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Thermal stability analysis of nonlinearly charged asymptotic AdS black hole solutions

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In this paper, the four-dimensional nonlinearly charged black hole solutions have been considered in the presence of the power Maxwell invariant electrodynamics. Two new classes of anti–de Sitter (AdS) black hole solutions have been introduced according to different amounts of the parameters in the nonlinear theory of electrodynamics. The conserved and thermodynamical quantities of either of the black hole classes have been calculated from geometrical and thermodynamical approaches, separately. It has been shown that the first law of black hole thermodynamics is satisfied for either of the AdS black hole solutions we just obtained. Through the canonical and grand canonical ensemble methods, the black hole thermal stability or phase transitions have been analyzed by considering the heat capacities with the fixed black hole charge and fixed electric potential, respectively. It has been found that the new AdS black holes are stable if some simple conditions are satisfied.

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I. INTRODUCTION

It is well known that there are many good reasons to consider Maxwell's electromagnetic theory as the successful theory of classical electrodynamics. But regarding the appearance of the infinite self-energy for the pointlike charges, it seems that this theory may be incomplete. It is for this reason that in recent years many authors are interested in generalizing the standard Maxwell electromagnetic theory. The initial idea to modify Maxwell's electromagnetic theory was apparently outlined, in order to overcome the problem of infinite self-energy of the point charges, by Born and Infeld [1]. Along the same line, the logarithmic, exponential, and other models of nonlinear electrodynamics have been introduced by some other authors [2–7]. Among alternative proposed models, the so-called extended theory of electrodynamics or the nonlinear electromagnetic actions, the logarithmic, exponential, quadratic, and power-law nonlinear theories of electrodynamics have provided interesting results [6–10]. These models are based on the actions that are constructed by nonlinear combinations of the Maxwell invariant $\mathcal{F} = F^{\alpha\beta}F_{\alpha\beta}$. Models of nonlinear electrodynamics can be considered as the effective models with the quantum corrections taken into account. Maxwell's theory of electrodynamics is a special case of the nonlinear theories of electrodynamics in the weak fields limit [7,9,11]. In the case of the high strength electromagnetic fields, when the self-interaction of the photons is important, the linear model of electromagnetic theory should be generalized to nonlinear models [5].

Furthermore, one of the outstanding achievements in the context of geometrical physics is that black holes are thermodynamical systems with temperature proportional to the surface gravity. According to the Hawking-Bekenstein entropy-area law, they have entropy proportional to the horizon surface area [12–14]. Although modification of the usual electrodynamics theory itself originates from the quest of establishing a new theory of electrodynamics that is able to produce a finite amount of self-energy for pointlike charges, the modified models of electrodynamics have extensively been used for characterizing the physical and thermodynamical properties of the various kinds of charged black holes [15]. If black holes have large amounts of electric charge, they can create a strong enough electric field. In this case, the nonlinear electrodynamics can lead to a more realistic physical description. Now, the nonlinear electrodynamics has been the subject of many interesting works, and a lot of papers have appeared in which the usual theory of electrodynamics is modified at the framework of gravitational physics [16].

The main objective here is to provide a detailed analysis of the thermodynamical properties of new four-dimensional electrically charged anti–de Sitter (AdS) black holes in the presence of a power-law Maxwell invariant. The motivation for studying black holes with a negative cosmological constant arises from the correspondence between the gravitating fields in an AdS spacetime and the conformal field theory living on the boundary of the AdS spacetime. It was argued that the thermodynamics of black holes in AdS spaces can be identified with that of a certain dual conformal field theory, the AdS/CFT correspondence [17].

This paper is organized based on the following order. In Sec. II, we obtain the gravitational and nonlinear electromagnetic field equations by varying the related four-dimensional action with respect to the metric and the electromagnetic potential, respectively. Making use of the power Maxwell invariant, as a model of nonlinear

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electrodynamics, we solve the field equations in a static spherically symmetric geometry. We consider the properties of the solutions and introduce two new classes of asymptotically AdS black hole solutions, according to the proper ranges of the allowed parameters. Section III is devoted to thermodynamics and stability analysis of the various black hole solutions we just obtained. We calculate the temperature, entropy, electric potential, conserved mass, and charge of the new asymptotic AdS black holes from both the geometrical and the thermodynamical approaches separately. Also we show that the first law of black hole thermodynamics is satisfied for either of the AdS black hole solutions. Finally, making use of the canonical and grand canonical ensemble methods, we study the thermal stability or phase transition of either of the new AdS black hole classes. We show that the AdS black hole solutions, introduced here, are stable if some simple conditions are satisfied. We summarize and discuss the results in Sec. IV.

