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# New inhomogeneous universes in scalar-tensor and $f(\mathcal{R})$ gravity

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A new family of spherically symmetric inhomogeneous solutions of Brans-Dicke gravity is generated using conformal transformation techniques and the Fonarev solution of general relativity as a seed. The latter is mapped from the Einstein to the Jordan conformal frame and this Jordan frame version constitutes a new solution of Brans-Dicke theory. The Brans-Dicke scalar field self-interacts with a power-law or inverse power-law potential in the Jordan frame. The new 4-parameter family of geometries thus generated, which is dynamical and asymptotically Friedmann-Lemaître-Robertson-Walker, contains as special cases two previously known classes of solutions and solves also the field equations of $f(\mathcal{R}) = \mathcal{R}^n$ gravity, which can also be seen as an $\omega = 0$ Brans-Dicke theory with a potential.

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## I. INTRODUCTION

General relativity (GR) has been tested with good precision and its prediction of gravitational waves has received a spectacular experimental confirmation with the recent LIGO detections [1–4]. However, the theory is not tested at most spatial and temporal scales and curvature regimes [5,6]. What is more, GR does not agree with quantum mechanics and all attempts to quantize it produce, in their low-energy limit, theories which deviate from GR. The most compelling motivation to go beyond Einstein theory, however, comes from cosmology: within the context of GR the present acceleration of the Universe discovered with type Ia supernovae can only be explained with an enormously fine-tuned cosmological constant or with a completely *ad hoc* dark energy fluid as a matter source in the Einstein equations. A viable alternative consists of modifying gravity at cosmological scales while leaving untouched the predictions of GR at small scales. The most popular class of theories achieving this goal is $f(\mathcal{R})$ gravity (where $\mathcal{R}$ is the Ricci scalar of the metric connection) [7], see [8] for reviews.

The prototype of alternative gravity is Brans-Dicke theory [9], which has been generalized to the wider class of scalar-tensor theories [10] and contains as a fundamental variable a gravitational scalar field $\phi$ in addition to the metric tensor $g_{ab}$. The wide class of $f(\mathcal{R})$ theories is a subclass of scalar-tensor gravity. When attempting to understand these theories, spherically symmetric analytic solutions play an important role. Alternative theories of gravity which attempt to explain the current acceleration of the cosmic expansion without dark energy have a built-in time-dependent cosmological "constant" and spherical objects in these theories are not isolated, but are asymptotically Friedmann-Lemaître-Robertson-Walker (FLRW) and are dynamical [11]. Even in GR, exact solutions of the field equations representing dynamical inhomogeneous universes are rare and their physical interpretation is often puzzling ([12,13] and references therein). Here we take one such solution of GR, the Fonarev inhomogeneous universe sourced by a matter scalar field with an exponential potential [14,15], and we use it as a seed to generate a family of new solutions of Brans-Dicke gravity with a power-law (or inverse power-law) potential. By "seed" we mean that the solution of GR is regarded as the Einstein conformal frame version of a Jordan frame solution of Brans-Dicke theory. By reversing the conformal transformation from Jordan to Einstein frame, a new solution of Jordan frame Brans-Dicke theory is generated.

We then show that this family of geometries is also a solution of a class of $f(\mathcal{R})$ theories. Extra motivation for this work comes from the old idea that the gravitational constants of nature may not be constant after all [16,17], and scalar-tensor gravity provides an arena in which the gravitational coupling strength is dynamical. In several scalar-tensor or $f(\mathcal{R})$ theories, a chameleon mechanism makes the effective range of a massive scalar degree of freedom of the theory dependent on the environment. At higher environmental densities this scalar field is short ranged, evading Solar System constraints on deviations from GR, while at cosmological densities the field becomes long ranged. In this context, inhomogeneous universes are useful to probe spatial variations of the gravitational "constant," which was the motivation behind Ref. [18] containing a geometry which corresponds to a special case of the new family of solutions that we introduce here.

Following the notation of Ref. [19] and using units in which Newton's constant $G$ and the speed of light are unity, the action of vacuum Brans-Dicke theory in the Jordan frame is [9]

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$$S_{\rm BD} = \int d^4x \frac{\sqrt{-g}}{16\pi}\left(\phi\mathcal{R} - \frac{\omega}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right), \quad (1.1)$$

where $\phi$ is the Brans-Dicke scalar field (approximately equivalent to the inverse of the gravitational coupling $G_{\rm eff}$) with potential $V(\phi)$, $\omega$ is the constant Brans-Dicke parameter, and $g$ is the determinant of the spacetime metric $g_{ab}$. Inspection of the Brans-Dicke action (1.1) suggest that the effective gravitational coupling present in the theory and obtained by writing the field equations as effective Einstein equations, is $G_{\rm eff} = \phi^{-1}$. A more careful analysis for Solar System experiments [20] or cosmological perturbation theory [21] yields

$$G_{\rm eff} = \frac{2(\omega+2)}{2\omega+3}\frac{1}{\phi}. \quad (1.2)$$

However, in the following it will suffice to consider $G_{\rm eff} \simeq 1/\phi$.

The variation of the action (1.1) generates the Brans-Dicke field equations *in vacuo* [9]

$$R_{ab} - \frac{\mathcal{R}}{2}g_{ab} = \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) - \frac{V}{2\phi}g_{ab}, \quad (1.3)$$

$$\Box\phi = \frac{1}{2\omega+3}\left(\phi\frac{dV}{d\phi} - 2V\right). \quad (1.4)$$

(The original Brans-Dicke theory [9] did not include a potential $V$ for the Brans-Dicke field $\phi$.) The more general class of scalar-tensor theories [10] promotes the Brans-Dicke parameter $\omega$, which is constant in the original Brans-Dicke theory, to a function of the scalar $\phi$.

Another representation of scalar-tensor gravity, the Einstein frame [22], is widely used. By performing the conformal transformation of the metric

$$g_{ab} \to \tilde{g}_{ab} = \phi g_{ab}, \quad (1.5)$$

and the scalar field redefinition

$$\phi \to \tilde{\phi} = \sqrt{\frac{|2\omega+3|}{16\pi}}\ln\left(\frac{\phi}{\phi_*}\right), \quad (1.6)$$

where $\phi_*$ is a constant and $\omega \neq -3/2$, the Brans-Dicke action (1.1) assumes its Einstein frame form

$$S_{\rm BD} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{\mathcal{R}}}{16\pi} - \frac{1}{2}\tilde{g}^{ab}\nabla_a\tilde{\phi}\nabla_b\tilde{\phi} - \tilde{V}(\tilde{\phi})\right], \quad (1.7)$$

where

$$\tilde{V}(\tilde{\phi}) = \left.\frac{V(\phi)}{\phi^2}\right|_{\phi=\phi(\tilde{\phi})}. \quad (1.8)$$

In the presence of matter, obtained by adding a Lagrangian density $\sqrt{-g}\mathcal{L}_{(m)}(\psi_{(m)}, g_{ab})$ [where $\psi_{(m)}$ collectively denotes the matter fields] to the Jordan frame action, there will be a difference. The Einstein frame action will then exhibit an anomalous coupling of the scalar field to matter, described by the term $\sqrt{-\tilde{g}}\mathcal{L}_{(m)}(\psi_{(m)}, g_{ab})/\phi^2(\tilde{\phi})$ in the action. Since in this work we consider only vacuum solutions, we will not be concerned with this anomalous coupling of the scalar to matter.

