Anti–de Sitter relativity

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The relative geodesic motion on $(1 + 3)$-dimensional anti–de Sitter spacetimes is studied in terms of conserved quantities by adapting the Nachtmann boosting method created initially for de Sitter spacetimes. In this approach the Lorentzian isometry is derived, relating the coordinates of the local chart of a fixed observer with the coordinates of a mobile chart considered as the rest frame of a massive mobile object moving on a timelike anti–de Sitter geodesic. The transformation of the conserved quantities is also investigated, constructing thus a complete theory of the relative geodesic motion on anti–de Sitter spacetimes. Some applications are discussed, among which the problems of twin paradox and Lorentz contraction are solved.

I. INTRODUCTION

In general relativity the relativistic covariance guarantees the consistency of the theory under any arbitrary coordinate transformations defined as diffeomorphisms. In this manner one creates the illusion of a universal symmetry but which is sterile in the sense that this is not able to produce conserved quantities. Another sterile general covariance is the gauge one, which assures the independence of the physical meaning of the theory on the arbitrary gauge transformations of the local orthogonal nonholonomic frames. Thus, the only symmetry which may produce conserved quantities remains the covariance under isometries. For this reason, it is worth studying how the conserved quantities are generated via the Noether theorem and, especially, the physical meaning and role of these quantities in describing the geodesic motion.

The simplest four-dimensional spacetimes of special or general relativity are vacuum solutions of the Einstein equations whose geometry is determined only by the value of the cosmological constant Λ. These are the Minkowski flat spacetime (with $\Lambda = 0$) and the hyperbolic spacetimes, de Sitter (dS) with $\Lambda > 0$ and anti–de Sitter (AdS) having $\Lambda < 0$. All these spacetimes have the highest possible isometries [1], thus representing a good framework for studying the role of the conserved quantities seen as the principal observables of the classical or quantum theory.

The AdS spacetime is the only maximally symmetric spacetime which does not have space translations [1] since its $\Lambda < 0$ produces an attraction of the elastic type such that the geodesic motion is oscillatory around the origins of the central charts (i.e., static and spherically symmetric) with ellipsoidal closed trajectories. This particular behavior was studied by many authors [2–8] for a long time and is still of actual interest [9,10], such that today we are able to understand the principal features of these spacetimes. On the other hand, the higher dimensional AdS manifolds were intensively studied in the framework of the AdS/CFT correspondence of the string theory [11,12]. However, there

are still unsolved problems as, for example, the relative geodesic motion in $(1 + 3)$-dimensional AdS spacetimes and the transformation of the conserved quantities along geodesics observed in different local charts related through isometries.

For the $(1 + 3)$-dimensional dS spacetime we solved this problem starting with the analysis of the physical meaning of the conserved quantities and invariants [13,14] that allowed us to find the Lorentzian isometries involved in the relative motion of the dS relativity [15]. However, for the AdS spacetime we do not have a similar theory, since we have succeeded so far in discussing only the physical meaning of the conserved quantities on the timelike geodesics, pointing out the existence of two conserved vectors that may play a similar role as the Runge-Lenz vector of the Keplerian motion [16]. With this starting point, we would like to continue here the construction of the AdS relativity, focusing on the relative motion for determining the isometries between different local charts considered as fixed or mobile inertial natural frames.

As in the dS case [15], we use here a method which was proposed initially by Nachtmann for constructing covariant representations of the de Sitter isometry group [17]. The idea is to introduce the coordinates of the local charts with the help of point-dependent isometry transformations, which are usually called "boosts". In the dS spacetimes we needed to complete the original Nachtmann boosts with suitable gauge transformations in order to give rise simultaneously to local coordinates and desired conserved quantities [15]. Fortunately, for the AdS spacetimes we may apply a similar method but without resorting to additional gauge transformations. By using such boosts we define the natural rest frames of the massive pointlike particles that will be seen as mobile frames when the attached particles are moving on timelike geodesics. In this manner we can identify any mobile frame with the help of the conserved quantities on the geodesic of the particle carrying this frame. Under such circumstances, we may derive the isometry transformation between the coordinates

of a mobile frame and those of a fixed one. Thus we are able to study the relative geodesic motion in terms of conserved quantities, applying on the AdS spacetime the methods of special relativity or of our dS relativity [15].

The principal new result presented here is the form of the Lorentzian isometries relating the coordinates of the moving and fixed natural frames on AdS backgrounds. It is remarkable that these have the same form as in the case of the usual Lorentz transformations of special relativity or of the Lorentzian isometries of the dS relativity [15], depending in the same manner on a conserved momentum. This may appear paradoxical as long as on the AdS spacetimes there are no space translations and, consequently, no conserved momentum. Nevertheless, on timelike AdS geodesics, one of the conserved vectors studied in Ref. [16] may take over the role of momentum being interpreted as the momentum of the particle carrying the mobile frame when this is passing through the origin of the fixed frame. However, the mechanisms that lead to this common form of the Lorentzian isometries are quite different since the motion on the AdS geodesics is oscillatory while on Minkowski or dS spacetimes the timelike geodesics are open.

For marking out the advantages of our approach, we give examples of simple relativistic effects that can be solved in terms of conserved quantities on AdS spacetime, focusing on the time dilation of the twin paradox and Lorentz contraction affecting the measurements performed in the origin of the mobile frame. The general method of investigating the relative motion based on our approach is also discussed, and the general transformations of coordinates and conserved quantities under Lorentzian isometries are written down.

We start in the second section with a short review of the properties of the AdS manifolds as hyperboloids in an embedding five-dimensional pseudo-Euclidean spacetime, introducing the principal local charts we need and studying the AdS isometries. The next section is devoted to the conserved quantities on the AdS geodesics, focusing on the timelike ones for which we discuss the physical meaning of these observables. Section IV is devoted to the AdS relativity based on the method of boosting coordinates that allows us to derive the Lorentzian isometries with different parametrizations. In the next section, we solve the twin paradox and Lorentz contraction and discuss the general method of relativity applied to the geodesic motion, giving the general rule of transforming coordinates and conserved quantities. The last section presents some concluding remarks.

II. ANTI–DE SITTER ISOMETRIES

Let us consider the CAdS spacetime (M, g), defined as the universal covering space of the $(1 + 3)$-dimensional AdS spacetime which is a vacuum solution of the Einstein equations with $\Lambda < 0$ and negative constant curvature. Thus, the AdS manifold is a hyperboloid of radius

$R = \frac{1}{\omega} = \sqrt{-\frac{3}{\Lambda}}$ embedded in the $(2 + 3)$-dimensional pseudo-Euclidean spacetime (M^5, η^5) of Cartesian coordinates z^A (labeled by the indices $A, B, \ldots = -1, 0, 1, 2, 3$) and metric $\eta^5 = \text{diag}(1, 1, -1, -1, -1)$. These coordinates are global, corresponding to the pseudo-orthonormal basis $\{\nu_A\}$ of the frame into consideration, whose unit vectors satisfy $\nu_A \cdot \nu_B = \eta^5_{AB}$. Any point $z \in M^5$ is represented by the five-dimensional vector $z = \nu_A z^A = (z^{-1}, z^0, z^1, z^2, z^3)^T$, which transforms linearly under the gauge group $G(\eta^5) = SO(2, 3)$, which leaves the metric η^5 invariant.

A. Coordinates on CAdS spacetimes

The local charts $\{x\}$ of coordinates x^μ ($\alpha, \ldots \mu, \nu \ldots = 0, 1, 2, 3$) can be introduced on (M, g) giving the set of functions $z^A(x)$ which solve the hyperboloid equation,

$$\eta^5_{AB} z^A(x) z^B(x) = \frac{1}{\omega^2}. \qquad (2.1)$$

The usual chart $\{t, \vec{x}\}$ with Cartesian spaces coordinates x^i ($i, j, k, \ldots = 1, 2, 3$) is defined by

$$z^{-1} = \omega^{-1} \chi(r) \cos(\omega t),$$
$$z^0 = \omega^{-1} \chi(r) \sin(\omega t),$$
$$z^i(x) = x^i, \qquad (2.2)$$

where we denote $r = |\vec{x}|$ and $\chi(r) = \sqrt{1 + \omega \vec{x}^2} = \sqrt{1 + \omega^2 r^2}$. Hereby one obtains the line element

$$ds^2 = \eta^5_{AB} dz^A(x) dz^B(x)$$
$$= \chi(r)^2 dt^2 - \left[\delta_{ij} - \omega^2 \frac{x^i x^j}{\chi(r)^2}\right] dx^i dx^j. \qquad (2.3)$$

The associated central chart $\{t, r, \theta, \phi\}$ with spherical coordinates, canonically related to the Cartesian ones, $\vec{x} \to (r, \theta, \phi)$, has the line element

$$ds^2 = \chi(r)^2 dt^2 - \frac{dr^2}{\chi(r)^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2.4)$$

The AdS spacetime is covered once by the time $t \in [0, 2\pi)$ because of the periodicity of the coordinates z^{-1} and z^0, but if we consider that $t \in \mathbb{R}^+$ then we are in its universal covering space, CAdS. Here any motion is returning periodically in the same state at times $t + n\frac{2\pi}{\omega}$ for all $n \in \mathbb{N}$. In what follows this fact is a matter of course, restricting ourselves to studying the motion during only a period.