II. SOLUTION TO THE FIELD EQUATIONS

Let us start with the following action for a nonlinearly charged four-dimensional black hole in the presence of the cosmological constant:

$$I = -\frac{1}{16\pi}\int \sqrt{-g}d^4x[R - 2\Lambda + \mathcal{L}(\mathcal{F})]. \tag{2.1}$$

Here, R is the Ricci scalar, and $\Lambda = -3\ell^{-2}$ is the AdS cosmological constant. $\mathcal{L}(\mathcal{F})$ denotes the electromagnetic Lagrangian density as a function of Maxwell's invariant $\mathcal{F} = F^{\mu\nu}F_{\mu\nu}$. It is chosen as a power law in the following form:

$$\mathcal{L}(\mathcal{F}) = (-\mathcal{F})^p, \tag{2.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the electromagnetic potential. By varying action (2.1) with respect to the gravitational field we get Einstein's field equations as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{3}{\ell^2}g_{\mu\nu} = \frac{1}{2}g_{\mu\nu}(-\mathcal{F})^p + 2p(-\mathcal{F})^{p-1}F_{\mu\alpha}F^{\alpha}_{\nu}. \tag{2.3}$$

Also, varying action (2.1) with respect to the electromagnetic field yields

$$\nabla_\mu[\mathcal{L}'(\mathcal{F})F^{\mu\nu}] = 0 \quad \text{or equivalently}$$
$$\partial_\mu[\sqrt{-g}\mathcal{L}'(\mathcal{F})F^{\mu\nu}] = 0, \tag{2.4}$$

where the prime denotes the derivative with respect to the argument. The only nonvanishing component of the electromagnetic field is that of F_{tr}. Assuming it is a function of r, that is $F_{tr} = -E(r) = h'(r)$, we have

$$\mathcal{F} = -2(F_{tr}(r))^2 = -2(h'(r))^2. \tag{2.5}$$

We would like to solve the gravitational and electromagnetic field equations, (2.3) and (2.4), in a one function four-dimensional spherically symmetric geometry. It can be written in the following form:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.6}$$

Now, making use of Eqs. (2.5) and (2.6) in the electromagnetic field equations (2.4) we have

$$r(h'(r))^{2p-2}[2h'(r) + (2p-1)rh''(r)] = 0, \qquad p \neq \frac{1}{2}. \tag{2.7}$$

The solution to the differential equation (2.7) can be obtained as

$$h(r) = \begin{cases} -q\ln\left(\frac{r}{\ell}\right) & \text{for } p = \frac{3}{2}, \\ -qr^{\frac{2p-3}{2p-1}} & \text{for } p \neq \frac{1}{2}, \frac{3}{2}, \end{cases} \tag{2.8}$$

from which we obtain the nonzero component of the electromagnetic field as

$$F_{tr} = \begin{cases} -\frac{q}{r} & \text{for } p = \frac{3}{2}, \\ q\left(\frac{3-2p}{2p-1}\right)r^{\frac{-2}{2p-1}}. & \text{for } p \neq \frac{1}{2}, \frac{3}{2}. \end{cases} \tag{2.9}$$

Note that q is an integration constant related to the black hole charge. Also, note that F_{tr} reduces to its Reissner-Nordström-anti-de Sitter (R-N-AdS) correspondence if we set $p = 1$.

To obtain the metric function $f(r)$, we use Eq. (2.6) in the gravitational field equation (2.3). It leads to the following differential equations:

$$e_{tt} = e_{rr} = \begin{cases} \frac{f'(r)}{r} + \frac{f(r)-1}{r^2} - \frac{3}{\ell^2} - 2\sqrt{2}\frac{q^3}{r^3} = 0 & \text{for } p = \frac{3}{2}, \\ \frac{f'(r)}{r} + \frac{f(r)-1}{r^2} - \frac{3}{\ell^2} + (2p-1)2^{p-1}\left(\frac{q(3-2p)}{2p-1}\right)^{2p} r^{\frac{-4p}{2p-1}} = 0 & \text{for } p \neq \frac{1}{2}, \frac{3}{2}, \end{cases} \tag{2.10}$$

$$e_{\theta\theta} = e_{\varphi\varphi} = \begin{cases} f''(r) + \frac{2f'(r)}{r} - \frac{6}{\ell^2} + 2\sqrt{2}\frac{q^3}{r^3} = 0 & \text{for } p = \frac{3}{2}, \\ f''(r) + \frac{2f'(r)}{r} - \frac{6}{\ell^2} - 2^p\left(\frac{q(3-2p)}{2p-1}\right)^{2p} r^{\frac{-4p}{2p-1}} = 0 & \text{for } p \neq \frac{1}{2}, \frac{3}{2}. \end{cases} \tag{2.11}$$

Making use of Eqs. (2.10) and (2.11), one can show that

$$e_{\theta\theta} = \left(2 + r\frac{d}{dr}\right)e_{rr}, \quad \text{for } p \neq \frac{1}{2}. \tag{2.12}$$

It means that Eqs. (2.10) and (2.11) are not independent. Therefore, we solve the first order differential equation (2.10) and ensure that the solution satisfies the second order differential equation (2.11). It must be noted that, in the case of $p = \frac{1}{2}$, one can obtain the components of the gravitational field equations similar to those of Eqs. (2.10) and (2.11), but they are not compatible for $p = \frac{1}{2}$. Thus the gravitational field equations do not have solutions in the spacetime geometry described by metric (2.6).

The solution to the field equations (2.10) and (2.11) are

$$f(r) = \begin{cases} 1 - \frac{m}{r} + \frac{r^2}{\ell^2} + \frac{2\sqrt{2}q^3}{r}\ln\left(\frac{r}{\ell}\right), & \text{for } p = \frac{3}{2}, \\ 1 - \frac{m}{r} + \frac{r^2}{\ell^2} + (2p-1)(2)^{p-1}q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p-1} r^{\frac{-2}{2p-1}}, & \text{for } \frac{1}{2} < p < \frac{3}{2}, \end{cases} \tag{2.13}$$

where m is the constant of integration related to the black hole mass. It is notable that in the case of $p = 1$ the power-law nonlinear electrodynamics (2.2) reduces to the usual electrodynamics and (2.13) to the R-N-AdS metric function.

