

Let us now reintroduce the logarithmic derivation

$$y(x) = -\frac{\psi'(x)}{\psi(x)}$$

and write a standard Riccati equation for the potential (2),

$$y' - y^2 = 2E - x^2 - 2a_3 gx^3 - 2a_4 g^2 x^4 - \ldots \tag{47}$$

[cf. Eq. (45)], instead of the Schrödinger equation, as in the *nonlinearization procedure* [12,13]. Now we develop a perturbation theory in powers of g,

$$E = \sum_0^\infty e_m g^m, \qquad y(x) = \sum_0^\infty y_m(x) g^m.$$

It is evident that all e_m with odd m should vanish [see Eq. (3)], $e_{2k+1} = 0, k = 0, 1, 2, \ldots$ The unperturbed solution of Eq. (47) at $g = 0$ is equal to $E_0 = e_0 = \frac{1}{2}$ and $y_0 = x$.

The equation for the correction of the order g reads

$$y_1' - 2xy_1 = -2a_3 x^3,$$

with the solution

$$y_1 = a_3 x^2 + a_3, \qquad e_1 = 0.$$

The equation for the correction of the order g^2 is

$$y_2' - 2xy_2 = e_2 - 2a_4 x^4 + y_1^2 = e_2 + a_3^2 - (2a_4 - a_3^2)x^4 + 2a_3^2 x^2,$$

with the solution

$$-y_2 = (a_4 - a_3^2/2)x^3 + a_3 x^2 + ax + a, e_1 = 0.$$

In general, the equation for the mth correction has the form

$$y_m' - 2y_0 y_m = e_m - Q_m - a_m x^m,$$

where $Q_m = -\sum_{p=1}^{m-1} y_p y_{m-p}$ for $m > 1$ plays the role of an effective perturbation potential: it is made from previous iterations. It can be easily demonstrated that *the correction* $y_m(x)$ *is a finite-order polynomial in* x and, in principle, it can be found by algebraic means,

$$y_m(x) = A_{m-1}^{(m)} x^{m-1} + \cdots + A_k^{(m)} x^k + \cdots + A_0^{(m)}. \tag{48}$$

A straightforward analysis leads to the remarkable property

$$A_k^{(m)} \sim \frac{m!}{k!}$$

(see Ref. [12]). In general, $A_k^{(m)}$ look like generalized Catalan numbers. Hence, the coefficient $A_k^{(m)}$ at fixed $k \sim m$ defines the convergent series in m, while at small fixed k the series is usually divergent. In particular,

$$A_1^{(m)} = e_m.$$

Let us change the variable in Eq. (48),

$$A_k^{(m)} \to A_{m-k}^{(m)}, \qquad k = 1, 2, \ldots (m-1).$$

It is natural to introduce the generating function

$$\tilde{y}_k(x; \{a\}) = \sum_{m=k}^\infty g^m A_{m-k}^{(m)}(\{a\}) x^{m-k}.$$

If the potential (2) is a polynomial, several leading generating functions can be found explicitly, at $k = 1, 2, 3, \ldots$ E.g., for the AHO ($a_3 = 0$, $a_4 = 1$ and $a_k = 0, k = 5, 6, \ldots$),

$$\tilde{y}_0 = \sum_{m=0}^\infty g^m A_m^{(m)} x^m = x(1 + g^2 x^2)^{1/2}$$

(see Ref. [12]), cf. Eq. (41). It can be immediately recognized as the classical momentum at zero energy! Hence, the sum of leading terms of the corrections $y_m(x), m = 0, 1, 2\cdots$ at $x \to \infty$ is the classical momentum at zero energy: it is similar to the leading log approximation in QFT. In the same way, one can calculate the sum of next-to-leading terms of the corrections $y_m(x), m = 1, 2\cdots$ at $x \to \infty$,

$$\tilde{y}_1 = \sum_{m=1}^{\infty} g^m A^{(m)}_{m-1} x^{m-1} = g^2 x \frac{1 + \frac{1}{2\sqrt{1+g^2x^2}}}{(1 + g^2x^2 + \sqrt{1+g^2x^2})},$$

(see Ref. [12]), cf. Eq. (41), which is the logarithmic derivative of the determinant! Hence, the sum of subleading (next-to-leading) terms of the corrections $y_m(x)$, $m = 0, 1, 2\cdots$ at $x \to \infty$ is the logarithmic derivative of the determinant: it is similar to the next-to-leading-log approximation in QFT. We can move even further and calculate next-to-next-to-leading terms in the corrections $y_m(x)$, $m = 1, 2\cdots$ at $x \to \infty$ and then sum them up and discover that

$$\tilde{y}_2 = \sum_{m=2}^{\infty} g^m A^{(m)}_{m-2} x^{m-2} = \frac{dB_1}{dx}$$

(see Ref. [12]), cf. Eq. (41). Thus, the result occurs as a derivative of the two-loop contribution (19) [see Eq. (29)], where $x = X_{\mathrm{AHO}}$. It can be checked (see below) that $\tilde{y}_3$ is the first derivative of the three-loop contribution (19).

The property that the first three generating functions $\tilde{y}_{0,1,2}$ are first derivatives of the first three terms in the loop expansion holds for the DWP. In general, the expansion in such generating functions,

$$y = \tilde{y}_0 + \tilde{y}_1 + \tilde{y}_2 + \ldots,$$

is a *new* semiclassical expansion, resembling, e.g., the logarithmic approximations of QFT.

2. Generalized Bloch equation case

Calculations of the loop expansion for the ground-state wave function phase performed earlier for the AHO, DWP, and SGP show that this expansion looks like a perturbation series in g^2 (starting from the $1/g^2$ term) if the classical coordinate $u = gx$ is introduced. It is natural to construct this perturbation theory for the phase in the generalized Bloch equation (46) and compare it with the loop calculation.

