Cosmological horizon and the quadrupole formula in de Sitter background

Ghanashyam Date[*] and Sk Jahanur Hoque[†]

The Institute of Mathematical Sciences, HBNI, CIT Campus, Chennai-600 113, India

An important class of observables for gravitational waves consists of the fluxes of energy, momentum, and angular momentum carried away by them and are well understood for weak gravitational waves in Minkowski background. In de Sitter background, the future null infinity, $\mathcal{J}^+$, is spacelike, which makes the meaning of these observables subtle. A spatially compact source in de Sitter background also provides a distinguished null hypersurface, its cosmological horizon, $\mathcal{H}^+$. For sources supporting the short wavelength approximation, we adopt the Isaacson prescription to define an effective gravitational stress tensor. We show that the fluxes computed using this effective stress tensor can be evaluated at $\mathcal{H}^+$, match with those computed at $\mathcal{J}^+$, and also match with those given by Ashtekar *et al.* at $\mathcal{J}^+$ at a coarse grained level.

I. INTRODUCTION

The weak gravitational field of a spatially compact source is identified as a perturbation about a background space-time which is a solution of the Einstein equation in the source-free region. In the presence of a positive cosmological constant, the background space-time is the de Sitter space-time. Unlike the Minkowski background for the zero cosmological constant, de Sitter space-time has different patches, e.g., the global patch ($\mathbb{R} \times S^3$), a Poincaré patch, and a static patch. In the cosmological context, a Poincaré patch is appropriate, which is what we focus on. A solution at the linearized level, valid throughout the Poincaré patch and extending to the future null infinity $\mathcal{J}^+$, is available in [1–3]. However, the spacelike character of the $\mathcal{J}^+$ poses challenges for defining energy, momentum, and their fluxes.

Let us recall that the cleanest articulation of "infinity" arises in the conformal completion of physical space-times. Conformal completion preserves the light cone structure of the physical space-time and naturally identifies boundary components, $\mathcal{J}^\pm$, where timelike and null geodesics "terminate." The causal nature of these boundary components is determined by the asymptotic form of "source-free" equations: $\mathcal{J}^\pm$ are null when $\Lambda = 0$ and spacelike for $\Lambda > 0$ (timelike for $\Lambda < 0$). These boundary components serve to define outgoing (incoming) fields as those solutions of the asymptotic equations that have suitably finite limiting values on $\mathcal{J}^+(\mathcal{J}^-)$. It is then a result that the Weyl tensor of outgoing fields evaluated along outgoing null geodesics has a definite pattern of falloff in inverse powers of an affine parameter along the geodesics (the peeling-off theorem) [4,5]. This enables one to identify the leading term as representing gravitational radiation (far field of a source) in a coordinate invariant manner. It is conveniently described in terms of the Weyl scalars which are defined with respect to a suitable null tetrad. When $\mathcal{J}^+$ is null, a null tetrad at a point $p \in \mathcal{J}^+$ is uniquely determined (modulo real scaling and rotation) by the tangent vector ℓ^μ of an outgoing null geodesic reaching p *and* the null normal n^μ, satisfying $\ell \cdot n = -1$. Clearly as the null geodesic changes its direction, ℓ changes but not n, and hence the Weyl scalar $\Psi_4 (:= C_{\mu\nu\rho\sigma} n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma)$ remains unchanged. Its nonzero value can be taken as showing the presence of gravitational radiation. This feature is lost when the $\mathcal{J}^+$ is spacelike. Now the null vector n^μ, with $\ell \cdot n = -1$, is chosen to be in the plane defined by ℓ^μ and the (timelike) normal N^μ. Clearly, as ℓ^μ changes, so does n^μ and *none* of the Weyl scalars is invariant. An invariant characterization of gravitational radiation is no longer available [5].

The de Sitter space-time also has the so-called observer horizons—the boundary of the causal past of an observer's end point on $\mathcal{J}^+$. In particular, for a spatially extended but compact source, the worldlines of different components of the source must reach the same point on $\mathcal{J}^+$ to maintain a *finite* physical separation among them. A spatially compact source then defines (its) cosmological horizon as the past light cone of the common point on $\mathcal{J}^+$ where the source world tube converges. Equally well, *any observer* who remains at a finite physical distance from the compact source for all times must necessarily lie within the cosmological horizon, i.e., within the static patch bounded by the cosmological horizon. Unlike the $\mathcal{J}^+$, the cosmological horizon is a null hypersurface but shares with $\mathcal{J}^+$ the property that whichever curve meets a point on it can never causally intersect the world tube of the spatially compact source. In other words, once any energy/momentum/angular momentum is carried away across the cosmological horizon, it is "lost" from the source forever. We would like to explore to what extent and under what conditions we may regard the cosmological horizon as a "substitute" for the future null infinity.

[*]shyam@imsc.res.in
[†]jahanur@imsc.res.in

It is obvious at the outset that the outgoing null geodesics emanating from the source intersect the cosmological horizon at a *finite* value of any affine parameter and it can be chosen to be 1 by a suitable normalization. Such a normalized affine parameter equals the ratio of the physical distance from the source to $\sqrt{3/\Lambda} \sim 10^{26} m \sim 10$ Gpc. All spatially compact sources may be taken to lie within a sphere of radius $\sim \Lambda^{-1/2}$. Furthermore, only sources varying over cosmological time scales will have comparable wavelengths. Thus, most sources producing gravitational waves would have wavelengths far smaller than $\Lambda^{-1/2}$ and any wave crossing the horizon may be taken to be a "far zone field." Cosmological horizon being a null hypersurface, a Ψ_4 can be defined on it, independent of the null geodesics meeting the horizon. A notion of radiation based on the asymptotic behavior of fields is physically useful, provided there are suitable definitions of fluxes of energy-momentum, and angular momentum in terms of these asymptotic fields. And there are many such definitions.

One of the definitions of such conserved quantities is based on the covariant phase space framework [6,7]. In the context of the linearized theory, it exploits the phase space structure of the space of solutions and defines a manifestly gauge invariant and conserved "Hamiltonian" corresponding to each of the seven isometries of the Poincaré patch. Although defined on each spacelike hypersurface of the Poincaré patch, the simplest expressions result for evaluation at $\mathcal{J}^+$. Thus, the conserved quantities are directly expressed in terms of the asymptotic fields.

For sources which are sufficiently rapidly varying (relative to the scale set by the cosmological constant), there is an alternative identification of gravitational waves as ripples on a background within the so-called "short wave approximation" [8,9]. Furthermore, it is possible to define an effective gravitational stress tensor, $t_{\mu\nu}$, for the ripples. For vanishing Λ, it is symmetric, conserved and gauge invariant. For nonzero Λ it is *not* gauge invariant but the gauge violations are suppressed by powers of $\sqrt{\Lambda}$. It is very convenient to have such a stress tensor to define and compute fluxes of energy and momenta carried by the ripples across *any* hypersurface.

We use the fluxes defined using the effective gravitational stress tensor and show that for the retarded solution given in [1–3], the fluxes of energy and momentum across the cosmological horizon exactly equal the corresponding fluxes across the $\mathcal{J}^+$. Furthermore, these fluxes computed at $\mathcal{J}^+$ also equal the fluxes defined in the covariant phase space framework [2], albeit at a coarse grained level [see Eq. (93)]. The instantaneous power received at infinity matches with that crossing the horizon. This is our main result.

The paper is organized as follows.

In Sec. II, we summarize various details needed to establish our result. Most are available in the cited literature and are collected here for self-contained reading. It is divided into three subsections. In Sec. II A, we recall the solution at the linearized level [1–3] for which the fluxes will be evaluated. We specify and denote the (spatial components of) the exact retarded solution by $\mathcal{X}_{ij}$. This is approximated when the source dimension is much smaller than the distance to the source. The leading term is the approximated retarded solution and is denoted by χ_{ij}. Physical solutions have to satisfy the gauge conditions imposed in simplifying the linearized equation. This is achieved by extracting the (spatial) transverse and traceless (TT) part of the solution, which is denoted by $\mathcal{X}_{ij}^{TT}$. For the approximated solution, the TT part is conveniently extracted by an algebraic projection to the same level of approximation. The algebraically projected transverse, traceless part of the approximated solution is denoted by χ_{ij}^{tt} and used throughout. We also collect relevant expressions for subsequent use. A table of notation is included at the end of this subsection. In Sec. II B we summarize the covariant phase space framework and recall the definitions of the fluxes and quadrupole power from [2]. The energy momentum fluxes defined here are compared to those defined in the next subsection. In Sec. II C, we discuss the Isaacson prescription adapted to the presence of the cosmological constant and present the definition of the ripple tensor in Eq. (46), which is used in the next section.

Section III is divided into three subsections. In Sec. III A, we present computations of the energy flux for the χ_{ij}^{tt} across various hypersurfaces. In particular we show that the fluxes across the outgoing null hypersurfaces are zero, implying, for example, that the energy propagation is sharp. Section III B contains the fluxes for the momentum and the angular momentum. In Sec. III C we discuss how the computations can be extended to $\chi_{ij}^{TT} := (\mathcal{X}_{ij}^{TT})_{\text{approx}}$.

In Sec. IV, we discuss applications of these flux computations and establish our main results. Finally, Sec. V concludes with a discussion. An Appendix is included to illustrate an averaging procedure.

II. PRELIMINARIES

In this section we summarize and assemble already available relevant details needed for our main result, with the main citations included in the subsection headings.

A. Weak gravitational field of interest

The summary in this subsection is based on references [1–3].

Weak gravitational fields are understood as perturbations about a background specified in the form $g_{\mu\nu} := \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$. The background $\bar{g}_{\mu\nu}$ is chosen to be a solution of the source-free Einstein equation with a positive cosmological constant. The Einstein equation for $g_{\mu\nu}$, expanded to first order in ϵ, gives the linearized Einstein equation for $h_{\mu\nu}$. The physical perturbations are understood as the equivalence classes of solutions $h_{\mu\nu}$, with respect to the gauge transformations: $\delta h_{\mu\nu}(x) = \mathcal{L}_\xi \bar{g}_{\mu\nu}(x) = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$. In
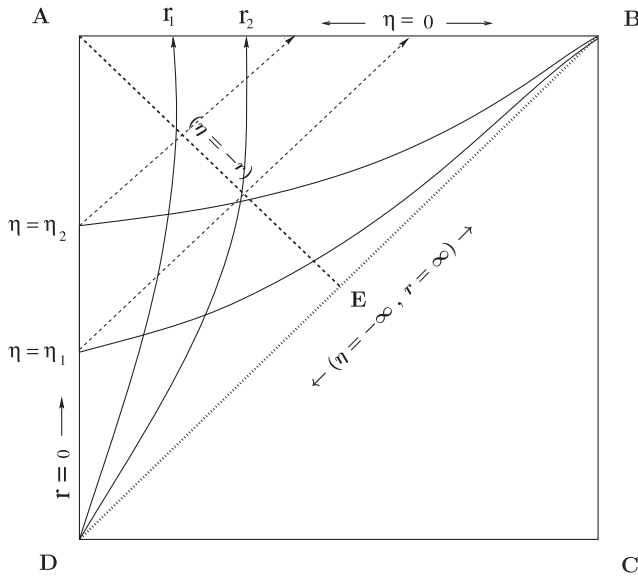
FIG. 1. The full square is the Penrose diagram of de Sitter space-time with a generic point representing a two-sphere. The world tube of a spatially compact source is taken to be centered on the line DA. The corresponding Poincaré patch is labeled ABD and is covered by the Poincaré chart (η, r, θ, ϕ). The line BD does not belong to the chart. The line AB denotes the future null infinity, $\mathcal{J}^+$, while the line AE denotes the cosmological horizon, $\mathcal{H}^+$, of the source. The region AED is the static patch admitting the stationary Killing vector, T, of Eq. (21). Two constant η spacelike hypersurfaces are shown with $\eta_2 > \eta_1$. The two constant r, timelike hypersurfaces have $r_2 > r_1$, while the two dotted lines at 45 degrees denote the outgoing null hypersurfaces emanating from $\eta = \eta_1, \eta_2$ on the worldline at $r = 0$.

terms of the trace reversed combination $\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\bar{h}_{\mu\nu}(\bar{g}^{\alpha\beta}h_{\alpha\beta})$, the linearized equation takes the form

$$\frac{1}{2}[-\bar{\Box}\tilde{h}_{\mu\nu} + \{\bar{\nabla}_\mu B_\nu + \bar{\nabla}_\nu B_\mu - \bar{g}_{\mu\nu}(\bar{\nabla}^\alpha B_\alpha)\}] + \frac{\Lambda}{3}[\tilde{h}_{\mu\nu} - \tilde{h}\bar{g}_{\mu\nu}]$$
$$= 8\pi T_{\mu\nu} \qquad (1)$$

where $B_\mu := \bar{\nabla}_\alpha \tilde{h}^\alpha{}_\mu$. The gauge freedom is exploited subsequently to simplify the equation.

In the present context, the background space-time is taken to be the Poincaré patch of the de Sitter space-time (see Fig. 1) which admits a conformally flat form of the background metric in coordinates (η, x^i):

$$ds^2 = \frac{1}{H^2\eta^2}\left[-d\eta^2 + \sum_i (dx^i)^2\right], \qquad \eta \in (-\infty, 0),$$

$$x^i \in \mathbb{R}, \qquad H := \sqrt{\frac{\Lambda}{3}}. \qquad (2)$$

The future null infinity is approached as $\eta \to 0_-$, while the $\eta \to -\infty$ corresponds to the Friedmann-Lemaître-Robertson-Walker (FLRW) singularity. The conformal factor is $a^2(\eta) := (H\eta)^{-2}$.