Apart from the above usual charts, it is useful to consider the central chart $\{\tilde{x}\} = \{t, \rho, \theta, \phi\}$ resulting after the substitution [5,6]

$$r = \frac{\rho}{\tilde{\chi}(\rho)}, \qquad \tilde{\chi}(\rho) = \sqrt{1 - \omega^2 \rho^2}, \qquad (2.5)$$

where $0 \le \rho < \frac{1}{\omega}$. Then the embedding equations become

$$z^{-1}(\tilde{x}) = \frac{1}{\omega \tilde{\chi}(\rho)} \cos(\omega t),$$

$$z^0(\tilde{x}) = \frac{1}{\omega \tilde{\chi}(\rho)} \sin(\omega t),$$

$$z^1(\tilde{x}) = \frac{\rho}{\tilde{\chi}(\rho)} \sin \theta \cos \phi,$$

$$z^2(\tilde{x}) = \frac{\rho}{\tilde{\chi}(\rho)} \sin \theta \sin \phi,$$

$$z^3(\tilde{x}) = \frac{\rho}{\tilde{\chi}(\rho)} \cos \theta, \qquad (2.6)$$

while the line element reads

$$ds^2 = \frac{1}{\tilde{\chi}(\rho)^2} \left[dt^2 - \frac{d\rho^2}{\tilde{\chi}(\rho)^2} - \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \qquad (2.7)$$

In this chart the components of the four-velocity are denoted as $\tilde{u}^\mu = \frac{d\tilde{x}^\mu}{ds}$.

B. Isometries

The CAdS spacetimes are homogeneous spaces of the gauge group $G(\eta^5) = SO(2,3)$ whose transformations leave invariant the metric η^5 of the embedding manifold (M^5, η^5) and implicitly Eq. (2.1). For this group we adopt the canonical parametrization

$$\mathfrak{g}(\xi) = \exp\left(-\frac{i}{2} \xi^{AB} \mathfrak{S}_{AB}\right) \in SO(2,3) \qquad (2.8)$$

with skew-symmetric parameters, $\xi^{AB} = -\xi^{BA}$, and the covariant generators of the fundamental representation of the $so(2,3)$ algebra carried by M^5 having the matrix elements

$$(\mathfrak{S}_{AB})^C_{\cdot D} = i(\delta^C_A \eta^5_{BD} - \delta^C_B \eta^5_{AD}). \qquad (2.9)$$

In any local chart $\{x\}$, defined by the functions $z = z(x)$, each transformation $\mathfrak{g} \in SO(2,3)$ gives rise to the isometry $x \to x' = \phi_{\mathfrak{g}}(x)$ derived from the system of equations $z[\phi_{\mathfrak{g}}(x)] = \mathfrak{g}z(x)$. The simplest isometries are the rotations $\mathfrak{r} \in SO(3) \subset SO(2,3)$ generated by $\mathfrak{J}_i = \frac{1}{2}\varepsilon_{ijk}\mathfrak{S}_{jk}$ that transform linearly the Cartesian coordinates, $x^i \to \phi^i_{\mathfrak{r}}(x) = R_{ij}x^j$, when these are proportional with z^i as in Eq. (2.2). Consequently, in this case the $SO(3)$ symmetry becomes global [13]. Other rotations in the plane $\{z^{-1}, z^0\}$ form the subgroup $T(1)_H$ generated by $\mathfrak{H} = \mathfrak{S}_{-1,0}$ as

$$\exp(-i\mathfrak{H}\alpha): \begin{array}{l} z^{-1} \to z^{-1} \cos\alpha - z^0 \sin\alpha \\ z^0 \to z^{-1} \sin\alpha + z^0 \cos\alpha \\ z^i \to z^i, \end{array} \qquad (2.10)$$

producing the time translations $t \to t + \frac{\alpha}{\omega}$. In addition, there are three generators of Lorentz transformations, $\mathfrak{K}_i = \mathfrak{S}_{i0}$, acting on M^5. For example, those generated by $\mathfrak{K}_1$,

$$\exp(-i\mathfrak{K}_1\alpha): \begin{array}{l} z^{-1} \to z^{-1} \\ z^0 \to z^0 \cosh\alpha + z^1 \sinh\alpha \\ z^1 \to z^1 \cosh\alpha + z^0 \sinh\alpha, \\ z^2 \to z^2 \\ z^3 \to z^3 \end{array} \qquad (2.11)$$

give the isometries

$$t \to \frac{1}{\omega} \arctan\left(\tan\omega t \cosh\alpha + x^1 \sec\omega t \sinh\alpha\right),$$

$$x^1 \to x^1 \cosh\alpha + \chi(r) \sin\omega t \sinh\alpha,$$

$$x^2 \to x^2,$$

$$x^3 \to x^3.$$

The last three generators, $\mathfrak{N}_i = \mathfrak{S}_{i,-1}$, are of the Lorentz type but involving the coordinate z^{-1} instead of z^0. The transformations along the z^1 axis,

$$\exp(-i\mathfrak{N}_1\alpha): \begin{array}{l} z^{-1} \to z^{-1} \cosh\alpha + z^1 \sinh\alpha \\ z^0 \to z^0 \\ z^1 \to z^1 \cosh\alpha + z^{-1} \sinh\alpha, \\ z^2 \to z^2 \\ z^3 \to z^3 \end{array} \qquad (2.12)$$

give rise to the isometries

$$t \to \frac{1}{\omega} \text{arccot}(\cot\omega t \cosh\alpha + x^1 \csc\omega t \sinh\alpha),$$

$$x^1 \to x^1 \cosh\alpha + \chi(r) \cos\omega t \sinh\alpha,$$

$$x^2 \to x^2,$$

$$x^3 \to x^3.$$

We specify that the generators $\{\mathfrak{J}_i, \mathfrak{K}_i\}$ are of the $so(1,3)$ subalgebra, while $\mathfrak{H}$ and $\mathfrak{N}_i$ are the specific ones of the $so(2,3)$ algebra.

We presented the above isometries in Cartesian coordinates but these can be rewritten at any time in spherical coordinates by using the above indicated substitutions.

III. CONSERVED QUANTITIES ON TIMELIKE GEODESICS

In general, after integrating the geodesic equations, one obtains the geodesic trajectories depending on some integration constants that must get a physical interpretation. This is possible only by expressing them in terms of conserved quantities on geodesics.

A. Conserved quantities

The conserved quantities are given by the Killing vectors associated to the $SO(2,3)$ isometries, which are defined (up to a multiplicative constant) as [13]

$$k_{(AB)\mu} = z_A \partial_\mu z_B - z_B \partial_\mu z_A, \qquad z_A = \eta^5_{AC} z^C. \quad (3.1)$$

In Ref. [16] we introduced ten independent conserved quantities associated to the $SO(2,3)$ generators, for any pointlike particle of mass m, freely falling on a CAdS background. By using the four-velocity $u^\mu = \frac{dx^\mu}{ds}$ we defined, in any chart, the energy

$$\mathfrak{H} \to E = m\omega k_{(-1,0)\mu} u^\mu, \quad (3.2)$$

and the angular momentum components

$$\mathfrak{J}_i \to L_i = m\frac{1}{2}\varepsilon_{ijk} k_{(j,k)\mu} u^\mu, \quad (3.3)$$

that have the traditional physical meaning. In addition, there are two more conserved vectors having the components

$$\mathfrak{K}_i \to K_i = mk_{(i,0)\mu} u^\mu, \quad (3.4)$$

$$\mathfrak{N}_i \to N_i = mk_{(i,-1)\mu} u^\mu. \quad (3.5)$$

These conserved quantities represent the components of a skew-symmetric tensor $\mathcal{K}_{(AB)} = mk_{(AB)\mu} u^\mu$ that can be written in matrix form as

$$\mathcal{K} = \begin{pmatrix} 0 & \frac{E}{\omega} & -N_1 & -N_2 & -N_3 \\ -\frac{E}{\omega} & 0 & -K_1 & -K_2 & -K_3 \\ N_1 & K_1 & 0 & L_3 & -L_2 \\ N_2 & K_2 & -L_3 & 0 & L_1 \\ N_3 & K_3 & L_2 & -L_1 & 0 \end{pmatrix}. \quad (3.6)$$

The transformations $\mathfrak{g} \in SO(2,3)$, generating the isometries $x \to x' = \phi_\mathfrak{g}(x)$, transform the conserved quantities according to the rule

$$\mathcal{K}'_{(AB)} = \mathfrak{g}^{\cdot C}_{A\cdot} \mathfrak{g}^{\cdot D}_{B\cdot} \mathcal{K}_{(CD)}, \quad (3.7)$$

where $\mathfrak{g}^{\cdot B}_{A\cdot} = \eta^5_{AC} \mathfrak{g}^{C\cdot}_{\cdot D} \eta^{5BD}$ are the matrix elements of the adjoint matrix $\bar{\mathfrak{g}} = \eta^5 \mathfrak{g} \eta^5$. Consequently, this transformation can be written simpler as $\mathcal{K}' = \bar{\mathfrak{g}} \mathcal{K} \bar{\mathfrak{g}}^T$. Thus any transformation $\mathfrak{g} \in SO(2,3)$ generates an isometry between two local charts and transforms, simultaneously, all the conserved quantities that can be measured in these charts.