Solving Eq. (46) iteratively, we generate the flucton loop expansion. Let us define the series

$$z(u) = \sum_{n=0} g^{2n} z_n(u), \tag{49}$$

which corresponds to the perturbative solution of this equation in powers of g^2 with

$$E = \sum_{n=0} g^{2n} E_n$$

given by standard perturbation theory in g^2.

At the zeroth order, in which all terms proportional to the coupling are ignored, the equation is very simple,

$$-u^2 z_0(u)^2 = -2\tilde{V}(u), \tag{50}$$

leading to

$$z_0(u) = \frac{\sqrt{2\tilde{V}(u)}}{u}. \tag{51}$$

This result (uz_0) is, in fact, the classical momentum at zero energy, and therefore, as one returns to the wave function, the zeroth-order term gives the well-known semiclassical action. So, at this stage, the result is the well-known $\psi \sim \exp(-\int^x p(x')dx')$ but at zero energy. It can be immediately checked that the classical flucton action for the AHO, DWP, and SGP [see Eq. (41)] is nothing but the semiclassical action at zero energy, $\int u z_0(u) du$ with $z_0(u)$ given by Eq. (51).

Moving to the next term of the expansion, one finds the following $O(g^2)$ equation:

$$u z_0'(u) + z_0(u) - 2u^2 z_0(u) z_1(u) = 2E_0. \tag{52}$$

Note that here the equation involves the known function z_0 of the previous order, and z_1 just appears linearly. A similar feature takes place at all orders!

An important point of the correct procedure is that the energy needs to be used in the form of the perturbative expansion in powers of g^2 as well but for the original potential (2),

$$E = \sum_{n=0} g^{2n} E_n. \tag{53}$$

The zeroth-order potential is for the harmonic oscillator, so $2E_0 = 1$. Hence, the first correction is

$$z_1(u) = \frac{u z_0'(u) + z_0(u) - 1}{2u^2 z_0(u)}. \tag{54}$$

It can be immediately checked that the logarithm of the determinant $\log \mathrm{Det}(O_{\mathrm{flucton}})$ [Eq. (17)] for all three potentials AHO, DWP, and SGP [see Eq. (41)] is nothing but $\int u z_1(u) du$ with $z_1(u)$ given by Eq. (54). Thus, in a very simple way we calculated determinant $\log \mathrm{Det}(O_{\mathrm{flucton}})$ explicitly in closed analytic form for the general potential $V(x)$ [Eq. (2)]. Or, in other words, we explicitly calculated

the one-loop diagram of Fig. 2. This result, written in terms of the classical fluctuon action and its derivatives, is one of the central results of this paper.

Moving to the next term of the expansion, one finds the following $O(g^4)$ equation:

$$uz_1'(u) + z_1(u) - u^2 z_1^2(u) - 2u^2 z_0(u) z_2(u) = 2E_1. \quad (55)$$

Note that here the equation involves the known functions of the previous orders $z_{0,1}$ nontrivially, but the new function z_2 appears only linearly. This feature is generic and is repeated at each order, so there is no difficulty in finding new corrections.

The perturbative coefficient E_1 is the perturbative correction $\sim g^2$ to the ground-state energy in the quartic part of the original potential (2),

$$V(x) = \frac{1}{2}x^2 + a_3 g x^3 + a_4 g^2 x^4,$$

which explicitly can be easily found, e.g., in the nonlinearization procedure [12],

$$2E_1 = \frac{3}{2}a_4 - \frac{11}{4}a_3^2.$$

Solving Eq. (55), we find the second correction

$$z_2(u) = \frac{uz_1'(u) + z_1(u) - u^2 z_1^2(u) - 2E_1}{2u^2 z_0(u)}, \quad (56)$$

which defines the two-loop contribution B_1. It can be immediately checked that for the AHO, DWP, and SGP [see Eq. (41)] it is nothing but $\int u z_2(u) du$ with $z_2(u)$ given by Eq. (56). Thus, in a very simple way we calculated the two-loop contribution explicitly in closed analytic form for the general potential $V(x)$ [Eq. (2)]. Or, in other words, we explicitly calculated the sum of the three two-loop diagrams in Fig. 3, weighted with symmetry factors. For the AHO, DWP, and SGP this sum $\int u z_2(u) du$ with $z_2(u)$ does *not* contain transcendental contributions. It must be emphasized again that it was difficult to guess that such a result could exist in general.

Moving to the next term of the expansion, one finds the following $O(g^6)$ equation:

$$uz_2'(u) + z_2(u) - 2u^2 z_1(u) z_2(u) - 2u^2 z_0(u) z_3(u) = 2E_2. \quad (57)$$

Note that here the equation involves the known functions of the previous orders $z_{0,1,2}$, and again z_3 appears linearly. The perturbative coefficient E_2 is the perturbative correction $\sim g^4$ to the ground-state energy in the sextic part,

$$V(x) = \frac{1}{2}x^2 + a_3 g x^3 + a_4 g^2 x^4 + a_5 g x^5 + a_6 g^2 x^6,$$

of the general potential (2), which can be straightforwardly found, e.g., in the nonlinearization procedure [12]. Eventually, the third correction

$$z_3(u) = \frac{uz_2'(u) + z_2(u) - 2u^2 z_1(u) z_2(u) - 2E_2}{2u^2 z_0(u)} \quad (58)$$

defines the three-loop contribution B_2 analytically. In the Feynman diagram technique (the fluctuon formalism) it corresponds in general to a sum of the 15 three-loop diagrams in Fig. 2 in Ref. [10], weighted with symmetry factors. So far we have been unable to analytically calculate any diagrams for AHO, DWP, and SGP potentials. It is needless to say that the next iterations will provide higher-loop contributions in the same straightforward way.