In the following, Einstein frame quantities will be denoted by a tilde. The action (1.7) is formally the Einstein-Hilbert action coupled to a matter scalar field which has canonical kinetic energy density. The Einstein frame field equations *in vacuo* are

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{\mathcal{R}} = 8\pi\left(\nabla_a\tilde{\phi}\nabla_b\tilde{\phi} - \frac{1}{2}\tilde{g}_{ab}\tilde{g}^{cd}\nabla_c\tilde{\phi}\nabla_d\tilde{\phi}\right) - \tilde{V}(\tilde{\phi})\tilde{g}_{ab}, \quad (1.9)$$

$$\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} - \frac{d\tilde{V}}{d\tilde{\phi}} = 0. \quad (1.10)$$

If we know a solution of the Einstein equations with a minimally coupled scalar field as the matter source, it is possible to regard it as the Einstein frame representation of a scalar-tensor solution and to map it back to the Jordan frame representation. In general, the scalar field potential thus obtained in the Jordan frame is not motivated by a physical theory and the corresponding spacetime does not carry much physical meaning. This is the reason why this solution-generating technique has seen only limited applications in cases where the scalar field potential is absent [18,23]. However, in the particular application to the Fonarev spacetime [14,15] studied here, the Jordan frame potential $V(\phi)$ turns out to be physically well motivated.

$f(\mathcal{R})$ theories of gravity [8] are a subclass of scalar-tensor theories described by the action

$$S = \int d^4x\frac{\sqrt{-g}}{16\pi}f(\mathcal{R}) \quad (1.11)$$

*in vacuo*, where $f(\mathcal{R})$ is a nonlinear function of the Ricci scalar $\mathcal{R}$. By setting $\phi = f'(\mathcal{R})$ and

$$V(\phi) = \phi\mathcal{R}(\phi) - f(\mathcal{R}(\phi)), \quad (1.12)$$

it can be shown that the action (1.11) is equivalent to the vacuum Brans-Dicke action [8]

$$S = \int d^4x\frac{\sqrt{-g}}{16\pi}[\phi\mathcal{R} - V(\phi)], \quad (1.13)$$

which has Brans-Dicke parameter $\omega = 0$ and the potential (1.12) for the Brans-Dicke scalar $\phi$.

The plan of this article is as follows. In Sec. II we review the Fonarev solution of GR. In Sec. III we obtain a new family of scalar-tensor solutions using the Fonarev spacetime as a seed. This family includes, as a special case, a solution previously reported in [18] which is conformal to the Husain-Martinez-Nuñez geometry of GR [24]. Another special case reproduces the Campanelli-Lousto solution [25], which describes a wormhole [26]. In Sec. IV we comment on the physical interpretation of the new family of solutions. Section V explains how this new family is also a solution of a subclass of $f(\mathcal{R})$ theories and Sec. VI contains a discussion and the conclusions.

## II. THE FONAREV SOLUTION OF GENERAL RELATIVITY

The Fonarev solution of the Einstein equations of GR [14] is a spherically symmetric, dynamical, inhomogeneous, and asymptotically FLRW geometry sourced by a minimally coupled scalar field $\tilde{\phi}$ self-interacting with an exponential potential. The line element is

$$d\tilde{s}^2 = -\mathrm{e}^{8\alpha^2 at}\left(1-\frac{2m}{r}\right)^{\delta} dt^2 + \mathrm{e}^{2at}\left[\frac{dr^2}{(1-\frac{2m}{r})^{\delta}} + \left(1-\frac{2m}{r}\right)^{1-\delta} r^2 d\Omega_{(2)}^2\right] \tag{2.1}$$

and the matter scalar field is

$$\tilde{\phi}(t,r) = \frac{1}{\sqrt{4\pi}}\left[2\alpha at + \frac{1}{2\sqrt{1+4\alpha^2}}\ln\left(1-\frac{2m}{r}\right)\right], \tag{2.2}$$

with the scalar field potential

$$\tilde{V}(\tilde{\phi}) = \tilde{V}_0 \mathrm{e}^{-8\sqrt{\pi}\alpha\tilde{\phi}} \tag{2.3}$$

and where $d\Omega_{(2)}^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the unit 2-sphere. This is a 3-parameter $(m, \alpha, a)$ family of solutions of the Einstein equations, where

$$\delta = \frac{2\alpha}{\sqrt{1+4\alpha^2}} < 1, \tag{2.4}$$

$$\tilde{V}_0 = \frac{a^2(3-4\alpha^2)}{8\pi}. \tag{2.5}$$

In order to guarantee a non-negative energy density for the scalar field it must be $\tilde{V}_0 \geq 0$, which implies that $|\alpha| \leq \sqrt{3}/2$.

This solution was introduced in [14] and studied in [15] and [27]. Five-dimensional Fonarev solutions were given in [28] and it was shown recently that Fonarev solutions can be generated via dimensional reduction from Fisher-like brane solutions in $4+n$ dimensions [27].

The Fonarev line element is conformal to the Fisher-Buchdahl-Janis-Newman-Winicour-Wyman scalar field solution of the Einstein equations [29] (hereafter referred to simply as "Fisher solution")

$$ds^2 = -A(r)^{\nu} dt^2 + A(r)^{-\nu} dr^2 + A(r)^{1-\nu} r^2 d\Omega_{(2)}^2, \tag{2.6}$$

$$\phi(r) = \phi_0 \ln A(r), \tag{2.7}$$

where $\nu$ and $\phi_0$ are constants and $A(r) \equiv 1\text{–}2m/r$. An interesting feature of the Fisher solution (pointed out in [27]) is that the redshift factor for light traveling radially outward from a radius approaching the singularity diverges and these solutions have the properties of "frozen stars" like the Schwarzschild spacetime. The Fisher solution is the limit of charged dilaton black holes [30] when the electric charge vanishes and, therefore, the Fonarev solutions can be seen as limits of a family of dilaton black holes embedded in FLRW space [27].

For the special parameter values $\alpha = \pm\sqrt{3}/2$ the potential $\tilde{V}(\tilde{\phi})$ vanishes identically and the Fonarev solution reduces to the Husain-Martinez-Nuñez solution of the Einstein equations [24], which is already known to be conformal to the Fisher solution [24]. For $\alpha \neq \pm\sqrt{3}/2$ and $\alpha a \neq 0$, consider the new time coordinate $\tau$ defined by

$$\tau = \frac{\mathrm{e}^{4\alpha^2 at}}{4\alpha^2 a}, \tag{2.8}$$

which turns the Fonarev line element and scalar field (2.1) and (2.2) into

$$d\tilde{s}^2 = -A(r)^{\delta} d\tau^2 + (4\alpha^2 a\tau)^{1/(2\alpha^2)}[A(r)^{-\delta} dr^2 + A(r)^{1-\delta} r^2 d\Omega_{(2)}^2], \tag{2.9}$$

$$\tilde{\phi}(\tau,r) = \frac{1}{4\sqrt{\pi}}\ln\left[(4\alpha^2 a\tau)^{1/\alpha} A(r)^{\frac{1}{\sqrt{1+4\alpha^2}}}\right]. \tag{2.10}$$

The further time redefinition

$$\eta = \frac{(a\tau)^{1-\frac{1}{4\alpha^2}}}{(4\alpha^2)^{\frac{1}{4\alpha^2}-1}(4\alpha^2-1)a} \tag{2.11}$$

for $\alpha^2 \neq 0, 1/4$ and $a \neq 0$ turns the geometry and scalar field into

$$d\tilde{s}^2 = (a\eta)^{\frac{2}{4\alpha^2-1}}[-A(r)^{\delta} d\eta^2 + A(r)^{-\delta} dr^2 + A(r)^{1-\delta} r^2 d\Omega_{(2)}^2], \tag{2.12}$$

$$\tilde{\phi}(\eta, r) = \frac{1}{4\sqrt{\pi}} \ln \left\{ [(4\alpha^2 - 1)a\eta]^{\frac{4\alpha}{4\alpha^2-1}} A(r)^{\frac{1}{\sqrt{1+4\alpha^2}}} \right\}. \quad (2.13)$$

The line element (2.12) is explicitly conformal to the Fisher one.

