AdS_3 to dS_3 transition in the near horizon of asymptotically de Sitter solutions

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We consider two solutions of Einstein-Λ theory which admit the extremal vanishing horizon (EVH) limit, odd-dimensional multispinning Kerr black hole (in the presence of cosmological constant) and cosmological soliton. We show that the near horizon EVH geometry of Kerr has a three-dimensional maximally symmetric subspace whose curvature depends on rotational parameters and the cosmological constant. In the Kerr-dS case, this subspace interpolates between AdS_3, three-dimensional flat and dS_3 by varying rotational parameters, while the near horizon of the EVH cosmological soliton always has a dS_3. The feature of the EVH cosmological soliton is that it is regular everywhere on the horizon. In the near EVH case, these three-dimensional parts turn into the corresponding locally maximally symmetric spacetimes with a horizon: Kerr-dS_3, flat space cosmology or BTZ black hole. We show that their thermodynamics match with the thermodynamics of the original near EVH black holes. We also briefly discuss the holographic two-dimensional CFT dual to the near horizon of EVH solutions.

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I. INTRODUCTION

Exploring the classical and semiclassical aspects of solutions to gravitational theories may provide a good framework to better understand the nature of gravitational fields and the origin of spacetime, such as the causal structure of spacetime in relativistic theories. In this way, studying the stationary black hole (brane) solutions with their nontrivial causal structures is distinguished. It is known that at the classical level they perform some thermodynamiclike behavior which can be promoted to the real thermodynamics at the semiclassical regime.

Studying black hole physics involves finding and classifying black hole solutions to different gravitational theories in diverse dimensions. Various limits of these black hole solutions and their thermodynamic properties are interesting. In particular, studying the near horizon limit of extremal black branes and black holes (which have vanishing surface gravity) have shed light on black hole physics and (quantum) gravity. The main property of this limit is based on the symmetry enhancement of near horizon geometries at the extremality. The most well-known example is AdS/CFT where the emergence of an AdS throat in the near horizon of extremal p-branes inspires a holographic duality between gravity on AdS_{p+2} space and a strongly coupled CFT [1]. Another example is the near horizon of the extremal black holes. In this case, the enhanced symmetry in the near horizon geometry provides a description for extremal black hole in the context of Kerr/CFT [2,3] or entropy function e.g. [4].

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In addition to symmetry enhancement, the near horizon of generic extremal black holes with smooth horizon enjoys more interesting features: Firstly, they can be considered as a new class of solutions to the same gravity theory of the original black hole. These solutions usually have a two-dimensional maximally symmetric subspace. As is proved in [5–7], this two-dimensional subspace is limited to be AdS_2 or two-dimensional flat by imposing strong energy condition (see [8] for a review.). Generically, in most studied extremal examples, it turns out that this two-dimensional subspace is AdS_2. Secondly, they show a thermodynamiclike behavior which is called near horizon extremal geometry (NHEG) dynamics [9,10].

In above discussion, there is an implicit assumption: Horizon is smooth and the horizon area remains nonzero in the extremal limit. However, there is another class of extremal black holes for which horizon area vanishes in the extremal limit such that

$$A, \kappa \to 0, \qquad \frac{A}{\kappa} = \text{finite}, \tag{1.1}$$

where A and κ denote area and surface gravity of horizon, respectively. These kinds of black holes are called extremal vanishing horizon (EVH) black holes, vanishing of the horizon comes from the vanishing of one-cycle on the horizon [11] (for an incomplete list of EVH examples and their common features see [12–25]). One can study the near horizon geometry of EVH black holes similar to the generic extremal black holes.

These near horizon EVH geometries are also solutions to the same original theory. Interestingly enough, most EVH black holes admit a three-dimensional subspace, which is generically pinching AdS_3, as the near horizon geometry. In the case of near EVH black holes, this AdS_3 should be replaced with a BTZ black hole [11,26,27]. Thermodynamics of these BTZ black holes represents the thermodynamics of original EVH black holes around the EVH limit [11,28].

It is worthwhile to mention that the existence of the AdS_3 throat suggests a dual description in terms of AdS_3/CFT_2 for the near horizon physics of the EVH black hole and its excitations. In fact, in the four-dimensional case [11], it has been proven that the near horizon limit of a (near) EVH black hole is a decoupling limit. These suggest that a dual description for near horizon physics of EVH black holes in terms of a two-dimensional CFT which is called EVH/CFT [11].

The near horizon structure of EVH black holes has been studied in [26,27]. A summary of these results is in the following theorems:

(i) *Theorem I.* The near horizon of any EVH black hole in Einstein-Maxwell-Scalar-Λ theory which has a finite energy momentum tensor at the horizon has a three-dimensional maximally symmetric subspace.

(ii) *Theorem II.* The strong energy condition.[1] excludes the three-dimensional de Sitter space (dS_3) in the near horizon of EVH black holes which are asymptotically flat or AdS.

These theorems do not exclude the existence of dS_3 and three-dimensional flat spacetime in the near horizon geometry once the spacetime is asymptotically dS. Indeed, exploring this possibility is one of the main motivation of this paper. In other words, we are mainly interested in the near horizon structure of asymptotically dS EVH solutions.

The presence of a subspace with positive curvature in the near horizon of asymptotically dS black holes is not new. For example, the near horizon of the dS-Schwarzschild black hole solution in the extremal limit (when the cosmological and black hole horizons coincide) is $dS_2 \times S^n$ denoted as Nariai solution [29–33].

This paper is organized as follow. In the first section, after a short review on the multispinning Kerr black hole in the presence of a cosmological constant, we will study the (near) EVH limit and its near horizon limit in odd dimensions. In particular we will show that the three-dimensional part of the near horizon geometry of EVH Kerr-dS can be either AdS_3, dS_3 or three-dimensional flat. In the case of the near EVH limit, we also study the relation between the thermodynamics of the EVH and the three-dimensional part of the near horizon. In Sec. II, we will study EVH limit of the cosmological soliton. In particular, we will show that the near horizon geometry enjoys a dS_3 subspace. The last section is devoted to discussions.

[1]We note that here the energy condition is on the matter energy momentum tensor *without* cosmological constant.