In the following subsection we investigate the mathematical and physical properties of the solutions we obtained here.

A. Properties of the solutions

To study the general structure of the solutions we just obtained, at first one must notice that, as a physical condition, the electric potential (2.8) should be finite as r goes to infinity. Therefore, the p-dependent power of r (i.e., $\frac{2p-3}{2p-1}$) must be negative. It restricts the allowed p values to the range $\frac{1}{2} < p < \frac{3}{2}$.

To investigate the asymptotic behavior of the solutions, we notice the metric function $f(r)$ for the limit of $r \to \infty$. One can show that the p dependent power of r (i.e., $\frac{-2}{2p-1}$) is negative for $p > \frac{1}{2}$, positive for $p < \frac{1}{2}$, and equal to 2 for $p = 0$. Thus it can be obtained from (2.13) that

$$\lim_{r\to\infty} f(r) = 1 + \frac{r^2}{\ell^2} \quad \text{for } \frac{1}{2} < p \leq \frac{3}{2}, \tag{2.14}$$

which confirms that the metric function $f(r)$ describes an asymptotically AdS spacetime, depending on the sign of cosmological parameter Λ, for the mentioned p values. Also the spacetime is a pure AdS for $p = 0$ with the following effective cosmological constant

$$\frac{1}{\ell_{\text{eff}}} = \frac{1}{\ell} + \frac{1}{2}. \tag{2.15}$$

In the case of $p < \frac{1}{2}$ the geometry of the spacetime is not asymptotically AdS nor asymptotically flat.

Now, we look for the curvature singularities. One can show that the Ricci and Kretschmann scalars can be written in the following forms [8]:

$$R = \begin{cases} -\frac{12}{\ell^2} - 2\sqrt{2}\frac{q^3}{r^3}, & \text{for } p = \frac{3}{2}, \\ -\frac{12}{\ell^2} + (p-1)(2)^{p+1}\left(\frac{q(3-2p)}{2p-1}\right)^{2p} r^{\frac{-4p}{2p-1}}, & \text{for } \frac{1}{2} < p < \frac{3}{2}, \end{cases} \tag{2.16}$$

$$R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} = \begin{cases} \frac{24}{\ell^4} + \frac{8}{\ell^2 r^2} + \frac{8\sqrt{2}q^3}{\ell^2 r^3} + \frac{4}{r^4} + \frac{8}{r^5}(2\sqrt{2}q^3\ln(r/\ell) - m) + \frac{12m^2}{r^6} + \frac{8A_0 q^3}{r^6}, & \text{for } p = \frac{3}{2}, \\ \frac{24}{\ell^4} + \frac{8}{\ell^2 r^2} + \frac{4}{r^4} + \frac{8m}{r^5} + \frac{12m^2}{r^6} + A_1(p)r^{\frac{-4p}{2p-1}} + A_2(p)r^{\frac{3-10p}{2p-1}} + A_3(p)r^{\frac{2-8p}{2p-1}} + A_4(p)r^{\frac{-8p}{2p-1}}, & \text{for } \frac{1}{2} < p < \frac{3}{2}, \end{cases} \tag{2.17}$$

where

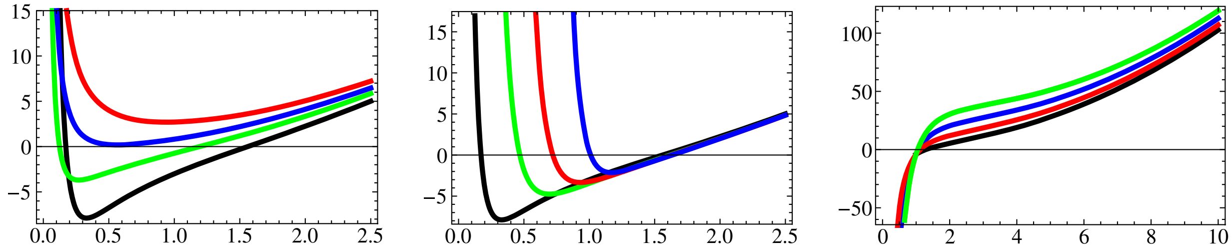

FIG. 1. $f(r)$ versus r. Left: $M = 3$, $Q = 1$, $\ell = 1$, and $p = 1, 1.28, 1.34, 1.38$, from bottom to top. Middle: $M = 3$, $Q = 1$, $\ell = 1$, and $p = 1, 0.75, 0.65, 0.58$, from left to right. They show black holes with two horizon, extreme black hole, and naked singularity. Right: $M = 3$, $p = 3/2$, $\ell = 1$, and $Q = 5, 10, 15, 20$, from bottom to top. All show nonextreme black holes with one horizon. Note that Eqs. (3.2) and (3.8) have been used.

$$A_0 = 5\sqrt{2}m + 13q^3 + 2[6q^3 \ln(r/\ell) - 10q^3 - 3\sqrt{2}m]\ln(r/\ell),$$

$$A_1(p) = (1-p)(2)^{p+3}\frac{q^{2p}}{\ell^2}\left(\frac{3-2p}{2p-1}\right)^{2p},$$

$$A_2(p) = m(2)^{p+3}q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p}\frac{p(2p+1)}{2p-3},$$

$$A_3(p) = (2p-1)(2)^{p+2}q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p-1},$$

$$A_4(p) = \frac{(2)^{2p+1}q^{4p}}{(3-2p)^2}\left(\frac{3-2p}{2p-1}\right)^{4p} \times (8p^4 - 16p^3 + 22p^2 - 10p + 3).$$

It is easily shown that the Ricci and Kretschmann scalars reduce to those of the R-N-AdS black hole by setting $p = 1$.