For $\alpha^2 = 1/4$ the coordinate transformation (2.11) ceases to be valid but the line element reduces to

$$d\tilde{s}^2 = e^{2at}[-A(r)^\delta dt^2 + A(r)^{-\delta} dr^2 + A(r)^{1-\delta} r^2 d\Omega^2_{(2)}], \quad (2.14)$$

which also is conformal to the Fisher line element (2.6).

Special cases of the Fonarev solution include the following.

### A. Vanishing mass parameter

When the mass parameter $m$ vanishes, the Fonarev solution reduces to the spatially flat FLRW metric

$$d\tilde{s}^2 = -e^{8\alpha^2 at} dt^2 + e^{2at}(dr^2 + r^2 d\Omega^2_{(2)}) \quad (2.15)$$

and the scalar field is linear,

$$\tilde{\phi}(t) = \frac{\alpha a t}{\sqrt{\pi}}. \quad (2.16)$$

The redefinition of the time coordinate $d\tau = e^{4\alpha^2 at} dt$ then recasts the line element as

$$d\tilde{s}^2 = -d\tau^2 + (4\alpha^2 a\tau)^{\frac{1}{2\alpha^2}}(dr^2 + r^2 d\Omega^2_{(2))}) \quad (2.17)$$

with scale factor $S(\tau) = (4\alpha^2 a\tau)^{\frac{1}{4\alpha^2}}$ and matter scalar field

$$\tilde{\phi}(\tau) = \frac{\alpha}{\sqrt{\pi}} \ln S(\tau). \quad (2.18)$$

This Universe is accelerated if $0 < |\alpha| < 1$.

### B. Vanishing a parameter

When the parameter $a$, which has the dimensions of an inverse time, vanishes the line element and scalar field reduce to

$$d\tilde{s}^2 = -A(r)^\delta dt^2 + A(r)^{-\delta} dr^2 + A(r)^{1-\delta} r^2 d\Omega^2_{(2)}, \quad (2.19)$$

$$\tilde{\phi}(t, r) = \frac{1}{4\sqrt{\pi(1+4\alpha^2)}} \ln A(r), \quad (2.20)$$

and the scalar field potential vanishes, $\tilde{V} \equiv 0$. This is a static Fisher solution (2.6) with $\nu = \delta$ and with areal radius

$$R(r) = A(r)^{\frac{1-\delta}{2}} r = \left(1 - \frac{2m}{r}\right)^{\frac{1-\delta}{2}} r. \quad (2.21)$$

As is well known [29], it exhibits a central singularity at $R = 0$ which, as is clear from Eq. (2.21), corresponds to $r = 2m$.

### C. Parameter $\alpha^2 = 3/4$

The parameter $\alpha$ is related to the slope of the scalar field potential (2.3). In the special cases $\alpha = \pm\sqrt{3}/2$, the potential $\tilde{V}(\tilde{\phi})$ vanishes identically and the line element and scalar field reduce to

$$d\tilde{s}^2 = -e^{6at} A(r)^{\pm\sqrt{3}/2} dt^2 + e^{2at}[A(r)^{\mp\sqrt{3}/2} dr^2 + A(r)^{1\mp\sqrt{3}/2} r^2 d\Omega^2_{(2)}], \quad (2.22)$$

$$\tilde{\phi}(t, r) = \frac{1}{2\sqrt{\pi}} \left[\pm\sqrt{3} at + \frac{1}{4} \ln A(r)\right]. \quad (2.23)$$

This is recognized as the Husain-Martinez-Nuñez solution of GR [24]. It is not obtained as the time development of regular Cauchy data because it contains a naked singularity part of the time and it exhibits an interesting phenomenology of apparent horizons which appear and disappear in pairs [24], and it constitutes one of two paradigmatic situations identified with time-evolving apparent horizons (the other situation is exemplified by McVittie-type solutions) [13]. The special subcase $a = 0$ eliminates the time dependence and reduces the Husain-Martinez-Nuñez solution to the Fisher spacetime. Assuming $a \neq 0$, the use of the time coordinate

$$\tau(t) = \frac{e^{3at}}{3a} \quad (2.24)$$

reduces the line element and scalar field to

$$d\tilde{s}^2 = -A(r)^\delta d\tau^2 + (3a\tau)^{2/3}[A(r)^{-\delta} dr^2 + A(r)^{1-\delta} r^2 d\Omega^2_{(2)}], \quad (2.25)$$

$$\tilde{\phi}(\tau, r) = \frac{1}{8\sqrt{\pi}} \ln [A(r)(3a\tau)^{\pm 4/\sqrt{3}}]. \quad (2.26)$$

The geometry is asymptotically FLRW as $r \to +\infty$. The scale factor $S(\tau) \sim \tau^{1/3}$ corresponds to the stiff equation of state $P = \rho$ for the cosmic fluid equivalent of the free [in the sense that $V(\phi) = 0$] scalar field.

### D. Parameter $\alpha = 0$

This special case implies $\delta = 0$ and the scalar field potential $\tilde{V} = \tilde{V}_0$ reduces to an effective cosmological constant $\Lambda = 8\pi\tilde{V}_0 = 3a^2$ with

$$d\tilde{s}^2 = -dt^2 + e^{2at}[dr^2 + A(r) r^2 d\Omega^2_{(2)}], \quad (2.27)$$

$$\tilde{\phi}(t,r)=\frac{1}{4\sqrt{\pi}}\ln A(r), \qquad (2.28)$$

which can be seen as a time-dependent generalization of a Fonarev solution with $\nu=0$ (to which it reduces if $a=0$), which is asymptotically de Sitter. Because of formal similarities, the Husain-Martinez-Nuñez solution and its Fonarev generalization could be superficially seen as time-dependent generalizations of the Fisher solution but they are qualitatively different in the parameter range in which apparent horizons exist (the Fisher solution, by contrast, has no apparent/trapping horizons to cover the central naked singularity).

## III. GENERATING NEW BRANS-DICKE SOLUTIONS

Following the method outlined in Sec. I, assume now that the Fonarev solution $\tilde{g}_{ab}$ of GR with matter scalar field $\tilde{\phi}$ is formally the Einstein frame representation of a solution of Brans-Dicke theory in the Jordan frame $(g_{ab},\phi)$, where the scalar field is now the gravitational Brans-Dicke field, which is related to the Fonarev geometry by

$$\tilde{g}_{ab}=\phi g_{ab}, \qquad (3.1)$$

$$\tilde{\phi}=\sqrt{\frac{|2\omega+3|}{16\pi}}\ln\left(\frac{\phi}{\phi_0}\right), \qquad (3.2)$$

where $\phi_0$ is a constant and $\omega\neq -3/2$. By inverting Eq. (3.2) and substituting Eq. (2.2) in it, one obtains

$$\phi(t,r)=\phi_0 e^{\frac{4\alpha at}{\sqrt{|2\omega+3|}}}\left(1-\frac{2m}{r}\right)^{\frac{1}{\sqrt{|2\omega+3|(1+4\alpha^2)}}} \qquad (3.3)$$

and the corresponding scalar field potential $V(\phi)$ is obtained from Eq. (1.8) as

$$V(\phi)=V_0\phi^{2\beta} \qquad (3.4)$$

with

$$\beta=1-\alpha\sqrt{|2\omega+3|}, \qquad V_0=\tilde{V}_0\phi_0^{2\alpha\sqrt{|2\omega+3|}}. \qquad (3.5)$$

As already remarked, in general the potential (1.8) generated by using the conformal transformation to the Jordan frame and a known GR solution as the seed is not of a form motivated by scalar field theories in particle physics or in cosmology. However, the power-law potential obtained in the Jordan frame from the Fonarev solution has been the subject of intensive studies in both cosmology and particle physics [31,32]. It includes as special cases the mass potential $m_\phi^2\phi^2/2$, the quartic potential $\lambda\phi^4$, and many quintessence potentials [31,32].