Now we can verify that all the conserved quantities carrying space indices $(i, j, \ldots)$ transform alike under rotations as $SO(3)$ vectors or tensors. Moreover, the condition $z^i \propto x^i$ fixes the same (common) three-dimensional basis $\{\vec{\nu}_1, \vec{\nu}_2, \vec{\nu}_3\}$ in both the Cartesian charts, of M^5 and respectively M. Then we say that the $SO(3)$ symmetry is global [13] and we use the vector notation for the conserved quantities as well as for the local Cartesian coordinates on M. However, this basis must not be confused with that of the local orthogonal frames on M which will be defined canonically later.

For studying these conserved quantities on the timelike geodesics it is convenient to work in the chart $\{t, \rho, \theta, \phi\}$, taking the angular momentum along the third axis, $\vec{L} = L\vec{\nu}_3 = (0, 0, L)$, for restricting the motion in the equatorial plane, with $\theta = \frac{\pi}{2}$ and $\tilde{u}^\theta = 0$. Then the nonvanishing conserved quantities can be written as

$$E = \frac{m}{\tilde{\chi}^2} \tilde{u}^t, \quad (3.8)$$

$$L = \frac{m\rho^2}{\tilde{\chi}^2} \tilde{u}^\phi, \quad (3.9)$$

$$K_1 = \frac{m}{\omega\tilde{\chi}^2}(-\omega\rho\tilde{u}^t \cos\omega t \cos\phi \\ + \tilde{u}^\rho \sin\omega t \cos\phi - \rho\tilde{u}^\phi \sin\omega t \sin\phi), \quad (3.10)$$

$$K_2 = \frac{m}{\omega\tilde{\chi}^2}(-\omega\rho\tilde{u}^t \cos\omega t \sin\phi \\ + \tilde{u}^\rho \sin\omega t \sin\phi + \rho\tilde{u}^\phi \sin\omega t \cos\phi), \quad (3.11)$$

$$N_1 = \frac{m}{\omega\tilde{\chi}^2}(\omega\rho\tilde{u}^t \sin\omega t \cos\phi \\ + \tilde{u}^\rho \cos\omega t \cos\phi - \rho\tilde{u}^\phi \cos\omega t \sin\phi), \quad (3.12)$$

$$N_2 = \frac{m}{\omega\tilde{\chi}^2}(\omega\rho\tilde{u}^t \sin\omega t \sin\phi \\ + \tilde{u}^\rho \cos\omega t \sin\phi + \rho\tilde{u}^\phi \cos\omega t \cos\phi), \quad (3.13)$$

while $K_3 = N_3 = 0$. Hereby we deduce the following obvious properties,

$$\vec{K} \cdot \vec{L} = \vec{N} \cdot \vec{L} = 0, \qquad \vec{K} \wedge \vec{N} = -\frac{E}{\omega}\vec{L}, \quad (3.14)$$

and verify the identity

$$E^2 + \omega^2(\vec{L}^2 - \vec{K}^2 - \vec{N}^2) = m^2\tilde{u}^2 = m^2, \qquad (3.15)$$

defining the principal invariant corresponding to the first Casimir operator of the $so(2,3)$ algebra. Notice that in the classical theory the second invariant of this algebra vanishes since there is no spin [14].

Finally, we must specify that a mobile of mass m can stay at rest in the origin, $\vec{x} = 0$, on a world line along the vector field ∂_t, only when $E = m$ and $\vec{L} = \vec{K} = \vec{N} = 0$. Then, the matrix (3.6) takes the simplest form

$$\mathcal{K}_o = \begin{pmatrix} 0 & \frac{m}{\omega} & 0 & 0 & 0 \\ -\frac{m}{\omega} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (3.16)$$

depending only on the mobile mass. We observe that the group $SO(3) \otimes T(1)_H \subset SO(1,4)$ is the stable group of this matrix since $\bar{\mathfrak{g}}\mathcal{K}_o\bar{\mathfrak{g}}^T = \mathcal{K}_o$ for any transformation $\mathfrak{g}$ of this group.

B. Timelike geodesics

In the case of the timelike geodesics we may exploit the identity $\tilde{u}^2 = 1$ and Eqs. (3.8) and (3.9) for obtaining the radial component,

$$\tilde{u}^\rho = \tilde{\chi}(\rho)^2 \left[\frac{E^2}{m^2}\tilde{\chi}(\rho)^2 + \frac{\omega^2 L^2}{m^2} - \frac{L^2}{m^2\rho^2} - 1\right]^{\frac{1}{2}}, \qquad (3.17)$$

that allows us to derive the following prime integrals [16],

$$\left(\frac{d\rho}{dt}\right)^2 + \omega^2\rho^2 + \frac{L^2}{E^2\rho^2} = 1 + \frac{\omega^2 L^2}{E^2} - \frac{m^2}{E^2}, \qquad (3.18)$$

$$\frac{d\phi}{dt} = \frac{L}{E\rho^2}, \qquad (3.19)$$

that give the geodesic equations

$$\rho(t) = [\kappa_1 + \kappa_2 \cos 2\omega(t - t_0)]^{\frac{1}{2}}, \qquad (3.20)$$

$$\phi(t) = \phi_0 + \arctan\left[\sqrt{\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}} \tan \omega(t - t_0)\right], \qquad (3.21)$$

where

$$\kappa_1 = \frac{\omega^2 L^2 + E^2 - m^2}{2\omega^2 E^2}, \qquad (3.22)$$

$$\kappa_2 = \frac{1}{2\omega^2 E^2}[(E + m)^2 - \omega^2 L^2]^{\frac{1}{2}}[(E - m)^2 - \omega^2 L^2]^{\frac{1}{2}} \qquad (3.23)$$

satisfy the identity

$$\kappa_1^2 - \kappa_2^2 = \frac{L^2}{\omega^2 E^2}. \qquad (3.24)$$

Thus we solve the geodesic equation in terms of conserved quantities which give a physical meaning to the principal integration constants. The remaining ones, t_0 and ϕ_0, determine only the initial position of the mobile and implicitly its trajectory.

In this manner, we recover the well-known behavior of the timelike geodesic motion which is oscillatory, with frequency ω, having a closed trajectory which has, in general, an ellipsoidal form in the domain $\rho \in [\rho_{\min}, \rho_{\max}]$, where

$$\rho_{\min} = \sqrt{\kappa_1 - \kappa_2}, \qquad \rho_{\max} = \sqrt{\kappa_1 + \kappa_2} \qquad (3.25)$$

satisfy $\rho_{\min} \le \rho_{\max} < \frac{1}{\omega}$ for any values of E and L. The trajectory is symmetric with respect the origin $\vec{x} = 0$ such that the mobile, moving in the trigonometric sense, reaches two opposite aphelions and perihelions during a period. From Eq. (3.20) we see that t_0 is the time when the mobile is passing through an aphelion while ϕ_0 is the angle between the axis $\vec{\nu}_1$ and the major semiaxis of the ellipsoidal trajectory (as in Fig. 1).

We specify that the axes $\vec{\nu}_1$ and $\vec{\nu}_2$ can be rotated at any time such that the major axis is along the direction of $\vec{\nu}_1$, which means that we may set $\phi_0 = 0$ without losing generality. Then the geodesic equations can be written easily even in Cartesian space coordinates,

$$x^1(t) = \frac{\rho_{\max}}{\sqrt{1 - \omega^2\rho(t)^2}} \cos \omega(t - t_0), \qquad (3.26)$$

$$x^2(t) = \frac{\rho_{\min}}{\sqrt{1 - \omega^2\rho(t)^2}} \sin \omega(t - t_0), \qquad (3.27)$$

$$x^3(t) = 0, \qquad (3.28)$$

as it results from Eqs. (2.5), (3.20), and (3.21) for $\phi_0 = 0$.

