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Thermodynamics of (2 + 1)-dimensional black holes in Einstein-Maxwell-dilaton gravity

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In this paper, the linearly charged three-dimensional Einstein's theory coupled to a dilatonic field has been considered. It has been shown that the dilatonic potential must be considered in a form of generalized Liouville-type potential. Two new classes of charged dilatonic black hole solutions, as the exact solutions to the Einstein-Maxwell-dilaton (EMd) gravity, have been obtained and their properties have been studied. The conserved charge and mass related to both of the new EMd black holes have been calculated. Through comparison of the thermodynamical extensive quantities (i.e., temperature and entropy) obtained from both, the geometrical and the thermodynamical methods, the validity of first law of black hole thermodynamics has been investigated for both of the new black holes we just obtained. At the final stage, making use of the canonical ensemble method and regarding the black hole heat capacity, the thermal stability or phase transition of the new black hole solutions have been analyzed. It has been shown that there is a specific range for the horizon radius in such a way that the black holes with the horizon radius in that range are locally stable. Otherwise, they are unstable and may undergo type one or type two phase transitions to be stabilized.

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I. INTRODUCTION

Although the Einstein tensorial theory of gravitation, known today as general relativity, was more successful to pass observational tests and became the standard theory of gravitation, it seems that this theory may be incomplete [1–5]. One of the main approaches for explanation of the related problems is to modify Einstein's theory of gravity. In this regard, various modifications were proposed in the literatures [6–13]. Among them the so-called scalar-tensor theories [14], as the modification arising from string theory, have provided interesting results [15]. At the sufficiently high energy scales, the Einstein's action is naturally modified by the scalar-tensor superstring terms. In the low energy limit of the string theory, the Einstein gravity is recovered with a dilatonic scalar field which is coupled to the gravity [16].

Black holes with scalar hair are interesting solutions of Einstein's theory of gravity and also of certain types of modified gravity theories. These solutions have been investigated by theoretical physicists in four and higher dimensional spacetimes for a long time [17]. The first studies on the three-dimensional black holes, as the interesting predictions of Einstein's theory of relativity in lower dimensional spacetimes, have been done originally by Banados, Teitelboim, and Zanelli (BTZ) [18]. After the discovery of the BTZ black holes, a large number of studies on different kinds of black holes in (2 + 1)-dimensional spacetimes have been done by many authors [19]. Apparently, the first attempts for investigation of the charged three-dimensional dilatonic black holes were made by Chan and Mann [20], in the presence of a minimally coupled logarithmic dilaton field with an exponentially potential term.

It is a commonly believed that study of the physics of black holes in lower dimensions is easier and can lead to a deeper insight into the fundamental ideas in comparison to higher dimensional black holes. Also, according to (A)dS/CFT correspondence, there is a dual between quantum gravity on A(dS) space and a Euclidean conformal field theory on the lower dimensional spacetimes [21,22]. Thus, study of physics in (2 + 1)-dimensional spacetimes can be useful for understanding of quantum field theory on A(dS) spacetimes. Although this subject area has been considered extensively, it still has many unknown and interesting parts to be studied [23].

On the other hand, after the discoveries of Bekenstein, Bardeen, Carter, and Hawking, it is well known that black holes can be considered as the thermodynamical systems with a temperature proportional to the surface gravity and having pure geometrical entropy equal to one-fourth of the horizon area [24–26]. When a dilatonic scalar field is coupled to the three-dimensional Einstein-Maxwell theory, it is expected to produce new and interesting consequences for the black hole solutions. Thus, it is worthwhile to find exact solutions of EMd gravity for an arbitrary coupling constant, and investigate how the thermodynamical properties of black holes are modified in the presence of a dilatonic scalar field. Also it is interesting to investigate the black holes remnant and find out the impacts of dilaton on the thermal stability of the black hole solutions.

The main object this paper is to introduce new EMd black holes as the exact solutions to the coupled scalar, vector, and tensor field equations and provide a detailed

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analysis of the thermodynamical properties of new three-dimensional electrically charged black holes in the presence of a dilatonic scalar field.

The paper is organized based on the following order. In Sec. II the scalar, electromagnetic and gravitational field equations have been solved and the static spherically symmetric black hole solutions have been obtained. The physical properties of the black hole solutions, we obtained here, have been considered. It has been found that their asymptotical behaviors are not like anti-de Sitter (AdS) black holes. Section III is devoted to checking the validity of the first law of black hole thermodynamics for both of the EMd black hole solutions obtained in the previous section. The black holes masses, charges, entropies, temperatures as well as the electric potentials have been calculated based on geometrical methods. The black holes masses have been written as functions of the charge and entropy, as the complete set of extensive thermodynamical quantities. Making use of the mass formulas, the temperatures and electric potentials have been calculated, as the intensive quantities conjugate to entropy and charge, respectively. Through comparison of the quantities obtained from both geometrical and thermodynamical methods, the validity of the first law of black hole thermodynamics has been confirmed for both of the EMd black hole solutions introduced here. In Sec. IV, the thermal stability or phase transition of the obtained black hole solutions has been analyzed, making use of the canonical ensemble method and regarding the black hole heat capacity. It has been found that based on the choice of the dilatonic parameter, it is possible for the EMd black holes to be locally stable or undergo phase transition in order to be stabilized. The results are summarized and discussed in Sec. V.