The linearized equation is simplified by imposing the generalized transverse gauge conditions: $B_\mu = \frac{2\Lambda}{3}\eta\tilde{h}_{0\mu}$. The conformal factor can be scaled out by using the fields $\tilde{\chi}_{\mu\nu} := a^{-2}\tilde{h}_{\mu\nu}$ and the linearized equation (with source included) then takes the form[1] [1]

$$-16\pi T_{\mu\nu} = \Box\tilde{\chi}_{\mu\nu} + \frac{2}{\eta}\partial_0\tilde{\chi}_{\mu\nu} - \frac{2}{\eta^2}(\delta^0_\mu\delta^0_\nu\tilde{\chi}_\alpha{}^\alpha + \delta^0_\mu\tilde{\chi}_{0\nu} + \delta^0_\nu\tilde{\chi}_{0\mu}),$$
$$(3)$$

$$0 = \partial^\alpha\tilde{\chi}_{\alpha\mu} + \frac{1}{\eta}(2\tilde{\chi}_{0\mu} + \delta^0_\mu\tilde{\chi}_\alpha{}^\alpha) \quad \text{(gauge condition)}. \qquad (4)$$

It is further shown in [1] that the residual gauge invariance is exhausted by imposing the additional gauge conditions: $\tilde{\chi}_{0\alpha} = 0 = \hat{\tilde{\chi}}(:= \tilde{\chi}_{00} + \tilde{\chi}_i{}^i)$. The gauge condition (4) then implies that physical perturbations may be characterized solutions of (3) which satisfy the spatial transverse, traceless condition, or "spatial TT" for short: $\partial^j\tilde{\chi}_{ji} = 0 = \tilde{\chi}^k{}_k$. Thanks to the decoupled equations, it suffices to focus on the spatial components of the equation.

The exact retarded solution is given by

$$\mathcal{X}_{ij}(\eta, x) = 4\int d^3x' \frac{\eta}{|x-x'|(\eta-|x-x'|)}T_{ij}(\eta', x')\Big|_{\eta'=\eta-|x-x'|}$$
$$+ 4\int d^3x'\int_{-\infty}^{\eta-|x-x'|} d\eta' \frac{T_{ij}(\eta', x')}{\eta'^2}. \qquad (5)$$

The spatial integration is over the matter source confined to a compact region and is finite. The second term in Eqs. (5) is the tail term. This particular solution does *not* satisfy the spatial TT conditions. Using the transverse, traceless decomposition of the tensor fields, the TT part $\mathcal{X}_{ij}^{TT}$ is extracted, which represents the physical retarded field due to the source.

For $|\vec{x}| \gg |\vec{x}'|$, we can approximate $|\vec{x} - \vec{x}'| \approx r := |\vec{x}|$. This allows us to separate out the $\vec{x}'$ dependence from the $\eta - |x - x'|$. The thus approximated retarded solution, χ_{ij}, is given by

$$\mathcal{X}_{ij} = \chi_{ij}(\eta, x) + o(r^{-1}), \quad \text{with}$$

$$\chi_{ij}(\eta, x) := 4\frac{\eta}{r(\eta-r)}\int d^3x' T_{ij}(\eta', x')\Big|_{\eta'=\eta-r}$$
$$+ 4\int_{-\infty}^{\eta-r} d\eta' \frac{1}{\eta'^2}\int d^3x' T_{ij}(\eta', x'). \qquad (6)$$

We will work with the approximated solution. Note that χ_{ij} depends on $\vec{x}$ only through $r = |\vec{x}|$. The spatial integral of T_{ij} can be simplified using moments. This is done through the matter conservation equation.

---

[1]From now on in this subsection, the tensor indices are raised/lowered with the Minkowski metric.

To define these moments, introduce the orthonormal tetrad $f^\alpha{}_{\underline{a}} := -H\eta\delta^\alpha{}_{\underline{a}}$ and denote the frame components of the source stress tensor as $\rho := H^2\eta^2 T_{\underline{0}\underline{0}}$, $\pi := H^2\eta^2 T_{\underline{ij}}\delta^{\underline{ij}}$. Define the moment variable $\bar{x}^{\underline{i}} = f^{\underline{i}}{}_\alpha x^\alpha = -(\eta H)^{-1}\delta^{\underline{i}}_j x^j := a(t)x^{\underline{i}}$. Two sets of moments are defined as functions of η (or of t defined through $\eta := -H^{-1}e^{-Ht}$) as

$$Q^{\underline{ij}}(t) := \int_{\text{Source}(t)} d^3x\, a^3(t)\rho(t,\vec{x})\bar{x}^{\underline{i}}\bar{x}^{\underline{j}},$$

$$\bar{Q}^{\underline{ij}}(t) := \int_{\text{Source}(t)} d^3x\, a^3(t)\pi(t,\vec{x})\bar{x}^{\underline{i}}\bar{x}^{\underline{j}}. \quad (7)$$

In terms of these, the approximated retarded solution is given by

$$\chi_{ij}(\eta,\vec{x}) = \frac{1}{r}f_{ij}(\eta_{ret}) + g_{ij}(\eta_{ret}) + \hat{g}_{ij}, \quad \text{with} \quad (8)$$

$$f_{ij}(\eta_{ret}) := \frac{2}{a(\eta_{ret})}[\mathcal{L}_T^2 Q_{ij} + 2H\mathcal{L}_T Q_{ij}$$
$$+ H\mathcal{L}_T\bar{Q}_{ij} + 2H^2\bar{Q}_{ij}], \quad (9)$$

$$g_{ij}(\eta_{ret}) := -2H[\mathcal{L}_T^2 Q_{ij} + H\mathcal{L}_T Q_{ij} + H\mathcal{L}_T\bar{Q}_{ij} + H^2\bar{Q}_{ij}],$$
$$\text{and} \quad (10)$$

$$\hat{g}_{ij} := -2H^2[\mathcal{L}Q_{ij} + H\bar{Q}_{ij}]|_{-\infty}. \quad (11)$$

All moments are evaluated at the retarded $\eta_{ret} := (\eta - r)$, $a(\eta_{ret}) := -(H\eta_{ret})^{-1}$ and $\mathcal{L}_T$ denotes the Lie derivative with respect to the time translation Killing vector defined in Eq. (21) below. On the moments, the Lie derivative is given by

$$\mathcal{L}_T Q_{ij} = -H(\eta\partial_\eta + r\partial_r)Q_{ij} - 2HQ_{ij}$$
$$= -H(\eta - r)\partial_\eta Q_{ij} - 2HQ_{ij}$$
$$= \partial_t Q_{ij}|_{t_{ret}} - 2HQ_{ij}. \quad (12)$$

In Eq. (8), the first term is the contribution of the so-called sharp term, while the second and the third terms denote the tail contributions. The tail contribution has separated into a term which depends on retarded time, $(\eta - r)$ only, just as the sharp term does, and the contribution from the history of the source is given by the limiting value at $\eta = -\infty$. This expression is valid as the leading term for $|\vec{x}| \gg |\vec{x}'|$. There is no TT label on these expressions. While the solution has a tail term, it will turn out that the energy propagation is sharp.

For future use in Sec. III A, we display the derivatives of χ_{ij}. Since χ_{ij} depends on $\vec{x}$ only through r, we need only the derivatives with respect to η and r. On functions of η_{ret}, $\partial_r = -\partial_\eta$ and we can replace the r derivatives in favor of η derivatives. Hence,

$$\partial_\eta\chi_{ij} = \frac{1}{r}\partial_\eta f_{ij} + \partial_\eta g_{ij},$$

$$\partial_r\chi_{ij}(\eta,r) = -\partial_\eta\chi_{ij} - \frac{f_{ij}}{r^2}. \quad (13)$$

There is a well-known algebraic projection method to construct spatial tensors which satisfy the spatial TT condition to the leading order in r^{-1}. Since the approximated solution is also valid to $o(r^{-1})$, we may use this convenient method.

For the unit vectors $\hat{x}$ denoting directions, define the projectors

$$P_i{}^j := \delta_i{}^j - \hat{x}_i\hat{x}^j, \qquad \Lambda_{ij}{}^{kl} := \frac{1}{2}(P_i{}^k P_j{}^l + P_i{}^l P_j{}^k - P_{ij}P^{kl}),$$

$$\chi_{ij}^{tt} := \Lambda_{ij}{}^{kl}\chi_{kl}. \quad (14)$$

We have used the notation tt to refer to the algebraically projected transverse, traceless part as in [2]. Noting that on χ_{ij} the spatial derivative is $\partial^j = \hat{x}^j\partial_r$, it follows that

$$\partial_\eta(\chi_{ij}^{tt}) = (\partial_\eta\chi_{ij})^{tt}, \qquad \partial_r(\chi_{ij}^{tt}) = (\partial_r\chi_{ij})^{tt}, \quad (15)$$

$$\partial_m(\chi_{ij}^{tt}) = (\partial_m\Lambda_{ij}^{kl})\chi_{kl} + \hat{x}_m(\partial_r\chi_{ij})^{tt}, \quad (16)$$

$$\therefore \quad \partial^j(\chi_{ij}^{tt}) = \hat{x}^j\Lambda_{ij}{}^{kl}\partial_r\chi_{ij} + (\partial^j\Lambda_{ij}{}^{kl})\chi_{kl} = 0 + o(r^{-1});$$
$$\text{where we used} \quad (17)$$

$$\partial_m\Lambda_{ij}{}^{kl} = -\frac{1}{r}[\hat{x}_i\Lambda_{mj}{}^{kl} + \hat{x}_j\Lambda_{mi}{}^{kl} + \hat{x}^k\Lambda_{ijm}{}^l + \hat{x}^l\Lambda_{ijm}{}^k]$$
$$= o(r^{-1}). \quad (18)$$

The tracelessness of χ_{ij}^{tt} is manifest and hence χ_{ij}^{tt} satisfies the spatial TT condition to $o(r^{-1})$.

Using the derivatives of χ_{ij} given in (13), we can write (the right-hand sides denote row vectors of the $\mu = \eta$ and $\mu = m$ components)

$$\partial_\mu\chi_{ij}^{tt} = (\partial_\eta\chi_{ij}^{tt}, \hat{x}_m\partial_r\chi_{ij}^{tt} + (\partial_m\Lambda_{ij}{}^{kl})\chi_{kl}) \quad (19)$$

$$= (\partial_\eta\chi_{ij}^{tt})(1, -\hat{x}_m) - \left(\frac{f_{ij}^{tt}}{r^2}\right)(0, \hat{x}_m)$$

$$- \frac{1}{r}(0, [\hat{x}_i\Lambda_{mj}{}^{kl} + \hat{x}_j\Lambda_{mi}{}^{kl} + \hat{x}^k\Lambda_{ijm}{}^l + \hat{x}^l\Lambda_{ijm}{}^k]\chi_{kl}). \quad (20)$$

The first term is proportional to a null vector. The second term is proportional to the spacelike, radial vector. The third is again a spacelike vector. Both the second and the third terms are down by a power of r relative to χ_{ij} and therefore also relative to the first term. We will see later in the calculation of the fluxes that for energy and momentum, the second and the third terms can be neglected. However for

the flux of angular momentum, the third term is crucial. When the second and the third terms can be neglected, the effective gravitational stress tensor turns out to correspond to an outgoing null dust with energy density proportional to $\langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle$.

Finally, we note the isometries of the Poincaré patch. There are seven globally defined Killing vectors on the Poincaré chart, corresponding to energy, three momenta, and three angular momenta [2,10]. They are given by (up to constant scaling)

$$\text{Generator of time translation}: T = -H(\eta \partial_\eta + x^i \partial_i) \tag{21}$$

$$\text{Generators of space translation}: S_{(i)} = \partial_i \tag{22}$$

$$\text{Generators of space rotations}: L_{(j)} = \epsilon_{jk}{}^i x^k \partial_i. \tag{23}$$

We focus on the time translation vector field, which is timelike in the static patch, null on the cosmological horizon, and spacelike beyond it. In particular it is spacelike and tangential to $\mathcal{J}^+$.

Many different symbols used for the retarded solution, its approximations, their TT parts, and different "radiation fields" are summarized below:

$\tilde{\chi}_{ij}$:	generic solution of linearized equation	
$\mathcal{X}_{ij}$:	exact, retarded solution	Eq. (5)
$\mathcal{X}_{ij}^{TT}$:	TT part of exact, retarded solution	Eq. (86)
χ_{ij} :	approximated, retarded solution	Eq. (6)
χ_{ij}^{tt} :	Λ-projection of approximated, retarded solution	Eq. (14)
χ_{ij}^{TT} :	approximation of $\mathcal{X}_{ij}^{TT}$ for $|\vec{x}'| \ll |\vec{x}|$	Above Eq. (87)
:	χ_{ij}^{TT} does not denote the TT part of χ_{ij}	See foot note 3.
$\mathcal{R}_{ij}^{TT}$:	radiation field defined in [2]	Eqs. (33), (34)
$\mathcal{Q}_{ij}^{tt}$:	radiation field used throughout	Eqs. (58), (59)
$\mathcal{Q}_{ij}^{TT}$:	defined to equal $2\partial_\eta \mathcal{M}_{ij}^{TT}$	Eq. (87)

B. Covariant phase space framework

Traditionally, the conserved energy, momentum, etc. are defined through pseudotensors, which have the shortcoming of not being covariant. The framework of covariant phase space provides manifestly gauge invariant definitions of the conserved quantities and is briefly recalled below [2,6,7].

Consider the space $\mathcal{C}$ of a class of solutions of the full Einstein equation, satisfying the stipulated boundary condition. At each point of this space, the linearized solutions provide tangent vectors. Under certain conditions, it is possible to define a presymplectic form on the tangent spaces. Every infinitesimal diffeomorphism of the spacetime, with suitable asymptotic behavior, induces a vector field on $\mathcal{C}$. Some of these lie in the kernel of the presymplectic form and constitute "gauge directions," while the remaining ones constitute (asymptotic) symmetries shared by the stipulated class of solutions. By modding out by the gauge directions (null space of the presymplectic form), one imparts a symplectic structure to the space of solutions, now denoted

as the $\Gamma \sim \mathcal{C}/\text{gauge}$. Under favorable conditions, the vector fields on $\mathcal{C}$ corresponding to the asymptotic symmetries descend to Γ and generate infinitesimal canonical transformations. Their generating functions, or "Hamiltonians," are candidates for representing energy, momenta, angular momenta, etc. [6].