II. KERR BLACK HOLES IN HIGHER DIMENSIONS

In this section, we will apply the (near) EVH limit to the multispinning Kerr black hole in higher dimensions and in the presence of a (possible) cosmological constant [34,35]. This spacetime generalizes the single spinning four-dimensional Kerr black hole to multispinning higher-dimensional black hole and in the vanishing cosmological constant case, it reduces to the Myers-Perry solution [36]. It solves the following Einstein equation,

$$R_{\mu\nu} = (d-1)\lambda g_{\mu\nu}, \tag{2.1}$$

in $d = 2n + 1 + \alpha$ dimensions and its metric is given by [34,35]

$$\begin{aligned} ds^2 = &-W(1-\lambda r^2)dt^2 + \frac{2m}{VF}\left(W dt - \sum_{i=1}^{n}\frac{a_i}{\Xi_i}\mu_i^2 d\varphi_i\right)^2 \\ &+ \sum_{i=1}^{n}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2 d\varphi_i^2 + \frac{VF dr^2}{V-2m} + \sum_{i=1}^{n+\alpha}\frac{r^2+a_i^2}{\Xi_i}d\mu_i^2 \\ &+ \frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{n+\alpha}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i\right)^2. \end{aligned} \tag{2.2}$$

Here, α is the "even-ness" parameter such that it equals 1 in even dimensions and 0 otherwise. In addition, one should set $a_{n+1} = 0$ in even dimensions. In general, metric functions are

$$\Xi_i \equiv 1 + \lambda a_i^2, \qquad W \equiv \sum_{i=1}^{n+\alpha}\frac{\mu_i^2}{\Xi_i}, \tag{2.3}$$

$$\begin{aligned} V &\equiv r^{\alpha-2}(1-\lambda r^2)\prod_{i=1}^{n}(r^2+a_i^2), \\ F &\equiv \frac{1}{1-\lambda r^2}\sum_{i=1}^{n+\alpha}\frac{r^2\mu_i^2}{r^2+a_i^2}. \end{aligned} \tag{2.4}$$

This solution is described by the parameters m and a_i's which are respectively related to the mass and rotations. In the case of $m = 0$, this solution is nothing but (A)dS. On the other hand, in the case of $a_i = 0$ with nonvanishing m, spherical symmetry restores and this solution reduces to the known Schwarzschild-(A)dS black hole. This geometry is written in a coordinate system with $(n+\alpha)$ number of latitudinal coordinates μ_i which are constrained by

$$\sum_{i}^{n+\alpha}\mu_i^2 = 1, \tag{2.5}$$

where $\mu_i \in [0,1]$ for $1 \le i \le n$, $\mu_{n+1} \in [-1,1]$ for even-dimensional cases and n number of azimuthal angular coordinates $\varphi_i \in [0, 2\pi]$. In general, this black hole solution may admit several horizons: inner and outer black hole horizons and a cosmological horizon for $\lambda > 0$. All of them are determined by the roots of the following equation,

$$V(r = r_h) = 2m. \tag{2.6}$$

Let us consider a typical horizon H which is specified by radius r_h. One may show this horizon is generated by the following Killing vector

$$\xi_H = \frac{\partial}{\partial t} + \Omega_H^i \frac{\partial}{\partial \varphi^i}, \qquad \Omega_H^i = \frac{a_i(1-\lambda r_h^2)}{(r_h^2 + a_i^2)}, \tag{2.7}$$

where Ω_H^i is the angular velocity of the horizon along φ_i. Then, the surface gravity computation gives

$$\kappa_H = \frac{(1-\lambda r_h^2)}{4m} V'(r = r_h). \tag{2.8}$$

Entropy and temperature of this horizon are

$$S = \frac{A_H}{4G_d} = \frac{\mathcal{A}_{d-2}}{4G_d} r_h^{\alpha-1} \prod_{i=1}^{n} \frac{r_h^2 + a_i^2}{\Xi_i},$$
$$T = \frac{\kappa_H}{2\pi} = \frac{1}{2\pi}\left[r_h(1-\lambda r_h^2)\left(\sum_{i=1}^{n} \frac{1}{r_h^2 + a_i^2} + \frac{\alpha}{2r_h^2}\right) - \frac{1}{r_h}\right], \tag{2.9}$$

in which $\mathcal{A}_n$ is the volume of a unit n-sphere. The mass and angular momenta of this solution are given by [37][2]

$$M = \frac{m\mathcal{A}_{d-2}}{4\pi G_d \prod_j \Xi_j}\left(\sum_{i=1}^{n} \frac{1}{\Xi_i} + \frac{\alpha - 1}{2}\right),$$
$$J_i = \frac{m\mathcal{A}_{d-2}}{4\pi G_d (\prod_j \Xi_j)} \frac{a_i}{\Xi_i}. \tag{2.10}$$

Using these quantities, one can check the first law holds

$$\delta M = T\delta S + \sum_i \Omega_H^i \delta J_i, \tag{2.11}$$

here, δ denotes all possible variations in the parameter space of Kerr-(A)dS solution, i.e. $\{m, a_1, a_2, \ldots, a_n\}$.

A. (Near) EVH limit

In the following, we explore the EVH limit of the Kerr black hole metric given by (2.2). From Eq. (2.8), it is clear that the extremal limit, $\kappa_H \to 0$, is simply given by the condition $V'(r = r_h) = 0$ while r_h is a root of (2.6). To find the extremal vanishing horizon limit, we need to take vanishing horizon limit, $A_H \to 0$, as well, such that A_H/κ_H is fixed. Note that, generically, an extremal limit happens when two horizons degenerate and we define the vanishing horizon limit for the corresponding horizons.

In the case of solution (2.2), one can check that there is no such EVH limit in even dimensions. Therefore, in what follows, we only consider odd dimensions and simply set $\alpha = 0$.