II. FIELD EQUATIONS AND THE CIRCULARLY SYMMETRIC SOLUTIONS

The action for three dimensional charged hairy black holes in the presence of nonlinear electrodynamics can be written in the following general form [20,27]

$$I = -\frac{1}{16\pi}\int \sqrt{-g}d^3x[\mathcal{R} - U(\phi) - 2g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \mathcal{F}e^{-2\alpha\phi}]. \tag{2.1}$$

Here, $\mathcal{R}$ is the Recci scalar. ϕ is a scalar field coupled to itself via the functional form $U(\phi)$. The parameter α is the scalar-electromagnetic coupling constant and $\mathcal{F} = F^{\mu\nu}F_{\mu\nu}$ being the Maxwell invariant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the electromagnetic potential. By varying the action (2.1) with respect to gravitational, electromagnetic and scalar fields, we get the related field equations as

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}U(\phi) = T^{(s)}_{\mu\nu} + T^{(em)}_{\mu\nu}, \tag{2.2}$$

$$T^{(s)}_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}(\nabla\phi)^2,$$
$$T^{(em)}_{\mu\nu} = -\frac{1}{2}\mathcal{F}e^{-2\alpha\phi}g_{\mu\nu} + 2e^{-2\alpha\phi}F_{\mu\alpha}F_\nu{}^\alpha,$$

$$\nabla_\mu[e^{-2\alpha\phi}F^{\mu\nu}] = 0, \tag{2.3}$$

$$4\Box\phi = \frac{dU(\phi)}{d\phi} - 2\alpha\mathcal{F}e^{-2\alpha\phi}, \qquad \phi = \phi(r). \tag{2.4}$$

Assuming as a function of r, the only nonvanishing component of the electromagnetic field is $F_{\mathrm{tr}} = -E(r) = h'(r)$, and we have

$$\mathcal{F} = -2E^2(r) = -2(h'(r))^2. \tag{2.5}$$

In overall the paper, prime means derivative with respect to the argument. The gravitational field Eq. (2.2) can be rewritten as

$$\mathcal{R}_{\mu\nu} = U(\phi)g_{\mu\nu} + 2\nabla_\mu\phi\nabla_\nu\phi - (\mathcal{F}g_{\mu\nu} - 2F_{\mu\alpha}F_\nu{}^\alpha)e^{-2\alpha\phi}. \tag{2.6}$$

We consider the following ansutz as the three dimensional spherically symmetric solution to the gravitational field Eq. (2.6)

$$ds^2 = -\Psi(r)dt^2 + \frac{1}{\Psi(r)}dr^2 + r^2R(r)^2d\theta^2. \tag{2.7}$$

It leads to the following independent differential equations

$$E_{00} \equiv \Psi'' + \left(\frac{1}{r} + \frac{R'}{R}\right)\Psi' + 2U = 0, \tag{2.8}$$

$$E_{11} \equiv E_{00} + 2\Psi\left(\frac{R''}{R} + \frac{2R'}{rR} + 2\phi'^2\right) = 0, \tag{2.9}$$

$$E_{22} \equiv \left(\frac{1}{r} + \frac{R'}{R}\right)\Psi' + \left(\frac{R''}{R} + \frac{2R'}{rR}\right)\Psi + U + 2F^2_{\mathrm{tr}}e^{-2\alpha\phi} = 0. \tag{2.10}$$

Noting Eqs. (2.8) and (2.9) we obtain

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + 2\phi'^2 = 0. \tag{2.11}$$

The differential Eq. (2.11) can be written in the following form

$$\frac{2}{r}\frac{d}{dr}\ln R(r) + \frac{d^2}{dr^2}\ln R(r) + \left(\frac{d}{dr}\ln R(r)\right)^2 + 2\phi'^2 = 0. \tag{2.12}$$

From Eq. (2.12), one can argue that $R(r)$ must be an exponential function of $\phi(r)$. Therefore, we can write $R(r) = e^{2\alpha\phi}$, in Eq. (2.12), and show that $\phi = \phi(r)$ satisfies the following differential equation

$$\alpha\phi'' + (1+2\alpha^2)\phi'^2 + \frac{2\alpha}{r}\phi' = 0. \tag{2.13}$$

It is easy to write the solution of (2.13) in terms of a positive constant b as $\phi(r) = \gamma \ln(\frac{b}{r})$, with $\gamma = \alpha(1+2\alpha^2)^{-1}$. Similar solutions have been used by Hendi *et al.* [27]. The authors of Ref. [20], have started with a power law of the form $R(r) \propto r^n$ and $\phi(r) \propto \ln r$, and showed that black hole solutions can exist if n is restricted in some ranges.