The relation $g_{ab}=\phi^{-1}\tilde{g}_{ab}$ gives the Jordan frame line element

$$\begin{aligned} ds^2 = {} & -A(r)^{\frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha-\frac{1}{\sqrt{|2\omega+3|}}\right)} e^{4\alpha at\left(2\alpha-\frac{1}{\sqrt{|2\omega+3|}}\right)} dt^2 \\ & + e^{2at\left(1-\frac{2\alpha}{\sqrt{|2\omega+3|}}\right)}\left[A(r)^{-\frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha+\frac{1}{\sqrt{|2\omega+3|}}\right)}dr^2 \right. \\ & \left. + A(r)^{1-\frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha+\frac{1}{\sqrt{|2\omega+3|}}\right)} r^2 d\Omega_{(2)}^2\right] \end{aligned} \qquad (3.6)$$

(neglecting an irrelevant overall multiplicative constant $\phi_0^{-1}$). We have a family of solutions of the vacuum Brans-Dicke field equations (1.3), (1.4) parametrized by the four parameters $(\omega,m,a,\alpha)$, of which $\omega$ is a parameter of the theory and the others are parameters of this specific family of solutions. By introducing the quantities

$$\gamma\equiv\frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha+\frac{1}{\sqrt{|2\omega+3|}}\right), \qquad (3.7)$$

$$\epsilon\equiv\frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha-\frac{1}{\sqrt{|2\omega+3|}}\right), \qquad (3.8)$$

the new time coordinate

$$\tau(t)=\frac{e^{2\alpha at\left(2\alpha-\frac{1}{\sqrt{|2\omega+3|}}\right)}}{2\alpha a\left(2\alpha-\frac{1}{\sqrt{|2\omega+3|}}\right)} \qquad (3.9)$$

(defined for $a\neq 0$ and $\alpha\neq 0,\frac{1}{2\sqrt{|2\omega+3|}}$), and the FLRW scale factor

$$S(\tau)=\left[2\alpha a\tau\left(2\alpha-\frac{1}{\sqrt{|2\omega+3|}}\right)\right]^{\frac{\sqrt{|2\omega+3|}-2\alpha}{2\alpha(2\alpha\sqrt{|2\omega+3|}-1)}}, \qquad (3.10)$$

one can write the new family of solutions as

$$ds^2=-A(r)^{\epsilon}d\tau^2+S^2(\tau)[A(r)^{-\gamma}dr^2+A(r)^{1-\gamma}r^2d\Omega_{(2)}^2], \qquad (3.11)$$

$$\phi(\tau,r)=\phi_0[S(\tau)]^{\frac{4\alpha}{\sqrt{|2\omega+3|}-2\alpha}}A(r)^{\frac{1}{\sqrt{|2\omega+3|(1+4\alpha^2)}}}. \qquad (3.12)$$

When the mass parameter $m$ vanishes, the line element reduces to the spatially flat FLRW one

$$ds^2=-d\tau^2+S^2(\tau)(dr^2+r^2d\Omega_{(2)}^2), \qquad (3.13)$$

while the Brans-Dicke scalar field is

$$\phi(\tau) = \phi_0[S(\tau)]^{\frac{4\alpha}{\sqrt{|2\omega+3|}-2\alpha}}. \tag{3.14}$$

Therefore (when it is defined) the coordinate $\tau$ has the meaning of comoving time of the FLRW space in which the inhomogeneity is embedded.

By contracting the Brans-Dicke field equations (1.3) and substituting $\Box\phi$ from Eq. (1.4) one obtains the Jordan frame Ricci scalar

$$\mathcal{R} = \omega\nabla^c \ln\phi\nabla_c \ln\phi + \frac{1}{2\omega+3}\left(3\frac{dV}{d\phi} + \frac{4\omega V}{\phi}\right) = \frac{\omega}{\phi^2}\nabla^c\phi\nabla_c\phi + 2V_0\left[\frac{3(\beta-1)}{2\omega+3}+1\right]\phi^{2\beta-1} \tag{3.15}$$

and, using Eq. (3.5),

$$\mathcal{R} = \omega\nabla^c \ln\phi\nabla_c \ln\phi + 2V_0\left[1 - \frac{3\alpha\mathrm{sign}(2\omega+3)}{\sqrt{|2\omega+3|}}\right]\phi^{2\beta-1}. \tag{3.16}$$

Since

$$\nabla_\mu \ln\phi = \frac{4\alpha a}{\sqrt{|2\omega+3|}}\delta_{\mu 0} + \frac{2m}{r^2(1-2m/r)\sqrt{|2\omega+3|(1+4\alpha^2)}}\delta_{\mu 1}, \tag{3.17}$$

one obtains

$$\begin{aligned}\mathcal{R} = {} & -A(r)^{\frac{1}{\sqrt{1+4\alpha^2}}\left(\frac{1}{\sqrt{|2\omega+3|}}-2\alpha\right)} \cdot \frac{16\alpha^2 a^2 \omega \mathrm{e}^{4\alpha a t\left(\frac{1}{\sqrt{|2\omega+3|}}-2\alpha\right)}}{|2\omega+3|} \\ & + \frac{4m^2\omega}{r^4|2\omega+3|(1+4\alpha^2)}A(r)^{\frac{1}{\sqrt{1+4\alpha^2}}\left(\frac{1}{\sqrt{|2\omega+3|}}+2\alpha\right)-2} \cdot \mathrm{e}^{2at\left(\frac{2\alpha}{\sqrt{|2\omega+3|}}-1\right)} \\ & + 2V_0\left[1-\frac{3\alpha}{\sqrt{|2\omega+3|}}\mathrm{sign}(2\omega+3)\right]\phi_0^{2\beta-1} \cdot \mathrm{e}^{-4\alpha a t\left(2\alpha-\frac{1}{\sqrt{|2\omega+3|}}\right)}A(r)^{\frac{1}{\sqrt{1+4\alpha^2}}\left(\frac{1}{\sqrt{|2\omega+3|}}-2\alpha\right)}.\end{aligned} \tag{3.18}$$

The three terms which add up to compose the Ricci scalar can vanish in special cases. The first term is absent if $\alpha = 0$ or $a = 0$ or $\omega = 0$. The second term is absent if $m = 0$ or $\omega = 0$. The third term drops out if $V_0 = 0$ (which happens if $a = 0$ or $\alpha^2 = 3/4$) or if $\phi_0 = 0$ (which is forbidden) or if $\alpha = \sqrt{|2\omega+3|}\mathrm{sign}(2\omega+3)/3 \equiv \alpha_*$. Unless these three terms disappear simultaneously, there is a spacetime singularity at $r = 2m$ for the parameter values for which the exponent of $A(r)$ in these terms is negative.

Regardless of the possible presence of a spacetime singularity at $r = 2m$, the scalar field (3.3) always vanishes, and the effective gravitational coupling $G_{\mathrm{eff}}$ diverges, as $r \to 2m^+$, which is another physical pathology to be avoided (e.g., [33,34]).

Assume that $\gamma \neq 1$: then the vanishing of the areal radius [given in Eq. (4.3) below] $R = 0$ corresponds to $r = 2m$ and to a central singularity if the Ricci scalar $\mathcal{R}$ diverges there. When present, the singularity at $R = 0$ is timelike. In fact, consider the surface of equation $\Psi(r) \equiv r - 2m$. The direction of its normal is $N_\mu = \nabla_\mu\Psi = \delta_{\mu r}$ and

$$N_c N^c = g^{ab}N_a N_b = g^{rr} = \mathrm{e}^{2at\left(\frac{2\alpha}{\sqrt{|2\omega+3|}}-1\right)}A(r)^\gamma \tag{3.19}$$

is positive for any $r > 2m$, hence it is non-negative in the limit $r \to 2m^+$, making $N^c$ spacelike or null and the surfaces $\Psi(r) = \mathrm{const.}$ timelike or null (further, if $\gamma = 0$ then $N^c N_c$ is strictly positive and the singularity is timelike). The singularity of the conformally related Fonarev solution is timelike [15] and the conformal map respects causality, hence the singularity obtained as the limit $r \to 2m^+$ is timelike. Therefore, when there is a naked singularity, the geometry cannot be obtained as the development of regular data on an initial Cauchy surface.