In [2,7], this strategy is applied to the space of fully gauge fixed solutions of the linearized equation and we summarize it below. Isometries of the background leave the covariant phase space itself invariant and constitute canonical transformations. In the present context, the Hamiltonians corresponding to the seven isometries are the proposed definitions of energy, linear momentum, and angular momentum.

Explicitly, $\mathcal{C}$ denotes the solutions of Eq. (24) together with the gauge fixing conditions (25):

$$\Box \tilde{\chi}_{ij} + \frac{2}{\eta} \partial_\eta \tilde{\chi}_{ij} = 0 \tag{24}$$

$$\partial^i \tilde{\chi}_{ij} = 0, \qquad \tilde{\chi}_{ij} \delta^{ij} = 0. \tag{25}$$

A symplectic form is defined by an integral over a cosmological slice Σ_η. A definition which has a smooth limit to $\mathcal{J}^+$ ($\eta \to 0_+$) is defined in terms of the electric part of the perturbed Weyl tensor, $\mathcal{E}_{ij} := -(H\eta)^{-1}[^{(1)}C^0_{j0i}] = \frac{1}{2H\eta^2}(\partial_\eta \tilde{\chi}_{ij} + \eta \nabla^2 \tilde{\chi}_{ij}) = \frac{1}{2H\eta}(\partial_\eta^2 - \frac{1}{\eta}\partial_\eta)\tilde{\chi}_{ij}$. For two elements $\tilde{\chi}, \underline{\tilde{\chi}} \in \mathcal{C}$, the symplectic form is defined by [2]

$$\omega(\tilde{\chi}, \underline{\tilde{\chi}}) = \frac{1}{16\pi H}\int_{\Sigma_\eta} d^3 x(\tilde{\chi}_{ij}\underline{\mathcal{E}}_{kl} - \underline{\tilde{\chi}}_{kl}\mathcal{E}_{ij})\delta^{ik}\delta^{jl}. \quad (26)$$

The TT label on the $\tilde{\chi}$'s is suppressed.

A Killing vector K of the de Sitter background defines a vector field $h^{(K)}_{ij} := \mathcal{L}_K h_{ij} = a^2(\mathcal{L}_K \tilde{\chi}_{ij} + 2(a^{-1}\mathcal{L}_K a)\tilde{\chi}_{ij}) =: a^2 \tilde{\chi}^{(K)}_{ij}$, on the space $\mathcal{C}$. This vector field generates a canonical transformation and the corresponding Hamiltonian function is given by

$$H_K := -\frac{1}{2}\omega(h, h^{(K)}) = -\frac{1}{2}\omega(\tilde{\chi}, \tilde{\chi}^{(K)}). \quad (27)$$

For the time translation Killing vector T, H_T ($=: E_T$), is obtained as

$$E_T := -\frac{1}{2}\omega(\tilde{\chi}, \tilde{\chi}^{(T)})$$
$$= -\frac{1}{32\pi H}\int_{\Sigma_\eta} d^3 x(\tilde{\chi}_{ij}\mathcal{E}^{(T)}_{kl} - \tilde{\chi}^{(T)}_{kl}\mathcal{E}_{ij})\delta^{ik}\delta^{jl} \quad (28)$$

$$= -\frac{1}{32\pi H}\int_{\Sigma_\eta} d^3 x(\tilde{\chi}_{ij}\mathcal{L}_T\mathcal{E}_{kl} - \mathcal{E}_{kl}\mathcal{L}_T\tilde{\chi}_{ij} - 3H\tilde{\chi}_{ij}\mathcal{E}_{kl})\delta^{ik}\delta^{jl}. \quad (29)$$

This integral is independent of the choice of Σ_η and is conveniently performed on $\mathcal{J}^+ = \Sigma_0$. The Killing vector T also has a smooth limit to $\mathcal{J}^+$, $T|_{\mathcal{J}^+} = -H(x\partial_x + y\partial_y + z\partial_z)$. Equation (29) simplifies to

$$E_T = \frac{1}{16\pi H}\int_{\mathcal{J}^+} d^3 x \mathcal{E}_{kl}(\mathcal{L}_T\tilde{\chi}_{ij} + 2H\tilde{\chi}_{ij})\delta^{ik}\delta^{jl}. \quad (30)$$

Now using $(\mathcal{L}_T\tilde{\chi}_{ij} + 2H\tilde{\chi}_{ij})|_{\mathcal{J}^+} = T^m\partial_m\tilde{\chi}_{ij}$,

$$E_T = \frac{1}{16\pi H}\int_{\mathcal{J}^+} d^3 x \mathcal{E}_{kl}(T^m\partial_m\tilde{\chi}_{ij})\delta^{ik}\delta^{jl} \quad (31)$$

$$= \frac{1}{32\pi H^2}\int_{\mathcal{J}^+} d^3 x\left[\frac{1}{\eta}\left(\partial_\eta^2 - \frac{1}{\eta}\partial_\eta\right)\tilde{\chi}_{kl}\right]^{TT}(T^m\partial_m\tilde{\chi}_{ij})^{TT}\delta^{ik}\delta^{jl}. \quad (32)$$

In the last line we have used the equation of motion and restored the TT label [2]. Both $\mathcal{E}_{kl}$ and $T^m\partial_m\tilde{\chi}_{ij}$ have a smooth limit on $\mathcal{J}^+$.

When evaluated at the approximated solution given in (6), the energy flux turns out to be given by [2]

$$E_T = \frac{1}{8\pi}\int_{\mathcal{J}^+} d\tau d^2 s[\mathcal{R}_{kl}\mathcal{R}^{TT}_{ij}]\delta^{ik}\delta^{jl}, \quad (33)$$

where $\mathcal{R}_{ij}$ denotes the "radiation field" on $\mathcal{J}^+$, expressed in terms of source moments, and is given by

$$\mathcal{R}^{TT}_{mn} := [\dddot{Q}_{mn} + 3H\ddot{Q}_{mn} + 2H^2\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn} + 3H^2\dot{\bar{Q}}_{mn} + 2H^3\bar{Q}_{mn}]^{TT}(t_{ret}), \quad (34)$$

with the overdot denoting the Lie derivative $\mathcal{L}_T$.

The instantaneous power received on $\mathcal{J}^+$ at τ is given by

$$P(\tau) := \frac{1}{8\pi}\int_{S^2} d^2 s[\mathcal{R}^{ij}\mathcal{R}^{TT}_{ij}](-r(\tau)). \quad (35)$$

This expression is not manifestly positive. Manifestly positive expressions for the flux and the power are given by [2]

$$E_T = \frac{1}{2\pi}\int_{\mathcal{J}^+} d\tau d^2 s[\partial_r\mathcal{M}^{TT}_{ij}][\partial_r\mathcal{M}^{ij}_{TT}], \quad \text{where} \quad (36)$$

$$\mathcal{M}^{TT}_{ij}(\eta - r) := \int d^3 x' T^{TT'}_{ij}(\eta - r, \vec{x}'); \quad (37)$$

$$P(\tau) = \frac{1}{2\pi}\int_{S^2} d^2 s[\partial_r\mathcal{M}^{TT}_{ij}][\partial_r\mathcal{M}^{ij}_{TT}](-r(\tau)). \quad (38)$$

In the definition of $\mathcal{M}^{TT}_{ij}$, the TT' on the stress tensor on the right-hand side denotes transversality with respect to the $\vec{x}'$ argument. The $\mathcal{M}^{TT}_{ij}$ has no simple relation to the various source moments and its radial derivative is distinct from the $\mathcal{R}^{TT}_{ij}$. For completeness, the momentum and angular momentum fluxes are given by [2]

$$P_j = \frac{1}{16\pi H}\int_{\mathcal{J}^+} d^3 x \mathcal{E}^{mn}\mathcal{L}_{\xi_j}\tilde{\chi}^{TT}_{mn} = 0; \quad (39)$$

$$J_j = -\frac{1}{8\pi H}\int_{\mathcal{J}^+} d^3 x \mathcal{E}^{mn}\mathcal{L}_{L_j}\tilde{\chi}^{TT}_{mn}$$
$$= \frac{1}{4\pi}\int_{\mathcal{J}^+} d\tau d^2 s\epsilon_{jmn}\mathcal{R}^{nl}[\ddot{Q}_l^{\ m} + H\dot{Q}_l^{\ m} + H\dot{\bar{Q}}_l^{\ m} + H^2\bar{Q}_l^{\ m}]^{TT}. \quad (40)$$

The momentum flux is zero because the integrand is linear in x_j (parity odd) and in the angular momentum flux, the second factor is proportional to the tail term.

C. Isaacson prescription

In the previous subsection we saw a definition of the total energy of the radiation field of compactly supported sources in Eq. (33). The radiated power, received at infinity, is given in Eq. (35). In this subsection we recall an alternative framework, based on a short wavelength expansion [8,9], for a restricted class of sources but with the benefit of a symmetric, conserved, and suitably gauge invariant effective gravitational stress tensor.

Conceptually, the framework is somewhat different from perturbation about a *fixed*, given background solution. It is designed to construct a class of solutions for which there exists a coordinate system in which the metric components display two widely separated temporal/spatial scales of variation. The slowly varying (or long wavelength $\sim L$) component is taken as the background component and the fast (or short wavelength $\sim \lambda$) component, whose amplitude is small compared to that of the background, is identified as the ripple component.[2] These statements are manifestly coordinate dependent, but the existence of a coordinate system with sufficiently large domain admitting such an identification is itself a physical property. The calculational scheme is again iterative but now allows for both the background and the ripple components to be corrected. To make such a separation, an averaging scheme is introduced. It splits the Einstein equation into two separate, coupled equations for the background and the ripple. These equations provide a definition of the effective gravitational stress tensor.

For the metric of the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$, the Einstein equation to $o(\epsilon^2)$ takes the form

$$R_{\mu\nu}(\bar{g} + \epsilon h) = \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) + 8\pi\epsilon\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}\right)$$

$$\therefore R_{\mu\nu}^{(0)}(\bar{g}) + \epsilon R_{\mu\nu}^{(1)}(\bar{g}, h) + \epsilon^2 R_{\mu\nu}^{(2)}(\bar{g}, h)$$

$$= \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) + 8\pi\left\{\epsilon T_{\mu\nu} - \frac{1}{2}(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu})\right.$$

$$\left. \times (\bar{g}^{\alpha\beta} - \epsilon h^{\alpha\beta} + \epsilon^2 h^{\alpha\rho}h_\rho{}^\beta)(\epsilon T_{\alpha\beta})\right\}. \qquad (41)$$

Introduce an averaging over an intermediate scale ℓ, $\lambda \ll \ell \ll L$, which satisfies the following properties: (i) the average of the odd powers of h vanish and (ii) the average of the space-time divergence of the tensors is subleading [9,11]. The average of course leaves the L-scale variations intact; in particular, the average of $g_{\mu\nu}$ equals $\bar{g}_{\mu\nu}$. For simplicity, we will assume that the average of the matter stress tensor is zero; i.e., it has only λ-scale variations.

---

[2]In the present context, $L \sim \Lambda^{-1/2}$, while λ could be taken as the inverse of the characteristic frequency. The length scale R denoting the extent of a spatially compact source satisfies $R \ll \lambda$.

Taking the average of the above equation and noting that $\langle R_{\mu\nu}^{(0)}\rangle = R_{\mu\nu}^{(0)}$ and $R_{\mu\nu}^{(2)} - \langle R_{\mu\nu}^{(2)}\rangle \approx (R_{\mu\nu}^{(2)})_{\lambda-\text{scale}}$, Eq. (41) can be separated into Eq. (43) for the background and Eq. (42) for the ripple:

$$8\pi T_{\mu\nu} = G_{\mu\nu}^{(1)} + \Lambda h_{\mu\nu}$$

$$= R_{\mu\nu}^{(1)} - \frac{1}{2}(\bar{g}_{\mu\nu}R^{(1)} - \bar{g}_{\mu\nu}h^{\alpha\beta}\bar{R}_{\alpha\beta} + h_{\mu\nu}\bar{R})$$

$$+ \Lambda h_{\mu\nu} \qquad (42)$$

$$8\pi t_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} + \Lambda\bar{g}_{\mu\nu}, \quad \text{with} \qquad (43)$$

$$t_{\mu\nu}(\bar{g}, h) := -\frac{\epsilon^2}{8\pi}\left[\langle R_{\mu\nu}^{(2)}\rangle - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\langle R_{\alpha\beta}^{(2)}\rangle\right]. \qquad (44)$$

Equation (42) is exactly the same linearized Einstein equation we had before for the weak field $h_{\mu\nu}$ and every term of it has a scale of variation λ. However, Eq. (43) for the background is different. Although it has terms of order ϵ^2, every term has a scale of variation L. If we now recognize that for λ-scale variation $\partial h \sim \lambda^{-1}h$, and $\epsilon' := \lambda/L$ is taken to be of the same order as ϵ, then the effective stress tensor, which has a leading term of the form $(\partial h)^2$, is of the order $(\epsilon/\epsilon')^2 \sim o(1)$ and is thus included in the equation.

The effective stress tensor defined in Eq. (44) is manifestly symmetric and is covariantly conserved with respect to the background covariant derivative, since the divergence of the right-hand side of (43) vanishes identically. For ripples over the Minkowski background, it is gauge invariant and the energy momentum computed using it agrees with the quadrupole formula obtained by other methods, thereby strengthening its interpretation as a gravitational stress tensor. An averaging procedure constructing a tensor has been given in [8,12] and an explicit illustrative computation is given in the Appendix.