D. AHO: Three-loop correction

Starting from the two-loop correction $z_2(u)$, the details of the specific example become relevant since one needs a concrete value for E_1. So, from this point on, we present one more term for the AHO case (4), i.e., the three-loop correction. In order to do this, we repeat the consideration of the previous section in brief for the case of the AHO.

For convenience, we introduce a new variable $s = u^2$. Then, the generalized Bloch equation (46) takes the form

$$2g^2 s z'(s) + g^2 z(s) - s z(s)^2 = 2Eg^2 - s(1+s), \quad (59)$$

where $s \in [0, \infty)$. At zeroth order $(g^2)^0$ we have the equation

$$z_0(s)^2 = (1+s), \quad (60)$$

cf. Eq. (51). Then, for the normalizability of the wave function it is required to take the positive solution $z_0(s) = \sqrt{1+s}$. The equation to the next order g^2 is given by

$$2sz_0'(s) + z_0(s)(1 - 2sz_1(s)) = 2E_0, \quad (61)$$

from which it follows that

$$z_1(s) = \frac{1 + 2s - 2E_0\sqrt{1+s}}{2s(1+s)}, \qquad 2E_0 = 1.$$

Note that the condition that the function $z_1(s)$ is not singular at the origin $(s = 0)$ also implies $2E_0 = 1$. Now, the vanishing of the coefficient in Eq. (59) of order g^4 leads to the equation for the second correction $z_2(s)$, which is equal to

$$z_2(s)=\frac{-E_1+z_1-sz_1^2+2sz_1'}{2sz_0}=\frac{4(-1+\sqrt{1+s})+s(-7-8s+8\sqrt{1+s})-4s(1+s)^2E_1}{8s^2(1+s)^{5/2}}, \tag{62}$$

where $z_1(s)$ is already known and $E_1=\frac{3}{4}$ is the well-known first energy correction to the AHO ground state. At small $s\to 0$ we obtain

$$z_2(s)\approx\frac{3-4E_1}{8s}+\frac{1}{4}(-6+E_1)-\frac{3}{64}s(-63+4E_1)+\ldots \tag{63}$$

The value $E_1=\frac{3}{4}$ leads to the disappearance of the first (singular) term in this expansion. Similarly, we obtain

$$z_3(s)=\frac{1}{32\sigma^8(1+\sigma)^3}[60+230\sigma+346\sigma^2+270\sigma^3+150\sigma^4+108\sigma^5+84\sigma^6+63\sigma^7+21\sigma^8], \tag{64}$$

where $E_2=-\frac{21}{16}$, and $\sigma=\sqrt{1+s}$. Of course, in the variable σ [which is nothing but $z_0(s)$ in Eq. (60)] it can be easily seen that all corrections $z_n(s)$ are meromorphic functions, and no transcendental terms occur. From Eq. (44) we immediately make the identifications

$$\begin{aligned}z_0(s)&=g^2\partial_s S_{\text{flucton}},\\ z_1(s)&=\partial_s\left(\frac{1}{2}\log\text{Det}(\text{flucton})\right),\\ z_2(s)&=g^{-2}\partial_s(\text{two-loop}),\\ z_n(s)&=g^{-2(n-1)}\partial_s(n\text{-loop}).\end{aligned} \tag{65}$$

E. ADWP: Classical action and one-, two-, and three-loop corrections

Finally, within the iteration method for the generalized Bloch equation (46), we consider the ADWP [see Eq. (7)],

$$V=\frac{1}{2}x^2(1+2tgx+g^2x^2),\qquad t\in[0,1],$$

which is, in fact, a general quartic potential.

In this case, one potential minimum is situated at $x=0$ and $V(0)=0$, while the second minimum (when it exists) is situated to the left of $x=0$ and $V(x_{\min})\geq 0$, see Fig. 4.

The generalized Bloch equation (46) takes the form

$$g^2uz'(u)+g^2z(u)-u^2z(u)^2-2Eg^2+u^2(1+2tu+u^2)=0. \tag{66}$$

After a straightforward calculation (see, e.g., Ref. [12]), we find explicitly the first three coefficients of the perturbative expansion in g^2 of the energy (53),

$$\begin{aligned}E_0&=1,\\ E_1&=\frac{1}{4}(3-11t^2),\\ E_2&=-\frac{3}{16}(7-114t^2+155t^4).\end{aligned} \tag{67}$$

These coefficients will be needed to find the one-, two-, and three-loop contributions in the iteration method applied for Eq. (66).

The zero iteration of Eq. (66) gives the classical momentum at zero energy.

$$z_0=u\sqrt{1+2tu+u^2}=gx\sqrt{1+2tgx+g^2x^2}\equiv(gx)X^{(\text{ADWP})},$$

where for convenience we will denote hereafter $X^{(\text{ADWP})}\equiv X_4$, while the classical flucton action $\sim\int uz_0(u)$ reads

$$S_{\text{flucton}}=\frac{-2+3t^2+3t(1-t^2)(\text{Log}[1+t]-\text{Log}[F+X_4])-3FtX_4+2X_4^3}{3g^2}, \tag{68}$$

where $F\equiv\sqrt{-1+t^2+X_4^2}$. As for the determinant, $\log\text{Det}(O_{\text{flucton}})$ is equal to

$$\frac{1}{2}\log\text{Det}(\text{flucton})=\text{Log}\left[\frac{X_4}{2}\right]+\text{Log}[1+(F-t)t+X_4]. \tag{69}$$

It can be immediately checked that by taking Eqs. (68) and (69) at $t=0$ we recover the results for the AHO and at $t=\pm1$ those of the DWP. As for the two-loop correction B_1, it takes the form

$$B_1^{(\mathrm{ADWP})} = \frac{1}{12(F-t)^2(-1+t^2)X_4^3}(5(1+Ft-2t^2-Ft^3+t^4)+6(1-t^2)X_4-2(1+Ft-t^2)X_4^2 + (-17-34Ft+134t^2+200Ft^3-285t^4-170Ft^5+170t^6)X_4^3+(-9+31Ft-20t^2-33Ft^3+33t^4)X_4^4 + (17-100t^2+85t^4)X_4^5).$$

In the limits $t=\pm 1$ and $t=0$, it coincides with the two-loop correction B_1 for the DWP and AHO, respectively.