## IV. INTERPRETATION OF THE SOLUTIONS

Let us interpret now the new solutions of Brans-Dicke theory found by mapping back the Fonarev solution of GR to the Jordan frame. We maintain the condition $|\alpha| \leq \sqrt{3}/2$ which now guarantees non-negativity of the Jordan frame potential $V(\phi)$. We look for possible apparent horizons and, if found, we attempt to identify them as black hole horizons or wormhole throats.[1] In general, dynamical spacetimes do not admit event horizons and the best substitute for the notion of horizon is the apparent or trapping horizon (see Refs. [13,36,37] for definitions and reviews of the related literature). A naked singularity is one that is not covered by apparent black hole horizons.

The areal radius which is read off the geometry (3.11) is

$$R(\tau, r) = S(\tau)A(r)^{\frac{1-\gamma}{2}}r. \tag{4.1}$$

[1] We identify a wormhole throat with an apparent horizon with radius corresponding to a double root of Eq. (4.2). Other authors (such as [35]) have more stringent definitions of wormhole throats.

Apart from the special parameter value $\gamma = 1$, the coordinate radius $r = 2m$ always corresponds to zero areal radius $R$. Therefore, for $\gamma \neq 1$ and assuming non-negative mass parameter $m$, the physical range of the radial coordinate is $r \geq 2m$ (or $R \geq 0$). The apparent horizons, when they exist, are located by the roots of the equation [38]

$$\nabla^c R \nabla_c R = 0. \tag{4.2}$$

In coordinates $(t, r)$ the areal radius is

$$R(t,r) = \mathrm{e}^{at\left(1-\frac{2\alpha}{\sqrt{|2\omega+3|}}\right)} A(r)^{\frac{1}{2}-\frac{1}{\sqrt{1+4\alpha^2}}\left(\alpha+\frac{1}{2\sqrt{|2\omega+3|}}\right)} r \tag{4.3}$$

and, in order for it to be well defined and positive, it must be $r > 2m$ [except for the special case $\gamma = 1$ in which the exponent of $A(r)$ vanishes]. Equation (4.2) takes the form

$$a^2\left(1-\frac{2\alpha}{\sqrt{|2\omega+3|}}\right)^2 \mathrm{e}^{2at(1-4\alpha^2)} r^2 A(r)^{-\frac{4\alpha}{\sqrt{1+4\alpha^2}}+2} = \left\{1-\frac{m}{r}\left[1+\frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha+\frac{1}{\sqrt{|2\omega+3|}}\right)\right]\right\}^2 \tag{4.4}$$

in coordinates $(t, r)$. This form of Eq. (4.2) is useful when the time coordinate $\tau$ cannot be used. For the parameter values for which $\tau$ is well defined, Eq. (4.2) can be written in the form

$$A(r)^{2\left(1-\frac{2\alpha}{\sqrt{1+4\alpha^2}}\right)} r^2 S_\tau^2 = \left[1-(\gamma+1)\frac{m}{r}\right]^2. \tag{4.5}$$

Equations (4.4) or (4.5) should be solved for $r$ (or $R$) in order to locate the radii of the apparent horizons (if these exist). It is not possible to solve these equations analytically except for special points in parameter space. Likewise, their numerical solution requires the complete specification of the values of the parameters $(\omega, m, a, \alpha)$. Let us examine the solutions of Eqs. (4.4) and (4.5) in special cases.

### A. Special case 1 ($\alpha = 0$)

Let us consider the parameter value $\alpha = 0$ which trivially satisfies the constraint $|\alpha| \leq \sqrt{3}/2$ and makes the coordinate transformation $t \to \tau$ invalid. In this case the scalar field potential $V = V_0\phi^{2\beta}$ reduces to a mass term $m_\phi^2\phi^2/2$ with

$$m_\phi = \sqrt{2V_0} = \sqrt{2\tilde{V}_0} = \sqrt{\frac{3a^2}{4\pi}}, \tag{4.6}$$

while $\gamma = |2\omega+3|^{-1/2}$. The Brans-Dicke spacetime is given by

$$ds^2 = -A(r)^{-\frac{1}{\sqrt{|2\omega+3|}}}dt^2 + \mathrm{e}^{2at}\left[A(r)^{-\frac{1}{\sqrt{|2\omega+3|}}}dr^2 + A(r)^{1-\frac{1}{\sqrt{|2\omega+3|}}} r^2 d\Omega_{(2)}^2\right], \tag{4.7}$$

$$\phi(r) = \phi_0 A(r)^{\frac{1}{\sqrt{|2\omega+3|}}}. \tag{4.8}$$

Equation (4.4) becomes

$$a^2\mathrm{e}^{2at}r^2A(r)^2 = \left[1-\frac{m}{r}\left(1+\frac{1}{\sqrt{|2\omega+3|}}\right)\right]^2, \tag{4.9}$$

which cannot be solved analytically for general values of the parameters $(a, m, \omega)$. For illustration, we consider $a \neq 0$ in conjunction with the special values of the Brans-Dicke parameter $\omega = -2, -1$ for which $|2\omega+3| = 1$. Then Eq. (4.9) admits the single positive root

$$r_{\rm AH} = \frac{\mathrm{e}^{-at}}{a} \tag{4.10}$$

(where the subscript "AH" denotes apparent horizons) or, since the areal radius is $R(t,r) = r\mathrm{e}^{at}$,

$$R_{\rm AH} = \frac{1}{a}. \tag{4.11}$$

The "background" cosmology is obtained by letting $m$ go to zero and it is a de Sitter space with Hubble parameter $a$, constant scalar field $\phi_0$, and cosmological constant $\Lambda = 8\pi V_0\phi_0 = 3a^2\phi_0$. Therefore, this apparent horizon always coincides with the de Sitter (cosmological) horizon of the background, which is a null surface.

The minimal physical requirement that the Brans-Dicke field

$$\phi = \phi_0\left(1-\frac{2m}{r}\right) \tag{4.12}$$

be positive imposes that $r > 2m$. Then the apparent horizons exists only at comoving times

$$t < a^{-1}\ln\left(\frac{1}{2ma}\right) \equiv t_*. \tag{4.13}$$

The effective coupling $G_{\rm eff}$ neither diverges nor vanishes on this apparent horizon because $r = 2m$ [where $A(r)$ vanishes] is distinct from $r_{\rm AH}$, except at the time $t_*$. At $t = t_*$ it is $r_{\rm AH} = 2m$ and $\phi$ vanishes while $G_{\rm eff}$ diverges.