At the zeroth iteration, we choose the Poincaré patch of the de Sitter space-time as the solution of (43), ignoring the effective gravitational stress tensor. Let us quickly verify the gauge invariance of $t_{\mu\nu}$ under $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$, to leading order in ϵ. Recall that the gauge transformation involves derivatives of ξ_μ and for a consistency with the background plus ripple split, the gauge transformation should also be restricted to preserve it. There are two possibilities for the generator: (i) ξ is comparable with h and slowly varying, and (ii) ξ is order ϵh but is rapidly varying so that its derivative becomes order h. The gauge transformation of $t_{\mu\nu}$, after dropping space-time divergences in the averaging, has leftover terms of the form $\Lambda\langle h\nabla\xi\rangle$. These vanish identically for the Minkowski background, making the $t_{\mu\nu}$ gauge invariant. For the ξ of type (i), the average vanishes since the enclosed quantity is rapidly varying and for ξ of type (ii), the averaged quantity is order ϵ. But $t_{\mu\nu}$

itself is $o(1)$ and hence gauge invariance of $t_{\mu\nu}$ is ensured to the leading order [8].

Using the properties of averaged quantities, the effective gravitational stress tensor for the gauge fixed solution of the ripple equation evaluates to

$$
\begin{aligned}
&- 8\pi t_{\mu\nu} \\
&= \epsilon^2 \left\langle \left[-\frac{1}{4}\bar{\nabla}_\mu \tilde{h}^{\alpha\beta}\bar{\nabla}_\nu \tilde{h}_{\alpha\beta} + \frac{\Lambda}{3}\tilde{h}_\mu{}^\alpha \tilde{h}_{\alpha\nu} - \frac{\Lambda}{4}\bar{g}_{\mu\nu}(\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\beta}) \right] \right\rangle.
\end{aligned}
\tag{45}
$$

This expression reduces to the stress tensor for the Minkowski background by taking $\nabla_\mu \to \partial_\mu$ and dropping the last two terms. However, for the ripple, $\partial \tilde{h} \sim \lambda^{-1}\tilde{h} \sim \epsilon^{-1}\tilde{h}$. The connection terms in the covariant derivatives are order $\tilde{h}$. Hence, to the leading order in $\epsilon \sim \lambda/L$, *all* terms without derivatives of the ripple can be dropped and we are back to the same expression for the Minkowski background. Notice that the leading term has no ϵ.

In the conformal coordinates, substituting $\tilde{h}_{\alpha\beta} = a^2 \tilde{\chi}_{\alpha\beta}$ and once again, keeping only the terms with derivatives of the ripple, the stress tensor for the fully gauge fixed solutions of (42) becomes

$$
t_{\mu\nu} = \frac{1}{32\pi}\langle \partial_\mu \tilde{\chi}_{ij}^{TT}\partial_\nu \tilde{\chi}_{TT}^{ij}\rangle.
\tag{46}
$$

We will refer to this as the ripple stress tensor. We will compute this for the tt projected, approximated retarded solution, χ_{ij}^{tt}. In Sec. III C, we will discuss how the computations change when $\chi_{ij}^{tt} \to \mathcal{X}_{ij}^{TT}$.

III. CONSERVED QUANTITIES

Given *any* symmetric, conserved stress tensor, for every Killing vector of the background space-time, ξ^μ, the current $J_\xi^\mu \sim T^\mu{}_\nu \xi^\nu$ is covariantly conserved. In order that for a future directed timelike Killing vector, the corresponding energy-momentum current is also timelike and future directed, we define $J_\xi^\mu := -T^\mu{}_\nu \xi^\nu$. We adopt this definition for the time translation Killing vector $T = -H(\eta\partial_\eta + x^i\partial_i)$.

The time translation Killing vector field, $T^\nu\partial_\nu$, involves only the η and r derivatives since $x^i\partial_i = r\partial_r$ and these pass through the Λ projector. For the space translation along the jth direction, we have ∂_j, which does act on the Λ projector. In the present context where derivatives of the ripple dominate over (ripple/r), the derivative of the projector can be neglected and we write $\partial_j \chi_{mn}^{tt} \approx \hat{x}_j \partial_r \chi_{mn}^{tt}$. For generators of rotation, however, the situation is different. Once again we get two terms from ∂_i, but now $\epsilon_{jki}x^k\hat{x}^i\partial_r\chi^{mn} = 0$! and we can no longer neglect the derivative of the projector. With these understood, we write the corresponding currents, $J_\xi^\mu = -\frac{a^{-2}}{32\pi}\langle \partial^\mu \chi_{tt}^{mn}\partial_\nu \chi_{mn}^{tt}\rangle \xi^\nu$. Note that the

ripple stress tensor has been defined as a covariant rank-2 tensor and hence there is the factor of $a^{-2} = H^2\eta^2$ since the index μ has been raised. The currents are given by

$$
a^2 J_T^\eta = -\frac{H}{32\pi}\{\eta\langle \partial_\eta \chi_{tt}^{mn}\partial_\eta \chi_{mn}^{tt}\rangle + r\langle \partial_\eta \chi_{tt}^{mn}\partial_r \chi_{mn}^{tt}\rangle\}
\tag{47}
$$

$$
a^2 J_T^i = \frac{H}{32\pi}\{\eta\langle \hat{x}^i\partial_r \chi_{tt}^{mn}\partial_\eta \chi_{mn}^{tt}\rangle + r\langle \hat{x}^i\partial_r \chi_{tt}^{mn}\partial_r \chi_{mn}^{tt}\rangle\}
\tag{48}
$$

$$
\begin{aligned}
a^2 J_{\xi_j}^\eta &= \frac{1}{32\pi}\langle \partial_\eta \chi_{tt}^{mn}\hat{x}_j \partial_r \chi_{mn}^{tt}\rangle, \\
a^2 J_{\xi_j}^i &= -\frac{1}{32\pi}\langle \hat{x}^i\partial_r \chi_{tt}^{mn}\hat{x}_j \partial_r \chi_{mn}^{tt}\rangle
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
a^2 J_{L_j}^\eta &= -\frac{1}{16\pi}\epsilon_{jmn}\hat{x}^m \langle \partial_\eta \chi_{tt}^{nl}\chi_{lk}\hat{x}^k\rangle, \\
a^2 J_{L_j}^i &= \frac{1}{16\pi}\epsilon_{jmn}\hat{x}^m \langle \hat{x}^i\partial_r \chi_{tt}^{nl}\chi_{lk}\hat{x}^k\rangle.
\end{aligned}
\tag{50}
$$

The unit vectors within the angular brackets have come from the spatial derivatives, while those outside the brackets come from the Killing vector. It is shown in the Appendix [Eq. (A9)] that for the averaging regions far away from the source, the unit vectors can be taken across the angular brackets and we will do so in the subsequent expressions.

Notice that for the energy and momentum currents (48), (49), both fields have the tt label, whereas for the angular momentum current (50), the second factor does *not* have the tt label. The entire contribution to the angular momentum current comes from the derivative of the Λ projector. The contribution from the derivative of the field vanishes since the field (without the projector) is spherically symmetric. In all these equations we may use $\partial_r\chi_{mn} = -\partial_\eta\chi_{mn} - \frac{f_{mn}}{r^2}$ from (13).

We note in passing that *if* $\frac{f_{mn}}{r^2}$ can be neglected compared to $\partial_\eta\chi_{mn}$, then the currents corresponding to the generators of time and space translations both become proportional to the vector $(1, x^i/r)$, which is a null vector. Both energy and momentum propagate along this direction.

Let $\mathcal{V}$ denote a space-time region with a boundary $\partial\mathcal{V}$. Then it follows that

$$
0 = \int_\mathcal{V} d^4x \sqrt{\bar{g}}\,\bar{\nabla}_\mu J_\xi^\mu = \int_\mathcal{V} d^4x \partial_\mu(\sqrt{\bar{g}}J_\xi^\mu) = \int_{\partial\mathcal{V}} d\sigma_\mu J_\xi^\mu,
\tag{51}
$$

where $d\sigma_\mu$ is the oriented volume element of the boundary $\partial\mathcal{V}$.

In the next subsection we evaluate the energy flux, $\mathcal{F}_\Sigma := \int_\Sigma d\sigma_\mu J_T^\mu$, for various hypersurfaces, Σ's. These, together with the conservation equation (51), will be used to relate power received at $\mathcal{J}^+$ to that crossing the

cosmological horizon. In the following subsection, we will present the fluxes for momentum and angular momentum.

A. Flux computations

We present flux calculations for three classes of hypersurfaces: (a) hypersurfaces of constant physical radial distance, (b) spacelike hypersurfaces of constant η, and (c) the outgoing and incoming null hypersurfaces.

The solution χ_{ij} in this and the next subsection stands for χ_{ij}^{tt}.

1. Hypersurface of constant physical radial distance

These hypersurfaces are timelike, null, or spacelike, respectively, according to the physical distance being less than, equal to, or greater than the physical distance to the cosmological horizon, namely, H^{-1}. They are spanned by the integral curves of the Killing vector T.

This Killing vector is special because in the static patch, it is timelike and its integral curves represent Killing observers. Denoting $x^i := r\hat{x}^i$, $\hat{x}^i\hat{x}^j\delta_{ij} = 1$, in general, its integral curves are given by $\eta(\tau) = \eta_* e^{-H\tau}$, $r(\tau) = r_* e^{-H\tau}$, $\hat{x}^i = \hat{x}_*^i$. Evidently, along each curve, $\rho := r/(-H\eta) = r_*/(-H\eta_*)$ is constant. This also represents the physical radial distance, $r_{phy} := |\Omega| r$. Each particular curve is labeled by ρ and the two angular coordinates $\hat{x}_*^i$. We compute the flux across the hypersurface Σ_ρ, defined by $r_{phy} = \rho$. This surface is coordinatized by the Killing parameter τ and the usual spherical angles θ, ϕ represented by the unit vectors $\hat{x}^i$. These hypersurfaces are topologically $\Sigma_\rho \sim \Delta\tau \times S^2$ and their embedding is given by

$$\eta(\tau,\theta,\phi) = \eta_* e^{-H\tau}, \qquad x = r_* e^{-H\tau}\sin\theta\cos\phi,$$
$$y = r_* e^{-H\tau}\sin\theta\sin\phi, \qquad z = r_* e^{-H\tau}\cos\theta,$$

with $r_* + H\rho\eta_* = 0$.

The induced metric is given by $h_{ab} = \text{diag}(H^2\rho^2 - 1, \rho^2, \rho^2\sin^2\theta)$. This has Lorentzian signature for $H\rho < 1$ (inside the static patch), is degenerate for $H\rho = 1$ (the cosmological horizon), and has Euclidean signature for $H\rho > 1$ (beyond the cosmological horizon). The measure factor for the non-null cases is given by $\sqrt{|\det h_{ab}|} = \sqrt{|1 - H^2\rho^2|}\rho^2\sin\theta$, while on the cosmological horizon it is given by $\sqrt{h_{ab}} = \rho^2\sin\theta$. Here $\underline{a}, \underline{b}$ denote the transverse coordinates θ, ϕ. In the non-null case, the unit normal is given by $n_\mu = \frac{\epsilon}{|H\eta|}|1 - H^2\rho^2|^{-1/2}(H\rho, x_i/r) \leftrightarrow n^\mu = \epsilon|H\eta||1 - H^2\rho^2|^{-1/2}(-H\rho, x^i/r)$. Here $\epsilon = +1$ for timelike Σ_ρ ($H\rho < 1$) and $\epsilon = -1$ for spacelike Σ_ρ ($H\rho > 1$). On the cosmological horizon, we choose the normal to be $n_\mu = -|H\eta|^{-1}(H\rho, x_i/r) \leftrightarrow n^\mu = -|H\eta|(-H\rho, x^i/r)$, so that $n^\mu = T^\mu$ is future directed. Introduce $N^\mu := (-H\rho, \hat{x}^i)$, so that the normal for non-null cases is expressed as $n^\mu = \epsilon|H\eta||1 - H^2\rho^2|^{-1/2}N^\mu$. Note that n^μ is the same for the spacelike and the null

hypersurfaces, $\Sigma_{\rho\geq H^{-1}}$. For the timelike hypersurface, the n^μ points in the opposite direction. However, the induced orientation on Σ_ρ is also reversed as the hypersurface changes from being spacelike to being timelike. Hence, in *all cases*, $H\rho > 0$, $n^\mu\sqrt{h} = -|H\eta|N^\mu\rho^2\sin\theta$, and the hypersurface integral is expressed as

$$\int_{\Sigma_\rho} d\Sigma_\alpha J_T^\alpha = -\int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2 s\rho^2(-|H\eta(\tau)|N^\mu)(-t_{\mu\nu}T^\nu)$$

with

$$N^\mu t_{\mu\nu}T^\nu = -Hr(\tau)\{t_{\eta\eta} + \hat{x}^i t_{ij}\hat{x}^j - \hat{x}^i t_{i\eta}((H\rho)^{-1} + H\rho)\}. \tag{52}$$

The minus sign in front of the hypersurface integral is because the orientation defined by the Killing parameter and the angles is *negative* relative to that defined by the r and the angles. The $\sin\theta$ is absorbed in $d^2 s$. The minus sign inside the last parentheses is due to the definition $J_\mu = -t_{\mu\nu}T^\nu$. In the second line, we have also used $-H\rho\eta = r$ valid on Σ_ρ.

Substituting for the ripple stress tensor, and taking the unit vectors $\hat{x}$ outside of the angular bracket as mentioned before, the expression within the braces becomes

$$\{\} = \frac{1}{32\pi}\left\{\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn} + \partial_r\chi_{mn}\partial_r\chi^{mn}\rangle - \frac{1 + H^2\rho^2}{H\rho}\langle\partial_r\chi_{mn}\partial_\eta\chi^{mn}\rangle\right\}. \tag{53}$$

The (implicit) tt projection introduces angle dependence in χ_{ij}^{tt}; however, Eq. (52) needs only η and r derivatives.