To find the EVH limit, we simplify the entropy using the equation $V(r_h) = 2m$ (2.6),

$$A_H = \frac{2m\mathcal{A}_{d-2}}{(1-\lambda r_h^2)}\left(\prod_{i=1}^{n} \frac{1}{\Xi_i}\right) r_h. \tag{2.12}$$

It is clear that entropy is not vanishing unless r_h goes to zero. This can not be compatible with the equation which gives the location of the horizon (2.6),

$$\frac{(1-\lambda r_h^2)}{r_h^2} \prod_{i=1}^{n} (r_h^2 + a_i^2) = 2m, \tag{2.13}$$

unless one of a_i's is zero. We assume that the zero rotation parameter is along the φ_1 direction and set $a_1 = 0$. This assumption simplifies the surface gravity expression to

$$\kappa_H = \frac{(1-\lambda r_h^2)}{4m} V'(r_h)\Big|_{a_1=0} = \left(\sum_{i=2}^{n} \frac{1-\lambda r_h^2}{r_h^2 + a_i^2} - \lambda\right) r_h, \tag{2.14}$$

which is also proportional to r_h. So, by setting $a_1 = 0$ and $r_h = 0$ one can obtain an extremal black hole with vanishing horizon area. To show that the ratio of A_H and κ_H remains finite, we need to do more careful analysis. Actually, from the above argument and Eq. (2.13), one may deduce it is necessary to set $a_1 = 0$ first and then $r_h = 0$. In other words, we should consider the following scaling limit[3]

$$r_h = \rho_0 \epsilon, \qquad a_1 = a_0 \epsilon^2, \qquad \epsilon \to 0. \tag{2.15}$$

Now, we can obtain surface gravity κ_H and area of the horizon A_H via these scaling limits

$$\kappa_H = -\left(\frac{a_0^2}{\rho_0^4} + \lambda_3\right)\rho_0 \epsilon, \qquad A_H = \mathcal{A}_{d-2}\left(\prod_{i=2}^{n} \frac{a_i^2}{\Xi_i}\right)\rho_0 \epsilon, \tag{2.16}$$

in which, for convenience, λ_3 is defined by

[2] Although the expressions of thermodynamic quantities are given for negative λ in that reference but those are valid for positive cosmological constant as well. This has been checked by calculating the charges via symplectic phase space method [38].

[3] In general, one can scale a_i with ϵ^β as far as $\beta \geq 2$. However, the near horizon limit of the near-EVH with $\beta = 2$ includes also $\beta > 2$ cases.

$$\lambda_3 \equiv \lambda - \sum_{i=2}^{n} \frac{1}{a_i^2}. \tag{2.17}$$

It manifestly shows $\frac{A_H}{\kappa_H}$ is finite in $\epsilon \to 0$ limit. One may note that the surface gravity may become negative for some values of parameters. However, as we will discuss later, it is not a serious issue and one can show either the negative temperature is due to cosmological horizon (for $\lambda > 0$) or violation of extremality bound.

Note that EVH limit (2.15) should be also compatible with $V(r_h) = 2m$. Therefore, we need to fix the value of m in an appropriate way. Consequently, we add the proper scaling of m to (2.15) and take the following limit as the EVH limit of Kerr black hole (2.2) in odd dimensions

$$\begin{aligned} r_h &= \rho_0 \epsilon, \qquad a_1 = a_0 1 \epsilon^2, \\ m &= \frac{1}{2} \prod_{i=2}^{n} a_i^2 + \tilde{m}\epsilon^2, \qquad \epsilon \to 0, \end{aligned} \tag{2.18}$$

where the parameter $\tilde{m}$ is given by

$$\tilde{m} = \frac{\rho_0^2}{2} \left(\frac{a_0^2}{\rho_0^4} - \lambda_3 \right) \prod_{i=2}^{n} a_i^2. \tag{2.19}$$

For later use, we also apply this limit to the angular velocity and the momentum along φ_1

$$\Omega^1 = \frac{a_0}{\rho_0^2} + \mathcal{O}(\epsilon^2), \qquad J_1 = \frac{\mathcal{A}_{d-2}}{8\pi G_d} \prod_{i=2}^{n} \frac{a_i^2}{\Xi_i^2} a_0 \epsilon^2 + \mathcal{O}(\epsilon^4). \tag{2.20}$$

One may note the expansions of J_i and Ω^i along the other directions ($i \neq 1$) start from the zeroth order of ϵ.

There is an interesting interpretation for $\tilde{m}$. Indeed, one can simply check that this term does not contribute to any quantities in the EVH limit. However, for the near EVH case, this term becomes important and changes the mass of the black hole above the EVH. In other words, we can interpret $\tilde{m}$ as excitations above the EVH surface in the parameter space [21,28]. Moreover, by eliminating λ_3 between $\tilde{m}$ and κ_H in (2.16), we find $\tilde{m}$ depends on the temperature $T = \frac{\kappa_H}{2\pi}$ and $\frac{a_0 1^2}{\rho_0^2} \sim \Omega_H^1 J_1$ in this way,

$$\tilde{m}\epsilon^2 = \frac{8\pi G_d \prod_{i=2}^{n} \Xi_i}{\mathcal{A}_{d-2}} \left(\frac{1}{2} TS + \Omega_H^1 J_1 \right) + \mathcal{O}(\epsilon^4). \tag{2.21}$$

Curiously, the expression inside the parentheses is exactly the Smarr mass formula for a three-dimensional BTZ black hole. It suggests that the excitations of EVH black holes may be governed by a three-dimensional gravity [11,28]. However, one should note that this result is independent of the λ_3. We will come back to this point later.

B. Horizon structure in the EVH limit

In the previous subsection, we considered the EVH limit for a typical horizon H. However, the metric (2.2) admits various types of horizons: cosmological, inner or outer horizon. In what follows, we investigate under which conditions cosmological horizon (if exists) coincides with a black hole horizon or two black hole horizons degenerate in the EVH limit. As a part of these conditions, the sign of λ_3 would be constrained in each situation.

Let us consider the horizon at $r = 0$, in the EVH limit,

$$a_1 = 0, \qquad 2m = \prod_{i=2}^{n} a_i^2. \tag{2.22}$$

In general, to specify the type of the horizon we analyze $V(r) - 2m$ whose roots determine the locations of the horizons. Applying the EVH conditions (2.22) to this expression gives

$$V(r) - 2m = (1 - \lambda r^2)\Pi - 2m, \qquad \Pi \equiv \prod_{i=2}^{n} (r^2 + a_i^2), \tag{2.23}$$

that Π is polynomial function of r and can be rewritten as a summation,

$$\Pi = \sum_{p=0}^{n-1} C_p (r^2)^p, \tag{2.24}$$

where the C_p coefficients are defined by

$$C_p \equiv \sum_{i_1 < \cdots < i_{(n-p-1)} = 2}^{n} a_{i_1}^2 a_{i_2}^2 \cdots a_{i_{(n-p-1)}}^2, \quad C_{n-1} \equiv 1. \tag{2.25}$$

In particular, for C_0 and C_1, we have

$$C_0 = \prod_{i=2}^{n} a_i^2, \qquad C_1 = C_0 \sum_{i=2}^{n} \frac{1}{a_i^2}. \tag{2.26}$$