### B. Special case 2 ($a = 0$)

The time scale of variation of the new Brans-Dicke solution is, roughly speaking, $a^{-1}$, therefore the limit $a \to 0$ makes this time scale infinite, yielding a family of static solutions. In this case the coordinate transformation (3.9) degenerates and, using Eqs. (3.6) and (3.3), one obtains the geometry and Brans-Dicke field in coordinates $(t, r)$

$$ds^2 = -A(r)^{\frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha - \frac{1}{\sqrt{|2\omega+3|}}\right)} dt^2 + A(r)^{-\frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha + \frac{1}{\sqrt{|2\omega+3|}}\right)} dr^2 + A(r)^{1-\frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha + \frac{1}{\sqrt{|2\omega+3|}}\right)} r^2 d\Omega_{(2)}^2, \tag{4.14}$$

$$\phi(r) = \phi_0 A(r)^{\frac{1}{\sqrt{|2\omega+3|(1+4\alpha^2)}}}, \tag{4.15}$$

which are static, while $V(\phi) = 0$. Equation (4.4) for the apparent horizons degenerates and admits the double root

$$r_{\text{AH}} = m\left[1 + \frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha + \frac{1}{\sqrt{|2\omega+3|}}\right)\right] \equiv (1+\gamma)m, \tag{4.16}$$

which corresponds to a wormhole apparent horizon provided that $r_{\text{AH}} > 2m$. This condition translates into

$$\alpha > \frac{\omega+1}{2\sqrt{2\omega+3}} \quad \text{if } \omega > -3/2, \tag{4.17}$$

$$\alpha > \frac{-(\omega+2)}{2\sqrt{|2\omega+3|}} \quad \text{if } \omega < -3/2. \tag{4.18}$$

We further impose the condition $|\alpha| \le \sqrt{3}/2$. Consider first the situation in which $\omega > -3/2$: then in order to satisfy both (4.17) and $|\alpha| \le \sqrt{3}/2$ it must be

$$\frac{\omega+1}{2\sqrt{2\omega+3}} < \frac{\sqrt{3}}{2}, \tag{4.19}$$

which is equivalent to $\omega + 1 < \sqrt{3(2\omega+3)}$. If $\omega < -1$ this inequality is always satisfied while, if $\omega \ge -1$ both sides of (4.19) are non-negative and we can square it, obtaining $\psi_1(\omega) \equiv \omega^2 - 4\omega - 8 < 0$. The parabola of equation $\psi_1(\omega)$ has concavity facing upward, crosses the $\omega$ axis at $\omega_\pm = 2(1 \pm \sqrt{3})$, and is negative if $\omega_- < \omega < \omega_+$. Therefore, the restriction $|\alpha| \le \sqrt{3}/2$ limits the range of the Brans-Dicke parameter to

$$-\frac{3}{2} < \omega < 2(1+\sqrt{3}). \tag{4.20}$$

Let us consider now the other situation $\omega < -3/2$: the restriction $|\alpha| \le \sqrt{3}/2$ is compatible with (4.18) only if

$$-\frac{(\omega+2)}{2\sqrt{2\omega+3}} < \frac{\sqrt{3}}{2}, \tag{4.21}$$

equivalent to $-(\omega+2) < \sqrt{3|2\omega+3|}$. If $-2 < \omega < -3/2$ the left-hand side of (4.21) is negative and its right-hand side is non-negative, hence (4.21) is always satisfied. If instead $\omega \le -2$, then both sides of (4.21) are non-negative and we can square this inequality, obtaining $\psi_2(\omega) \equiv \omega^2 + 10\omega + 13 < 0$. The parabola $\psi_2(\omega)$ has concavity facing upward, crosses the $\omega$ axis at $\omega_\pm = -5 \pm 2\sqrt{3}$, and is negative if $-5 - 2\sqrt{3} < \omega \le -2$. Therefore the condition $|\alpha| \le \sqrt{3}/2$ imposes the restriction on the range of the Brans-Dicke parameter

$$-5 - 2\sqrt{3} < \omega < -3/2. \tag{4.22}$$

The wormhole apparent horizon has areal radius

$$R_{\text{AH}} = m(1+\gamma)\left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{1-\gamma}{2}}. \tag{4.23}$$

Let us discuss the causal nature of this apparent horizon, which is the surface of equation $f(r) = 0$, where $f(r) \equiv r - (\gamma+1)m$. The normal to the surfaces $f = \text{const.}$ has components

$$N_\mu = \nabla_\mu f = \delta_{\mu 1}, \tag{4.24}$$

its norm squared is

$$N_a N^a = g^{ab}\nabla_a f \nabla_b f = g^{rr} = A(r)^\gamma, \tag{4.25}$$

and on the apparent horizon it is

$$N_a N^a|_{r_{\text{AH}}} = \left(\frac{\gamma-1}{\gamma+1}\right)^\gamma. \tag{4.26}$$

If $\gamma > 1$ the normal $N^a$ is spacelike and the apparent horizon is a timelike surface, while it is null if $\gamma = 1$. The fact that this static apparent horizon is timelike for $\gamma > 1$ is in apparent contradiction with the well-known statement of GR [39] that in stationary situations apparent horizons and event horizons (which are null) coincide. However there is no real contradiction here because the proof of this statement requires the dominant energy condition [39], which cannot be imposed on the Brans-Dicke scalar field.[2] The Brans-Dicke field $\phi$ does not diverge nor vanish on the apparent horizon (4.16).

The new family of static Brans-Dicke solutions obtained for $a = 0$ contains, as a special case, another class of known solutions, the Campanelli-Lousto class [25], which is obtained when

[2]The effective stress-energy tensor of $\phi$ in the right-hand side of Eq. (1.3) contains second order derivatives which have indefinite sign, contrary to the canonical products of first order derivatives. In addition to the fact that the sign of the coefficient $\omega$ can be negative, this fact makes the sign of the effective energy density indefinite.

$$\omega > -\frac{3}{2}, \qquad \alpha = \frac{1}{2\sqrt{2\omega+3}}. \tag{4.27}$$

The Campanelli-Lousto family is given by

$$ds^2 = -A(r)^{b_0+1}dt^2 + A(r)^{-a_0-1}dr^2 + A(r)^{-a_0}r^2 d\Omega_{(2)}^2, \tag{4.28}$$

$$\phi = \phi_0 A(r)^{\frac{a_0-b_0}{2}}, \tag{4.29}$$

where $a_0$, $b_0$, and $\phi_0$ are constants, with the first two related to the Brans-Dicke parameter by

$$\omega(a_0,b_0) = \frac{-2(a_0^2+b_0^2-a_0b_0+a_0+b_0)}{(a_0-b_0)^2} \tag{4.30}$$

and $V(\phi)=0$ [25]. The metric and Brans-Dicke field satisfy the field equations

$$R_{ab} = \frac{\omega}{\phi^2}\nabla_a\phi\nabla_b\phi + \frac{\nabla_a\nabla_a\phi}{\phi}, \tag{4.31}$$

$$\Box\phi = 0. \tag{4.32}$$

The correspondence with our $a=0$ line element (4.14) gives

$$a_0 = -1 + \frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha + \frac{1}{\sqrt{|2\omega+3|}}\right), \tag{4.33}$$

$$b_0 = -1 - \frac{1}{\sqrt{1+4\alpha^2}}\left(2\alpha - \frac{1}{\sqrt{|2\omega+3|}}\right), \tag{4.34}$$

while the correspondence with our $a=0$ Brans-Dicke field (4.15) yields

$$\frac{a_0-b_0}{2} = \frac{2\alpha}{\sqrt{1+4\alpha^2}}. \tag{4.35}$$

Setting, for consistency, this value equal to the value of $(a_0-b_0)/2$ obtained from Eqs. (4.33) and (4.34), which is $1/\sqrt{|2\omega+3|(1+4\alpha^2)}$, gives the special value of the $\alpha$-parameter

$$\alpha = \frac{1}{2\sqrt{|2\omega+3|}}. \tag{4.36}$$

Then it must be

$$a_0 = -1 + \frac{2}{\sqrt{1+|2\omega+3|}}, \tag{4.37}$$

$$b_0 = -1. \tag{4.38}$$

Finally, the relation (4.30) must also be reproduced. Using the values (4.37) and (4.38) of $a_0$ and $b_0$, one has

$$\frac{-2(a_0^2+b_0^2-a_0b_0+a_0+b_0)}{(a_0-b_0)^2} = \frac{|2\omega+3|-3}{2}. \tag{4.39}$$

If $\omega > -3/2$ this expression reduces[3] to $\omega$. Therefore, the Campanelli-Lousto family of static, spherically symmetric, asymptotically flat solutions of Brans-Dicke gravity is reproduced by our new solutions when $a=0$, $\omega > -3/2$, and $\alpha = \frac{1}{2\sqrt{2\omega+3}}$. It is now established that the Campanelli-Lousto spacetimes describe wormhole geometries [26].