The three-loop correction B_2—which in the path-integral formalism is given by the sum of 15 weighted Feynman integrals (running from one-dimensional up to six-dimensional integrals, weighted with symmetry factors)—in the iterative approach to the generalized Bloch equation (46) can be easily calculated:

$$B_2^{(\mathrm{ADWP})} = \frac{1}{64(F-t)^4}\left[\frac{40(1+2Ft-2t^2)(-1+t^2)}{X_4^6}-\frac{40(1+Ft-t^2)}{X_4^5}+\frac{8(8+8Ft-13t^2)}{X_4^4} + \frac{64}{X_4^3}-\frac{8(5-6Ft+17t^2+22Ft^3-22t^4)}{X_4^2}+\frac{8(-5+11Ft-11t^2)}{X_4} + ((207-3598t^2+5639t^4)(F-t)^4+8(-3+11t^2))+4(35+426t^2-1833t^4+1860t^6 - 3Ft(69-301t^2+620t^4))X_4-4(21+406t^2-1395t^4+Ft(-187+465t^2))X_4^3\right]. \quad (70)$$

It is rather surprising that $B_2^{(\mathrm{ADWP})}$ is given by such a compact expression. In particular,

$$B_2^{(\mathrm{AHO})} = -\frac{40+120X_{\mathrm{AHO}}+136X_{\mathrm{AHO}}^2+88X_{\mathrm{AHO}}^3+80X_{\mathrm{AHO}}^4+112X_{\mathrm{AHO}}^5-39X_{\mathrm{AHO}}^6-330X_{\mathrm{AHO}}^7-207X_{\mathrm{AHO}}^8}{64X_{\mathrm{AHO}}^6(1+X_{\mathrm{AHO}})^2},$$
$$B_2^{(\mathrm{DWP})} = \frac{X_{\mathrm{DWP}}(128+300X_{\mathrm{DWP}}+248X_{\mathrm{DWP}}^2+71X_{\mathrm{DWP}}^3)}{4(1+X_{\mathrm{DWP}})^4}. \quad (71)$$

For completeness we present the three-loop correction for the SGP:

$$B_2^{(\mathrm{SGP})} = \frac{7-(6+\cos(X_{\mathrm{SGP}}))\sec^4(X_{\mathrm{SGP}})}{1024}. \quad (72)$$

We note that is has an amazingly simple form.

F. Strong coupling expansion

So far we have studied the weak coupling expansion for the generalized Bloch equation (46),

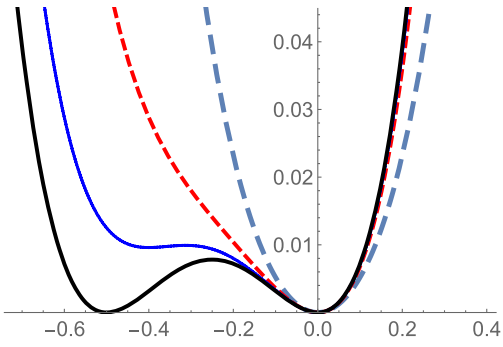

FIG. 4. The ADWP for $t=0$ (long dashed, grey, anharmonic oscillator case), 0.8 (short dashed, red), 0.95 (solid blue), and 1 (solid black, double well case) at the coupling $g=2$.

$$z(u)=\sum_{n=0} g^{2n} z_n(u), \qquad E=\sum_{n=0} g^{2n} E_n,$$

which corresponds to perturbation theory in g^2. Now we will study the strong coupling expansion in $1/g$. It is convenient to consider a particular potential to break the idea of generality. We present here the results for the AHO case (4),

$$V=\frac{1}{2}x^2(1+g^2x^4).$$

First we will introduce the classical coordinate $u=gx$, and then introduce the new variable $s=u^2$. Then the generalized Bloch equation (46) takes the form (59),

$$2g^2 s z'(s)+g^2 z(s)-s z(s)^2 = 2Eg^2 - s(1+s),$$

where $s\in[0,\infty)$. Since we know (functionally) the strong coupling expansion for energy (see, e.g., Ref. [14]), we can develop the perturbation theory,

$$E=g^{2/3}\sum b_n g^{-\frac{4n}{3}}, \qquad z=g^{2/3}\sum F_n(s) g^{-\frac{4n}{3}}, \quad (73)$$

where b_n, $n=0,1,2\ldots$ are strong coupling coefficients for energy, a few of which have been found numerically with

high accuracy. The equation for finding the zeroth-order $O(g^{2+\frac{2}{3}})$ is of the form

$$2sF_0' + F_0 = b_0,$$

which does *not* depend on the potential explicitly, and

$$F_0 = b_0. \tag{74}$$

The equation for the first-order correction $O(g^{2-\frac{2}{3}})$ is

$$2sF_1' + F_1 - sF_0^2 = b_1,$$

which also does *not* depend on the potential explicitly, and

$$F_1 = \frac{b_0^2}{3}s + b_1. \tag{75}$$

It can be easily shown that the nth correction is a polynomial of degree n,

$$F_n = \alpha_0(b)s^n + \alpha_1(b)s^{n-1} + \cdots + b_n.$$

IV. THE ACCURACY OF THE PERTURBATIVE AND SEMICLASSICAL LOOP EXPANSIONS

In this section we address the issues of convergence and accuracy of the expressions derived above, comparing the terms of the expansion to each other and to the wave functions obtained numerically. In particular, this comparison will quantify the meaning of the "large classical coordinate" $y = gx$ and "small coupling" g. For definiteness, we discuss these issues for the case of the AHO system.