Substituting (2.24) into (2.23), we obtain

$$\begin{aligned} V(r) - 2m = (C_0 - 2m) &+ \sum_{p=1}^{n-1} (C_p - \lambda C_{p-1}) r^{2p} \\ &- \lambda C_{n-1} r^{2n}. \end{aligned} \tag{2.27}$$

Again, using EVH conditions (2.22), we have $C_0 = 2m$ and by defining $C_n = 0$, we arrive at the following form for the exact expansion of $V(r) - 2m$,

$$V(r) - 2m = \sum_{p=1}^{n} c_p r^{2p}; \qquad c_p \equiv (C_p - \lambda C_{p-1}). \tag{2.28}$$

Near the origin, $r = 0$, the most dominant term of $V(r) - 2m$ comes from the smallest power of r, i.e. r^2, which is

$$c_1 = C_1 - \lambda C_0 = C_0\left(\sum_{i=2}^{n}\frac{1}{a_i^2} - \lambda\right) = -2m\lambda_3, \tag{2.29}$$

and far from the origin, the term r^{2n} is dominant with the coefficient $c_n = -\lambda$. Depending on the sign of λ, these coefficients can be positive or negative. In the following, we explore each case of positive/negative λ separately.

1. Kerr-dS ($\lambda > 0$)

In the presence of positive cosmological constant, the Einstein equation admits spacetimes with a cosmological horizon. So, the type of multispinning Kerr-dS EVH black hole horizons are more complicated. There are three types of degenerate horizon: (i) the outer horizon of the black hole coincides with the cosmological horizon, (ii) the outer horizon comes to the inner horizon and (iii) the cosmological, outer and inner horizons coincide. Following the nomenclature of [39] for the extremal four-dimensional Kerr-dS, we also call them the Nariai, cold and ultracold limit, respectively.

Nariai limit For the asymptotically dS spaces, c_n is negative, then for the ranges of parameters a_i's where all c_p's are negative, $V(r) - 2m$ has no positive root except at $r = 0$ (see Fig. 1). This condition,

$$c_p < 0 \quad \forall\ p = 1, 2, \ldots, n, \tag{2.30}$$

also includes $c_1 < 0$ which translates to $\lambda_3 > 0$. As we will show in the next section it implies that the near horizon geometry have a dS$_3$ part.

Cold limit This limit only happens when the spacetime has a cosmological horizon at $r > 0$ (see Fig. 1). As we will show in the following, it indicates $\lambda_3 < 0$. Descartes' rule of signs implies the existence of the cosmological horizon is only possible when at least a positive c_p exists. For the case of

$$c_1 > 0(\lambda_3 < 0), \qquad c_p < 0, \quad \forall\ p > 1, \tag{2.31}$$

we have the possibility of at most a positive root for $V(r) - 2m$. Accordingly, the slope and concavity of this function is also positive at the origin ($c_1 > 0$) and it goes to minus infinity in the large r region ($c_n < 0$), then it certainly has that positive root, at $r > 0$. As we mentioned, this root is corresponding to the cosmological horizon. Therefore, in this case, the degenerate horizon at $r = 0$ comes from the coincidence of black hole horizons (cold limit).

Now let us assume $c_1 < 0$ ($\lambda_3 > 0$) and that the cosmological horizon exists. So, c_p's must be positive for some $p > 1$. Using the asymptotic behavior of $V(r) - 2m$ and its concavity at the origin and the cosmological horizon, one can deduce there is another root between the origin and the cosmological horizon. In this case, two inner black hole horizons are degenerate at $r = 0$ (we do not study this case anymore since we are interested in the thermodynamics of outer black hole horizon).

Ultracold limit(s) Up to here, we have assumed that none of the c_p's are zero and the horizon at $r = 0$ is just a double root or equivalently $r^2 = 0$ is a simple root (since $a_{i\neq 1}^2 > 0$). However, Kerr-dS admits $c_p = 0$, then $r^2 = 0$ can be n-tuple root by requiring some exact relations between the parameters a_i and λ. The simplest case is $c_1 = 0$. In this case,

$$c_1 = 0 = \lambda_3 \Rightarrow \lambda = \sum_{i=2}^{n}\frac{1}{a_i^2}. \tag{2.32}$$

Again, the condition,

$$c_p < 0, \quad \forall\ p > 2, \tag{2.33}$$

guarantees that the root at $r = 0$, is the largest horizon (see Fig. 1). The only difference with the condition (2.30) is that $r^2 = 0$ is a double root now, instead of being a simple root. Thus, a cosmological horizon coincides with black hole inner and outer horizons which gives the ultracold limit. We note that if $c_1 \neq 0$ and one of the other c_p's vanishes, the root at $r^2 = 0$ does not change its type because the largest term in the origin comes from the smallest power of r. Therefore, to have higher q-tuple root at $r^2 = 0$, all c_p's for $p \leq q$ should be zero.

The summary of the results is

$$\begin{aligned} &\text{Double root at } r^2 = 0\colon\ c_1 = 0, \\ &\text{Triple root at } r^2 = 0\colon\ c_1 = 0 \quad \text{and} \quad c_2 = 0, \\ &\text{q-tuple root at } r^2 = 0\colon\ c_p = 0 \quad \forall\ p < q. \end{aligned} \tag{2.34}$$

2. Kerr-(AdS) ($\lambda \leq 0$)

In these cases, c_p's are always positive and there is no horizon for $r > 0$ (see Fig. 2). The degenerate horizon at origin is due to the coincidence black hole outer and inner horizons.

C. Near horizon geometries

The near horizon geometry of extremal four-dimensional Kerr-(A)dS (nonvanishing horizon area) has been studied in [40,41] and in the asymptotically AdS case in higher dimensions [42].[4] For completion, we also mention the near horizon geometry of extremal Kerr for generic λ in

[4] Recently, the near horizon of extremal Kerr-(A)dS-NUT solution in even dimensions has been studied in [43]. By setting the NUT charge to zero, one can recover the extremal limit of the even-dimension extremal Kerr.

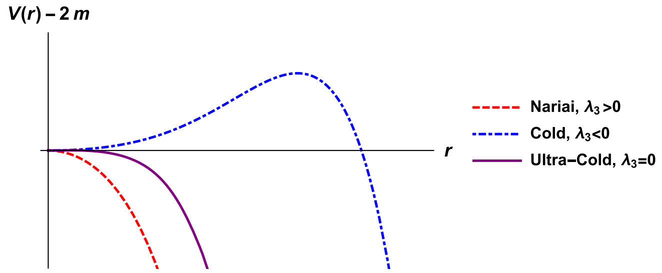

FIG. 1. Roots of Kerr-dS black hole ($\lambda > 0$) in the EVH limit.