Now it remains to analyze the role of the conserved vectors $\vec{K}$ and $\vec{N}$ that depends on E and L, as well as on t_0 and ϕ_0. In the simpler case of $\phi_0 = 0$ their nonvanishing components read

$$K_1 = E\rho_{\max} \cos \omega t_0, \qquad (3.29)$$

$$K_2 = -E\rho_{\min} \sin \omega t_0, \qquad (3.30)$$

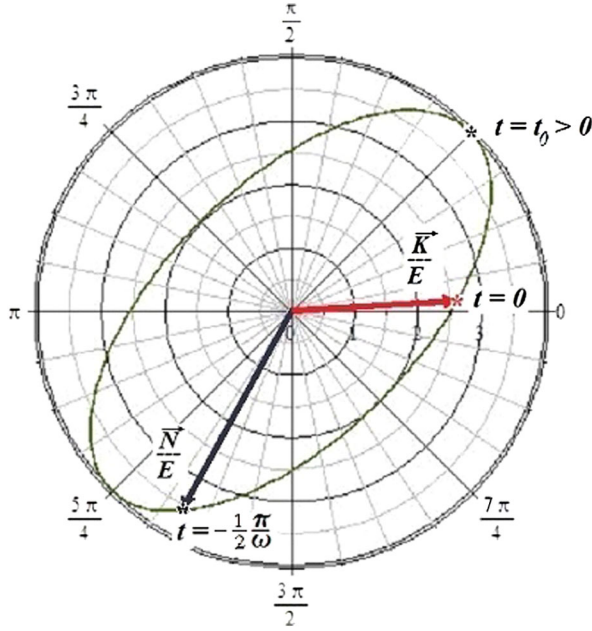$$N_1 = -E\rho_{\max} \sin \omega t_0, \qquad (3.31)$$

FIG. 1. Anti–de Sitter timelike geodesic with arbitrary initial conditions, $\phi_0 = \frac{\pi}{4}$ and $t_0 = \frac{1}{3}\frac{\pi}{\omega}$. When $t = t_0$, the mobile reaches the aphelion. The vector $\frac{\vec{K}}{E}$ indicates the position of the mobile at $t = 0$, while $\frac{\vec{N}}{E}$ gives its position at $t = -\frac{1}{2}\frac{\pi}{\omega}$.

$$N_2 = -E\rho_{\min}\cos\omega t_0, \qquad (3.32)$$

complying with the specific properties

$$|\vec{K}| = E[\kappa_1 + \kappa_2 \cos 2\omega t_0]^{\frac{1}{2}}, \qquad (3.33)$$

$$|\vec{N}| = E[\kappa_1 - \kappa_2 \cos 2\omega t_0]^{\frac{1}{2}}, \qquad (3.34)$$

$$\vec{K} \cdot \vec{N} = -E^2 \kappa_2 \sin 2\omega t_0. \qquad (3.35)$$

In the general case of $\phi_0 \neq 0$ these components have the form given in the Appendix, where we discuss the inverse problem.

Hereby we understand that the vectors $\frac{\vec{K}}{E}$ and $\frac{\vec{N}}{E}$ are radial, indicating the mobile positions at $t = 0$ and, respectively, $t = -\frac{1}{2}\frac{\pi}{\omega}$, as in Fig. 1. For $t_0 = 0$ and $t_0 = \frac{1}{2}\frac{\pi}{\omega}$ these vectors become orthogonal, playing the role of Runge-Lenz vectors [16] as in Figs. 2 and 3. More specifically, for $t_0 = 0$ the vector $\frac{\vec{K}}{E}$ lays over the major semiaxis and $\frac{\vec{N}}{E}$ gives the minor one, but for $t_0 = \frac{1}{2}\frac{\pi}{\omega}$ these vectors change their roles between themselves.

There are two important particular cases. (I) When $L = \frac{1}{\omega}(E - m)$ then $\kappa_2 = 0$ and the motion becomes uniform, $\phi = \phi_0 + \omega(t - t_0)$, on a circle of radius $\rho_c = \sqrt{2\kappa_1} = \frac{1}{\omega}\sqrt{\frac{E-m}{E}}$. Then the radial vectors $\frac{\vec{K}}{E}$ and $\frac{\vec{N}}{E}$, having the same norm ρ_c, are orthogonal regardless the value of t_0 which determines only their position. (II) For $L = 0$ we
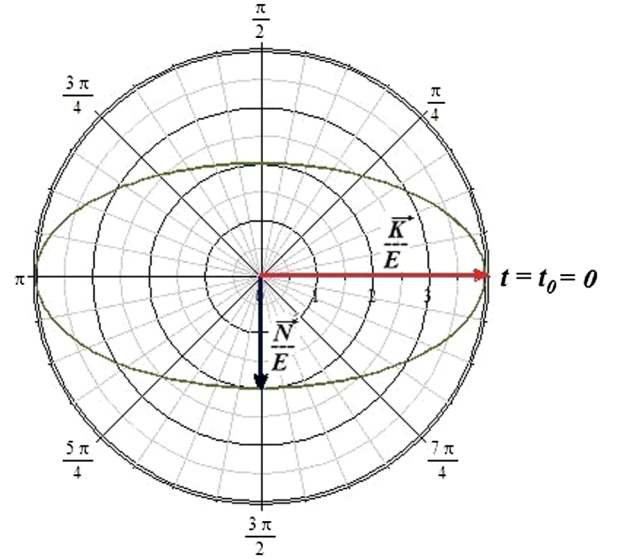
have $\kappa_2 = \kappa_1$ such that the oscillation becomes rectilinear, since $\phi = \phi_0$, and harmonic, with the amplitude a,

$$\rho(t) = a\sin\omega(t - t_0), \quad a = \sqrt{2\kappa_1} = \frac{\sqrt{E^2 - m^2}}{\omega E}, \quad (3.36)$$

on the direction $\vec{n} = \vec{\nu}_1 \cos\phi_0 + \vec{\nu}_2 \sin\phi_0$. In this case the vectors $\vec{K}$ and $\vec{N}$ are relevant only for the following particular choices of the initial conditions:

$$(\text{IIa}): \; t_0 = 0 \rightarrow \begin{cases} \vec{K} = E\rho_{\max}\vec{n}, \\ \vec{N} = 0, \end{cases} \qquad (3.37)$$

FIG. 2. The vectors $\frac{\vec{K}}{E}$ and $\frac{\vec{N}}{E}$ playing the role of Runge-Lenz vectors for $t_0 = 0$ ($\phi_0 = 0$).
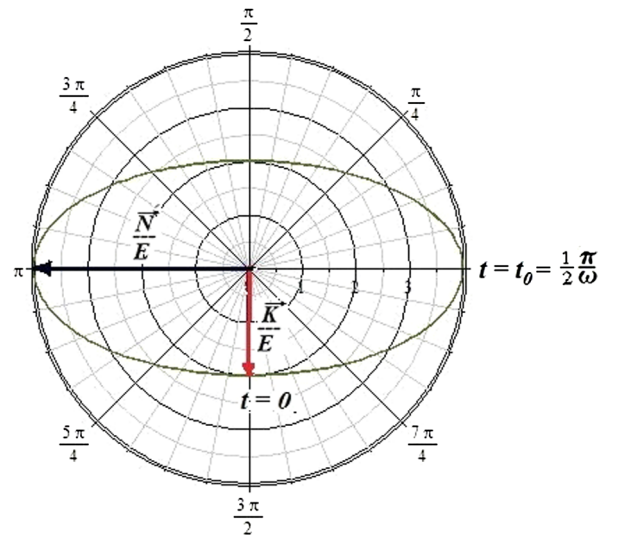

FIG. 3. The vectors $\frac{\vec{K}}{E}$ and $\frac{\vec{N}}{E}$ playing the role of Runge-Lenz vectors for $t_0 = \frac{1}{2}\frac{\pi}{\omega}$ ($\phi_0 = 0$).

$$\text{(IIb):} \quad t_0 = \frac{1}{2}\frac{\pi}{\omega} \rightarrow \begin{cases} \vec{K} = 0, \\ \vec{N} = E\rho_{max}\vec{n}. \end{cases} \tag{3.38}$$

Thus the conserved vectors $\vec{K}$ and $\vec{N}$ are related with the trajectory parameters and the initial condition such that they could be useful in analyzing the CAdS kinematics.