Let us first address the issue of convergence of the perturbative series in g at weak coupling. In Fig. 5 we plot four subsequent terms of the expansion of the wave-function phase and their sum, for "small" ($g^2 = 1/3$) and "large" ($g^2 = 2$) couplings. In all cases there is a clear dominance of the classical $O(g^{-2})$ term at large values of the coordinates, $x \gg 1$. This happens because only the classical term grows with x. But if one excludes the leading term and compares the subsequent loop corrections themselves, the series seem to converge at *all* x rather well.

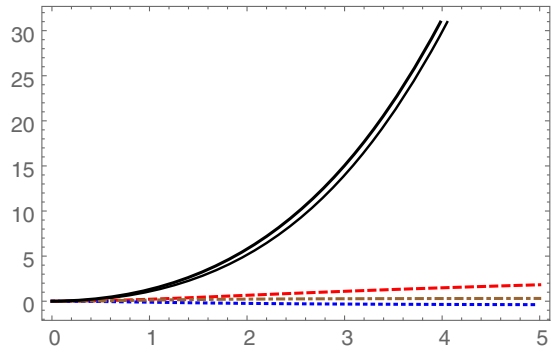

FIG. 5. The (double) phase of the wave function 2ϕ for the anharmonic oscillator versus the coordinate x, for the coupling $g^2 = 1/3$. The (lower) thin black solid line is the leading term, corresponding to the classical fluctuon action. The red dashed, blue dotted, and brown dot-dashed lines show the magnitude of the one-, two-, and three-loop corrections. Their sum is shown by the (upper) thick black solid line.

We now proceed to the case of strong coupling, $g > 1$, and ask whether the semiclassical theory is still applicable in this domain, at large values of the coordinate. In Fig. 6 we compare four terms of the loop expansion with the extremely accurate variational wave function derived previously by one of us [15], for rather strong coupling $g^2 = 2$.

Two observations come from this plot. The first is that at strong coupling the convergence at small $x < 1$ is gone. However, the second is that in the semiclassical domain $x > 1$ one can see that higher-loop corrections do in fact improve the classical result. In fact, for $x > 1.5$ the difference between the fluctuon series (up to the fourth term) and the variational curve is smaller than the width of the line!

The last issue we discuss in this section is that of the overall normalization constant. The fluctuon method, by construction, is designed to give the *relative* probability to find a particle at different locations. Rather arbitrarily, in the discussion above we have selected the "normalization point" to be located at the potential minimum. Indeed, our fluctuon and its action are both zero, for a particle located there. So the same value (taken to be one) is used for all wave functions at the maximum $x = 0$.

The left panel of Fig. 7 shows a comparison of the semiclassical density matrix (the sum of three terms) with the exact (numerically calculated at energy $E_0 = 0.69617575$) wave function squared, for $g = 1$. While such normalization is natural for the semiclassical approach used, it is in fact inadequate in the following sense. As it is clear from the left panel of Fig. 7, this normalization does not

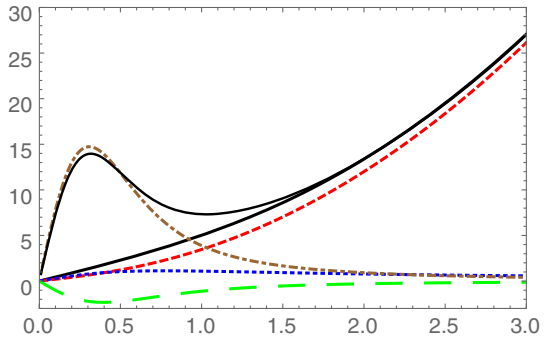

FIG. 6. The comparison between the variational wave function squared [15] (thick black solid line) with the fluctuon loop expansion, at zero to three loops, at the coupling $g^2 = 2$. The red dashed, blue dotted, green long-dashed, brown dot-dashed, and thin solid black lines are for the classical action, the one-, two-, and three-loop contributions, and their sum, respectively.

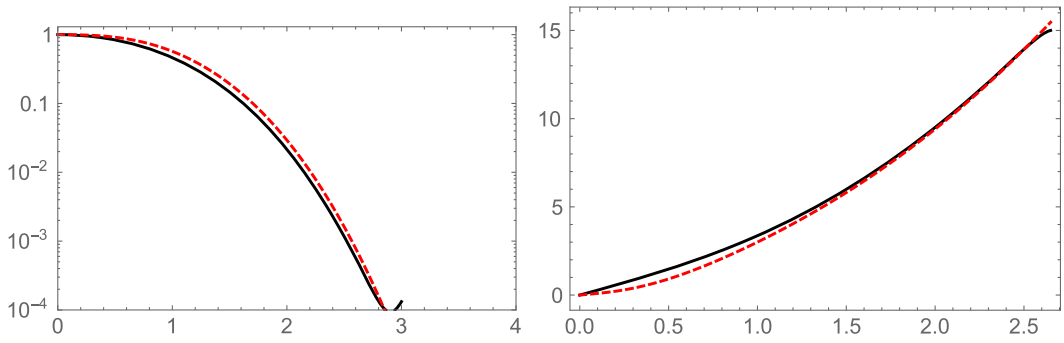

FIG. 7. The comparisons for the anharmonic oscillator with the coupling $g^2 = 1$ of the numerically calculated wave function (black solid lines), with the flucton expansion (zero, one, and two loops summed) shown by the red dashed lines. The left and right panels show the density in the ground state and its logarithmic derivative versus the coordinate x, respectively.

provide a good description at large x, which is the semiclassical domain.