Appendix A. In the following, we study the near horizon geometry of Kerr-dS black hole in the EVH limit.

It is more convenient to use the Kerr-Schild form of the metric (B2). To obtain the near horizon limit of EVH, we apply the EVH limit (2.18) and the following transformation

$$r = \gamma\rho + r_h, \qquad \bar{\tau} = \frac{v}{\gamma}, \qquad \bar{\varphi}_1 = \frac{\psi}{\gamma},$$
$$\bar{\varphi}_{i\geq 2} = \phi_i - \Omega_H^i \bar{\tau}, \qquad \gamma \to 0, \tag{2.35}$$

to the metric (B2) and assume $\epsilon \ll \gamma$. In this case, the near horizon of EVH black hole reads as

$$ds^2_{NH} = \mu_1^2\left[\sigma \frac{\rho^2}{\ell_3 3^2} dv^2 + 2dv d\rho + \rho^2 d\psi^2\right] + h_{ij}(r_h) d\phi^i d\phi^j + k_{ij}(r_h) d\mu^i d\mu^j, \tag{2.36}$$

where h_{ij} and k_{ij} can be read from (2.2) or (B2) whose explicit forms are as follows,

$$h_{ij}(r_h) = \frac{a_i^2\mu_i^2}{\Xi_i}\delta_{ij} + \frac{\mu_i^2\mu_j^2}{\mu_1^2}\frac{a_i a_j}{\Xi_i \Xi_j},$$
$$k_{ij}(r_h) = \frac{a_i^2}{\Xi_i}\delta_{ij} + \lambda \frac{\mu_i \mu_j}{W}\frac{a_i^2 a_j^2}{\Xi_i \Xi_j}, \tag{2.37}$$

where i, j run from 2 to n. Also, μ_i's are restricted by $\sum_{i=2}^n \mu_i^2 = 1 - \mu_1^2$. In addition, we introduced the

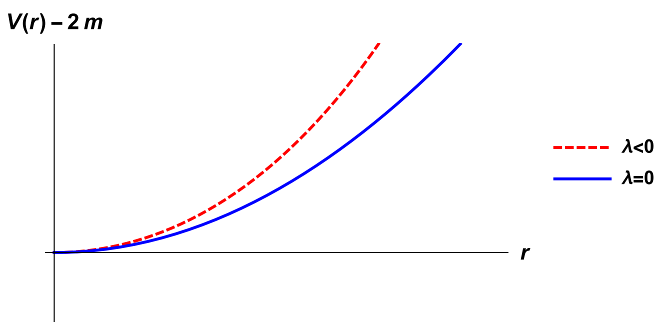

FIG. 2. Root of Kerr black hole for $\lambda \leq 0$ in the EVH limit. There is no horizon for $r > 0$. In these cases $\lambda_3 < 0$.

three-dimensional length scale $\ell_3 3$ for convenience and a sign bookkeeper σ via

$$\lambda_3 = \frac{\sigma}{\ell_3 3^2}; \qquad \sigma = (0, \pm 1). \tag{2.38}$$

One may note that three coordinates (v, ρ, ψ) make a three-dimensional maximally symmetric spacetime and λ_3 determines its curvature. The three-dimensional part is dS_3 when $\lambda_3 > 0$, three-dimensional flat for $\lambda_3 = 0$ and AdS_3 once $\lambda_3 < 0$. This three-dimensional spacetime comes form joining two-dimensional maximally symmetric subspace of near horizon of the extremal black hole and extra coordinate φ_1 due to vanishing horizon limit.

Using Eq. (2.17), it is clear that λ_3 is always negative or $\sigma = -1$, for $\lambda \leq 0$. This result is in agreement with theorems of [26,27] which imply that for nonpositive cosmological constant($\lambda \leq 0$) EVH black hole in the Einstein-Λ theory has an AdS_3 in the near horizon geometry. In this case, the near horizon geometry (2.36) lies in the classified solutions with SO(2,2) symmetry in [44]. Besides, from Eq. (2.17), one can see the positivity of the cosmological constant $\lambda > 0$ since Kerr-dS admits three-dimensional flat and dS_3 along with AdS_3. One may note that theorems of [26,27] do not exclude these possibilities. Actually, one of the main motivations of this paper is to study this possibility.

Another interesting result is that $\lambda_3 = 0$ is exactly the condition (2.32) for $r^2 = 0$ to be a double root of $V(r) - 2m = 0$. Since all higher n-tuple roots assume vanishing of the second derivative, the near horizon EVH geometries for all of those cases include three-dimensional flat spacetime.

We emphasize that this geometry is regular everywhere except at $\mu_1 = 0$ on which the Kretschmann scalar blows up. This is a typical features of EVHs [11,12].

1. Near horizon of near EVH case

In the previous discussion, we consider the near horizon limit for the case $\epsilon \ll \gamma$. However, in the $\gamma \sim \epsilon$ limit, we find a more general near horizon geometry which is called near horizon near EVH geometries. So, once again, we apply (2.18) and (2.35), but this time we assume $\epsilon \sim \gamma$. Up to a shift $\rho \to \rho - \rho_0$, it leads to

$$ds^2_{NH} = \mu_1^2 ds_3^2 + h_{ij}(r_h) d\phi^i d\phi^j + k_{ij}(r_h) d\mu^i d\mu^j, \tag{2.39}$$

where ds_3^2 is defined by

$$ds_3^2 = -f(\rho) dv^2 + 2d\rho dv + \rho^2\left(d\psi - \frac{a_0}{\rho^2} dv\right)^2, \tag{2.40}$$

and $f(\rho)$ is given by

$$f(\rho) = \frac{(\rho^2-\rho_0^2)(-\sigma\rho^2 - \frac{a_0^2\ell_3 3^2}{\rho_0^2})}{\rho^2\ell_3 3^2}. \tag{2.41}$$

This metric reduces to the near horizon metric of EVH (2.36) for $a_0 1 = \rho_0 = 0$. The corresponding three-dimensional part of the above geometry, depending on σ, is BTZ or Kerr-dS$_3$ [45] or three-dimensional flat space cosmology (FSC) [46,47]. Metric functions $f(\rho)$ suggests all of these three-dimensional spacetimes have horizon and one can attribute temperature or entropy to their horizon.[5] In what follows, we survey the thermodynamics of this three-dimensional part.