### C. Special case 3 ($\alpha^2 = 3/4$)

In the special case $\alpha^2 = 3/4$ it is $\alpha = \delta = \pm\sqrt{3}/2$ and

$$\begin{aligned} ds^2 = &-A(r)^{\alpha - \frac{1}{2\sqrt{|2\omega+3|}}}d\tau^2 \\ &+ S^2(\tau)[A(r)^{-\left(\alpha+\frac{1}{2\sqrt{|2\omega+3|}}\right)}dr^2 \\ &+ A(r)^{1-\left(\alpha+\frac{1}{2\sqrt{|2\omega+3|}}\right)}r^2 d\Omega_{(2)}^2], \end{aligned} \tag{4.40}$$

$$\phi(\tau,r) = \phi_0[S(\tau)]^{\frac{4\alpha}{\sqrt{|2\omega+3|}-2\alpha}}A(r)^{\frac{1}{2\sqrt{|2\omega+3|}}}. \tag{4.41}$$

For $\alpha = \pm\sqrt{3}/2$ the Fonarev solution reduces to the Husain-Martinez-Nuñez geometry [24], as already noted. The Jordan frame counterpart of the Husain-Martinez-Nuñez solution of GR, which is a solution of Brans-Dicke theory without potential, was derived in Ref. [18] as

$$\begin{aligned} ds^2 = &-A(r)^{\alpha\left(1-\frac{1}{\sqrt{3(2\omega+3)}}\right)}d\tau^2 \\ &+ \tau^{\frac{2\left(\sqrt{2\omega+3}-\sqrt{3}\right)}{3\sqrt{2\omega+3}-\sqrt{3}}}[A(r)^{-\alpha\left(1+\frac{1}{\sqrt{3(2\omega+3)}}\right)}dr^2 \\ &+ A(r)^{1-\alpha\left(1+\frac{1}{\sqrt{3(2\omega+3)}}\right)}r^2 d\Omega_{(2)}^2], \end{aligned} \tag{4.42}$$

$$\phi(\tau,r) = \tau^{\frac{2}{\sqrt{3(2\omega+3)}-1}}A(r)^{\pm\frac{1}{2\sqrt{2\omega+3}}}. \tag{4.43}$$

It is straightforward to check that, if $\omega > -3/2$ and $\alpha = +\sqrt{3}/2$, Eqs. (4.42) and (4.43) coincide with our

[3] Since there is no potential, the restriction $|\alpha| \le \sqrt{3}/2$ does not apply.

Eqs. (4.40) and (4.41), respectively (our solution is slightly more general as it allows for $\omega < -3/2$ and $\alpha = -\sqrt{3}/2$). Therefore, for these parameter values, the Brans-Dicke solution which is the Jordan frame counterpart of the Fonarev solution of GR reduces to the Einstein frame sibling of the Husain-Martinez-Nuñez geometry found in [18] and discussed in Refs. [13,40], in which it is found that only wormholes or naked singularities appear, as the remaining parameters $\omega$ and $a$ vary.

### D. The $\omega \to \infty$ limit

Let us analyze now the limit $\omega \to \infty$, in which Brans-Dicke theory is usually believed to converge to GR [41]. A complete and rigorous analysis of the limit of a family of solutions of a theory of gravity as a parameter of the family diverges would require coordinate-independent methods [42], but a more standard approach suffices here. Let us discuss the situation in which the parameters $\alpha$ and $\omega$ are independent of each other. Then, in the limit $\omega \to \infty$ the scalar field (3.3) becomes constant, $\phi \to \phi_0$, and the potential reduces to a constant, $V(\phi) \to \tilde{V}_0\phi_0^2$. This introduces the cosmological constant $\Lambda = 8\pi\tilde{V}_0\phi_0^2 = a^2(3-4\alpha^2)\phi_0^2$. The line element (3.6) reduces to the Fonarev line element (2.1) as $\omega \to \infty$. Therefore, one obtains the Fonarev geometry and a cosmological constant as the only effective matter source. This is not a solution of the vacuum Einstein equations (we know well that the Fonarev solution corresponds to a minimally coupled scalar field with an exponential potential and no cosmological constant as the matter source). The fact that vacuum or electrovacuum solutions of the Brans-Dicke field equations fail to reproduce the corresponding solution of GR is well known [43] and the reason for this behavior has been discussed in the literature [44].

The failure of a Brans-Dicke solution to reproduce the corresponding GR solution as $\omega \to \infty$ has been linked to the fact that, in this limit, one expects the Brans-Dicke scalar field to have the asymptotics $\phi = \phi_0 + O(1/\omega) + \cdots$ while the solutions giving the "incorrect" limit exhibit instead the asymptotics $\phi = \phi_0 + O(1/\sqrt{|\omega|}) + \cdots$. Therefore, while one would normally expect $\phi$ to become constant as $\omega \to \infty$ and all its gradients to disappear from the Brans-Dicke field equations (1.3), when $\phi$ has the "anomalous" behavior the term

$$A_{ab} \equiv \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) = \omega\left(\nabla_a \ln\phi\nabla_b \ln\phi - \frac{1}{2}g_{ab}\nabla^c \ln\phi\nabla_c \ln\phi\right) \quad (4.44)$$

on the right-hand side of these equations does not disappear but remains of order O(1). This is exactly what happens with the conformal cousin of the Fonarev solution. In fact, Eq. (3.3) yields

$$A_{\mu\nu} = \frac{16\alpha^2a^2\omega\delta_{\mu0}\delta_{\nu0}}{|2\omega+3|} + \frac{8m\alpha a\omega(\delta_{\mu1}\delta_{\nu0}+\delta_{\mu0}\delta_{\nu1})}{|2\omega+3|\sqrt{1+4\alpha^2}r^2(1-2m/r)} + \frac{4m^2\omega\delta_{\mu1}\delta_{\nu1}}{|2\omega+3|(1+4\alpha^2)r^4(1-2m/r)^2}. \quad (4.45)$$

As $\omega \to \infty$ the tensor $A_{\mu\nu}$ tends to

$$A_{\mu\nu} \approx 8\alpha^2a^2\mathrm{sign}(\omega)\delta_{\mu0}\delta_{\nu0} + \frac{4m\alpha a\,\mathrm{sign}(\omega)}{\sqrt{1+4\alpha^2}r^2(1-2m/r)}(\delta_{\mu1}\delta_{\nu0}+\delta_{\mu0}\delta_{\nu1}) + \frac{2m^2}{(1+4\alpha^2)r^4(1-2m/r)^2}\delta_{\mu1}\delta_{\nu1}, \quad (4.46)$$

which is of order unity.

In the special case 1 (corresponding to Sec. IV A) with $\omega = -2, -1$ and in the special case 2 (corresponding to Sec. IV B) previously examined, the parameter $\omega$ can only assume values in a finite range and the limit $\omega \to \infty$ cannot be taken.