Such an outcome is of course not unexpected. Our derivation from the Schrödinger equation in Sec. III B is based on the *logarithmic derivative* of the wave function (44), which does not depend on the normalization constant. Therefore, a more meaningful comparison between the semiclassical expansion and the exact wave function can be provided by the plot of the corresponding logarithmic derivatives. Such a comparison is shown in the right panel of Fig. 7: now the agreement between the two curves is observed for $x > 1$, in the semiclassical domain. Outside it, at $x < 1$, the agreement is not expected, but it is not too bad either. (Note that this figure corresponds to a coupling that is not small, $g^2 = 1$).

V. THE SEMICLASSICAL EXPANSION FOR POTENTIALS WITH MULTIPLE MINIMA

In general, one may also think of potentials with N minima, and ask how the flucton-based approximation for the path integral we develop should be applied in this case.

In our first case all minima are *degenerate*, corresponding to the same energy (which then can always be put to zero). Since in this case all of the minima can be used for the "long-time relaxation" of the flucton paths, one can think of a $N \times N$ matrix of fluctons $x^f_{ij}(\tau)$, starting at $\tau \to -\infty$ in the ith vacuum and ending at $\tau \to \infty$ in the jth one. Of course, for a given "observation point" x_0 one only needs to consider those paths which pass through it. The DWP is an example of such a degenerate situation, to be discussed in Sec. V A.

However the problems with the *nondegenerate* minima, such as in the ADWP case, obviously cannot be treated in this way. There are no nondiagonal paths $i \neq j$ between different maxima which can "relax" at both ends, as those have different energies. Long-time "relaxation" is obviously only possible at the global minima. This situation, which we discuss in Sec. V B, requires complexified classical paths.

A. The density between the minima for the symmetric double well

So far, we have only discussed the "outer region" $|x_0| > x_{\min} = 1/g$ outside its two minima. Now let us discuss the intermediate region, around the middle point $x \sim 0$. For sufficiently small g—and thus well-separated minima—it should also be amenable to a semiclassical treatment.

Following the discussion above, the double-well problem should have four flucton paths. We have discussed the fluctons $x^f_{11}(\tau)$ and $x^f_{22}(\tau)$ associated with, say, the left and the right potential minima. Their contributions generate two familiar maxima in the ground-state wave function.

In the outer region $|x_0| > x_{\min}$ the $x^f_{12}(\tau)$ and $x^f_{21}(\tau)$ fluctons are a combination of the $x^f_{11}(\tau)$ and $x^f_{22}(\tau)$ fluctons, *plus* the *instanton* or *anti-instanton* paths. So the actions are just

$$S^f_{12} = S^f_{11}(x_0) + S_{\text{instanton}}. \tag{76}$$

This means that the density matrix due to the flucton (11) is just corrected by an exponentially small and x_0-independent term,

$$\psi^2_0(x_0) \sim \exp(-S^f_{11}(x_0))(1 + O(e^{-S_{\text{instanton}}})). \tag{77}$$

If $|x_0| < x_{\min}$, in the inner region the $x^f_{12}(\tau)$ and $x^f_{21}(\tau)$ fluctons are nothing other than the *instanton* and *anti-instanton* paths. Their timing can be selected so that at $\tau = 0$ their value, as for all other fluctons, should be x_0. Furthermore, their classical actions

$$S^f_{12} = S^f_{21} = \int_{-x_{\min}}^{x_0} p(x')dx' + \int_{x_0}^{x_{\min}} p(x')dx' = \int_{-x_{\min}}^{x_{\min}} p(x')dx' \tag{78}$$

do not depend on the x_0. Therefore, their contribution to the density matrix in the inner region is (somewhat surprisingly) independent of the observation point x_0.

It may appear strange that the flucton theory has such unusual contributions in the inner region, since in the familiar WKB-like semiclassical theory one does not have those. Note, however, that the WKB method is applied to the wave function, while the flucton theory is applied to the density matrix, or its *square* ψ_0^2: the instanton terms thus come from the product of the two semiclassical contributions in the WKB-like approaches.

To demonstrate its validity, we will use the Turbiner trial function $\cosh(A)$ (see Ref. [15]),

$$\cosh^2(A) = (1/2)(\cosh(2A) + 2) \tag{79}$$

B. The density for the asymmetric double well: Complex fluctons

As we already noted at the beginning of this section, when the minima of the potential are nondegenerate there are no classical solutions going from one maximum to the other of the potential in Euclidian time and "relaxing" at both ends, simply because for that one needs two conflicting values of the energy. In particular, there are no "instanton" and "anti-instanton" solutions available.

It was argued in Refs. [16,17] that by complexification of the coordinate $x(t) \to z(t) = x(t) + i \cdot y(t)$, and thus by generalizing equations of motion to the so-called *holomorphic Newton's equation* (still for inverted potential)

$$\frac{d^2z}{dt^2} = +\frac{\partial V}{\partial z}, \tag{80}$$

one can find a complex generalization of those. Specifically, in those works the contribution of a periodic path with a finite action was discussed, called the "complexified bion" (CB), which is an extension to the instanton–anti-instanton pair solutions for the symmetric potential.

For continuity of notation, let us use the following (Euclidean time) Lagrangian:

$$\mathcal{L}_{\rm ADWP} = \frac{1}{2}\dot{x}(\tau)^2 + \frac{1}{2}(x(\tau)^2 - 1)^2 + pgx(\tau), \tag{81}$$

with the asymmetry parameter p. If p is nonzero but small, the left and right maxima located at x_+ and x_- have different heights, $E_+ = V(x_+) \neq E_- = V(x_-)$ [18].