Making use of these solutions together with Eqs. (2.3) and (2.7), we have

$$\begin{cases} h(r) = -q\ln(r/\ell), \\ F_{\rm tr} = -\frac{q}{r}, \end{cases} \tag{2.14}$$

where, q is an integration constant related to the total electric charge on black hole. It will be calculated in the following section. The specific length ℓ is related to the three-dimensional cosmological constant Λ through $\Lambda = -\ell^{-2}$.

Now, Eq. (2.10) can be rewritten as

$$\Psi' - \frac{2\alpha\gamma}{r}\Psi + \frac{r}{1-2\alpha\gamma}[U(\phi) + 2F_{\rm tr}^2 e^{-2\alpha\phi}] = 0. \tag{2.15}$$

To solve this equation for the metric function $\Psi(r)$, we need to calculate the functional form of $U(\phi(r))$ as the function of radial coordinate. For this purpose we proceed to solve the scalar field Eq. (2.4), which can be written as

$$\frac{dU(\phi)}{d\phi} - 4\beta U(\phi) - 4\alpha F_{\rm tr}^2 e^{-2\alpha\phi} = 0. \tag{2.16}$$

Now, the first order differential (2.16) can be solved as

$$U(\phi) = \begin{cases} (C \pm \frac{4q^2}{b^2}\phi)e^{\pm 4\phi}, & \text{for } \alpha = \pm 1, \\ [C + \frac{2\alpha^2 q^2}{b^2(1-\alpha^2)} e^{\frac{2}{\alpha}(1-\alpha^2)\phi}]e^{4\alpha\phi}, & \text{for } \alpha \neq \pm 1, \end{cases} \tag{2.17}$$

where C is an integration constant related to the cosmological constant Λ. Since, in the absence of the dilaton field (i.e., $\phi = 0$), the action (2.1) reduces to the action of Einstein-Λ-Maxwell gravity, one can obtain the integration constant C by imposing the condition $U(\phi = 0) = 2\Lambda$. It leads to $C = 2\Lambda$ and the solutions (2.17) can be written as the following versions of the Liouville dilaton potential

$$U(\phi) = \begin{cases} (2\Lambda \pm \frac{4q^2}{b^2}\phi)e^{\pm 4\phi}, & \text{for } \alpha = \pm 1, \\ 2\Lambda_0 e^{4\alpha_0\phi} + 2\Lambda e^{4\alpha\phi}, & \text{for } \alpha \neq \pm 1, \end{cases} \tag{2.18}$$

with

$$\alpha_0 = \frac{1+\alpha^2}{2\alpha}, \quad \text{and} \quad \Lambda_0 = \frac{\alpha^2 q^2}{b^2(1-\alpha^2)}. \tag{2.19}$$

Now, making use of Eqs. (2.15), (2.18), and (2.19) the metric function $\Psi(r)$ can be obtained as

$$\Psi(r) = \begin{cases} -mr^{2/3} - 6(\frac{r}{b})^{2/3}[\Lambda b^2 + q^2(1 + \frac{1}{3}\ln\frac{b^2}{r\ell})]\ln(\frac{r}{\ell}), & \text{for } \alpha = \pm 1, \\ -mr^{2\alpha\gamma} + \frac{(1+2\alpha^2)^2}{\alpha^2-1}[\Lambda r^2(\frac{b}{r})^{4\alpha\gamma} + \frac{2q^2}{1+2\alpha^2}(\frac{b}{r})^{-2\alpha\gamma}\ln(\frac{r}{\ell})], & \text{for } \alpha \neq \pm 1. \end{cases} \tag{2.20}$$

In the absence of the coupling constant α (i.e., $\alpha = 0$) we have

$$\Psi(r) = -m - \Lambda r^2 - 2q^2\ln(r/\ell), \tag{2.21}$$

which is nothing but the metric function of the charged BTZ black hole. All the field equations are satisfied by the solutions given in this section. Note that m is an integration constant related to the black hole mass.