Note that the Ricci and Krishmann scalars diverge at $r = 0$. There is singularity at $r = 0$ (i.e., $r = 0$ is an essential singularity) for the asymptotically AdS black holes introduced here. Otherwise ($p < \frac{1}{2}$), $r = 0$ is not a singular point. For more clarifying of the properties of the nonlinearly charged AdS black hole solutions as well as the effects of nonlinear electrodynamics theory, we have plotted the metric function $f(r)$ versus r in Fig. 1. As is clear from Fig. 1, the solutions with permitted p values in the range $\frac{1}{2} < p < \frac{3}{2}$ cannot present single horizon black holes. However, they can present two horizon, extreme black holes, and naked singularity depending on the parameter p. On the other hand, it shows that in the case of $p = \frac{3}{2}$ only nonextreme black holes with a single horizon can be presented.

III. THERMODYNAMICS

In this section we explore the thermodynamics properties of the four-dimensional nonlinearly charged AdS black hole solutions we just introduced. Also we consider separately the black hole stability or phase transitions regarding the black hole heat capacity for either the $p = \frac{3}{2}$ case or the $\frac{1}{2} < p < \frac{3}{2}$ case.

A. The conserved quantities and first law of black hole thermodynamics

In this subsection, we seek satisfaction of the first law of thermodynamics for our four-dimensional AdS black hole solutions. Let us start with the calculation of the black hole electric charge Q, as a conserved quantity, in terms of the integration constant q. Making use of Gauss's law, the electric charge can be found by calculating the flux of the electric field at infinity (i.e., $r \to \infty$), that is [7,9,16]

$$Q = \frac{1}{4\pi}\int \sqrt{-g}\mathcal{L}'(\mathcal{F})F_{\mu\nu}n^\mu u^\nu d\Omega, \tag{3.1}$$

where n^μ and u^ν are the unit spacelike and timelike normals to the hypersurface of radius r defined through the following relations:

$$n^\mu = \frac{dt}{\sqrt{-g_{tt}}} = \frac{dt}{\sqrt{f(r)}}, \qquad u^\nu = \frac{dr}{\sqrt{g_{rr}}} = \sqrt{f(r)}dr.$$

Making use of Eq. (2.9) after some simple calculations we arrived at

$$Q = \begin{cases} \frac{3}{\sqrt{2}}q^2 & \text{for } p = \frac{3}{2}, \\ p(2)^{p-1}\left[\frac{q(3-2p)}{2p-1}\right]^{2p-1} & \text{for } \frac{1}{2} < p < \frac{3}{2}, \end{cases} \tag{3.2}$$

from which we can write

$$q = \frac{2p-1}{3-2p}\left[\frac{Q}{p(2)^{p-1}}\right]^{1/(2p-1)} \quad \text{for } \frac{1}{2} < p < \frac{3}{2}. \tag{3.3}$$

The black hole charge coincides with that of the R-N-AdS black hole if one sets $p = 1$ in Eq. (3.3).

The second conserved quantity to be calculated is the black hole mass M, which is related to the other integration constant m. Since the spacetime under consideration is an asymptotically AdS one, we can use the counterterm method [18] to obtain the conserved mass. In the counterterm method the divergence free stress tensor is written in the following form:

$$T^{\alpha\beta} = \frac{1}{8\pi}\left[\Theta^{\alpha\beta} - \Theta\gamma^{\alpha\beta} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{\alpha\beta}}\right], \tag{3.4}$$

where $\Theta^{\alpha\beta}$ is the extrinsic curvature of the boundary and Θ is its trace. S_{ct} is the counterterm action that has been added to obtain a finite stress tensor. It is a local function of intrinsic geometry, $\gamma_{\alpha\beta}$, of the boundary

$$S_{ct}(\gamma_{\alpha\beta}) = \int_{\mathcal{B}} d^4x\sqrt{-\gamma}\left(\frac{\ell}{2}R - \frac{2}{\ell}\right). \tag{3.5}$$

The tt component of the stress tensor is written as

$$8\pi T_{tt} = -\frac{2r^2}{\ell^3} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tt}}, \tag{3.6}$$

and the black hole mass M in terms of the mass parameter m may be calculated as

$$M = \int d^2x\frac{r}{\ell}T_{tt}, \quad \text{for large } r. \tag{3.7}$$

In the present case $x_1 = \ell\theta$ and $x_2 = \ell\sin\theta d\varphi$. It is matter of calculation to show that

$$m = 2M. \tag{3.8}$$

One can obtain the Hawking temperature associated with the black hole horizon $r = r_+$, which is the root(s) of $f(r_+) = 0$, in terms of the surface gravity κ as

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\frac{d}{dr}f(r)|_{r=r_+} = \frac{1}{4\pi r_+}\times\begin{cases} 1 + \frac{3r_+^2}{\ell^2} + 2\sqrt{2}\frac{q^3}{r_+} & \text{for } p = \frac{3}{2}, \\ 1 + \frac{3r_+^2}{\ell^2} - (3-2p)(2)^{p-1}q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p-1} r_+^{\frac{-2}{2p-1}} & \text{for } \frac{1}{2} < p < \frac{3}{2}, \end{cases} \tag{3.9}$$

which reduces to the temperature of R-N-AdS in the case $p = 1$.