## V. GENERATING NEW SOLUTIONS OF $f(\mathcal{R})$ GRAVITY

As is well known, $f(\mathcal{R})$ gravity is equivalent to an $\omega = 0$ Brans-Dicke theory with Brans-Dicke scalar $\phi = f'(\mathcal{R})$ subject to the potential (1.12), where $\mathcal{R} = \mathcal{R}(\phi)$ is a function of $\phi = f'(\mathcal{R})$ usually defined implicitly [8]. One wonders whether the Jordan frame counterpart of the Fonarev solution can also be an analytic solution of $f(\mathcal{R})$ gravity.[4] For this to be true, one must set $\omega = 0$ and [cf. Eq. (3.5)]

$$V_0[f'(\mathcal{R})]^{2\beta} = \mathcal{R}f'(\mathcal{R}) - f(\mathcal{R}), \beta = 1 - \alpha\sqrt{3}. \quad (5.1)$$

It is easy to see that the functional form $f(\mathcal{R}) = \mu\mathcal{R}^n$ (where $\mu$ and $n$ are constants) satisfies Eq. (5.1) provided that

$$\beta = \frac{n}{2(n-1)}, \quad (5.2)$$

$$V_0 = \frac{n-1}{n^{2\beta}}\mu^{1-2\beta}, \quad (5.3)$$

which require $n \neq 1$ (for $n = 1$ the theory reduces to GR).

By comparing Eq. (5.2) with $\beta = 1 - \alpha\sqrt{3}$ it follows that the parameter $\alpha$ of the family of solutions is

[4] In some respect similar, a correspondence between solutions of $f(R) = R^n$ gravity and Einstein-conformally invariant Maxwell theory (in $D$ dimensions) was pointed out in Ref. [45].

TABLE I. A summary of the special cases studied.

| Case | $\alpha$ | $\omega$ | $a$ | Apparent horizons |
|---|---|---|---|---|
| Special case 1 (Sec. IV A) examples; $V = m_\phi^2\phi^2/2$ | 0 | $-2, -1$ | $\neq 0$ | $R_{\rm AH} = 1/a$ de Sitter horizon exists for $t < t_*$ |
| Special case 2 (Sec. IV B) static solutions, $V = 0$; contains Campanelli-Lousto wormholes as the subcase $\omega > -3/2$, $\alpha = (2\sqrt{2\omega+3})^{-1/2}$ | any | $-5 - 2\sqrt{3} < \omega < -3/2$ | 0 | wormhole throat at $R_{\rm AH} = m\frac{(\gamma-1)^{(1-\gamma)/2}}{(\gamma+1)^{-(1+\gamma)/2}}$ timelike if $\gamma > 1$, null if $\gamma = 1$ |
| Special case 3 (Sec. IV C) contains solutions of [18,40] as the subcase $\omega > -3/2$, $\alpha = \sqrt{3}/2$ | $\pm\sqrt{3}/2$ | $\neq -3/2$ | $\neq 0$ | wormhole throat or naked singularity |

$$\alpha = \frac{n-2}{2\sqrt{3}(n-1)}. \tag{5.4}$$

In particular, the conformal cousin of the Husain-Martinez-Nuñez solution obtained for $\alpha = \pm\sqrt{3}/2$ is a solution of $f(\mathcal{R}) = \mu\mathcal{R}^n$ gravity for $n = 1/2, 5/4$. For these values of the parameter $n$, $\mathcal{R}^n$ gravity is ruled out by Solar System experiments [46], but it is anyway interesting to add one more formal solution to the very scarce catalogue of analytic inhomogeneous solutions of $f(\mathcal{R})$ gravity.

The potential (1.12) is no longer required to satisfy $|\alpha| \leq \sqrt{3}/2$, but one has $V_0 > 0$ if $n > 1$. Solar System constraints require $n = 1 + \sigma$ with $\sigma = (-1.1 \pm 1.2) \times 10^{-5}$ [46,47], while any $f(\mathcal{R})$ theory is required to satisfy $f' > 0$ in order for the graviton to carry positive energy and $f'' \geq 0$ for local stability [8,48]. In the cosmological setting these requirements are satisfied if $n = 1 + \sigma$ with $\sigma \geq 0$. Then

$$\alpha = -\frac{(1-\sigma)}{2\sqrt{3}\sigma}, \qquad \beta = \frac{1+\sigma}{2\sigma} \tag{5.5}$$

(with $\alpha < 0$ for realistic theories), which gives the line element in the form

$$ds^2 = -A(r)^{-\frac{1}{\sqrt{1-2\sigma+4\sigma^2}}} e^{\frac{2(1-\sigma)at}{\sqrt{3}\sigma}} dt^2 + e^{\frac{2(1+2\sigma)at}{3\sigma}} [A(r)^{\frac{1-2\sigma}{\sqrt{1-2\sigma+4\sigma^2}}} dr^2 + A(r)^{\frac{1-2\sigma}{\sqrt{1-2\sigma+4\sigma^2}}-1} r^2 d\Omega_{(2)}^2]. \tag{5.6}$$

For consistency it must then be

$$\phi = f'(\mathcal{R}) = n\mu\mathcal{R}^{n-1} = (1+\sigma)\mu\mathcal{R}^\sigma. \tag{5.7}$$

This equation can be checked using the expression (3.3) of the Jordan frame Brans-Dicke field obtained by setting $\omega = 0$,

$$\phi(t,r) = \phi_0 e^{\frac{4\alpha at}{\sqrt{3}}} A(r)^{\sqrt{\frac{1}{3(1+4\alpha^2)}}} \tag{5.8}$$

$$= \phi_0 e^{\frac{-2(1-\sigma)at}{3\sigma}} A(r)^{\frac{\sigma}{\sqrt{4\sigma^2-2\sigma+1}}}, \tag{5.9}$$

where in the last equality we used Eq. (5.5) and we note that $4\sigma^2 - 2\sigma + 1 > 0$ for any value of $\sigma$. By imposing that this scalar field be equal to $f'(\mathcal{R}) = (1+\sigma)\mu\mathcal{R}^\sigma$ one obtains the expression of the Ricci scalar

$$\mathcal{R} = \left[\frac{\phi_0}{(1+\sigma)\mu}\right]^{1/\sigma} e^{\frac{-2(1-\sigma)at}{3\sigma^2}} A(r)^{\frac{1}{\sqrt{4\sigma^2-2\sigma+1}}}, \tag{5.10}$$

which can be compared with the expression (3.18) of the Ricci scalar already computed. For $\omega = 0$ the latter reduces exactly to Eq. (5.10) upon use of Eqs. (5.3) and (5.5).

## VI. CONCLUSIONS

The Fonarev solution of the Einstein equations which has a scalar field with exponential potential as the matter source has been mapped to the Jordan frame of Brans-Dicke theory, generating a new 4-parameter family of solutions of the vacuum Brans-Dicke field equations [$\omega$ is a parameter of the theory and $(m, a, \alpha)$ are parameters of the specific solution of this family]. Notably, the potential for the Brans-Dicke field in the Jordan frame is a power-law or inverse power-law potential, which is physically well motivated and is used extensively in cosmology and particle physics [31]. The solutions are spherically symmetric, inhomogeneous, time-dependent, and asymptotically FLRW. Special cases include the conformal version of the Husain-Martinez-Nuñez solution of GR with free scalar field [24] found in Ref. [18] using the same technique employed here, and the Campanelli-Lousto solution [25], which is now known to describe a wormhole [26], in agreement with our more general discussion of the case $a = 0$. Table I summarizes the cases and subcases studied in the previous sections.

It turns out that the conformal relative of the Fonarev geometry thus obtained is also a solution of $f(\mathcal{R}) = \mu\mathcal{R}^n$ gravity. To the best of our knowledge only one other analytic solution of this theory with the same properties (i.e., spherical, inhomogenous, dynamical, and asymptotically FLRW) is known [49].

In order to interpret physically the conformal relative of the Fonarev solution, it is necessary to solve the equation $\nabla^c R \nabla_c R = 0$ locating the apparent horizons and assess when solutions exist. Unfortunately, this is a transcendental equation which would require the complete specification of the values of the four parameters and, even then, it cannot be solved analytically. We have, nevertheless, considered special cases for illustration, in which the geometry describes a wormhole throat or a naked singularity.

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