Now, we look for the curvature singularities. One can show that the Ricci and Kretschmann scalars can be written in the following forms

$$R = \begin{cases} \frac{2}{9r^2} + (6\Lambda + \frac{2q^2}{b^2})(\frac{b}{r})^{4/3} + \frac{4q^2}{b^2}\ln(\frac{b}{r}), & \text{for } \alpha = \pm 1, \\ 6\Lambda(\frac{b}{r})^{\frac{4\alpha^2}{1+2\alpha^2}} + \frac{2\alpha^2}{b^2(1+2\alpha^2)^2}(\frac{b}{r})^2 - \frac{2q^2(1+2\alpha^2)}{b^2(\alpha^2-1)}(\frac{b}{r})^{\frac{2(1+\alpha^2)}{1+2\alpha^2}}, & \text{for } \alpha \neq \pm 1, \end{cases} \tag{2.22}$$

$$R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} = \begin{cases} r^{-8/3}[\xi_0 + \xi_1\ln r + \xi_2(\ln r)^2 + \xi_3(\ln r)^3], & \text{for } \alpha = \pm 1, \\ r^{-\frac{4(1+\alpha^2)}{1+2\alpha^2}}[A_0 + A_1 r^{\frac{2(1-\alpha^2)}{1+2\alpha^2}} + A_1 r^{\frac{4(1-\alpha^2)}{1+2\alpha^2}} + (A_3 + A_4 r^{\frac{2(1-\alpha^2)}{1+2\alpha^2}})\ln r + A_5(\ln r)^2], & \text{for } \alpha \neq \pm 1, \end{cases} \tag{2.23}$$

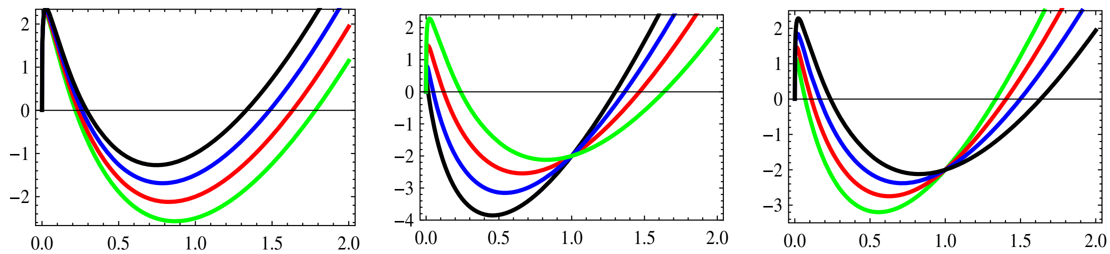

FIG. 1. $\Psi(r)$ versus r for $\alpha = \pm 1$ and $\Lambda = -1$. Left: $q = 1.5$, $b = 2$ and $m = 1, 1.5, 2, 2.5$ for black, blue, red and green curves, respectively. Middle: $m = 2$, $b = 2$ and $q = 1.2, 1.3, 1.4, 1.5$ for black, blue, red and green curves, respectively. Right: $q = 1.5$, $m = 1$ and $b = 2, 2.1, 2.2, 2.3$ for black, blue, red and green curves, respectively.

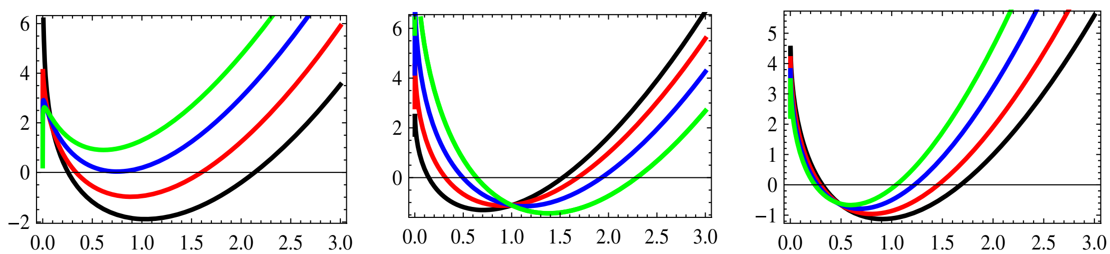

FIG. 2. $\Psi(r)$ versus r for $m = 3$ and $\Lambda = -1$. Left: $q = 1$, $b = 2$ and $\alpha = 0.15, 0.32, 0.42, 0.48$ for black, red, blue and green curves, respectively. Middle: $b = 2$, $\alpha = 0.3$ and $q = 0.8, 1.0, 1.2, 1.4$ for black, red, blue and green curves, respectively. Right: $q = 1$, $\alpha = 0.3$ and $b = 2, 3, 5, 8$ for black, red, blue and green curves, respectively.

where ξ_i's and A_i's are functions b, Λ, q, m, and α. Noting Eqs. (2.22) and (2.23) one can argue that Ricci and Kretschmann scalars are finite for finite values of the redial component r. There is an essential singularity located at $r = 0$. Also the asymptotical behavior of the geometry under consideration is not A(dS). The plot of metric functions (2.20) for different values of α, q, m, and b have been shown in Figs. 1 and 2. It is clear that the EMd black holes, we obtained here, by suitable choice of the parameter α can show two horizon, extreme and naked singularity black holes.

In the following section we explore the thermodynamics of the new EMd black hole solutions presented in Eq. (2.20).