D. Thermodynamics of the EVH near horizon

As we mentioned before, the near horizon of EVH black hole admits a three-dimensional maximally symmetric subspace which is replaced by a more general three-dimensional spacetime in the near EVH. We take this three-dimensional subspace as a solution to a three-dimensional gravity which is obtained by a Kaluza-Klein reduction over the $\mathcal{M}_{d-3}$ manifold via reduction ansatz (2.39). It is easy to show that the three-dimensional Newton constant, G_3, is given in terms of the d-dimensional Newton constant, G_d, as

$$G_3 = \frac{2\pi G_d}{\mathcal{A}_{d-2}}\prod_{i=2}^{n}\frac{\Xi_i}{a_i^2}. \tag{2.43}$$

Since the three-dimensional metric has horizon, we can study its thermodynamic in the context of the mentioned thee-dimensional gravity. Using the standard methods, charges and chemical potentials of this three-dimensional spacetime are obtained as

$$M_3 = \frac{a_0^2\ell_3 3^2 - \sigma\rho_0^4}{8G_3\ell_3 3^2\rho_0^2}\epsilon, \qquad S_3 = \frac{\pi\rho_0}{2G_3}\epsilon, \qquad J_3 = \frac{a_0}{4G_3}\epsilon,$$
$$T_3 = \frac{1}{2\pi\ell_3 3^2}\frac{-\sigma\rho_0^4 - a_0^2\ell_3 3^2}{\rho_0^3}, \qquad \Omega_3 = \frac{a_0}{\rho_0^2}, \tag{2.44}$$

where ϵ factor comes from periodicity of $\psi \in [0, 2\pi\epsilon]$ due to the near horizon limit. Interestingly enough, there is one to one correspondence between these quantities and the thermodynamic quantities of original black hole

$$T = \epsilon T_3, \qquad \Omega_3 = \Omega_H^1,$$
$$S = S_3, \qquad J_1 = \epsilon J_3. \tag{2.45}$$

In addition, the mass of three-dimensional spacetime is related to the excitation of mass parameter m above the EVH limit, i.e. $\tilde{m}$ given in (2.19) through

$$M_3 = \frac{\mathcal{A}_{d-2}}{8\pi G_d\prod_{i=2}^{n}\Xi_i}\tilde{m}\epsilon. \tag{2.46}$$

As we mentioned below Eq. (2.21), it represents Smarr mass formula $M_3 = \frac{1}{2}T_3S_3 + \Omega_3 J_3$. It is straightforward to see these quantities satisfy the first law of thermodynamics for the three-dimensional solution,

$$\delta_\perp M_3 = T_3\delta_\perp S_3 + \sum_i \Omega_3\delta_\perp J_3, \tag{2.47}$$

where $\delta_\perp$ refers to variations in the parameter space of the three-dimensional solution, i.e. $\{a_0, \rho_0\}$. This is a subclass of parametric variations of Kerr-dS which controls the distance from the EVH surface in the black hole parameter space, whereas varying $a_{i\neq 1}$ would maintain the solution EVH in the EVH limit (2.18). In this sense, following [21,28], we call the first subclass "normal" variations $\delta_\perp$ and the latter, "parallel" variations $\delta_\parallel$. In other words, it means the generic variation δ has a decomposition as $\delta = \delta_\perp + \delta_\parallel$. Using these, it is easy to see that $T\delta_\perp S = T_3\delta_\perp S_3$, which implies the following relation between the first law of the near EVH black hole and the corresponding first law for three-dimensional space in the near horizon,

$$T\delta_\perp S = \delta_\perp M - \sum_i \Omega_H^i \delta_\perp J_i,$$
$$\Downarrow \tag{2.48}$$

$$T_3\delta_\perp S_3 = \delta_\perp M_3 - \Omega_3\delta_\perp J_3. \tag{2.49}$$

Indeed, it generalizes the relation between near EVH black holes thermodynamics and BTZ thermodynamics [21,28] to general three-dimensional locally maximally symmetric spaces.

One may note the temperature of the three-dimensional spacetime is always negative for $\sigma \geq 0$ ($\lambda_3 \geq 0$). From the three-dimensional point of view, these cases present cosmological horizon of a Kerr-dS$_3$ and cosmological flat solution. For these spacetimes, the negativity of the cosmological horizon temperature is known [48]. Besides, from the original EVH black hole perspective, as we mentioned before, $\lambda_3 \geq 0$ is corresponding to Nariai and ultracold limits where in the both of them the cosmological horizon is involved. Therefore, one can relate the negativity of the temperature to cosmological nature of

[5] We note that using the transformation $dv = dt + \frac{d\rho}{f(\rho)}$, the metric takes the standard form of BTZ, Kerr-dS$_3$ and flat space cosmology metric,

$$ds^2 = -f(\rho)dt^2 + \frac{d\rho^2}{f(\rho)} + \rho^2(d\tilde{\psi} - \Omega dt)^2. \tag{2.42}$$

the horizon. In addition, for $\sigma = -1$ where the three-dimensional part is a BTZ black hole, one must assume $\rho_0^2 > a_0 1\ell_3 3$ to preserve the extremality bound of the near EVH black hole and the corresponding BTZ black hole. Clearly, it implies the positivity of temperature.

E. Cardy-like formula and black hole entropy

We have seen the correspondence between thermodynamic quantities of the near horizon geometry and the EVH black hole. For $\lambda_3 < 0$, the existence of AdS_3 geometry suggests a CFT_2 dual description for the physics of the near horizon geometry. In particular, one can apply Cardy formula to obtain the entropy of the near horizon BTZ black hole [49] which equals to the original near EVH black hole entropy [11]. In Appendix C, we study this case.

For $\lambda_3 \geq 0$, where the three-dimensional part of the near horizon is either locally flat or dS_3, there are several proposals for dual description of these geometries in the context of dS/CFT [50–52] and flat space holography [53,54]. Although these proposal generically have some problems with unitarity, but one may apply their procedures and use the Cardy-like formula to Kerr-dS_3 or three-dimensional FSC and obtain the entropy.