Eliminating $\partial_r\chi_{ij}$ using Eq. (13), we write

$$\{\} = \frac{1}{32\pi}\left[\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn}\rangle\frac{(1 + H\rho)^2}{H\rho}\right.$$
$$\left. + \left\langle\frac{f_{mn}}{r^2}\partial_\eta\chi^{mn}\right\rangle\frac{(1 + H\rho)^2}{H\rho} + \left\langle\frac{f_{mn}f^{mn}}{r^4}\right\rangle\right]$$
$$= \frac{1}{32\pi}\frac{(1 + H\rho)^2}{H\rho}\left[\left\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn} + \frac{f_{mn}}{r^2}\partial_\eta\chi^{mn}\right\rangle\right.$$
$$\left. + \frac{H\rho}{(1 + H\rho)^2}\left\langle\frac{f_{mn}f^{mn}}{r^4}\right\rangle\right]$$

$$\therefore \int_{\Sigma_\rho} d\Sigma_\alpha J_T^\alpha = \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2 s[-\rho^2 H^2\eta(\tau)r(\tau)]\{\}. \tag{54}$$

The approximated solution χ_{ij} is valid for (source dimension)/(distance to the source) $\ll 1$. This is consistent with the assumption that $\lambda/r \ll 1$. Furthermore, the source being rapidly changing, $\lambda H \ll 1$, it follows that $f_{mn}/r^2 \ll \dot{f}_{mn}/r$. Hence we drop f_{mn}/r^2 terms. With this, $\{\}$ takes a simple quadratic form $\frac{1}{32\pi}(1 + H\rho)^2(H\rho)^{-1}\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn}\rangle$.

To compute ∂_η we recall that $\eta_{ret} = \eta - r :=$ $-H^{-1}e^{-Ht_{ret}} := -(Ha(t_{ret}))^{-1}$ and use

$$\partial_\eta f_{ij}(\eta_{ret}) = \partial_{\eta_{ret}} f_{ij}(\eta_{ret}) = a(\eta_{ret}) \partial_{t_{ret}} f_{ij}(t_{ret})$$
$$= a(\eta_{ret})(\mathcal{L}_T + 2H) f_{ij}(t_{ret}).$$

This leads to (overdot denoting $\mathcal{L}_T$)

$$\partial_\eta \chi_{mn}(\eta_{ret}) = \frac{1}{r} \partial_\eta f_{mn}(\eta_{ret}) + \partial_\eta g_{mn}(\eta_{ret})$$

$$= \frac{a(t_{ret})}{r}(\mathcal{L}_T f_{mn} + 2H f_{mn})$$
$$+ a(t_{ret})(\mathcal{L}_T g_{mn} + 2g_{mn}) \qquad (55)$$

$$\mathcal{L}_T f_{mn} = \frac{2}{a(t_{ret})}[\dddot{Q}_{mn} + H\ddot{Q}_{mn} - 2H^2\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn}$$
$$+ H^2\dot{\bar{Q}}_{mn} - 2H^3\bar{Q}_{mn}] \qquad (56)$$

$$\mathcal{L}_T g_{mn} = -2H[\ddot{Q}_{mn} + H\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn} + H^2\dot{\bar{Q}}_{mn}] \Rightarrow \qquad (57)$$

$$\partial_\eta \chi_{mn}^{tt}(\eta_{ret}) = \frac{2}{r} \frac{\eta}{\eta - r} Q_{mn}^{tt} \quad \text{with} \quad , \qquad (58)$$

$$Q_{mn}^{tt} := [\dddot{Q}_{mn} + 3H\ddot{Q}_{mn} + 2H^2\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn}$$
$$+ 3H^2\dot{\bar{Q}}_{mn} + 2H^3\bar{Q}_{mn}]^{tt}(t_{ret}). \qquad (59)$$

Here we have also used $(1 - a(t_{ret})rH) = \frac{\eta}{\eta - r}$. Collecting all expressions, we write the flux through a segment of the $r_{phy} = \rho$ hypersurface in a convenient form as

$$\int_{\Sigma_\rho} d\Sigma_\alpha J_T^\alpha = \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s [-\rho^2 H^2 \eta(\tau) r(\tau)]$$
$$\times \left[\frac{(1 + H\rho)^2}{32\pi H\rho}\right] \left\langle \left[\frac{2}{r}\frac{\eta}{\eta - r}\right]^2 Q_{ij}^{tt} Q_{tt}^{ij} \right\rangle$$
$$= \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s \left[\frac{1}{8\pi}\right] \langle Q_{ij}^{tt} Q_{tt}^{ij} \rangle (t_{ret}). \qquad (60)$$

In the Appendix, we show that for large ρ, the expression within the square brackets inside the angular brackets can be taken outside. Then, using $r = -H\rho\eta$, which is valid over the hypersurface $\forall \rho \in \mathbb{R}^+$, we see that the explicit dependence on ρ (for large enough ρ) disappears from the integrand but there is an implicit dependence on ρ and τ through t_{ret}. If, however, the τ integration is extended over its full range, $(-\infty, \infty)$, then the integral *is* independent of ρ as well. Hence, for sufficiently large ρ, all Killing observers infer the same energy flux in the limit $(\tau_1, \tau_2) \to (-\infty, \infty)$.

The ρ independence of the full flux integral in particular means that the total flux across $\mathcal{J}^+$ equals the total flux across the cosmological horizon, $\mathcal{H}^+$:

$$\lim_{\rho \to \infty} \int_{\Sigma_\rho} d\Sigma_\mu J_T^\mu = \int_{\Sigma_{(H\rho=1)}} d\Sigma_\mu J_T^\mu \Leftrightarrow \int_{\mathcal{J}^+} d\Sigma_\mu J_T^\mu = \int_{\mathcal{H}^+} d\Sigma_\mu J_T^\mu. \qquad (61)$$

2. Flux through a constant η slice

The hypersurface Σ_{η_0} defined by $\eta = \eta_0$ is a cosmological slice $\sim \mathbb{R}^3$. It is spacelike, with a normal $n_\mu = -|H\eta_0|^{-1}(1, \vec{0}) \leftrightarrow n^\mu = |H\eta_0|(1, \vec{0})$, which is future directed. We choose a finite portion of it with $r \in [r_1, r_2]$. The hypersurface is topologically $\Delta r \times s^2$. Choosing the (r, θ, ϕ) coordinates on the hypersurface, the embedding is given by

$$\eta(r, \theta, \phi) = \eta_0, \qquad x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

The induced metric is given by $h_{ab} = (H\eta_0)^{-2}diag(1, r^2, r^2\sin^2\theta)$, giving $\sqrt{|\det h_{ab}|} = |H\eta_0|^{-3}r^2\sin\theta$. Denoting $N^\mu := (1, \vec{0})$, the hypersurface integral is given by

$$\int_{\Sigma_{\eta_0}} d\Sigma_\mu J_T^\mu = \int_{r_1}^{r_2} dr \int_{S^2} d^2s r^2 a^2(\eta_0)(-N^\mu t_{\mu\nu} T^\nu), \quad \text{with}$$
$$\qquad (62)$$

$$N^\mu t_{\mu\nu} T^\nu = (-H)(t_{\eta\eta}\eta + t_{\eta i}x^i)$$
$$= \frac{-H}{32\pi}(\eta\langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn}\rangle + x^i\langle \partial_\eta \chi_{mn} \partial_i \chi^{mn}\rangle) \qquad (63)$$
$$= \frac{-H}{32\pi}\left((\eta - r)\langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn}\rangle - \left\langle \frac{f_{mn}}{r} \partial_\eta \chi^{mn}\right\rangle\right) \qquad (64)$$

$$\therefore \int_{\Sigma_{\eta_0}} d\Sigma_\mu J_T^\mu = -\frac{1}{32\pi H^2 \eta_0^2} \int_{r_1}^{r_2} dr \int_{S^2} d^2s r^2$$
$$\times \left\{\frac{1}{a(\eta_{ret})}\langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn}\rangle\right\} \qquad (65)$$

$$\approx \int_{r_1}^{r_2} \frac{dr}{H(r - \eta_0)} \int_{S^2} d^2s \left[\frac{-1}{8\pi}\right] \langle Q_{mn}^{tt} Q_{tt}^{mn}\rangle. \qquad (66)$$

By the same reasoning as before, we have dropped $\frac{f_{mn}}{r}$ and also used Eq. (A12). In the limit $\eta_0 \to 0$ with $(r_1, r_2) \to (0, \infty)$, the hypersurface becomes $\mathcal{J}^+$ and the integration measure becomes $\frac{dr}{Hr}$. The limit $\eta \to 0$ is thus finite.

As noted earlier, the hypersurface integral when expressed in terms of the Killing parameter has a minus sign due to the reversal of the induced orientation. The measures (positive) themselves are related as $\frac{dr}{Hr} = d\tau$, leading to $\int_0^\infty dr/Hr = -\int_{-\infty}^\infty d\tau$, and we get

$$\lim_{\eta_0 \to 0} \int_{\Sigma_{\eta_0}} d\Sigma_\mu J_T^\mu = \lim_{\rho \to \infty} \int_{\Sigma_\rho} d\Sigma_\mu J_T^\mu. \qquad (67)$$
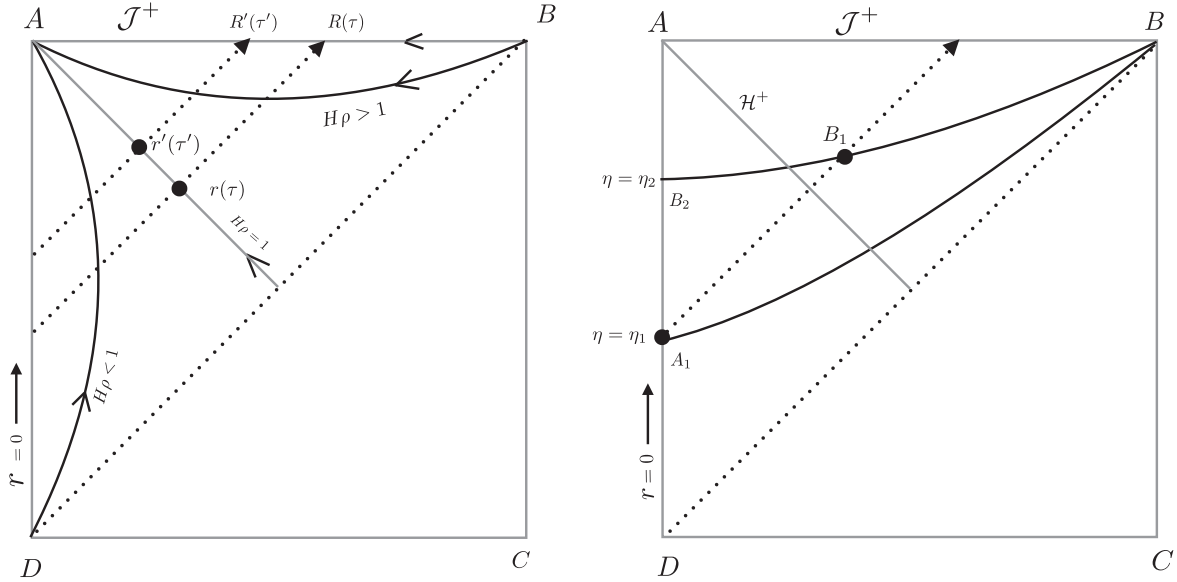
FIG. 2. Left: The $\rho = $ constant hypersurfaces which are timelike for $H\rho < 1$, null for $H\rho = 1$, and spacelike for $H\rho > 1$. The two 45 degree outgoing null hypersurfaces intersecting $\mathcal{H}^+$ and $\mathcal{J}^+$ in the spheres at $r(\tau), r'(\tau'), R(\tau), R'(\tau')$ bound a space-time region. Right: The spacelike hypersurfaces with a constant value of η. The fluxes across the outgoing null hypersurfaces turn out to be zero, signifying sharp propagation of energy-momentum and angular momentum. Hence the energy flux across the portion of the horizon bounded by the spheres at $r(\tau), r'(\tau')$ equals the flux across the portion of the future infinity bounded by the spheres at $R(\tau), R'(\tau')$.

3. Flux through null hypersurfaces

There are two families of future directed null hypersurfaces given by $\eta + \epsilon r + \sigma = 0$; see Fig. 2. For $\epsilon = +1$, these 45 degree lines in the Penrose diagram are parallel to the cosmological horizon while for $\epsilon = -1$, the lines are parallel to the null boundary of the Poincaré patch. We refer to these as the incoming ($\epsilon = 1$) and outgoing ($\epsilon = -1$) null hypersurfaces. The parameter σ labels members of these families.