IV. RELATIVITY

Recently we have studied the relativity on the dS manifolds [15], applying the Nachtmann method [17] of boosting coordinates which takes over the Wigner theory [18] but in configurations instead of momentum representation. This method can be used for the CAdS spacetimes too since these manifolds may be seen as spaces of left cosets $SO(2,3)/L_+^\uparrow$, where $L_+^\uparrow$ is the Lorentz group that plays the role of the gauge group of (M, g), leaving invariant the metric $\eta = \text{diag}(1, -1, -1, -1)$ of the Minkowskian pseudo-Euclidean model of this manifold. In what follows we develop this formalism, denoting for brevity $G = G(\eta) = L_+^\uparrow$ and $G_5 = G(\eta^5) = SO(2,3)$.

A. Boosting coordinates

Any AdS spacetime can be constructed as a space of left cosets G_5/G starting with the fixed point $z_o = (\omega^{-1}, 0, 0, 0, 0)^T \in M^5$ [of local coordinates $(0,0,0,0)$]. Then, the whole AdS manifold can be built as the orbit $M = \{\mathfrak{g}z_o | \mathfrak{g} \in G_5/G\} \subset M^5$ since the subgroup G is just the stable group of z_o obeying $\mathfrak{g}z_o = z_o$, $\forall \mathfrak{g} \in G$. Consequently, any point $z(x) \in M$ can be reached by performing the boost transformation $\mathfrak{b}(x): z_o \rightarrow z(x) = \mathfrak{b}(x)z_o$ which defines the functions $z^A(x)$ of the local coordinates $\{x\}$. In fact, these boosts are sections in the

principal fiber bundle on $(M, g) \sim G_5/G$ whose fiber is just the isometry group G_5. Once the AdS spacetime is defined we can take an open time axis for obtaining the CAdS one.

This formalism offers one, in addition, the advantage of defining the *canonical* five-dimensional 1-forms

$$\hat{\omega}(x) = \mathfrak{b}^{-1}(x)d\mathfrak{b}(x)z_o, \tag{4.1}$$

whose components,

$$\hat{\omega}^{\hat{\alpha}}(x) = \hat{e}_\mu^{\hat{\alpha}}(x)dx^\mu, \qquad \hat{\omega}^{-1}(x) = 0, \tag{4.2}$$

give the canonical gauge fields (or tetrads) $\hat{e}^{\hat{\mu}}$ of the local coframes associated to the fields $e_{\hat{\mu}}$ of the orthogonal local frames [17]. These fields are labeled by the local indices $\hat{\alpha}, \hat{\mu}, ...$, having the same range as the natural ones. They have the duality and orthogonality properties,

$$e_{\hat{\alpha}}^\mu \hat{e}_\nu^{\hat{\alpha}} = \delta_\mu^\nu, \qquad e_{\hat{\alpha}}^\mu \hat{e}_\mu^{\hat{\beta}} = \delta_{\hat{\alpha}}^{\hat{\beta}}, \qquad g_{\mu\nu}e_{\hat{\alpha}}^\mu e_{\hat{\beta}}^\nu = \eta_{\hat{\alpha}\hat{\beta}}, \tag{4.3}$$

giving the line element of any chart $\{x\}$ as

$$ds^2 = \hat{\eta}_{AB}\hat{\omega}^A\hat{\omega}^B = \eta_{\hat{\alpha}\hat{\beta}}\hat{e}_\mu^{\hat{\alpha}}\hat{e}_\nu^{\hat{\beta}}dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu. \tag{4.4}$$

In general, the boosts are defined up to an *arbitrary* gauge, $\mathfrak{b}(x) \rightarrow \mathfrak{b}(x)\lambda^{-1}(x)$, $\lambda(x) \in G$, that does not affect the functions $z^A(x)$ but changes the gauge fields, transforming the 1-forms as $\hat{\omega}(x) \rightarrow \lambda(x)\hat{\omega}(x)$ [17].

The boost may be chosen for determining the type of coordinates we desire. For example, the coordinates of the chart $\{t, \vec{x}\}$ with Cartesian space coordinates can be introduced by using the boost

$$\mathfrak{b}(t, \vec{x}) = \exp{(-i\mathfrak{H}\omega t)}$$

$$\times \exp\left(-i\mathfrak{N}_i \frac{x^i}{r}\text{arctanh}\left(\frac{\omega r}{\chi}\right)\right) = \begin{pmatrix} \chi\cos\omega t & -\sin\omega t & \omega x^1\cos\omega t & \omega x^2\cos\omega t & \omega x^3\cos\omega t \\ \chi\sin\omega t & \cos\omega t & \omega x^1\sin\omega t & \omega x^2\sin\omega t & \omega x^3\sin\omega t \\ \omega x^1 & 0 & 1+\lambda x^{12} & \lambda x^1 x^2 & \lambda x^1 x^3 \\ \omega x^2 & 0 & \lambda x^1 x^2 & 1+\lambda x^{22} & \lambda x^2 x^3 \\ \omega x^3 & 0 & \lambda x^1 x^3 & \lambda x^2 x^3 & 1+\lambda x^{32} \end{pmatrix}, \tag{4.5}$$

where we denote $\lambda = \frac{\chi-1}{r^2}$. Indeed, calculating $z(x) = \mathfrak{b}(t, \vec{x})z_o$, we obtain just Eq. (2.2). Moreover, we may derive the canonical tetrads defined by Eq. (4.1), obtaining the nonvanishing components

$$\hat{e}_0^0 = \chi, \qquad \hat{e}_j^i = \delta_j^i - \frac{x^i x^j}{r^2}\left(1 - \frac{1}{\chi}\right), \tag{4.6}$$

$$e_0^0 = \frac{1}{\chi}, \qquad e_j^i = \delta_j^i - \frac{x^i x^j}{r^2}(1 - \chi). \tag{4.7}$$

We recover thus the Cartesian gauge [13] which preserves the global $SO(3)$ symmetry.

Other charts can be introduced in the same manner by using suitable boosts. For defining the coordinates of the chart $\{\tilde{x}\}$ we chose the boost

$$\mathfrak{b}(t,\rho,\theta,\phi) = \exp\left(-i\mathfrak{J}_3\phi\right)\exp\left(-i\mathfrak{J}_2\left(\theta-\frac{\pi}{2}\right)\right)\exp\left(-i\mathfrak{H}\omega t\right)\exp\left(-i\mathfrak{N}_1\beta\right)\exp\left(\frac{i\pi}{2}\mathfrak{J}_1\right), \qquad (4.8)$$

$$\beta = \operatorname{arctanh}\omega\rho, \qquad (4.9)$$

where the last rotation is a gauge transformation determining convenient 1-forms. After a little calculation we obtain

$$\mathfrak{b}(t,\rho,\theta,\phi) = \begin{pmatrix} \frac{1}{\tilde{\chi}}\cos\omega t & -\sin\omega t & \frac{\omega\rho}{\tilde{\chi}}\cos\omega t & 0 & 0 \\ \frac{1}{\tilde{\chi}}\sin\omega t & \cos\omega t & \frac{\omega\rho}{\tilde{\chi}}\sin\omega t & 0 & 0 \\ \frac{\omega\rho}{\tilde{\chi}}\sin\theta\cos\phi & 0 & \frac{1}{\tilde{\chi}}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \frac{\omega\rho}{\tilde{\chi}}\sin\theta\sin\phi & 0 & \frac{1}{\tilde{\chi}}\sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \frac{\omega\rho}{\tilde{\chi}}\cos\theta & 0 & \frac{1}{\tilde{\chi}}\cos\theta & -\sin\theta & 0 \end{pmatrix}, \qquad (4.10)$$

and we may convince ourselves that $z(\tilde{x}) = \mathfrak{b}(t,\rho,\theta,\phi)z_o$ gives the embedding equations (2.6). In this case the canonical 1-forms read

$$\hat{\omega}^0 = \frac{1}{\tilde{\chi}(\rho)}\,dt, \qquad (4.11)$$

$$\hat{\omega}^1 = \frac{1}{\tilde{\chi}(\rho)^2}\,d\rho, \qquad (4.12)$$

$$\hat{\omega}^2 = \frac{\rho}{\tilde{\chi}(\rho)}\,d\theta, \qquad (4.13)$$

$$\hat{\omega}^3 = \frac{\rho\sin\theta}{\tilde{\chi}(\rho)}\,d\phi, \qquad (4.14)$$

defining a familiar diagonal gauge.

B. Lorentzian isometries

Let us consider now the problem of the relative motion that studies how a geodesic motion can be measured by two different observers. The above introduced local charts play the role of inertial frames related through isometries. Each observer may have a proper frame in which he stays at rest in the origin on the world line along the vector field ∂_t. In general, these frames move along geodesics such that we need to introduce supplemental hypotheses in order to describe their motion.