Note that the relation $f(r_+) = 0$ has been used to eliminate the mass parameter m from Eq. (3.9). Extreme black holes occur if q and r_+ are chosen such that $T = 0$. With this issue in mind, making use of Eq. (3.9) we have

$$q_{\mathrm{ext}} = \frac{2p-1}{3-2p}\left(\frac{1+3r_{\mathrm{ext}}^2/\ell^2}{(2p-1)(2)^{p-1}}\right)^{\frac{1}{2p}} r_{\mathrm{ext}}^{\frac{1}{p(2p-1)}} \quad \text{for } \frac{1}{2} < p < \frac{3}{2}. \tag{3.10}$$

As shown in Fig. 1, our solutions produce extreme black holes if $q = q_{\mathrm{ext}}$, two horizon black holes for $q < q_{\mathrm{ext}}$, and naked singularities provided $q > q_{\mathrm{ext}}$.

Next, we calculate the entropy of the black hole. It can be obtained from Hawking-Bekenstein entropy-area law, that is,

$$S = \frac{A}{4} = \pi r_+^2. \tag{3.11}$$

Also, the black hole's electric potential can be obtained in terms of the null generator of the horizon as [7,9,16]

$$U = A_\mu\chi^\mu|_{\mathrm{reference}} - A_\mu\chi^\mu|_{r=r_+}, \tag{3.12}$$

where χ^μ is the null generator of the horizon [19]. It is the black hole's electric potential measured by an observer located at infinity relative to the horizon. Noting Eq. (2.8), we have

$$U = \begin{cases} q\ln\left(\frac{r_+}{\ell}\right) & \text{for } p = \frac{3}{2}, \\ q(r_+)^{\frac{2p-3}{2p-1}} & \text{for } \frac{1}{2} < p < \frac{3}{2}, \end{cases} \tag{3.13}$$

which is consistent with the electric potential of the R-N-AdS black hole in the case $p = 1$.

Here, we check the first law of black hole thermodynamics for the conserved and thermodynamic quantities obtained from the geometrical methods. At first we obtain the black hole mass M as a function of the extensive quantities entropy S and charge Q. For this purpose we use Eqs. (3.3), (3.8), and (3.12) in the relation $f(r_+) = 0$ and find the following Smarr-type mass formula:

$$M(Q,S) = \begin{cases} \frac{1}{2}\left(\frac{S}{\pi}\right)^{\frac{1}{2}} + \frac{1}{2\ell^2}\left(\frac{S}{\pi}\right)^{\frac{3}{2}} + \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{3}Q\right)^{\frac{3}{2}}\ln\left(\frac{S}{\pi\ell^2}\right) & \text{for } p = \frac{3}{2}, \\ \frac{1}{2}\left(\frac{S}{\pi}\right)^{\frac{1}{2}} + \frac{1}{2\ell^2}\left(\frac{S}{\pi}\right)^{\frac{3}{2}} + (2)^{p-2}\frac{(2p-1)^2}{3-2p}\left(\frac{Q}{p2^{p-1}}\right)^{\frac{2p}{2p-1}}\left(\frac{S}{\pi}\right)^{\frac{2p-3}{2(2p-1)}} & \text{for } \frac{1}{2} < p < \frac{3}{2}. \end{cases} \tag{3.14}$$

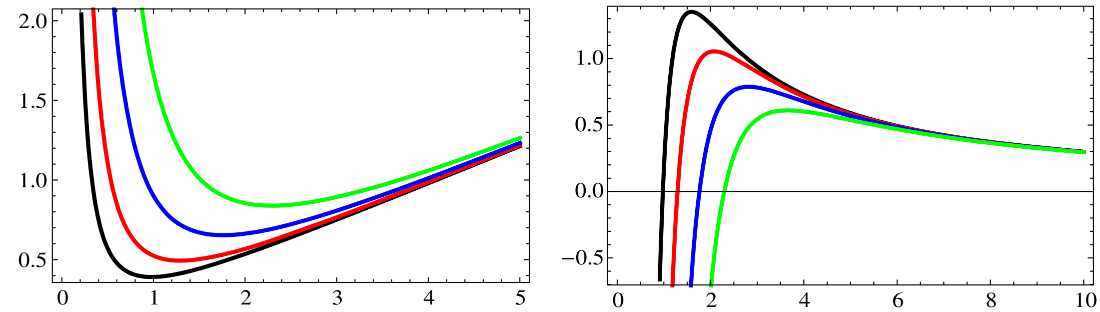

FIG. 2. (Left) T versus r_+ and (right) $8\pi^2(\partial^2 M/\partial S^2)_Q$ versus r_+ for $p = 3/2$ and $\ell = 1$. Black, red, blue, and green curves correspond to $Q = 1, 2, 4, 7$, respectively.

From the thermodynamic point of view, the mass of a black hole can be interpreted as the internal energy. Therefore, in the classical black hole thermodynamics, it is necessary for the physical black holes to have positive mass [20]. Now, the last term in the black hole mass given by the last line of Eq. (3.14) is negative for $p > \frac{3}{2}$. It can lead to a negative black hole mass. Therefore, by imposing the confinement $\frac{1}{2} < p < \frac{3}{2}$, the black hole mass is always positive, which confirms the validity of the range of allowed p values.