III. FIRST LAW OF BLACK HOLE THERMODYNAMICS

In this section, we would like to check the validity of the first law of black hole thermodynamics for the new EMd black holes introduced here. At first it must be noted that the conserved charge of the black hole can be obtained by calculating the total electric flux measured by an observer located at infinity with respect to the horizon (i.e., $r \to \infty$) [28–30], that is

$$Q = \frac{1}{4\pi} \int \sqrt{-g} \mathcal{L}'(\mathcal{F}) F_{\mu\nu} n^{\mu} u^{\nu} d\Omega, \tag{3.1}$$

where n^{μ} and u^{ν} are the unit spacelike and timelike normals to the hypersurface of radius r defined through the following relations

$$n^{\mu} = \frac{1}{\sqrt{-g_{tt}}} = \frac{dt}{\sqrt{\Psi(r)}}, \qquad u^{\nu} = \frac{1}{\sqrt{g_{rr}}} = \sqrt{\Psi(r)} dr.$$

Making use of Eq. (3.3) after some simple calculations we arrived at

$$q = -2Q. \tag{3.2}$$

The other conserved quantity to be calculated is the black hole mass. As mentioned before, it can be obtained in terms of the mass parameter m. The Abbott-Deser total mass of the EMd black holes introduced here can be obtained as [27,31]

$$m = \begin{cases} \frac{24M}{b^{2/3}}, & \text{for } \alpha = \pm 1, \\ \frac{8M}{1-2\alpha\gamma} b^{-2\alpha\gamma}, & \text{for } \alpha \neq \pm 1, \end{cases} \tag{3.3}$$

which is compatible with the mass of charged BTZ black hole when the dilatonic potential disappears.

One can obtain the Hawking temperature associated with the black hole horizon $r = r_+$, which is the root(s) of $\Psi(r_+) = 0$, in terms of the surface gravity κ as

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\frac{d}{dr}\Psi(r)|_{r=r_+} = \begin{cases} \frac{3q^2}{2\pi}(r_+ b^2)^{-1/3}[(\frac{b}{q\ell})^2 - 1 - \frac{2}{3}\ln(\frac{b}{r_+})], & \text{for } \alpha = \pm 1, \\ \frac{2\alpha^2+1}{2\pi(1-\alpha^2)}[\frac{b(1-\alpha^2)}{\ell^2}(\frac{b}{r_+})^{\frac{2\alpha^2-1}{2\alpha^2+1}} - \frac{q^2}{b}(\frac{b}{r_+})^{\frac{1}{2\alpha^2+1}}], & \text{for } \alpha \neq \pm 1, \end{cases} \quad (3.4)$$

in which, the mass parameter m has been eliminated by use of the relation $\Psi(r_+) = 0$.

In the case $\alpha = 0$, the black hole temperature coincides with that of charged BTZ black hole. Since, the terms in the brackets have opposite sign, from thermodynamical point of view, the physical (i.e., black holes with positive temperature) and unphysical black holes (i.e., black holes with negative temperature) can appear. Also, it must be noted that extreme black holes occur if q and r_+ be chosen such that $T = 0$. With this issue in mind, making use of Eq. (3.4) we have

$$q_{\text{ext}} = \begin{cases} \frac{b}{\ell}[1 + \frac{2}{3}\ln(\frac{b}{r_{\text{ext}}})]^{-1/2}, & \text{for } \alpha = \pm 1, \\ \frac{b}{\ell}\sqrt{1-\alpha^2}(\frac{b}{r_{\text{ext}}})^{\frac{\alpha^2-1}{2\alpha^2+1}}, & \text{for } -1 < \alpha < 1. \end{cases} \quad (3.5)$$

In order to investigate the effects of scalar hair on the horizon temperature the plot of black hole temperature versus horizon radius, for different values of α, has been shown in Figs. 3 and 4. The physical black holes with positive temperature are those for which $r_+ > r_{\text{ext}}$ and unphysical black holes, having negative temperature, occur if $r_+ < r_{\text{ext}}$.

Next, we calculate the entropy of the black hole. It can be obtained from Hawking-Bekenstein entropy-area law, that is

$$S = \frac{A}{4} = \begin{cases} \frac{\pi b}{2}(\frac{r_+}{b})^{1/3}, & \text{for } \alpha = \pm 1, \\ \frac{\pi b}{2}(\frac{r_+}{b})^{1-2\alpha\gamma}, & \text{for } \alpha \neq \pm 1. \end{cases} \quad (3.6)$$

Also, the black hole's electric potential on the horizon, measured by an observer at the reference point, can be obtained in terms of the null generator of the horizon $\chi^\mu = C\partial^\mu$, as [28,29,30]

$$\Phi = A_\mu \chi^\mu|_{\text{reference}} - A_\mu \chi^\mu|_{r=r_+}. \quad (3.7)$$