We assume that one observer, O, is fixed in his proper frame $\{x\}$ observing what happens in a mobile frame $\{x'\}$ which is the proper frame of the observer O'. The problem is how this relative motion can be described using isometries. We start with the hypothesis that the mobile frame is the rest frame of a particle of mass m which stays at rest in O', having $E' = m$ and $\vec{L}' = \vec{K}' = \vec{N}' = 0$. The fixed

observer O sees this particle moving on a timelike geodesic with given parameters that mark the relative motion. Under such circumstances, the previously presented boosting method allows us to derive the isometry transformation between these frames in terms of the geodesic parameters related with the conserved quantities.

This relativity does make sense only if we can compare the measurements of these observers, imposing the condition of the synchronization of their clocks. This means that, at a given common initial time, the origins of these frames must coincide. However, this condition is restrictive in the CAdS spacetimes since this forces the geodesic of the carrier particle of the mobile frame to across the origin of the fixed frame O. This means that its trajectory is rectilinear (with $\vec{L} = 0$) in a given direction $\vec{n}$ on which this oscillates with frequency ω.

With these preparations we can apply the boosting method in order to relate the five-dimensional vectors of M^5 corresponding to the fixed and mobile frames. In the mobile frame the observer O' stays at rest in $\vec{x}' = 0$, measuring the time t'. There are no restrictions to choose an arbitrary time t_0' of O', giving the point

$$z'(x_0') = \left(\frac{1}{\omega}\cos\omega t_0', \frac{1}{\omega}\sin\omega t_0', 0, 0, 0\right)^T \in M^5, \qquad (4.15)$$

while in the fixed frame the corresponding point

$$z(x_0) = \left(\frac{1}{\omega}\cos\omega t_0, \frac{1}{\omega}\sin\omega t_0, x_0^1, x_0^2, x_0^3\right)^T \in M^5 \qquad (4.16)$$

depends on the position $\vec{x}_0$ at time t_0 of the particle of mass m. The boosts (4.5) allow us to reach both of these points starting from z_o as $z(x) = \mathfrak{b}(t,\vec{x})z_o$ and $z'(x') = \mathfrak{b}(t',0)z_o$. This gives us the opportunity of defining

$\mathfrak{g}_* = \mathfrak{b}(t_0, \vec{x}_0)\mathfrak{b}(t'_0, 0)^{-1} \in SO(2,3)$, which relates the above defined points in M^5 by the transformation $z(x_0) = \mathfrak{g}_* z'(x'_0)$.

Furthermore, we assume that the isometry transformation between the frames $\{x\}$ and $\{x'\}$ is generated by $\mathfrak{g}_*$. However, the synchronization condition requires the origins to coincide for $t = t' = 0$ when we must have $\vec{x} = \vec{x} = 0$ such that O and O' overlap in the same point $z(0) = z'(0) = z_o \in M^5$. This happens only if we chose an initial condition of type (II.b), given by Eq. (3.38), setting

$$t_0 = t'_0 = \frac{1}{2}\frac{\pi}{\omega}. \qquad (4.17)$$

Thus we obtain the desired transformation,

$$\mathfrak{g}_* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \chi_0 & \omega n^1 r_0 & \omega n^2 r_0 & \omega n^3 r_0 \\ 0 & \omega n^1 r_0 & 1 + \lambda' n^{12} & \lambda' n^1 n^2 & \lambda' n^1 n^3 \\ 0 & \omega n^2 r_0 & \lambda' n^1 n^2 & 1 + \lambda' n^{22} & \lambda' n^2 n^3 \\ 0 & \omega n^3 r_0 & \lambda' n^1 n^3 & \lambda' n^2 n^3 & 1 + \lambda' n^{32} \end{pmatrix},$$
$$(4.18)$$

that results after we substituted $\vec{x}_0 = \vec{n} r_0$, laying out the unit vector $\vec{n}$ fixing the direction of the linear trajectory and the amplitude r_0 in the chart $\{t, \vec{x}\}$ that gives $\chi_0 = \sqrt{1 + \omega^2 r_0^2}$ and $\lambda' = \chi_0 - 1$.

It is remarkable that $\mathfrak{g}_*$ is in fact a genuine Lorentz transformation of the $SO(2,3)$ group, having the form $\mathfrak{g}_* = \exp(-i\mathfrak{K}_i n^i \sigma)$, where

$$\sigma = \operatorname{arctanh}\left(\frac{\omega r_0}{\chi_0}\right) = \operatorname{arctanh}(\omega \rho_0). \qquad (4.19)$$

Thus we obtained a Lorentz transformation even though we started with boosts generated by $\mathfrak{N}_i$. The explanation is that by imposing the condition (4.17) we perform the transformation

$$\exp\left(-\frac{i\pi}{2}\mathfrak{H}\right)\mathfrak{N}_i\exp\left(\frac{i\pi}{2}\mathfrak{H}\right) = \mathfrak{K}_i, \qquad (4.20)$$

changing the generators of $\mathfrak{g}_*$.

The transformation $\mathfrak{g}_*$ allows us to find the conserved quantities on the geodesic of the particle of mass m. Taking into account that in its rest frame O' the mobile has $E' = m$ and $\vec{L}' = \vec{K}' = \vec{N}' = 0$, corresponding to the matrix $\mathcal{K}_o$, we may apply the general rule for finding the conserved quantities observed by O, encapsulated in the matrix $\mathcal{K} = \bar{\mathfrak{g}}_* \mathcal{K}_o \bar{\mathfrak{g}}_*^T$. After a little calculation we obtain $\vec{L} = \vec{K} = 0$, as expected, and the nonvanishing conserved quantities

$E = m\chi_0$ and $\vec{N} = m\vec{x}_0$, with whose help we can express all the other constants as

$$\chi_0 = \frac{E}{m} \to r_0 = \frac{\sqrt{E^2 - m^2}}{\omega m} \to \rho_0 = a = \frac{\sqrt{E^2 - m^2}}{\omega E}. \qquad (4.21)$$

These formulas suggest to us to define a momentum vector

$$\vec{P} = \omega\vec{N} = m\omega\vec{x}_0 = \vec{n}P, \qquad P = \sqrt{E^2 - m^2}, \qquad (4.22)$$

which satisfies formally the usual dispersion relation of special relativity and whose meaning will be discussed later. Now we use this vector for writing down the definitive expression of the Lorentz transformation (4.18) as

$$\mathfrak{g}_* = \mathfrak{g}(\vec{P}) = \exp\left(-i\vec{P} \cdot \vec{\mathfrak{K}} \frac{1}{P} \operatorname{arcsinh}\left(\frac{P}{m}\right)\right), \qquad (4.23)$$

which can be obtained by substituting $\chi_0 = \frac{E}{m}$ and $r_0 = \frac{P}{m}$ into Eq. (4.18). We observe that this matrix has the same form as the corresponding Lorentz boost of special relativity [19] or as the transformation giving the Lorentzian isometry of the dS relativity [15].

The last step is to define the Lorentzian isometry $x = \phi_{\mathfrak{g}(\vec{P})}(x')$, between the coordinates of the mobile and fixed frames, as the transformation resulting from the system of equations $z(x) = \mathfrak{g}_* z'(x)$ that can be derived using the new parametrization (4.23). The direct transformation reads

$$t(t', \vec{x}') = \frac{1}{\omega}\arctan\left(\frac{E}{m}\tan\omega t' + \frac{\omega}{m}\frac{\vec{x}' \cdot \vec{P}}{\sqrt{1 + \omega^2|\vec{x}'|^2}}\sec\omega t'\right), \qquad (4.24)$$

$$\vec{x}(t', \vec{x}') = \vec{x}' + \frac{\vec{P}}{m}\left(\frac{\vec{x}' \cdot \vec{P}}{E + m} + \frac{1}{\omega}\sqrt{1 + \omega^2|\vec{x}'|^2}\sin\omega t'\right), \qquad (4.25)$$

while the inverse one has to be obtained by changing $x \leftrightarrow x'$ and $\vec{P} \to -\vec{P}$. We obtained thus the CAdS analogous of the Lorentz isometries of special relativity. Obviously, in the limit of $\omega \to 0$ we recover the usual Lorentz transformations of special relativity.