By treating Q and S as a complete set of extensive parameters for the mass $M(S, Q)$ and defining the intensive parameters conjugate to them as temperature T and electric potential U, we obtain

$$T = \left(\frac{\partial M}{\partial S}\right)_Q, \qquad U = \left(\frac{\partial M}{\partial Q}\right)_S, \tag{3.15}$$

which are compatible with the temperature and electric potential given in Eqs. (3.9) and (3.13). It means that the thermodynamics quantities we obtained in this section satisfy the first law of black hole thermodynamics in the form

$$dM = TdS + UdQ, \tag{3.16}$$

for either of the black hole solutions we just obtained.

B. Stability analysis in the canonical ensemble method

In this stage, we study the local stability or phase transitions of the introduced black holes in the canonical ensemble method. It is well known that the black hole, as a thermodynamical system, is locally stable if its heat capacity is positive. A nonstable black hole may undergo a phase transition to be stabilized. The phase transition points are where the heat capacity vanishes or diverges. In the vanishing points (roots of heat capacity) the phase transition is named conventionally as the type one phase transition. The points where the heat capacity diverges are known as the type two phase transition points. Therefore, the positivity of heat capacity $C_Q = T(\partial S/\partial T)_Q = T/(\partial^2 M/\partial S^2)_Q$ or equivalently the positivity of $(\partial S/\partial T)_Q$ or $(\partial^2 M/\partial S^2)_Q$ with $T > 0$ are sufficient to ensure the local stability of the black hole. It is a matter of calculation to show that

$$\left(\frac{\partial^2 M}{\partial S^2}\right)_Q = \frac{1}{8\pi^2 r_+^3} \times \begin{cases} 3\frac{r_+^2}{\ell^2} - 4\sqrt{2}\frac{q^3}{r_+} - 1 & \text{for } p = \frac{3}{2}, \\ (2)^{p-1}(2p+1)q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p}(r_+)^{-\frac{2}{2p-1}} - 1 + \frac{3r_+^2}{\ell^2} & \text{for } \frac{1}{2} < p < \frac{3}{2}. \end{cases} \tag{3.17}$$

It is obvious from Eq. (3.8) that for the case of $p = 3/2$ the black hole temperature is positive (i.e., $T > 0$), and no type one phase transition takes place. For discussing the type two phase transition we must consider the real roots of the denominator in the heat capacity. That is,

$$3r_+^3 - \ell^2 r_+ - 4\sqrt{2}q^3\ell^2 = 0. \tag{3.18}$$

It has a real root of the form

$$r_+ \equiv r_0 = \frac{1}{3}\left(\Gamma + \frac{\ell^2}{\Gamma}\right), \quad \text{with}$$
$$\Gamma = q\left[18\sqrt{2}\ell^2\left(1 + \sqrt{1 - \ell^2/(648q^6)}\right)\right]^{\frac{1}{3}}, \tag{3.19}$$

if the condition $\ell \leq 18\sqrt{2}q^3$ is satisfied. With this condition the heat capacity diverges and the black hole undergoes a type two phase transition to be stabilized. The heat capacity of the black hole is positive for $r_+ > r_0$ and negative for $r_+ < r_0$. Thus the black hole is

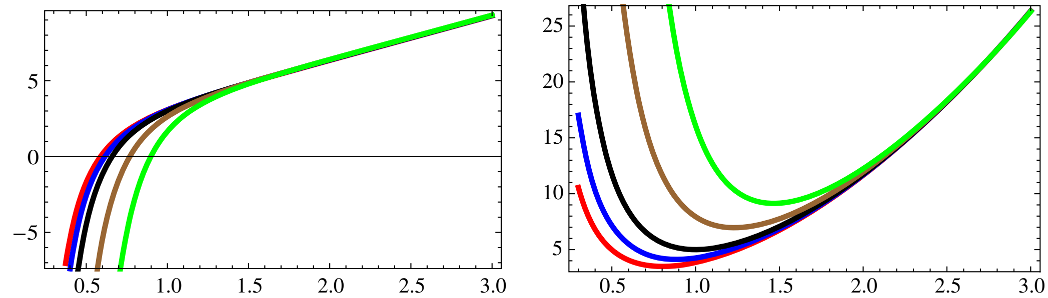

FIG. 3. (Left) $4\pi T$ versus r_+ and (right) $D(r_+)$ versus r_+ for $Q = 1$ and $\ell = 1$. Red, blue, black, brown, and green curves correspond to $p = 1.488,\ 1.2,\ 1,\ 0.8,\ 0.7$, respectively.

thermodynamically stable if its horizon radius is greater than r_0. The plots of T and $(\partial^2 M/\partial S^2)_Q$ versus r_+ are shown in Fig. 2.