III. COSMOLOGICAL SOLITON

In this section, we will study the EVH limit of cosmological soliton. This geometry is asymptotically dS and as we will show, and it also admits a dS_3 in the near horizon region of its EVH limit. The metric of the cosmological soliton in odd $d(= 2n + 1)$ dimensions is given by [55,56]

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)f(r)} + \left(\frac{r}{n}\right)^2 f(r)\left(d\psi + \sum_{i=1}^{n-1}\cos\theta_i d\phi_i\right)^2 + \frac{r^2}{2n}\sum_{i=1}^{n-1} d\Sigma_i^2, \tag{3.1}$$

where

$$g(r) = 1 - \frac{r^2}{\ell^2}, \quad f(r) = 1 - \frac{a^{2n}}{r^{2n}}, \quad d\Sigma_i^2 = d\theta_i^2 + \sin^2\theta_i d\phi_i^2, \tag{3.2}$$

θ_i and ϕ_i parametrize $(n-1)$ numbers of 2-spheres, so $\theta_i \in [0, \pi]$ and $\phi_i \in [0, 2\pi]$. This metric solves the Einstein Eq. (2.1) with

$$\lambda = \frac{\sigma}{\ell^2}, \qquad \sigma = 0, \pm 1, \tag{3.3}$$

where $\sigma = 0, -1$ represent asymptotically flat and AdS spaces respectively which do not admit EVH limit. Thus, we only consider the $\sigma = +1$ case in the following. This solution is described by two parameters a and ℓ. Since the factor $f(r)$ takes both negative and positive values and ψ is periodic, one may worry about the existence of closed timelike curve (CTC). To avoid this issue, one needs to fix the range of parameters and coordinates. Besides, we are also interested in the static patch of the solution. Thus, to find a static CTC-free patch, we will analyze the metric in the following. Let us consider two cases where $a > \ell$ or $a < \ell$ and determine the sign of each metric components. (We will come back to the case $a = \ell$ later when we study the EVH limit.) The summary of results is given in Tables I and II for $a < \ell$ and $a > \ell$ cases, respectively.

TABLE I. The $a < \ell$ case.

r	0		a		ℓ		∞
$g(r)$		+		+		−	
f(r)		−		+		+	
$\Vert\partial_r\Vert^2$		−		+		−	
$\Vert\partial_t\Vert^2$		−		−		+	
$\Vert\partial_\psi\Vert^2$		−		+		+	

As is clear from Table I, in the case of $a < \ell$, this metric has timelike Killing vector (∂_t) in the region $r \leq \ell$ and so is static in that region. Meanwhile, it has CTC along ψ direction in the region $r < a$, therefore we restrict our study to the region $a \leq r \leq \ell$ (it has also been studied in [57]). In this region, r is a spacelike coordinate and t is timelike one. Evidently, ψ and ϕ_i's are spacelike coordinates in this region.

For the case of $\ell < a$, the region which does not include CTC is $r \geq a$. While the metric is static in the region $r \leq \ell$ and does not have overlap with no-CTC region. Therefore, we do not consider this case anymore.

The cosmological horizon of this solution is located at $r = \ell$, and the horizon topology is $S^1 \times (S^2)^{n-1}$. One may note that when $\ell \to \infty$, this horizon disappears and the metric goes to the d-dimensional Eguchi-Hanson metric [55,56].

A. Thermodynamic quantities

Using the symplectic phase space method [38,58,59], the mass for this solution can be computed,

TABLE II. The $a > \ell$ case.

r	0		ℓ		a		∞
$g(r)$		+		−		−	
f(r)		−		−		+	
$\Vert\partial_r\Vert^2$		−		+		−	
$\Vert\partial_t\Vert^2$		−		+		+	
$\Vert\partial_\psi\Vert^2$		−		−		+	

$$M = \frac{k_n a^{2n}}{8\pi G_d \ell^2}; \qquad k_n \equiv \frac{1}{2}\left(\frac{2\pi}{n}\right)^n, \tag{3.4}$$

where G_d is the d-dimensional Newton's constant. The advantage of the symplectic phase space method is that it is not sensitive to the sign of cosmological constant and enables us to compute charges on codimension-two surfaces at any radius, not necessarily at infinity. In addition, straightforward calculations for temperature and entropy reveal that

$$T = -\frac{1}{2\pi\ell}\sqrt{1-\left(\frac{a^2}{\ell^2}\right)^n}, \qquad S = \frac{k_n \ell^{(2n-1)}}{2G_d}\sqrt{1-\left(\frac{a^2}{\ell^2}\right)^n}. \tag{3.5}$$

Similar to Kerr-dS in the previous section, we assume the temperature of cosmological horizon is negative. This justifies the minus sign in above temperature. Given these thermodynamic quantities, it is easy to check the first law of thermodynamics,[6]

$$\delta M = T\delta S. \tag{3.6}$$

B. Extremal vanishing horizon limit

As is clear from the above, the relation between entropy and temperature of this solution is given by $\frac{S}{T} = -\frac{\pi k_n \ell^{2n}}{G_d}$. Then, the EVH limit is simply obtained by $a \to \ell$, or more precisely

$$a = \ell\left(1 - \frac{b^2}{2}\epsilon^2\right), \qquad \epsilon \to 0. \tag{3.7}$$

The minus sign before ϵ^2 ensures that we are taking $a \to \ell$ limit for $a < l$. It is worthwhile to mention, this solution does not admit an extremal limit with nonzero entropy, in contrast to what we usually expect in black hole physics. Near the EVH limit, temperature and entropy behave as

$$T = \tilde{T}\epsilon + \mathcal{O}(\epsilon^2), \qquad \tilde{T} = -\frac{b\sqrt{n}}{2\pi\ell},$$
$$S = \tilde{S}\epsilon + \mathcal{O}(\epsilon^2), \qquad \tilde{S} = \frac{k_n b\sqrt{n}}{2G_d}\ell^{(2n-1)}. \tag{3.8}$$

From the above, it is clear that the ratio of temperature and entropy is finite in the $\epsilon \to 0$ limit. In addition, the mass has the following expansion in the EVH limit,

$$M = M^{(0)} + M^{(2)}\epsilon^2 + \mathcal{O}(\epsilon^4), \tag{3.9}$$

where

$$M^{(0)} = \frac{k_n \ell^{2(n-1)}}{8\pi G_d}, \qquad M^{(2)} = -nb^2 M^{(0)}, \tag{3.10}$$

which remains nonzero ($M^{(0)} \neq 0$) when $\epsilon \to 0$. Before closing this part, we would like to mention in this limit, vanishing of horizon area is a result of vanishing one-cycle on the horizon along the Killing direction ∂_ψ. It can be inferred by looking at the metric of the horizon in EVH limit

$$ds_H^2 \sim \frac{\ell^2 b^2}{n}\epsilon^2\left(d\psi + \sum_{i=1}^{n-1}\cos(\theta_i)d\phi_i\right)^2 + \frac{\ell^2}{2n}\sum_{i=1}^{n-1} d\Sigma_i^2 + \mathcal{O}(\epsilon^2). \tag{3.11}$$