Like for the symmetric case discussed before, for a generic point x_0 in between the two maxima $x_+ < x_0 < x_-$ there are two flucton solutions, also denoted by $\pm$, which start at x_0 and "relax" for an infinitely long time near either x_+ or x_-. But now there is no symmetry $x \to -x$; thus, these two fluctons have different energy and the issue of the relative normalization of their contributions is rather nontrivial.

The flucton path $f_+(\tau)$, starting from x_+, can reach any point we discuss. But, an additional problem indicating troubles with such an approach is that the flucton path f_- cannot reach *all* points x_0 in the interval $x_+ < x_0 < x_-$ since it has insufficient energy E_- to "climb" all the way to x_+. This path can only reach the turning point and get reflected back. A periodic path starting and ending at x_- is known as a "bounce" solution.

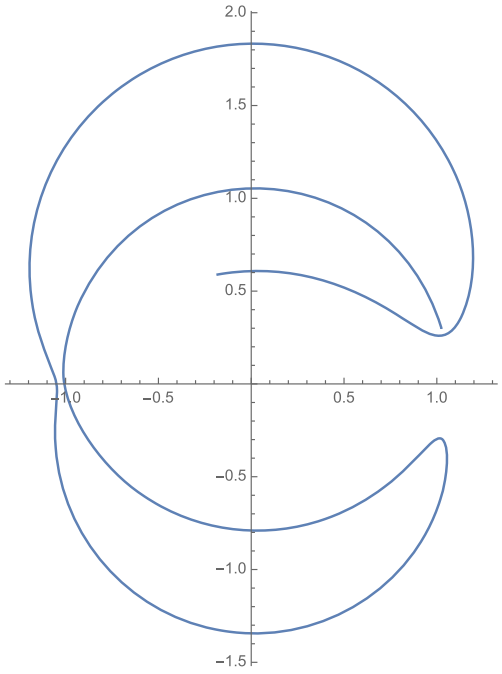

FIG. 8. Two examples of solutions to the holomorphic Newton's equation (80). Both start at zero velocity and are slightly displaced from the maximum of $-V$ located at x_+. The one going upward has a displacement phase tuned so that it goes to the turning point z_1 and is reflected back on the same path, so that one see a single curve. This is the "complex bion" of Ref. [16]. The one going downward, from a slightly different location, traces an infinite path going around both turning paths $z_{1,2}$.

A complexification of the paths opens many new options. Let us start with the generalization of the flucton path $f_+(\tau)$, starting from x_+ with energy E_+. The initial velocity at the top is zero, but—as for a skier at the top of a mountain—there is freedom to slide in any direction.

Where would one like to go? Along the real axis no other point is as high as the inverted potential, so the real flucton path f_+ can reach any x_0 inside or outside of the interval indicated. The classical action $S_+ = \int_{x_+}^{x_0} dx p(x)$ is a monotonous function since $p(x) > 0$; therefore, the corresponding amplitude $\exp(-S_+(x_0))$ decreases monotonously in both directions from x_+. This contribution must be included in the density $P(x_0)$, but it cannot be the only one.

Going in an arbitrary direction from the global maximum leads to a family of paths, two of which are shown in Fig. 8. They get reflected at (or in the vicinity of) two turning points back to the maximum x_+. Since at the top "relaxation" takes a long time, such paths could be periodic, with an infinite period but finite action.

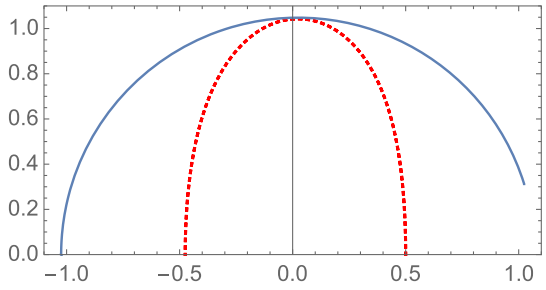

FIG. 9. Two solutions to the holomorphic Newton's equation (80). The blue solid line is the same "complex bion" of Ref. [16] as in the previous figure. The red dashed line starts at some generic point on the real axis, in this case $x_0 = 0.5$, with a velocity tuned so that it touches the "complex bion."

Let us find the two turning points. In the particular case of the ADWP the potential is the fourth-order polynomial in the coordinate, and thus it must have four roots. Hence, it can be rewritten in a convenient form for motion with the maximal energy $E_+ = V(x_+)$ as

$$V - E_+ \sim (z - x_+)^2 (z - z_1)(z - z_2). \tag{82}$$

Note that x_+ must be a double zero, and the two others should be the complex-conjugate $z_1^* = z_2$ pair of two *turning points*. (In our concrete example their location is at $z_{1,2} = 1.02412 \pm 0.312482i$.) At these turning points the velocity on the path vanishes, but since it is a not a maximum (a double zero) no long-time "relaxation" is possible, and the "complexified bion" path bounces back.

One such periodic path with a finite action was pointed out in Refs. [16,17] and named the "complex bion." The complex action of these paths contributes the factors $\exp(-\mathrm{Re}S(z_{CB}))\cos(\mathrm{Im}S(z_{CB}))$ to the amplitude, producing the cosine of certain nontrivial phases.

These phases violate the positivity of the amplitudes present for any real paths, and produce interesting oscillations/cancellations. Some known puzzles associated with the energy spectra of quasi-exactly-solvable [19] and/or supersymmetric examples have been explained in these works.

Our aim is to find "complex fluctons,", i.e., classical paths connecting a generic point x_0 on the real axis with the global maximum at x_+. However, the paths belonging to the family just described cross the real axis *only* at x_+.