For the range $\frac{1}{2} < p < \frac{3}{2}$ the black hole temperature can be written as

$$T = \frac{1}{4\pi r_+}\left[1 + 3\frac{r_+^2}{\ell^2} - (2p-1)(2)^{p-1} q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p} r_+^{\frac{-2}{2p-1}}\right]. \tag{3.20}$$

It is evident that for a nonextreme black hole to be physically reasonable the temperature must be positive. That is,

$$1 + 3\frac{r_+^2}{\ell^2} > (2p-1)(2)^{p-1} q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p} r_+^{\frac{-2}{2p-1}}. \tag{3.21}$$

It must be noted that the inequality (3.21) restricts the black hole charge and size to some allowed ranges. The plot of $4\pi T(r_+)$ versus r_+, for different allowed p values, has been shown in Fig. 3. It is clear from Fig. 3 that there is a minimum horizon radius $r_1 = r_{\text{ext}}$, such that the black hole temperature is positive for $r_+ > r_1$.

Now we investigate the divergent points of the black hole heat capacity. From Eq. (3.17), one can say that the black holes are stable if

$$D(r_+) \equiv (2)^{p-1}(2p+1) q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p}(r_+)^{-\frac{2}{2p-1}} - 1 + \frac{3r_+^2}{\ell^2} > 0. \tag{3.22}$$

It is a matter of calculation to combine inequalities (3.21) and (3.22) and show that

$$(2p+1)(1 + 3r_+^2/\ell^2) > (2p-1)(1 - 3r_+^2/\ell^2). \tag{3.23}$$

The inequality (3.23) is always fulfilled. It means that the denominator of the black hole heat capacity is positive and does not vanish. Therefore, the AdS black holes are thermally stable for $r_+ > r_1$. For more clarifying, we have plotted $D(r_+)$ versus r_+ for different allowed p values in Fig. 3. It shows that the denominator of the black hole heat capacity is always positive. The plots of Fig. 3 confirm that the black holes with the horizon radius greater than r_{ext} are thermodynamically stable.

C. Stability analysis in the grand canonical ensemble method

In the grand canonical ensemble method the black hole, as the thermodynamical system, is locally stable provided that the Hessian matrix $\mathbf{H}^M_{S,Q}$ is positive definite [21]. It can be written as

$$\mathbf{H}^M_{S,Q} = \begin{pmatrix} M_{QQ} & M_{QS} \\ M_{SQ} & M_{SS} \end{pmatrix}, \tag{3.24}$$

where $M_{XY} = \frac{\partial^2 M}{\partial X \partial Y}$ and the explicit form of $M(S,Q)$ has been given in Eq. (3.14). The Hessian matrix $\mathbf{H}^M_{S,Q}$ is positive definite provided that all its pivots are positive [21]. The pivots are

$$M_{QQ} = \frac{\partial U}{\partial Q} \quad \text{and} \quad \frac{\det(\mathbf{H}^M_{S,Q})}{M_{QQ}}. \tag{3.25}$$

Making use of Eqs. (3.3) and (3.13) one can show that

$$M_{QQ} = \begin{cases} \left(\frac{\sqrt{2}}{12Q}\right)^{\frac{1}{2}} \ln\left(\frac{r_+}{\ell}\right) & \text{for } p = \frac{3}{2}, \\ \frac{(r_+)^{\frac{2p-3}{2p-1}}}{Q(3-2p)}\left[\frac{Q}{p(2)^{p-1}}\right]^{1/(2p-1)} & \text{for } \frac{1}{2} < p < \frac{3}{2}, \end{cases} \tag{3.26}$$

which is clearly positive, and the first condition of thermal stability is always fulfilled. Therefore, the positivity of $\det(\mathbf{H}^M_{S,Q})/M_{QQ}$ ensures the black hole thermal stability. Now, we proceed to examine the second condition of the black hole local stability. Following the work of Peca and Lemos [21] we have

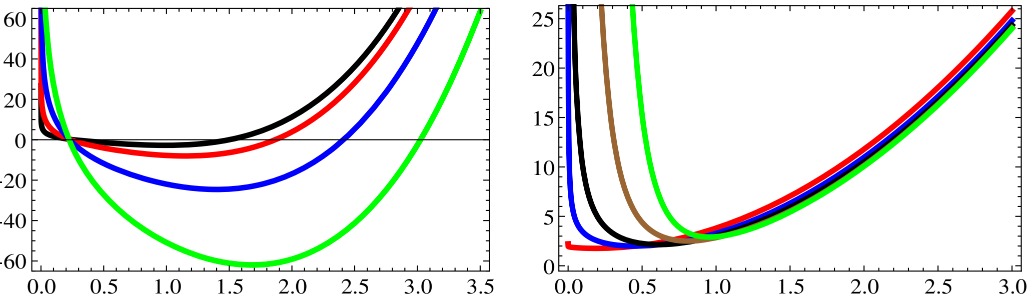

FIG. 4. Left: $\mathcal{M}(r_+)$ versus r_+ for $p = 3/2$ and $\ell = 1$. Black, red, blue, and green curves correspond to $Q = 1, 2, 4, 7$, respectively. Right: $\mathcal{M}^{(p)}(r_+)$ versus r_+ for $Q = 1$ and $\ell = 1$. Red, blue, black, brown, and green curves correspond to $p = 1.488, 1.2, 1, 0.8, 0.7$, respectively.