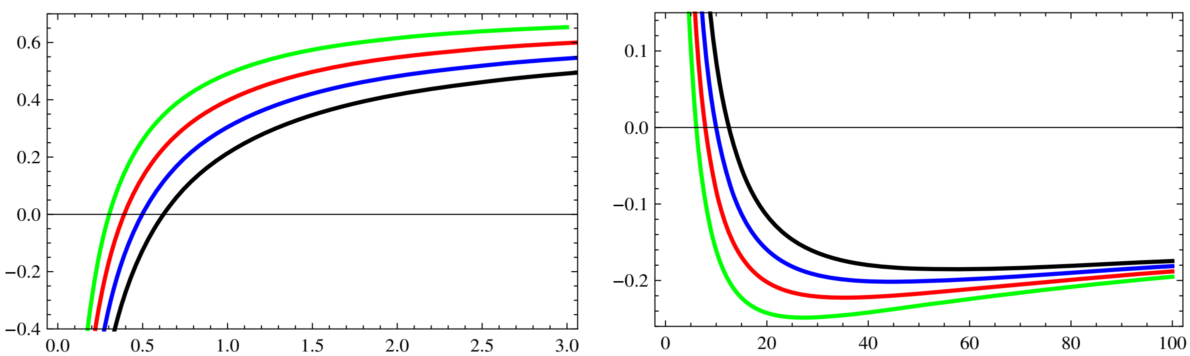

FIG. 3. Assuming $\alpha = \pm 1$, $\Lambda = -1$, $q = 1.5$ and $b = 2$, 2.1, 2.2, 2.3, for Black, blue, red, and green curves, respectively. Left: T versus r_+. Right: $10(\partial^2 M/\partial S^2)_Q$ versus r_+.

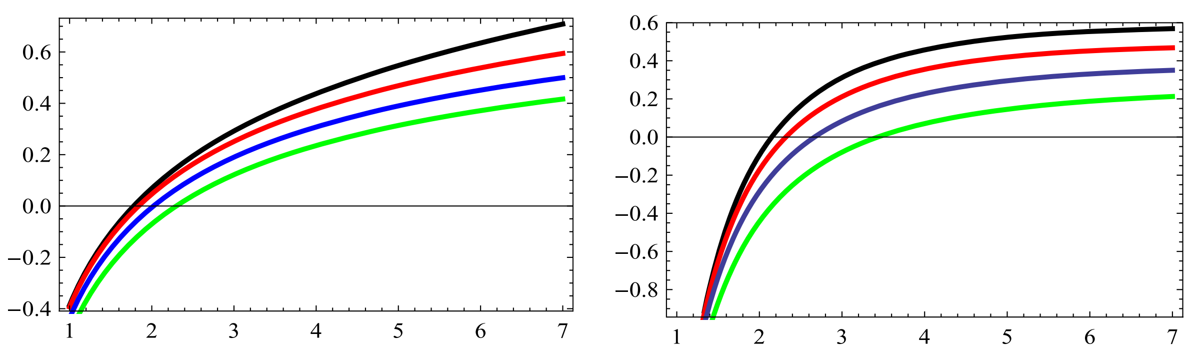

FIG. 4. Assuming $\Lambda = -1$, $q = 2$ and $b = 3$. Left: T versus r_+. Black, red, blue, and green curves are correspond to $\alpha = 0.5$, 0.6, 0.66, 0.7, respectively. Right: $\pi^2(\partial^2 M/\partial S^2)_Q$ versus r_+. Black, red, blue, and green curves are correspond to $\alpha = 0.4$, 0.45, 0.5, 0.55, respectively.

Noting Eq. (2.14) we have

$$\Phi = Cq \ln(r_+/\ell), \tag{3.8}$$

where C is a constant coefficient.

In order to investigate the consistency of these quantities with the thermodynamical first law, from Eqs. (2.20), (3.2), (3.3), and (3.6), we can obtain the black hole mass as the function of extensive parameters S and Q that is

$$M(S,Q) = \begin{cases} \frac{1}{4}[\frac{b^2}{\ell^2} - 4Q^2(1+\frac{1}{3}\ln\frac{b^2}{\ell r_+(S)})]\ln\frac{r_+(S)}{\ell}, & \text{for } \alpha = \pm 1, \\ \frac{2\alpha^2+1}{8(\alpha^2-1)}[\Lambda b^2(\frac{b}{r_+(S)})^{2(3\alpha\gamma-1)} + \frac{8Q^2}{2\alpha^2+1}\ln(\frac{r_+(S)}{\ell})], & \text{for } \alpha \neq \pm 1. \end{cases} \tag{3.9}$$