The null normals of these families are of the form $n_\mu = \gamma(1, \epsilon \hat{x}_i) \leftrightarrow n^\mu = (H\eta)^2\gamma(-1, \epsilon \hat{x}^i)$, where γ is to be chosen suitably and should be negative for future directed hypersurfaces. Choosing coordinates (λ, θ, ϕ) on a null hypersurface, its embedding may be taken as $\eta(\lambda), r(\lambda)$ with identity mapping of the angles. Here λ is an affine parameter of the null geodesics generating the null hypersurfaces. The induced metric is obtained as $h_{ab} = (H\eta)^{-2}diag(0, r^2, r^2 \sin^2 \theta)$. Note that the orientation of the hypersurfaces, relative to that defined by (r, θ, ϕ), is the same for the outgoing hypersurfaces and opposite for the incoming hypersurfaces. The hypersurface integral is then given by $[N^\mu := (-1, \epsilon \hat{x}^i)]$

$$\int_{\Sigma_{(\epsilon,\sigma)}} d\Sigma_\mu J_T^\mu = -\epsilon \int_{\lambda_1}^{\lambda_2} d\lambda \int_S^2 d^2s \left[\frac{r^2}{H^2\eta^2}\right](H^2\eta^2)\gamma(-N^\mu t_{\mu\nu}T^\nu), \tag{68}$$

$$N^\mu t_{\mu\nu}T^\nu = -H(-t_{\eta\eta}\eta - t_{\eta j}r\hat{x}^j + \epsilon\hat{x}^i t_{i\eta}\eta + \epsilon r\hat{x}^i t_{ij}\hat{x}^j)$$

$$= -\frac{H}{32\pi}(-\eta\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn}\rangle + (\epsilon\eta - r)\langle\partial_\eta\chi_{mn}\partial_r\chi^{mn}\rangle + \epsilon r\langle\partial_r\chi_{mn}\partial_r\chi^{mn}\rangle)$$

$$= -\frac{H}{32\pi}(1 + \epsilon)(r - \eta)\langle\partial_\eta\chi_{mn}^{tt}\partial_\eta\chi_{tt}^{mn}\rangle + o(r^{-2}) \tag{69}$$

$$= -\frac{H}{32\pi}(1 + \epsilon)(r - \eta)\left[\frac{4}{r^2}\frac{\eta^2}{(\eta - r)^2}\right]\langle\mathcal{Q}_{mn}^{tt}\mathcal{Q}_{tt}^{mn}\rangle + o(r^{-2}) \tag{70}$$

$$\therefore \int_{\Sigma_{(\epsilon,\sigma)}} d\Sigma_\mu J_T^\mu = \epsilon \int_{\lambda_1}^{\lambda_2} d\lambda \int_{S^2} d^2s[\gamma H]\left[\frac{(1 + \epsilon)}{8\pi}\frac{\eta^2}{\eta - r}\right]\langle\mathcal{Q}_{mn}^{tt}\mathcal{Q}_{tt}^{mn}\rangle. \tag{71}$$

As before, we have dropped the f_{mn}/r^2 terms from $\partial_r\chi_{mn}$ and used Eq. (A12).

It is immediately clear that the flux through the outgoing null hypersurfaces (ρ or r increase along these) vanishes. In the $\epsilon = 1$ family, only the cosmological horizon is of interest. For this we have ($\eta = -r$) and we *choose* the factor $\gamma = -(Hr)^{-1}$ so the null normal matches with the Killing vector (γ is negative as desired for future orientation) and the affine parameter λ matches with the Killing parameter τ. With this choice, the flux in Eq. (71) matches with that given in Eq. (60) for $H\rho = 1$. Thus, once again, the full flux through the cosmological horizon is exactly the same as that of the $r_{\text{physical}} = \text{const}$ hypersurfaces.

Remarks.—All three calculations consistently have the same $[+1/8\pi]$ factor, with integrals oriented along the stationary Killing vector.

It is surprising at first that the flux through the $\eta - r = \text{constant}$ hypersurfaces is zero, which indicates sharp propagation of the energy, even though the retarded solution has a tail contribution. This can be seen more directly as follows. Let us recast Eq. (54) as

$$\int_{\Sigma_\rho} d\Sigma_\alpha J_T^\alpha$$
$$= \frac{1}{32\pi} \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s \left\langle r^2(\eta - r)^2 \left(\frac{1}{\eta}\partial_\eta \chi_{ij}\right)\left(\frac{1}{\eta}\partial_\eta \chi_{ij}\right) \right\rangle,$$
(72)

where we have neglected the $1/r^2$ terms and have used $r = -H\rho\eta$.

Now in taking the η derivative, contribution of the tail term in (5) cancels out, leaving only the contribution from the sharp term:

$$\frac{1}{\eta}\partial_\eta \chi_{ij} = \frac{4}{r\eta_{ret}}\partial_\eta \int d^3x' T_{ij}(\eta - r, x').$$
(73)

B. Momentum and angular momentum fluxes

For the same three classes of hypersurfaces, we present the momentum and angular momentum fluxes. We already have the measures for these hypersurfaces as well as the currents given in (49), (50). The full fluxes, only to the leading order in r^{-1}, are given by

$$\Sigma_\rho : -\int_{-\infty}^{\infty} d\tau \int_{S^2} d^2s \rho^2 \frac{1}{H\eta}(H\rho, \hat{x}_i)J^\mu$$
(74)

$$\Sigma_{\eta_0} : \int_0^{\infty} dr \int_{S^2} d^2s \left[\frac{r^2}{|H\eta_0|^3}\right]\left[\frac{1}{H\eta_0}\right](1, \vec{0})J^\mu$$
(75)

$$\Sigma_{(\epsilon,\sigma)} : -\epsilon \int_{\lambda_1}^{\lambda_2} d\lambda \int_{S^2} d^2s \left[\frac{r^2\gamma}{|H\eta_0|^2}\right](1, \epsilon\hat{x}_i)J^\mu.$$
(76)

Momentum fluxes: The momentum current is given by

$$J_{\xi_j}^\mu = -\frac{a^{-2}}{32\pi}\left[\frac{2}{r}\frac{\eta}{\eta - r}\right]^2 \langle \mathcal{Q}_{tt}^{mn}\mathcal{Q}_{mn}^{tt}\rangle(\hat{x}_j)(1, -\hat{x}^i).$$
(77)

Dotting with the n_μ produces a rotational scalar and the average is a rotational scalar too. Then the angular integration with $\hat{x}_j$ vanishes, in all three cases. Hence, the momentum flux is zero across the three classes of hypersurfaces.

Angular momentum fluxes: Replacing $\partial_r \chi_{tt}^{mn} \approx -\partial_\eta \chi_{tt}^{mn}$, we can write the angular momentum current as

$$J_{L_j}^\mu = -\frac{a^{-2}}{16\pi}\left[\frac{2}{r}\frac{\eta}{\eta - r}\right][\epsilon_{jmn}\hat{x}^m\hat{x}^k\langle \mathcal{Q}_{tt}^{nl}\chi_{kl}\rangle](1, \hat{x}^i).$$

The fluxes then take the form

$$\Sigma_\rho : -\frac{\rho}{8\pi}\int_{-\infty}^{\infty} d\tau \int_{S^2} d^2s[\epsilon_{jmn}\hat{x}^m\hat{x}^k\langle \mathcal{Q}_{tt}^{nl}\chi_{kl}\rangle]$$
(78)

$$\Sigma_{\eta_0} : \frac{1}{8\pi H^2|\eta_0|}\int_0^{\infty} dr \frac{r}{r - \eta_0} \int_{S^2} d^2s[\epsilon_{jmn}\hat{x}^m\hat{x}^k\langle \mathcal{Q}_{tt}^{nl}\chi_{kl}\rangle]$$
(79)

$$\Sigma_{(\epsilon,\sigma)} : -\epsilon\frac{1+\epsilon}{8\pi}\int_{\lambda_1}^{\lambda_2} d\lambda(-\gamma)\frac{r\eta}{\eta - r}\int_{S^2} d^2s[\epsilon_{jmn}\hat{x}^m\hat{x}^k\langle \mathcal{Q}_{tt}^{nl}\chi_{kl}\rangle].$$
(80)

Consider the average. The function enclosed in averaging is the product of the Λ projector containing angular dependence and a function having dependence on (η, r). The averaging can then be split into averaging over a cell $\Delta\omega$ in the angular coordinates around the direction $\hat{r}$ and averaging over a cell in the (η, r) plane; see Eq. (A9). Thus, we write

$$\langle \mathcal{Q}_{tt}^{nl}\chi_{kl}\rangle(\eta, r, \hat{r}) = \left[\frac{1}{\Delta\omega}\int_{\Delta\omega} d^2s'\Lambda_{rs}^{nl}(\hat{r}')\right][\langle \mathcal{Q}^{rs}\chi_{kl}\rangle(\eta, r)]$$
(81)

$$= \Lambda_{rs}^{nl}(\hat{r})\langle \mathcal{Q}^{rs}\chi_{kl}\rangle(\eta, r).$$
(82)

The angular integration over the sphere can be done explicitly:

$$\int_{S^2} d^2s \epsilon_{jmn}\hat{x}^m\hat{x}^k\Lambda_{rs}^{nl}(\hat{r})\langle \mathcal{Q}^{rs}\chi_{kl}(\eta, r)\rangle$$
$$= \frac{8\pi}{15}\epsilon_{jmn}\langle \mathcal{Q}^{nl}\chi_l^m\rangle(\eta, r).$$
(83)

This is to be integrated over the Killing parameter τ or r or λ for the three classes of hypersurfaces. The average is now over an (η, r) cell.

This integration in the flux expressions above can be expressed in terms of the Killing parameter τ and then they all take the same form *provided* for Σ_{η_0} we consider the $\eta_0 \approx 0 \rightarrow \mathcal{J}^+$ and for the null hypersurface we choose the cosmological horizon, $\mathcal{H}^+$ ($\epsilon = +1, \eta = -r$):

Flux of the (Angular Momentum)$_j$

$$= -\frac{1}{15} \int_{-\infty}^{\infty} d\tau a(\eta(\tau)) r(\tau) \epsilon_{jmn} \langle \mathcal{Q}^{nl} \chi_l^m \rangle. \quad (84)$$

The radiation field $\mathcal{Q}^{nl}$ is given in Eq. (34) but without the *tt* label and

$$\chi_{lm} = \frac{2}{ra(\eta)} [\dddot{Q}_{lm} + 2H\ddot{Q}_{lm} + H\dot{\bar{Q}}_{lm} + 2H^2\bar{Q}_{lm}](\eta_{ret})$$
$$+ 2H^2[\dot{Q}_{lm} + H\bar{Q}_{lm}](\eta_{ret}) - 2H^2[\dot{Q}_{lm} + H\bar{Q}_{lm}](-\infty). \quad (85)$$

This flux does *not* have a finite limit to $\mathcal{J}^+$ due to the tail term in χ_l^m and does *not* match with the flux given by [2]. It is finite along the $\mathcal{H}^+$ though. It does *not* match with the correct angular momentum flux in the flat space limit as well and it is well known [9,13] that the Isaacson effective stress tensor does not suffice to capture the flux of angular momentum. The sharp propagation property still holds in the sense that the flux across the outgoing null hypersurface is zero.

C. Extending from tt to TT

We have used the algebraic tt projection on the approximated, retarded solution. How would the results change if we were to use the TT decomposition of the exact solution prior to the $|\vec{x}'|/|\vec{x}| \ll 1$ approximation? For this we note a few points.

It is easy to see that the TT part of the retarded solution is given by [2]

$$\mathcal{X}_{ij}^{TT}(\eta, x)$$
$$= 4 \int d^3x' \frac{\eta}{|x - x'|(\eta - |x - x'|)} T_{ij}^{TT'}(\eta', x')\big|_{\eta'=\eta-|x-x'|}$$
$$+ 4 \int d^3x' \int_{-\infty}^{\eta-|x-x'|} d\eta' \frac{T_{ij}^{TT'}(\eta', x')}{\eta'^2}, \quad (86)$$

where the TT' refers to the *second* argument of the stress tensor. This follows by checking that the divergence, ∂_x^i, of the right-hand side converts into the divergence, $\partial_{x'}^i$, on the second argument of the stress tensor. For this relation, it is important to have the exact $|x - x'|$ dependence and for the source to have compact support. The TT part of the approximated solution cannot be similarly expressed in terms of the TT part of the source stress tensor.

We can now consider the solution (86) for $|x| \gg |x'|$ and replace $|x - x'| \approx r$, which simplifies the source integral. We denote this approximated expression as χ_{ij}^{TT}. This satisfies the transversality condition to $o(r^{-1})$ only.[3] Furthermore, since the transverse, traceless part of the stress tensor drops out of its conservation equation, we cannot directly express $\int_{\text{source}} T_{ij}^{TT'}$ in terms of correspondingly defined moments. Nevertheless, we do get

$$\partial_\eta \chi_{ij}^{TT}(\eta, x) = 4\frac{\eta}{r(\eta - r)} \partial_\eta \mathcal{M}_{ij}^{TT},$$
$$\mathcal{M}_{ij}^{TT}(\eta - r) := \int d^3x' T_{ij}^{TT'}(\eta - r, x') \quad (87)$$

$$\partial_m \chi_{ij}^{TT}(\eta, x) = 4\frac{\hat{x}_m}{r}\left(\frac{\eta}{\eta - r}\partial_r\mathcal{M}_{ij}^{TT} - \frac{\mathcal{M}_{ij}^{TT}}{r}\right)$$
$$= -\hat{x}_m \partial_\eta \chi_{ij}^{TT} - 4\frac{\hat{x}_m}{r^2}\mathcal{M}_{ij}^{TT} \quad (88)$$

$$\therefore \partial_r \chi_{ij}^{TT} = -\partial_\eta \chi_{ij}^{TT} - 4\frac{\mathcal{M}_{ij}^{TT}}{r^2}. \quad (89)$$

Equation (89) has the same form as Eq. (13). Equation (87) has the same form as Eq. (58), which introduced the radiation field $\mathcal{Q}_{ij}^{tt}$. We can thus introduce a new radiation field, $\mathcal{Q}_{ij}^{TT} := 2\partial_\eta\mathcal{M}_{ij}^{TT}$. With this, the form of the expressions for fluxes will remain the same with $\mathcal{Q}_{ij}^{tt} \rightarrow \mathcal{Q}_{ij}^{TT}$. Note that unlike $\mathcal{Q}_{ij}^{tt}$, $\mathcal{Q}_{ij}^{TT}$ does *not* have a simple relation to the source moments defined earlier. Nevertheless, it shares an important property with $\mathcal{Q}_{ij}^{tt}$; namely, it too is a function of $\eta - r$ alone. This enables the space-time averaging to be reduced to averaging over $\rho = \text{constant}$ hypersurfaces, as shown in Eq. (A13):

$$\langle \partial_\eta \chi_{mn}^{TT} \partial_\eta \chi_{TT}^{mn}\rangle(t, r, \hat{r}) = 4\frac{a^2(\bar{t}_0)}{\rho_0^2}\langle \mathcal{Q}_{ij}^{TT} \mathcal{Q}_{TT}^{ij}\rangle(\bar{t}_0, \hat{r}). \quad (90)$$

In the next section we restrict to the energy fluxes and see two applications of the conservation equation and the sharp propagation property.