$$\frac{\det(\mathbf{H}^M_{S,Q})}{M_{QQ}} = \frac{1}{C_U}, \tag{3.27}$$

where $C_U = T(\frac{\partial S}{\partial T})_U$ is the black hole heat capacity at the constant electric potential. It can be calculated as follows:

$$C_U = T\left(\frac{\partial S}{\partial r_+}\right)_U \left(\frac{\partial T}{\partial r_+}\right)^{-1}_U = 8\pi^2\ell^2 r_+^4 T \begin{cases} \left[(3r_+^3 - \ell^2 r_+ - 4\sqrt{2}q^3\ell^2)\ln\left(\frac{r_+}{\ell}\right) - 6\sqrt{2}q^3\ell^2\right]^{-1}\ln\left(\frac{r_+}{\ell}\right), & \text{for } p = \frac{3}{2}, \\ \left[3r_+^3 - \ell^2 r_+ + (2)^{p-1}q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p}(2p-1)^2 r_+^{\frac{2p-3}{2p-1}}\right]^{-1} & \text{for } \frac{1}{2} < p < \frac{3}{2}. \end{cases} \tag{3.28}$$

In the case of $p = \frac{3}{2}$, T is positive (see Fig. 2). Therefore, positivity of the denominator of C_U guarantees the black hole to be locally stable. That is, the black hole is locally stable if

$$\mathcal{M}(r_+) = (3r_+^3 - \ell^2 r_+ - 4\sqrt{2}q^3\ell^2)\ln\left(\frac{r_+}{\ell}\right) - 6\sqrt{2}q^3\ell^2 > 0. \tag{3.29}$$

The plot of $\mathcal{M}(r_+)$ versus r_+ has been shown in Fig. 4 (left). It shows that there is a minimum value for the horizon radius such that the black holes with the horizon radius greater than this minimum value are thermodynamically stable. A similar result has been obtained when the canonical ensemble method was used.

Since the black holes with p in the range $\frac{1}{2} < p < \frac{3}{2}$ have a positive temperature for $r_+ > r_{\rm ext}$ (Fig. 3, left), one can argue from Eq. (3.28) that they have a positive heat capacity and are thermodynamically stable if the denominator is positive. That is,

$$\mathcal{M}^{(p)}(r_+) = 3r_+^3 - \ell^2 r_+ + (2)^{p-1}q^{2p}\left(\frac{3-2p}{2p-1}\right)^{2p} \times (2p-1)^2 r_+^{\frac{2p-3}{2p-1}} > 0. \tag{3.30}$$

As is shown in Fig. 4(right), the inequality (3.30) is always fulfilled. Therefore, the heat capacity is positive definite and AdS black holes with a horizon radius greater than $r_{\rm ext}$ are locally stable. It means that the results of canonical and grand canonical ensemble methods are compatible and the new AdS black holes, introduced in this work, are stable.

IV. CONCLUSION

Here, we studied the four-dimensional charged black hole solutions within the nonlinear electrodynamics. Making use of the power Maxwell invariant, as the generalization of the usual classical theory of electrodynamics, we solved the coupled electromagnetic and gravitational equations and obtained two new classes of black hole solutions. Through consideration of the physical properties of the black hole solutions we just obtained, we found that they behave asymptotically like the AdS black holes if we fix the power of the Maxwell invariant in the nonlinear theory of the electrodynamics to the range $\frac{1}{2} < p \leq \frac{3}{2}$. Also we found that Ricci and Kretschmann scalars diverge at $r = 0$. It means that $r = 0$ is an essential (not coordinate) singularity for either of the AdS black hole solutions. Furthermore, we showed that one of the solutions corresponding to $p = \frac{3}{2}$ presents black holes with only a single horizon, while the other that corresponds to $\frac{1}{2} < p < \frac{3}{2}$ presents naked singularity, extreme, and two horizon black holes if the parameter p is chosen properly (see Fig. 1).

Next, we proceed to explore the thermodynamical properties of the new AdS black hole solutions. At first we obtained the electric charge and mass of the black hole, as the conserved quantities, making use of Gauss's law and counterterm method, respectively. Also, we calculated the entropy, temperature, and electric potential by using the geometrical methods. On the other hand, through a Smarr-type mass formula, we constructed the black hole mass as a function of both the charge and the entropy, as the thermodynamical extensive quantities, from which we calculated the electric potential and temperature, as the thermodynamical intensive quantities, for either of the asymptotic AdS black holes. We found that the thermodynamical quantities obtained from geometrical and thermodynamical approaches are identical for either of the black hole classes. It confirms the validity of the first law of black hole thermodynamics in the form of Eq. (3.16).

Finally, we analyzed the local stability of either of the new asymptotic AdS black holes, making use of the black hole heat capacity with the fixed black hole charge. For the case $p = \frac{3}{2}$ we found that no type one phase transition takes place. The black hole undergoes a type two phase transition to be stabilized if the condition $\ell \leq 18\sqrt{2}q^3$ is satisfied. A black hole with a horizon radius greater than r_0, identified in Eq. (3.19), is thermodynamically stable (see Fig. 2). Furthermore, in the case $\frac{1}{2} < p < \frac{3}{2}$, we found that no type two phase transition takes place. There is a point of type one phase transition located at $r_1 = r_{\rm ext}$, where the black hole temperature vanishes. Since $(\partial^2 M/\partial S^2)_Q$ is positive everywhere, the two horizon AdS black holes with the outer horizon radius greater than the minimum value, $r_{\rm ext}$, are thermodynamically stable (see Fig. 3). Also, we studied the thermal stability of the new AdS black holes, making use of the grand canonical ensemble method and regarding the black hole heat capacity with the fixed electric potential. We found that the results of these two alternative approaches are compatible [Eq. (3.28) and Fig. 4].

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