Note that, if we set $\alpha = 0$ the mass of the black hole reduces to

$$m = \frac{r_+^2}{\ell^2} - 2q^2 \ln\left(\frac{r_+}{\ell}\right), \tag{3.10}$$

which is compatible with that of charged BTZ black hole. Now, we obtain the intensive parameters T and Φ, conjugate to the black hole entropy and charge, respectively. It is a matter of calculation to show that

$$\left(\frac{\partial M}{\partial S}\right)_Q = T \quad \text{for both} \quad \alpha = \pm 1 \quad \text{and} \quad \alpha \neq \pm 1, \tag{3.11}$$

and

$$\left(\frac{\partial M}{\partial Q}\right)_S = \Phi, \tag{3.12}$$

provided that C be equal to $(1-\alpha^2)^{-1}$ in Eq. (3.8) for the cases $\alpha \neq \pm 1$. Also Eq. (3.12) is valid for the case $\alpha = \pm 1$ if C be chosen equal to one (i.e., $C = 1$) and the horizon radius be restricted through the relation $b^2 = \ell r_+$ [32]. Therefore, we proved that the first law of black hole thermodynamics is valid for both classes of the new EMd black holes in the following form

$$dM(S,Q) = \left(\frac{\partial M}{\partial S}\right)_Q dS + \left(\frac{\partial M}{\partial Q}\right)_S dQ. \tag{3.13}$$

IV. HEAT CAPACITY AND STABILITY ANALYSIS

In this stage, we would like to study the local stability or phase transitions of the introduced black holes in the canonical ensemble method. It is well known that black holes, as the thermodynamical systems, are locally stable if their heat capacity is positive. The nonstable black holes may undergo a phase transition to be stabilized. The phase transition points are where the heat capacity vanishes or diverges. In the vanishing points (roots of heat capacity) the phase transition is named conventionally as the type one phase transition. The points where the heat capacity diverges are known as the type two phase transition points. Therefore, the positivity of heat capacity $C_Q = T(\partial S/\partial T)_Q = T/(\partial^2 M/\partial S^2)_Q$ or equivalently the positivity of $(\partial S/\partial T)_Q$ or $(\partial^2 M/\partial S^2)_Q$ with $T > 0$ are sufficient to ensure the local stability of the black holes. Here, we analyze the thermal stability or phase transition of the either of the new EMd black hole solutions, separately.

A. Black holes with $\alpha = \pm 1$

The denominator of the black hole heat capacity can be calculated as

$$\left(\frac{\partial^2 M}{\partial S^2}\right)_Q = \frac{2q^2}{\pi^2 b^2}\left(\frac{b}{r_+}\right)^{2/3}\left[\frac{2}{3}\ln\frac{b}{r_+} + 3 - \left(\frac{b}{q\ell}\right)^2\right]. \tag{4.1}$$

It is understood from Eq. (4.1) that if

$$r_+ \doteq r_0 = b\exp\left[\frac{3}{2}\left(3 - \left(\frac{b}{q\ell}\right)^2\right)\right], \tag{4.2}$$

the denominator of the black hole heat capacity vanishes and black holes with the size satisfying this condition undergo type two phase transition. Note that $r_0 > r_{\text{ext}}$ and as a result the type two phase transition takes place only for the physical black holes. In addition, if $r_+ > r_0$ the heat capacity of the physical black holes is negative and they will be thermodynamically unstable. On the other hand, if $r_+ < r_0$ the denominator of the heat capacity as well as the heat capacity s positive in the range $r_{\text{ext}} < r_+ < r_0$. It means that the physical black holes with the horizon radius in this range are thermodynamically stable. For the unphysical black holes ($r_+ < r_{\text{ext}}$), the denominator is positive, C_Q is negative and they will be locally unstable. The plots of T and $(\partial^2 M/\partial S^2)_Q$ are shown in Fig. 3. The plots show that there is a minimum $r_+ = r_0$ such that $(\partial^2 M/\partial S^2)_Q$ is positive for $r_+ < r_0$. It is evident that the physical black holes with the horizon radius in the range $r_+ < r_0$ are locally stable. Otherwise they are thermally unstable and can undergo phase transition to be stabilized.

B. Black holes with $\alpha \neq \pm 1$

First of all, it must be noted that, according to Eq. (3.4), it is possible for the black hole temperature to vanish at $r_+ = r_{\rm ext}$. Therefore, the black hole heat capacity will vanish too, and they can undergo type one phase transition at this point.