The first application of this isometry is to recover the geodesic trajectory of the carrier particle of mass m from the parametric equations in t' obtained by substituting $\vec{x}' = 0$ into Eqs. (4.24) and (4.25). Then, according to Eqs. (4.21) and (4.22), we obtain the trajectory of the origin O', denoted as

$$\vec{x}_*(t) = \vec{n} r_*(t) = \vec{n} \frac{a \sin \omega(t)}{\sqrt{1 - \omega^2 a^2 \sin^2 \omega(t)}}$$

$$= \frac{\vec{P} \sin(\omega t)}{\omega \sqrt{E^2 - P^2 \sin^2 \omega t}}, \qquad (4.26)$$

which corresponds to our initial condition $\vec{x}_*(0) = 0$. The components of the four-velocity,

$$u_*^0(t) = \frac{dt}{ds} = \frac{m^2 + P^2 \cos^2 \omega t}{mE}, \qquad (4.27)$$

$$\vec{u}_*(t) = \frac{d\vec{x}_*(t)}{ds} = \frac{E\vec{P}\cos(\omega t)}{m\sqrt{m^2 + P^2\cos^2 \omega t}}, \qquad (4.28)$$

show that the momentum we introduced above is just the canonical momentum of the mobile m at $t = 0$ since $E = mu_*^0(0)$ and $\vec{P} = m\vec{u}_*(0)$.

Now we may use as a principal parameter the velocity of the carrier particle at $t = 0$, defined usually as $\vec{V} = \frac{\vec{P}}{E} = \vec{n}V$, if we desire to bring the above formulas into forms closer to those of special relativity, eliminating the mass m. This can be done by changing the parametrization,

$$\frac{E}{m} = \gamma, \qquad \frac{P}{m} = \gamma V, \qquad \gamma = \frac{1}{\sqrt{1 - V^2}}, \qquad (4.29)$$

and substituting $\chi_0 = \gamma$ and $r_0 = \frac{\gamma V}{\omega}$ into Eq. (4.18). Then we can rewrite

$$\mathfrak{g}(\vec{P}) \to \mathfrak{g}(\vec{V}) = \exp\left(-i\vec{V} \cdot \vec{\mathfrak{K}} \frac{1}{V} \text{arctanh}(V)\right), \qquad (4.30)$$

obtaining the new expression of the Lorentzian isometry

$$t(t', \vec{x}') = \frac{1}{\omega} \arctan \gamma \left(\tan \omega t' + \frac{\omega \vec{x}' \cdot \vec{V}}{\sqrt{1 + \omega^2 |\vec{x}'|^2}} \sec \omega t' \right), \qquad (4.31)$$

$$\vec{x}(t', \vec{x}') = \vec{x}' + \gamma \vec{V} \left(\vec{x}' \cdot \vec{V} \frac{\gamma}{1 + \gamma} + \frac{1}{\omega} \sqrt{1 + \omega^2 |\vec{x}'|^2} \sin \omega t' \right), \qquad (4.32)$$

that may be used in applications.

The transformations $\mathfrak{g}(\vec{V})$ generating these isometries transform simultaneously all the conserved quantities. If those of the mobile frame are encapsulated in the matrix $\mathcal{K}'$ as in Eq. (3.6), then the corresponding ones measured in the fixed frame are the matrix elements of the matrix

$$\mathcal{K} = \bar{\mathfrak{g}}(\vec{V})\mathcal{K}'\bar{\mathfrak{g}}(\vec{V})^T. \qquad (4.33)$$

Thus we obtain the principal tools in studying the relative motion on CAdS spacetimes, or simply, CAdS relativity.

V. RELATIVISTIC EFFECTS

This approach gives us the opportunity of solving various applications. In what follows we present some interesting consequences that can be deduced from Eqs. (4.31), (4.32), and (4.33) but avoiding the complicated technical details.

A. Time dilation and Lorentz contraction

The simplest interesting effects are the time dilation (observed in the so-called twin paradox) and the Lorentz contraction, which in this case are quite complicated since these effects are strongly dependent on the position where the time and length are measured. For this reason, we restrict ourselves to giving a mere simple example, assuming that the measurements are performed in a small neighborhood of $\vec{x}' = 0$. Here we consider the general relations

$$\delta t = \frac{\partial t(t', \vec{x}')}{\partial t'}\bigg|_{\vec{x}'=0} \delta t' + \frac{\partial t(t', \vec{x}')}{\partial x'^i}\bigg|_{\vec{x}'=0} \delta x'^i, \qquad (5.1)$$

$$\delta x^j = \frac{\partial x^j(t', \vec{x}')}{\partial t'}\bigg|_{\vec{x}'=0} \delta t' + \frac{\partial x^j(t', \vec{x}')}{\partial x'^i}\bigg|_{\vec{x}'=0} \delta x'^i, \qquad (5.2)$$

among the quantities $\delta t, \delta x^j$ and $\delta t', \delta x'^j$ supposed to be measured by the observers O and O', respectively.

First we consider a clock in O' indicating $\delta t'$ without changing its position such that $\delta x'^i = 0$. Then, after a little calculation, we obtain the time dilation observed by O,

$$\delta t = \delta t' \tilde{\gamma}(t), \tilde{\gamma}(t) = \frac{\gamma}{\gamma^2 \sin^2 \omega t + \cos^2 \omega t}. \qquad (5.3)$$

Similarly, but with the supplemental simultaneity condition $\delta t = 0$, we derive the Lorentz contraction along the direction of $\vec{P}$ that reads

$$\delta x_{\parallel} = \delta x'_{\parallel} \frac{1}{\tilde{\gamma}(t)}. \qquad (5.4)$$

It is remarkable that here we have $\delta t \delta x_{\parallel} = \delta t' \delta x'_{\parallel}$ just as in the flat case.

The difference is that now the function $\tilde{\gamma}(t)$ is oscillating in the domain $[\frac{1}{\gamma}, \gamma]$, producing a time dilation for $\tilde{\gamma} > 1$ and a time contraction if $\tilde{\gamma} < 1$, during the same period (as in Fig. 4). This example shows how interesting may be the kinematics of the free motion on the CAdS spacetime.

B. Relative geodesic motion

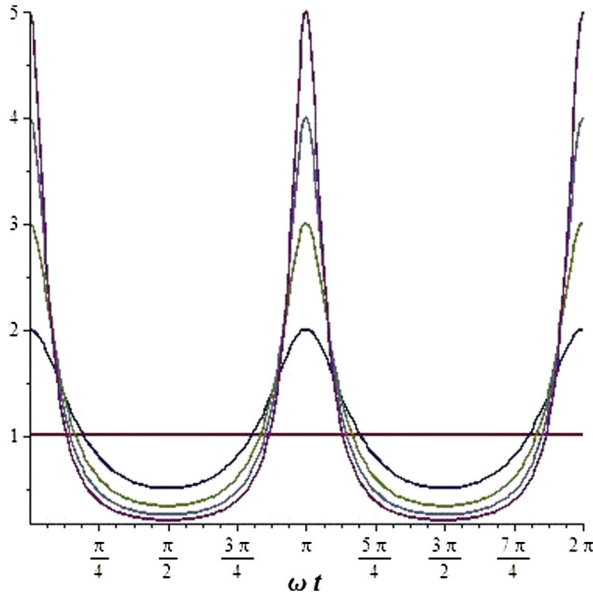However, the principal challenge of relativity is to find how an arbitrary geodesic motion with respect to the

FIG. 4. The function $\tilde{\gamma}(t)$ for different values, $\gamma = 1, \gamma = 2, \ldots$ $\gamma = 5$.

mobile frame can be measured by the fixed observer O. Let us consider the mobile frame $\{x'\}$ as the local chart of Cartesian space coordinates defined with respect to the orthogonal basis $\{\vec{\nu}_1', \vec{\nu}_2', \vec{\nu}_3'\}$, assuming that its origin has the velocity $\vec{V}$ when it is passing through the origin of a fixed frame. In this mobile frame, a test particle of mass m moves in the plane $\{\vec{\nu}_1', \vec{\nu}_2'\}$ on an ellipsoidal geodesic trajectory whose major axis is oriented along the unit vector $\vec{\nu}_1'$, obeying equations of the form (3.26)–(3.28) that now read x'^1

$$x'^1(t') = \frac{\rho'_{max}}{\sqrt{1 - \omega^2 \rho'(t')^2}} \cos \omega(t' - t'_0), \qquad (5.5)$$

$$x'^2(t') = \frac{\rho'_{min}}{\sqrt{1 - \omega^2 \rho'(t')^2}} \sin \omega(t' - t'_0), \qquad (5.6)$$

$$x'^3(t') = 0, \qquad (5.7)$$

where, for simplicity, we choose the initial condition $t'_0 = 0$ (when the test particle reaches the aphelion in the mobile frame). The trajectory parameters in the mobile frames are determined by the energy E' and angular momentum $\vec{L}' = (0, 0, L')$ of the test particle. Thus, we may calculate ρ'_{max} and ρ'_{min} according to Eqs. (3.25), (3.22), and (3.23), while the other conserved quantities result from Eqs. (3.29)–(3.32) as