IV. IMPLICATIONS OF CONSERVATION EQUATION AND SHARP PROPAGATION

In the previous subsection, we assembled fluxes through various hypersurfaces, all having the topology $\Delta \times S^2$. We considered Δ to be a finite interval and also the cases with $\Delta = \mathbb{R}$. The relevant hypersurfaces have $\rho = \text{constant}$. In all cases, the energy flux integral had the form

---

[3]Extracting the TT part and making the approximation for $|\vec{x}| \gg |\vec{x}'|$ do not commute, i.e., $[(\mathcal{X}_{ij})_{\text{approx}}]^{TT} \neq ([\mathcal{X}_{ij}]^{TT})_{\text{approx}}$. This is so because the ∂^j of the lhs is always zero by definition while that of the rhs is nonzero in general. We are using the rhs.

$$\mathcal{F}(a,b) := \int_a^b d\tau \int_{S^2} d^2 s \left[\frac{1}{8\pi} \right] \langle \mathcal{Q}_{mn}^{tt} \mathcal{Q}_{tt}^{mn} \rangle$$

$$=: \int_a^b d\tau \langle F \rangle(\tau). \tag{91}$$

As shown in the Appendix, Eq. (A12), the angular brackets denote averaging over τ intervals and a trivial averaging over the angular intervals. Since the angular average is trivial, we have taken the angular integration across the averaging and denoted the integration over the sphere by $\langle F \rangle(\tau)$. Using the mean value theorem, we write

$$\int_a^b d\tau \langle F \rangle(\tau) = \langle F \rangle(c)(b-a)$$

$$= \int_{c-\delta}^{c+\delta} d\tau F \frac{(b-a)}{2\delta}, \qquad c \in (a,b). \tag{92}$$

Let us choose (a,b) to be an averaging interval, i.e., $(b-a) = 2\delta$. Recall that the averaged quantities are slowly varying; i.e., $\langle F \rangle$ is varying only over the scale $L \gg 2\delta$ and is thus essentially constant over the averaging interval. Therefore, we can choose $c = (a+b)/2$, possibly making a small error. But then the right-hand side of the last equality in the above equation becomes $\int_a^b d\tau F(\tau)$. In effect, for the integral over an averaging interval, we can drop the angular brackets in Eq. (91).

For $a \ll 0$, $b \gg 0$, the τ integral can be replaced by a sum with each subinterval $[a_k, b_k]$ being an averaging interval. Using the above argument, we can write

$$\mathcal{F}(a,b) \approx \sum_k \int_{a_k}^{b_k} d\tau \int_{S^2} d^2 s \left[\frac{1}{8\pi} \right] \mathcal{Q}_{mn}^{tt} \mathcal{Q}_{tt}^{mn}. \tag{93}$$

However, the averaging τ intervals cannot be made arbitrarily finer and the Riemann sum cannot be taken to the integral. Hence, the flux integral over an averaged integrand matches with the flux integral over an *unaveraged* integrand only at a coarse grained level. The same arguments also hold for $\mathcal{Q}_{mn}^{tt} \to \mathcal{Q}_{mn}^{TT}$ and *then* the fluxes defined using the averaged stress tensor match with the expressions (36) at a coarse grained level.

By judicious choices of hypersurfaces comprising the boundary $\partial \mathcal{V}$ of a space-time region $\mathcal{V}$, we can relate different fluxes using the conservation equation (51). The sharp propagation of energy comes in very handy. We note two of its implications.

(1) The flux across two hypersurfaces Σ_{η_1} and Σ_{η_2} *cannot* be equal; see the right panel of Fig. 2.

Let $\eta_2 > \eta_1$. Let Σ_{η_1} meet the $r = 0$ line at A_1. Let the outgoing null hypersurface through A_1 intersect the Σ_{η_2} in S^2 at B_1 with the radial coordinate being r_1. The three hypersurfaces Σ_{η_1}, the outgoing null hypersurface, and the hypersurface Σ_{η_2} bounded by the sphere at B_1 enclose a space-time region, A_1BB_1.

By the conservation equation (51), the sum of the fluxes through these bounding hypersurfaces must vanish. But the flux through the outgoing null hypersurface vanishes as shown before. Hence the fluxes through Σ_{η_1} and the partial hypersurface Σ_{η_2} between B_1 and B must be equal. However, this leaves the contribution of the flux through the remaining portion of the Σ_{η_2} hypersurface between B_2 and B_1. Hence the result. Alternatively, one can also see this explicitly by writing the full flux through the two hypersurfaces using the expression given in Eq. (66) and matching the integrands along the outgoing null hypersurface. Evidently, the full flux through $\Sigma_{\eta \neq 0}$ is also not equal to that through $\mathcal{J}^+$. Physically this is understandable since the hypersurface at a later value of η receives energy emitted *after* the earlier value of η. The null infinity of course records *all* the energy emitted by the source and so does the cosmological horizon. We also conclude that the total flux at $\mathcal{J}^+$ computed by Ashtekar *et al.*, as given in Eq. (36), matches (at a coarse grained level) with that given in Eq. (66) (with $\mathcal{Q} \to \mathcal{R}$) *only* for $\eta = 0$. Note that unlike the spatial slices Σ_η, all hypersurfaces $\Sigma_{\rho > 0}$ intercept all the emitted energy.

(2) The sharp propagation of energy can also be used to infer the instantaneous emitted power. Consider two outgoing null hypersurfaces intersecting the cosmological horizon in spheres with radii $r(\tau)$ and $r'(\tau')$. The same hypersurfaces intersect the null infinity at corresponding spheres at $R(\tau)$ and $R'(\tau')$; see the left panel in Fig. 2. For $\tau' > \tau$, we have $r'(\tau') < r(\tau)$ and $R'(\tau') < R(\tau)$. By the conservation equation and sharp propagation, the flux integral over the portion bounded by the spheres R, R' on $\mathcal{J}^+$ and the flux integral over the portion bounded by the spheres $r(\tau), r'(\tau')$ on $\mathcal{H}^+$ are equal. Taking $\tau' = \tau + \delta\tau$, the integral becomes $\delta\tau \times$ the integral over the sphere at $r(\tau)$. The emitted power is then defined by dividing the flux integral by $\delta\tau$ and taking the limit. Thus we get the instantaneous power as

$$\mathcal{P}(\tau) := \lim_{\delta\tau \to 0} \frac{\mathcal{F}(\tau + \delta\tau, \tau)}{\delta\tau} = \frac{1}{8\pi} \int_{S^2} d^2 s \langle \mathcal{Q}_{ij}^{TT} \mathcal{Q}_{TT}^{ij} \rangle. \tag{94}$$

This is manifestly positive.

This is very similar to the definition given by Ashtekar *et al.* [2] in the form of Eq. (38) *except that* the integrand is an average over τ and angular windows. The power is usually averaged over a few periods. If this is done to the power expression in [2], it will match with the above expression, again at a coarse grained level.

The upshot is that the quadrupole power defined above is gauge invariant and can be computed at the cosmological horizon.

V. DISCUSSION AND SUMMARY

We have dealt with two aspects, namely, the role of the cosmological horizon and the use of the ripple stress tensor in the limited context of rapidly changing, distant sources.

A question regarding the validity of the short wavelength approximation near $\mathcal{J}^+$ arises due to the understanding that the physical wavelength will diverge near the future null infinity thanks to the scale factor $a(t)$. Let us recall that the background plus ripple decomposition is premised over the expectation: $\partial_\alpha \bar{g}_{\mu\nu} \sim \bar{g}_{\mu\nu}/L$ and $\partial_\alpha h_{\mu\nu} \sim h_{\mu\nu}/\lambda$. In the cosmological chart, the nonzero coordinate derivatives of the background are $\partial_t \bar{g}_{ij} = 2H\bar{g}_{ij} \sim \bar{g}_{ij}/L$. For the retarded solution we have

$$\frac{\partial_t h_{ij}}{h_{ij}} = \partial_t[\ell n(a^2(t)\chi_{ij})] = 2H + \partial_t \eta \partial_\eta \ell n(\chi_{ij})$$

$$= 2H + \frac{\partial_\eta \ell n(\chi_{ij})}{a(t)} \sim \frac{1}{L} + \frac{1}{a(t)\lambda}, \tag{95}$$

$$\frac{\partial_k h_{ij}}{h_{ij}} = \partial_k \ell n(\chi_{ij}) = \hat{r}_k \partial_r \ell n(\chi_{ij})$$

$$\approx -\hat{r}_k \partial_\eta \ell n(\chi_{ij}) \sim \frac{\hat{r}_k}{\lambda}. \tag{96}$$

The first equation shows that the t derivative of the perturbation does *not* satisfy the premise near $\mathcal{J}^+$ thanks to the presence of the scale factor. The second equation however does *not* have the scale factor and the ripple indeed has a short scale of spatial variation. Interestingly, in the calculation of the fluxes, spatial components of the ripple stress tensor (and hence the spatial derivatives of the perturbation) do contribute since all Killing vectors are spacelike near $\mathcal{J}^+$ and the short wavelength approximation can justifiably be used.

As noted in the Introduction, the cosmological horizon is unambiguously defined for a spatially compact source. This follows because worldlines with finite physical separation at every η must converge to i^+, the point A of Fig. 1. If Δ denotes the physical radial distance corresponding to the radial coordinate difference δ, then $\Delta^2 = \frac{\delta^2}{H^2\eta^2}$. To maintain Δ^2 to be finite as $\eta \to 0_-$, we must have $\delta^2 \sim \alpha^2\eta^2 + 0(|\eta|^3)$ near i^+. This identifies δ with $-\alpha\eta$ or $\alpha = H\rho$. Thus, the worldlines approach i^+ along the $\rho = $ constant hypersurfaces. The cosmological horizon is then the past light cone of i^+. The same argument also shows that any observer who remains at finite physical distance away from the source must remain confined within the cosmological horizon. Furthermore, neither any such observer nor the source has any access to energy/momentum which has

crossed the horizon. Hence the cosmological horizon does share physically relevant properties with the future infinity. Incidentally, any future directed causal curve reaching $\mathcal{J}^+$ also registers on $\mathcal{H}^+$.

Further support for the role of the cosmological horizon as future null infinity comes from the computations of the energy momentum fluxes. For these, we employed the effective ripple stress tensor and showed that the fluxes defined at $\mathcal{J}^+$ match with those defined at $\mathcal{H}^+$. Furthermore, these fluxes also matched (at a coarse grained level) with the energy momentum fluxes defined by the more geometric methods of the covariant phase space framework. This provides further support to the utility of the ripple stress tensor. The quadrupole power too matches likewise. The ripple stress tensor, although limited to short wavelength regimes (which cover most common sources), provides a convenient picture of energy momentum flows much like the flows for matter. There is a shortcoming of the ripple stress tensor—it does not capture the angular momentum flux correctly. A clearer understanding of this failure is lacking at present.

It should be noted that the definition of fluxes is not necessarily *unique*. Apart from a definition being well defined, its correctness should be tested in conjunction with the definition of the Bondi-type quantities having a loss formula relating to flux. Recent work within a Bondi-type framework may be seen in [14–16]. The observation that the cosmological horizon is a Killing horizon and hence an isolated horizon should be helpful in this regard.

APPENDIX: AN AVERAGING PROCEDURE

In the main body we specified an averaging procedure by stipulating its properties, namely, (i) the average of the odd powers of h vanishes and (ii) the average of the space-time divergence is subleading. This was then used to simplify the expression for the ripple stress tensor. An averaging procedure satisfying these properties was indeed given by Isaacson [8]. We will use the same one and give more explicit details in the present context.

Isaacson defines the space-time average of a tensor by using the parallel propagator bitensor, $g_\mu^{\mu'}(x, x')$, as

$$\langle X_{\mu\nu}\rangle(x) := \frac{\int_{\text{cell}} d^4x' \sqrt{|g(x')|} g_\mu^{\mu'}(x, x') g_\nu^{\nu'}(x, x') X_{\mu'\nu'}(x')}{\int_{\text{cell}} d^4x' \sqrt{|g(x')|}}. \tag{A1}$$

In the present context, we need the average of the stress tensor for ripples due to a retarded solution which has a certain explicit form. We will use this information to choose

suitable integration variables and a corresponding "cell" denoting the averaging region. Because of this, we have not used any weighting function as given by Isaacson [8].

To keep track of the powers of H, we begin by going from the conformal chart (η, x^i) to the cosmological chart (t, x^i), $\eta := -H^{-1}e^{-Ht}$, with the spatial coordinates unchanged. In the cosmological chart,

$$\text{Metric: } ds^2 = -dt^2 + a^2(t)(\delta_{ij}dx^i dx^j), \qquad a(t) := e^{Ht}$$
$$\text{(A2)}$$

$$\text{Connection: } \Gamma^t{}_{tt} = 0, \qquad \Gamma^t{}_{tj} = 0, \qquad \Gamma^t{}_{ij} = Ha^2(t)\delta_{ij}$$
$$\Gamma^i{}_{tt} = 0, \qquad \Gamma^i{}_{ij} = H\delta^i{}_j, \qquad \Gamma^i{}_{jk} = 0. \qquad \text{(A3)}$$

The parallel propagator is computed in terms of the parallel transport of an arbitrary cotetrad (or tetrad): $g_\mu{}^{\mu'}(x, x') := e_\mu{}^a(x)e_e{}^{\mu'}(x')$. The averaging region is small enough that for a cell around a point P with coordinates x^α, there is a unique geodesic to points P' with coordinates x'^α. The parallel transported cotetrad is obtained using Taylor expansions of the cotetrad, the affine connection, and the coordinates along the geodesic, in terms of its affine parameter, and eliminating the affine parameter afterwards in favor of the coordinate differences $\Delta x^\alpha := x'^\alpha - x^\alpha$. Details may be seen in Appendix B of [3]. There is a slight difference from [3], since that calculation was given in the context of Fermi normal coordinates where the connection is already of order H^2, while in the cosmological chart, the connection is of order H. The final expressions are

$$e_\mu{}^a(x') = \hat{e}_\lambda{}^a\Big[\delta_\mu{}^\lambda + \hat{\Gamma}_{\mu\alpha}{}^\lambda \Delta x^\alpha$$
$$+ \frac{1}{2}(\widehat{\partial_\rho\Gamma}_{\mu\sigma}{}^\lambda + \hat{\Gamma}_{\mu\sigma}{}^\alpha\hat{\Gamma}_{\alpha\rho}{}^\lambda)\Delta x^\rho \Delta x^\sigma\Big] \qquad \text{(A4)}$$

$$g_\mu{}^{\mu'}(x, x') = \delta_\mu{}^{\mu'} - \hat{\Gamma}_{\mu\alpha}{}^{\mu'}\Delta x^\alpha$$
$$- \frac{1}{2}(\widehat{\partial_\rho\Gamma}_{\mu\sigma}{}^{\mu'} - \hat{\Gamma}_{\alpha\rho}{}^{\mu'}\hat{\Gamma}_{\sigma\mu}{}^\alpha)\Delta x^\rho \Delta x^\sigma. \qquad \text{(A5)}$$

In the above, the hatted quantities are evaluated at x.