It is a matter of calculation to show that

$$\left(\frac{\partial^2 M}{\partial S^2}\right)_Q = \frac{1+2\alpha^2}{\pi^2}\left[\Lambda(2\alpha^2-1)\left(\frac{b}{r_+}\right)^{\frac{2\alpha^2}{1+2\alpha^2}} - \frac{q^2}{b^2}\left(\frac{b}{r_+}\right)^{\frac{2}{1+2\alpha^2}}\right]. \tag{4.3}$$

The statement given in Eq. (4.3) is the denominator of the black hole heat capacity. It is consist of two terms which, apart from scalar hair, show the contributions from Λ and black hole charge, separately. The charge-term is negative, while the signature of the Λ-term ($\Lambda < 0$) depends on that of $1-2\alpha^2$. If α be chosen such that $1-2\alpha^2 > 0$, one can show that the heat capacity of the EMd black holes diverges at

$$r_+ \doteq r_1 = b\left[\frac{b^2(1-2\alpha^2)}{q^2\ell^2}\right]^{\frac{1+2\alpha^2}{2(\alpha^2-1)}}, \qquad -\frac{1}{\sqrt{2}} < \alpha < \frac{1}{\sqrt{2}}. \tag{4.4}$$

Regarding Eq. (3.4), one can show that $r_1 > r_{\rm ext}$. It means that the heat capacity of the physical EMd black holes diverges and they undergo type one phase transition, while the unphysical black holes do not. Also, the physical black holes with the horizon radius in the range $r_{\rm ext} < r_+ < r_1$ ($T > 0$ and $(\partial^2 M/\partial S^2)_Q < 0$), have negative heat capacity and are unstable. In addition if $r_+ > r_1$, both T and $(\partial^2 M/\partial S^2)_Q$ are positive and physical black holes are locally stable.

On the other hand if $r_+ < r_{\rm ext}$ the denominator of the black hole heat capacity will be negative and the un-physical black holes (having negative temperature) are locally stable. The un-physical black holes do not undergo type two phase transition. The plot of $\pi^2(\partial^2 M/\partial S^2)_Q$ versus r_+ has been shown in Fig. 4, for some alternative values of α.

V. CONCLUSION

Here, we studied the three-dimensional charged black holes as the solutions to the Einstein-Maxwell theory coupled to a dilatonic scalar field. By introducing a circularly symmetric static geometry, we solved the coupled scalar, electromagnetic, and gravitational field equations and obtained two new classes of black hole solutions. Through the consideration of the physical properties of the black hole solutions, obtained here, we found that they do not behave asymptotically as the AdS black holes do. Also, we found that Ricci and Kretschmann scalars diverge at $r = 0$, that means $r = 0$ is an essential (not coordinate) singularity for either classes of the EMd black hole solutions. Furthermore, we showed that one of the black hole solutions, corresponding to $\alpha = \pm 1$, presents extreme and two horizon black holes while the other one which corresponds to $\alpha \neq \pm 1$ presents naked singularity, extreme and two horizon black holes if the parameter α is chosen suitably (see Figs. 1 and 2).

Next, we considered the thermodynamical properties of the new EMd black holes. We calculated the electric charges and masses of the black holes, as conserved quantities, making use of Gauss's law and the Abbott-Deser proposal, respectively. Also, we calculated the entropy, temperature and electric potential using the geometrical methods. On the other hand, through a Smarr-type mass formula, we constructed out the black holes masses as functions of both charge and entropy, as the thermodynamical extensive quantities, from which we calculated the electric potential and temperature, as the thermodynamical intensive quantities, for both of the new EMd black holes. We found that the thermodynamical quantities obtained from geometrical and thermodynamical approaches are identical for either of the black hole classes, confirming the validity of the first law of black hole thermodynamics in the form of Eq. (3.13).

Finally, we analyzed the local stability of both new EMd black holes, using the black hole heat capacity with fixed black hole charge. For the case $\alpha = \pm 1$ we found that the heat capacity vanishes at $r_+ = r_{\rm ext}$. Thus, physical and unphysical black holes can undergo type one phase transition. Also, it is possible for the denominator of the heat capacity to vanish at $r_+ = r_0 > r_{\rm ext}$ for the physical black holes and they may undergo type two phase transition. The physical black holes with the horizon radius in the range $r_{\rm ext} < r_+ < r_0$ are thermodynamically stable. They are unstable in the range $r_+ > r_0$ too. The denominator of the heat capacity does not vanish for the unphysical black holes ($r_+ < r_{\rm ext}$) and they do not undergo type two phase transition. The unphysical black holes are thermodynamically unstable, since they have negative heat capacity [see Eq. (4.1) and Fig. 3]. Furthermore, in the case $\alpha \neq \pm 1$, we showed that type two phase transition can take place only for the physical EMd black holes at $r_+ = r_1$. There is a point of type one phase transition, for both physical and unphysical black holes, located at $r_+ = r_{\rm ext}$ where the black hole temperature vanishes. The physical black holes with $r_+ > r_1$ are stable, they are unstable if their horizon radius is in the range $r_{\rm ext} < r_+ < r_1$. The heat capacity of the unphysical black holes does not diverge and they do not undergo type two phase transition. For the unphysical EMd black holes having $r_+ < r_{\rm ext}$, both T and $(\partial^2 M/\partial S^2)_Q$ are negative and they are stable. [see Eq. (4.4) and Fig. 4].

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