$$\vec{K}' = (E'\rho'_{max}, 0, 0), \qquad \vec{N}' = (0, -E'\rho'_{min}, 0), \qquad (5.8)$$

since $t'_0 = 0$. Under such circumstances, the fixed observer O measures the geodesic trajectory of the test particle

that results after applying the Lorentzian transformation, substituting Eqs. (5.5)–(5.7) into Eqs. (4.31)–(4.32). In this manner, we obtain the parametric equations of the geodesic trajectory in the fixed frame,

$$t(t') = \frac{1}{\omega} \arctan \gamma [\omega \rho'_{max} V^1 + (1 + \omega \rho'_{min} V^2) \tan \omega t'], \qquad (5.9)$$

$$x^1(t') = \frac{1}{\sqrt{1 - \omega^2 \rho'(t')^2}} \left[\left(1 + (V^1)^2 \frac{\gamma^2}{1 + \gamma} \right) \rho'_{max} \cos \omega t' + \frac{\gamma}{\omega} V^1 \left(1 + \omega \rho'_{min} V^2 \frac{\gamma}{1 + \gamma} \right) \sin \omega t' \right], \qquad (5.10)$$

$$x^2(t') = \frac{1}{\sqrt{1 - \omega^2 \rho'(t')^2}} \left[V^1 V^2 \frac{\gamma^2}{1 + \gamma} \rho'_{max} \cos \omega t' + \left(\frac{\gamma}{\omega} V^2 + \left(1 + (V^2)^2 \frac{\gamma^2}{1 + \gamma} \right) \rho'_{min} \right) \sin \omega t' \right], \qquad (5.11)$$

$$x^3(t') = \frac{1}{\sqrt{1 - \omega^2 \rho'(t')^2}} \left[V^1 V^3 \frac{\gamma^2}{1 + \gamma} \rho'_{max} \cos \omega t' \times \frac{\gamma}{\omega} V^3 \left(1 + V^2 \frac{\gamma}{1 + \gamma} \omega \rho'_{min} \right) \sin \omega t' \right], \qquad (5.12)$$

depending on the parameter t' that can be eliminated for deriving the equations $\vec{x}(t)$ of the geodesic of the test particle in this frame.

The basis $\{\vec{\nu}_1', \vec{\nu}_2', \vec{\nu}_3'\}$ which is fixed rigidly in the mobile frame, determining its Cartesian space coordinates, is seen by the fixed observer as a moving basis $\{\vec{\nu}_1(t), \vec{\nu}_2(t), \vec{\nu}_3(t)\}$. The time dependence of this basis can be found by using the Lorentzian isometry. We assume that each unit vector $\vec{\nu}_i'$ is observed at its own time t'_i, such that we may impose the simultaneity condition

$$t = t[t'_1, \vec{\nu}_1'] = t[t'_2, \vec{\nu}_2'] = t[t'_3, \vec{\nu}_3'], \qquad (5.13)$$

giving the functions $t'_i(t)$ that we need for writing down the form of the unit vectors of the mobile basis as

$$\vec{\nu}_i(t) = \vec{x}[t'_i(t), \vec{\nu}_i'], \qquad i = 1, 2, 3. \qquad (5.14)$$

The last task is to find the conserved quantities measured by the fixed observer as resulting from the transformation (4.33). Let us consider the general case of an arbitrary geodesic orbit of a test particle in the mobile frame whose kinematics is determined by the conserved quantities. The problem is how many such independent quantities we need for determining the kinematic parameters. We observe that six parameters, say the components of $\vec{K}$ and $\vec{N}$, are enough for finding the energy and angular momentum components

from Eqs. (3.15) and (3.14b). Moreover, Eqs. (3.33)–(3.35) give the trajectory parameters, including the time t_0 when the test particle is passing through the aphelion.

However, here we consider seven parameters, the above mentioned vectors, and, in addition, the energy E' instead of the mass m, since thus we avoid the difficulties arising when one uses explicitly Eq. (3.15). Therefore, starting with E', $\vec{K}'$, and $\vec{N}'$ in the mobile frame and applying the transformation (4.33), we derive the general rules

$$E = \gamma(E' + \omega \vec{V} \cdot \vec{N}'), \qquad (5.15)$$

$$\vec{K} = \gamma \vec{K}' + \frac{\omega\gamma}{E'}[\vec{K}'(\vec{V} \cdot \vec{N}') - \vec{N}'(\vec{V} \cdot \vec{K}'))] - \frac{\gamma^2}{1+\gamma}\vec{V}(\vec{V} \cdot \vec{K}'), \qquad (5.16)$$

$$\vec{N} = \vec{N}' + \frac{\gamma}{\omega}E'\vec{V} + \frac{\gamma^2}{1+\gamma}\vec{V}(\vec{V} \cdot \vec{N}'), \qquad (5.17)$$

giving the conserved quantities measured in the fixed frame where the angular momentum results from Eq. (3.14b) as

$$\vec{L} = -\frac{\omega}{E}\vec{K} \wedge \vec{N}. \qquad (5.18)$$

One can verify that this transformation preserves the invariant (3.15) which defines the mass of the test particle. Thus the problem of CAdS relativity is completely solved in terms of conserved quantities.

VI. CONCLUDING REMARKS

We presented here the principal methods of studying the relative geodesic motion on CAdS spacetimes, based on the $SO(2,3)$ transformation generating the Lorentzian isometry that may be expressed in terms of conserved parameters. Starting with this transformation, we derived the general rule of transforming the conserved quantities from the mobile natural frame to the fixed one, obtaining thus information about all the kinematic parameters we need without resorting to the complicated algebraic calculations of the geodesic equations in a particular local chart.

Technically speaking, the CAdS relativity is somewhat similar to the dS one [15] since both of these spacetimes are hyperboloidal and can be constructed as spaces of left cosets on the gauge groups of their embedding flat manifolds. However, the similarity stops here since their embedding manifolds and isometry groups are different, determining different types of geodesic trajectories or

quantum modes. Thus on dS spacetimes the geodesics are open and the quantum modes correspond to continuous energy spectra as it happens also in the Minkowski spacetime. On the contrary, on the AdS spacetimes the geodesic motion is oscillatory and the quantum modes are square-integrable wave functions corresponding to equidistant discrete energy spectra with the same quanta $\hbar\omega$ (in IS units) regardless of the spin [20,21].

Finally, we note that the embedding method may be used for solving common dS-CAdS problems, in a $(2+4)$-dimensional common embedding flat spacetime whose gauge group is just the conformal one, $SO(2,4)$. In this geometry we may study new problems as, for example, what happens in a CAdS sphere moving on the dS spacetime. We hope that our methods presented in Ref. [15] and here will be useful for solving such new problems in an extended (special) relativity of hyperbolic spacetimes, considering all 15 conserved quantities corresponding to the $SO(2,4)$ symmetry.

APPENDIX: INVERSE PROBLEM

There are situations when we know the integration constants κ_1, κ_2, ϕ_0, and t_0 and we need to find the physical conserved quantities. Then from Eqs. (3.22) and (3.24) we deduce

$$E = \frac{m}{\sqrt{(\kappa_1\omega^2 - 1)^2 - \kappa_2^2\omega^4}}, \qquad (A1)$$

$$L = \frac{m\omega\sqrt{\kappa_1^2 - \kappa_2^2}}{\sqrt{(\kappa_1\omega^2 - 1)^2 - \kappa_2^2\omega^4}}. \qquad (A2)$$

Furthermore, from Eqs. (3.10)–(3.13), calculated in the aphelion, at $t = t_0$, where $\tilde{u}^\rho = 0$ and $\tilde{u}^t$ and $\tilde{u}^\phi$ result from Eqs. (3.8) and (3.9), we derive the nonvanishing components

$$K_1 = E\rho_{\max}\cos\phi_0\cos\omega t_0 + E\rho_{\min}\sin\phi_0\sin\omega t_0, \qquad (A3)$$

$$K_2 = E\rho_{\max}\sin\phi_0\cos\omega t_0 - E\rho_{\min}\cos\phi_0\sin\omega t_0, \qquad (A4)$$

$$N_1 = -E\rho_{\max}\cos\phi_0\sin\omega t_0 + E\rho_{\min}\sin\phi_0\cos\omega t_0, \qquad (A5)$$

$$N_2 = -E\rho_{\max}\sin\phi_0\sin\omega t_0 - E\rho_{\min}\cos\phi_0\cos\omega t_0, \qquad (A6)$$

while $K_3 = N_3 = 0$. These components satisfy the properties (3.33)–(3.35).

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