The connection dependent terms are linear and quadratic in $H\Delta x$. Although the coordinate differences are much larger than the length scale λ, they are much smaller than H^{-1}. Hence, these terms can be neglected and effectively the parallel propagator reduces to just the Kronecker delta. For purposes of illustration of averaging, this suffices. It remains to integrate the $X_{\mu'\nu'}$ over the cell and as noted in the main text in the paragraph below Eq. (60), the components of the ripple stress tensor are essentially determined in terms of $\partial_\eta\chi^{tt}_{ij} = 2\frac{\eta}{r}\frac{Q^{tt}_{ij}(\eta - r)}{\eta - r}$ or alternatively in terms of $\partial_\eta\chi^{TT}_{ij} = 2\frac{\eta}{r}\frac{R^{TT}_{ij}(\eta - r)}{\eta - r}$.

The angular dependence is introduced due to the tt part, e.g., as is explicit in the $\Lambda^{kl}_{ij}(\hat{r})$ projector. The (η, r) dependence has a convenient factorized form. It is thus natural to change the integration variables from (η, r) to $(\bar{t}, \rho)$, where $\bar{t}$ is the retarded synchronous time defined through $\eta - r := -H^{-1}e^{H\bar{t}}$, and $H\rho := -\frac{r}{\eta}$ defines ρ. For definiteness, consider the average,

$$\langle\partial_\eta\chi^{tt}_{mn}\partial_\eta\chi^{mn}_{tt}\rangle(t, r, \hat{r})$$
$$:= \frac{\int_{\text{cell}} dt dr r^2 d^2 s a^3(t)\partial_\eta\chi^{tt}_{mn}(\bar{t})\partial_\eta\chi^{mn}_{tt}(\bar{t})}{\int_{\text{cell}} dt dr r^2 d^2 s a^3(t)}. \qquad \text{(A6)}$$

Here $\hat{r}$ denotes a point on S^2 (a spatial direction). We will specify the cell after changing over to $(\bar{t}, \rho, \hat{r})$.

From the definitions, we arrive at the coordinate transformations,

$$\eta(\bar{t}, \rho) = -\frac{1}{H(1 + H\rho)}e^{-H\bar{t}}, \qquad r(\bar{t}, \rho) = \frac{\rho}{1 + H\rho}e^{-H\bar{t}} \Rightarrow$$
$$a(t) = a(\bar{t})(1 + H\rho), \quad \text{with} \quad a(t) := e^{Ht}. \qquad \text{(A7)}$$

The Jacobian of transformation is $\frac{\partial(t, r)}{\partial(\bar{t}, \rho)} = \{a(\bar{t})(1 + H\rho)\}^{-1}$. We choose the cell so that $\bar{t} \in [\bar{t}_0 - \delta, \bar{t}_0 + \delta]$, $\rho \in [\rho_0 - \Delta, \rho_0 + \Delta]$, and $\hat{r} \in \Delta\omega$. The coordinate windows δ, Δ, and $\sqrt{r^2\Delta\omega}$ are several times the ripple scale, while $(\bar{t}_0, \rho_0)$ are the transforms of (t, r). In terms of these choices, the average becomes

$$\langle\partial_\eta\chi^{tt}_{mn}\partial_\eta\chi^{mn}_{tt}\rangle(t, r, \hat{r})$$
$$:= \frac{\int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho\rho^2 \int_{\Delta\omega} d^2 s[4\frac{a^2(\bar{t})}{\rho^2}Q^{tt}_{mn}(\bar{t})Q^{mn}_{tt}(\bar{t})]}{\int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho\rho^2 \int_{\Delta\omega} d^2 s}.$$
$$\text{(A8)}$$

Consider the angular integration. The angular dependence arises in taking the tt part of the solution $\chi_{ij}(\eta, r)$. For illustration purposes, consider r to be sufficiently large so that we can use the $\Lambda^{kl}_{ij}(\hat{r})$ projector, giving $\partial_\eta\chi^{tt}_{ij}\partial_\eta\chi^{ij}_{tt} \sim \Lambda^{kl}_{ij}\partial_\eta\chi^{ij}\partial_\eta\chi_{kl}$. For large r, the angular coordinate windows are $\sim\lambda/r \ll 1$. Using the mean value theorem in the angular integration in the numerator, we get

$$\frac{\int_{\Delta\omega} d^2 s(\hat{r}')\Lambda^{kl}_{ij}(\hat{r}')}{\int_{\Delta\omega} d^2 s(\hat{r}')} \approx \Lambda^{kl}_{ij}(\hat{r}). \qquad \text{(A9)}$$

In effect, the Λ projector comes out of the averaging and the angular average trivializes. Of the remaining integrations, the ρ integration can be done explicitly and is independent of Δ to the leading order in Δ/ρ_0. Thus, in the numerator of (A8) we get

$$\int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho 4\, a^2(\bar{t}) \mathcal{Q}^{ij}(\bar{t}) \mathcal{Q}_{kl}(\bar{t})$$

$$\approx 8\Delta \int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} a^2(\bar{t}) \mathcal{Q}^{ij}(\bar{t}) \mathcal{Q}_{kl}(\bar{t})$$

$$= 8\Delta \int_{-\delta}^{\delta} dy a^2(\bar{t}_0+y) \mathcal{Q}^{ij} \mathcal{Q}_{kl}(\bar{t}_0+y)$$

$$= 8\Delta a^2(\bar{t}_0) \int_{-\delta}^{\delta} dy a^2(y) \mathcal{Q}^{ij} \mathcal{Q}_{kl}(\bar{t}_0+y)$$

$$:= (8\Delta) a^2(\bar{t}_0)(2\delta)\langle \mathcal{Q}^{ij} \mathcal{Q}_{kl}\rangle_{\bar{t}}(\bar{t}_0). \tag{A10}$$

In the third line, we have used a^2 being an exponential function. In the last line we have defined the average over the retarded time around $\bar{t}_0$ and put the suffix on the angular bracket as a reminder.

The $\langle \rangle_{\bar{t}}$ averaging has the extra factor of $a^2(y)$. However, over the integration domain $(-\delta, \delta)$, we can approximate $a^2(y) \approx 1 + 2Hy + \cdots$ and neglect $o(Hy)$ terms since $H\delta \sim k\lambda/L \sim k\epsilon \ll 1$. The extra factor thus introduces a small deviation from the usual averaging without the extra factor. We neglect it henceforth and the reminder suffix $\bar{t}$ is also suppressed.

In the denominator we get

$$\int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho \rho^2 \approx 2\rho_0^2 \Delta \int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} = (2\Delta)(2\delta)\rho_0^2. \tag{A11}$$

Combining Eqs. (A8)–(A11), we get

$$\langle \partial_\eta \chi_{mn}^{tt} \partial_\eta \chi_{tt}^{mn}\rangle(t,r,\hat{r}) = 4\frac{a^2(\bar{t}_0)}{\rho_0^2}\langle \mathcal{Q}_{ij}^{tt} \mathcal{Q}_{tt}^{ij}\rangle(\bar{t}_0,\hat{r}). \tag{A12}$$

In the last equation, we have combined the averaging over retarded time and the (trivial) angular average. We have also inserted the Λ projector. The averaging over

a space-time cell has been reduced to averaging over a three-dimensional cell on a $\rho =$ constant hypersurface. The prefactor on the right-hand side of the above equation exactly equals the last square bracket in the first line of Eq. (60). In effect, the $(\frac{2\eta}{r(\eta-r)})^2$ factor has come out of the averaging.

We can also reduce the space-time average to a hypersurface average for $\partial_\eta \chi_{ij}^{TT} \partial_\eta \chi_{TT}^{ij}$. Following the same steps as from Eq. (A6) onwards, we will arrive at Eq. (A8) with $\mathcal{Q}_{mn}^{tt} \to \mathcal{Q}_{mn}^{TT}$. We cannot do the angular averaging as before, but we do not need to. Crucially, the ρ dependence has factored out exactly as before and the average over ρ gives ρ_0^{-2} as before. The $\bar{t}$ averaging too gives $a^2(\bar{t}_0)$ and we get the desired result,

$$\langle \partial_\eta \chi_{mn}^{TT} \partial_\eta \chi_{TT}^{mn}\rangle(t,r,\hat{r}) = 4\frac{a^2(\bar{t}_0)}{\rho_0^2}\langle \mathcal{Q}_{ij}^{TT} \mathcal{Q}_{TT}^{ij}\rangle(\bar{t}_0,\hat{r}). \tag{A13}$$

We can relate the averaging over the retarded time $\bar{t}$ to the averaging over the Killing time τ along the $\rho = \rho_0$ curve. From the coordinate transformation, we have $\eta - r = -H^{-1}e^{-H\bar{t}}$, while along the $\rho = \rho_0$ Killing trajectory, $\eta - r = (\eta_* - r_*)e^{-H\tau} = -(H^{-1}e^{-H\bar{t}_*})e^{-H\tau}$. Hence, $\bar{t} = \tau + \bar{t}_*$ and the temporal averaging is related to averaging over a Killing time. Note that since the averaging cell is bounded by two hypersurfaces of constant retarded times, the temporal averaging may be evaluated along the source worldline, $r = 0$, or along the Killing trajectory on $\mathcal{J}^+$.

We also have mixed and spatial components of the ripple stress tensor. These involve $\partial_i \chi_{mn}^{tt} \approx \hat{x}_i \partial_r \chi_{mn}^{tt} \approx -\hat{x}_i \partial_\eta \chi_{mn}^{tt}$. While taking the average, the $\hat{x}_i$ can be taken out of the average since the angular coordinate windows are of very small size $\sim \lambda/r$. This allows us to take $\hat{x}^i$ across the angular averages and replace all components of the ripple stress tensor by $t_{\eta\eta}$ in the conformal chart or by t_{00} in the cosmological chart.

---

[1] H. J. de Vega, J. Ramirez, and N. Sanchez, Generation of gravitational waves by generic sources in de Sitter spacetime, Phys. Rev. D **60,** 044007 (1999).

[2] A. Ashtekar, B. Bonga, and A. Kesavan, Asymptotics with a positive cosmological constant: III. The quadrupole formula, Phys. Rev. D **92,** 10432 (2015).

[3] G. Date and S. J. Hoque, Gravitational waves from compact sources in de Sitter background, Phys. Rev. D **94,** 064039 (2016).

[4] R. Sachs, Gravitational waves in general relativity VI. The outgoing radiation condition, Proc. R. Soc. A **264,** 309 (1961).

[5] R. Penrose, Zero rest-mass fields including gravitation: Asymptotic behaviour, Proc. R. Soc. A **284,** 159 (1965).

[6] A. Ashtekar, L. Bombelli, and O. Reula, Covariant phase space of asymptotically flat gravitational fields, in *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, edited by M. Francaviglia and D. Holm (North Holland, Amsterdam, 1991).

[7] A. Ashtekar, B. Bonga, and A. Kesavan, Asymptotics with a positive cosmological constant: II. Linear fields on de Sitter space-time, Phys. Rev. D **92,** 044011 (2015).

[8] R. A. Isaacson, Gravitational radiation in the limit of high frequency I: The linear approximation and geometrical optics, Phys. Rev. **166,** 1263 (1968); Gravitational radiation in the limit of high frequency II: Non-linear terms and the effective stress tensor, Phys. Rev. **166,** 1272 (1968).

[9] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (MacMillan, London, 1973).

[10] A. Ashtekar, B. Bonga, and A. Kesavan, Asymptotics with a positive cosmological constant: I. Basic framework, Classical Quantum Gravity **32**, 025004 (2015).

[11] L. C. Stein and N. Yunes, Effective gravitational wave stress-energy tensor in alternative theories of gravity, Phys. Rev. D **83**, 064038 (2011).

[12] Y. Choquet-Bruhat, Construction de solutions radiatives approchees des equations d'Einstein, Commun. Math. Phys. **12**, 16 (1969).

[13] E. Poisson and C. M. Will, *Gravity Newtonian, Post-Newtonian, Relativistic* (Cambridge Univ. Press, Cambridge, England, 2014).

[14] N. T. Bishop, Gravitational waves in a de Sitter universe, Phys. Rev. D **93**, 044025 (2016).

[15] P. T. Chruściel and L. Ifsits, The cosmological constant and the energy of gravitational radiation, Phys. Rev. D **93**, 124075 (2016).

[16] V.-L. Saw, Mass-loss of an isolated gravitating system due to energy carried away by gravitational waves with a cosmological constant, Phys. Rev. D **94**, 104004 (2016).