In general, starting with the real axis, one see that the kinetic energy $K = E - V(x_0)$ is real. Therefore $\mathrm{Im}(K) = \dot{x}\,\dot{y} = 0$, which means that one of the factors must vanish. Thus there are two sets of paths: they either go along the real axis, or they are normal to it. The answer obviously is defined by the sign of the kinetic energy $2K = \dot{x}^2 - \dot{y}^2$. Taking a generic initial point on the real axis as a starting point $x(t = 0) = x_0$ and various initial values of $\dot{y}(t = 0)$, one can obtain families of solutions to the holomorphic EOM.

In Fig. 9 we show only one of them (red dotted line, for $x_0 = 0.5$) tuned to be touching the "complex bion" path of Unsal *et al.* (blue solid line). We propose to use a combination of two segments of those two curves, before and after the touching point z_{cross}, as a "*complex flucton*," leading from a generic point x_0 to the global maximum x_+. Note, however, that while both curves at the touching point z_{cross} have the same directions of the velocities, its magnitude needs to jump, as the two curves correspond to two different energies.

For a generic paths one can think of them as classical ones, solving the EOM with an appropriate external force term $z(\tau)f(\tau)$ added to the Lagrangian and to the EOM. The advantage of the complex flucton path just introduced is that in this case the force should only be applied at only the crossing-time moment $f(\tau) \sim \delta(\tau - \tau_{\mathrm{cross}})$, as an instantaneous kick adjusting the total energy. There is a finite (although small in the example considered) contribution of this kick to the action, which should not be omitted.

To summarize the construction, there are two paths, both of which go from x_0 to the maximum x_+: the original real flucton and the complex one. The corresponding contributions to the density at x_0 are now both normalized in the same way, and their sum has the form

$$P(x_0) \sim 2\exp(-S_{\text{real fl}}) + \exp[-\mathrm{Re}(S_{\text{complex fl}})]2\cos(\mathrm{Im}(S_{\text{complex fl}}), \tag{83}$$

where we have added the complex-conjugate part of the path in the lower hemisphere $\mathrm{Im}(z) < 0$. Like for the complex bion contribution to the ground-state energy, the contribution of the complex flucton may be positive or negative, depending on the particular value of the imaginary part of its action.

VI. CONCLUSIONS

In our previous paper I we outlined a new *flucton-based* semiclassical theory, based on the path-integral representation of the density matrix. Corrections to leading semiclassical results take the form of Feynman diagrams, which are well defined to any order by standard Feynman rules. As examples of its applications, we calculated one- and two-loop corrections for the ground-state density (square of the wave function) of the AHO.

In this second paper we describe the foundations of the method in more details, and also presented a number of new results. At the start of the paper, we summarize the completed one- and two-loop calculations based on Feynman diagrams, for all three physically important examples: the AHO, DWP, and SGP [see Eq. (41)].

We showed that in the case of polynomial potentials the perturbation corrections to the imaginary phase of the wave function $\phi(x)$ are finite-degree polynomials. We demonstrated that generating functions of their leading degrees coincide with corresponding terms in a loop expansion. Eventually, we found the Taylor expansion at small distances,

$$\phi(x) = A_0 x^2 + A_1 x^3 + \cdots,$$

while the loop expansion is nothing but the expansion at large x. It is sufficiently straightforward to attempt to interpolate between these two regimes. For the AHO, the interpolating trial function

$$\phi_{\rm int}(x) = \frac{A + Bx^2 + Cg^2x^4}{\sqrt{D + g^2x^2}} + \frac{1}{2}\log \mathrm{Det}(O_{\rm flucton})$$

(with parameters A, B, C, and D) gives extremely high local accuracy for all x, in practice solving the problem, cf. Ref. [15]. Similar approximants, interpolating between the small-x series and our loop expansion, can be made for other quantum-mechanical problems.

We also were able to relate these results to the iterative solution of Eq. (46) for the reduced logarithmic derivative of the wave function. As we showed explicitly, all our results from the Feynman diagrams are reproduced exactly! Among other insights, this way of deriving it explains why all irrational functions (such as logs and polylogs) which appeared in expressions for an individual Feynman diagram are always canceled in their sum. This method of calculation allows us to go to higher orders; in particular, we calculated one more term of the expansion, corresponding to the sum of the 15 three-loop Feynman diagrams, none of which have analytically evaluated so far. (Needless to say, this method of calculation—starting from the Schrödinger equation—is not generalizable to QFT applications, at least at present.)

We have studied the issues of convergence and the accuracy of this version of the semiclassical theory. It was shown that in the case of weak coupling, the series for the density are well convergent. Even if the coupling is not small, in the semiclassical domain it seems to be convergent to the exact answer (provided one uses the logarithmic derivative of the wave function, removing sensitivity to the normalization constant.)

Finally, we discussed the generalization of the flucton theory to the case of more than one minimum of the potential using the example of the DWP problem. In the case when two minima are degenerate, the density in between them is defined via a sum of four contributions: the left- and right-side fluctons, and the instanton and anti-instanton solutions. Interestingly, the latter produce a constant (x-independent) contribution to the density matrix in this region.

The case of (slightly) nondegenerate minima is much more involved. The authors of Refs. [16,17], in which the ground-state energy of the asymmetric double-well potential was studied, proposed complexifying the coordinate $x(\tau) \to z(\tau) = x(\tau) + iy(\tau)$ and including contributions of a certain finite action solution to the holomorphic EOM (80). We studied such paths and found large families of such solutions, starting from the maximum of the inverted potential. We also showed how "complexified fluctons"—leading from a generic point to the global maximum of the (inverted) potential—can be constructed using (segments of) *two* such paths. While quantitative studies of these "complex fluctons" are deferred to future works, the main qualitative point is made here: since one of them has a complex action, its contribution to the density matrix has a nontrivial phase. It can be either positive or negative, depending on the magnitude of the imaginary part of their action, which in turn depends on the asymmetry parameter.

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