C. Near horizon geometries

In the EVH limit, $a \to \ell$, the width of static region goes to zero. For that narrow region, we study the near horizon geometry using

$$r = a\left(1 + \frac{n}{2\ell^2}\gamma^2\rho^2\right), \quad t = \frac{\tau}{\gamma}, \quad \psi = \frac{\Psi}{\gamma}, \quad \gamma \to 0, \tag{3.12}$$

along with the EVH limit (3.7). We note that this transformation is built such that we are taking the near horizon limit while we are still between $r = a$ and $r = \ell$. This gives a geometry which includes two small parameters ϵ and γ. For $\epsilon \ll \gamma$, the resulting geometry is

$$ds_{NH}^2 = \left(\frac{\rho^2}{\ell_3 3^2}d\tau^2 - \ell_3 3^2\frac{d\rho^2}{\rho^2} + \rho^2 d\Psi^2\right) + \frac{\ell^2}{2n}\sum_{i=1}^{n-1} d\Sigma_i^2, \tag{3.13}$$

where $\ell_3^2 = \frac{\ell^2}{n}$. The expression in the parentheses describes a locally dS$_3$ spacetime whose radius is $\ell_3 3$. Using the transformation (3.12) in the near EVH limit ($\epsilon \sim \gamma$), we get the near horizon near-EVH geometry

$$ds_{NH}^2 = \left(f(\rho)d\tau^2 - \frac{d\rho^2}{f(\rho)} + \rho^2 d\Psi^2\right) + \frac{\ell^2}{2n}\sum_{i=1}^{n-1} d\Sigma_i^2, \tag{3.14}$$

where

$$f(\rho) = \frac{\rho^2}{\ell_3 3^2} - b^2. \tag{3.15}$$

One can compare this geometry with the generic three-dimensional near horizon in (2.40): they match by setting $a_0 = 0$, $\rho_0^2 = \ell_3 3^2 b^2$ and changing the coordinates as $dv = d\tau + \frac{d\rho}{f(\rho)}$, along with setting $\sigma = +1$ (since the geometry is locally dS$_3$).

We note that in both near horizon metrics (3.13) and (3.14), the range of coordinate Ψ is $[0, 2\pi\gamma]$. However, in

[6]In this paper, we fix λ (cosmological constant) and do not consider the contribution of $\delta\lambda$ to thermodynamics. However, the authors of [57] use a different approach and take λ as a variable.

contrast to what happens for the EVH limit of Kerr, near horizon metric for (near) EVH soliton is regular everywhere except at the origin (where the conical singularity of ψ occurs).

D. Thermodynamics of the EVH near horizon

Analogous to what we have done for the near horizon of EVH Kerr-dS in the previous section, we can define thermodynamic quantities for the three-dimensional part of the near horizon metric. Therefore, we reduce the Einstein-Hilbert action on the $(d-3)$-dimensional manifold to three dimensions via the metric ansatz (3.13). After this reduction, the three-dimensional Newton constant G_3 is obtained in terms of G_d as

$$G_3 = \frac{\pi G_d}{n k_n} \ell^{-2(n-1)}. \tag{3.16}$$

Then the mass, temperature and entropy of the remaining three-dimensional space are

$$M_3 = -\frac{b^2}{8G_3}\epsilon, \quad T_3 = -\frac{b}{2\pi\ell_3}, \quad S_3 = \frac{\pi b \ell_3}{2G_3}\epsilon. \tag{3.17}$$

It is easy to check that the first law of thermodynamics holds for these quantities:

$$\delta M_3 = T_3 \delta S_3. \tag{3.18}$$

Here, $\delta_\perp$ is the same as δ, because the parameter space of three-dimensional subspace in the near horizon and soliton are the same. In other words, thermodynamics of this three-dimensional subspace is induced by the cosmological soliton thermodynamics near the EVH limit, (3.9), (3.8), with these scaling

$$M_3 = \epsilon M^{(2)}, \qquad T_3 = \tilde{T}, \qquad S_3 = \epsilon \tilde{S}. \tag{3.19}$$

IV. SUMMARY AND DISCUSSION

All the so-far studied examples of EVH black holes, e.g. [12–22] and black rings [23–25] are asymptotically AdS or flat spacetimes and they have the AdS_3 factor in their near horizon geometries. Based on the theorems studied in [26,27], near horizon EVH geometries of asymptotically de Sitter spaces, unlike anti–de Sitter spaces, can have either dS_3, three-dimensional flat space or AdS_3 as a subspace.[7] Motivated by this, in this paper, we analyzed the extremal vanishing horizon limit of two asymptotically de Sitter spacetimes.

In the first example, we studied the EVH limit of a d-dimensional Kerr-(A)dS black hole and see that, this limit can only occur in odd dimensions (similar to the EVH case of Myers-Perry black hole [18,23]). Then we studied the near horizon behavior in this limit and observed that the near horizon EVH geometry enjoys a three-dimensional maximally symmetric subspace. Depending on the sign of curvature of the three-dimensional subspace, $\lambda_3 = \lambda - \sum_{i=2} a_i^{-2}$, it can be dS_3, AdS_3 or flat. Thus, for asymptotically *AdS* black holes ($\lambda < 0$), the λ_3 is also negative and the three-dimensional subspace can only be AdS_3. In the asymptotically flat case, this solution reduces to the known d-dimensional Myers-Perry black hole, so its near horizon contains AdS_3 ($\lambda_3 < 0$). However, in asymptotically de Sitter case ($\lambda > 0$), λ_3 can be positive, zero or negative. For the highly spinning Kerr-dS black hole, λ_3 is positive and near horizon includes a dS_3 factor.

In general, the Kerr-dS solution has at most $[\frac{d+1}{2}]$ roots for the horizon Eq. (2.6). The largest one is the cosmological horizon. The EVH limit could be either the result of the degeneracy of a black hole outer horizon with its cosmological horizon (Nariai limit) or with its inner horizon (cold limit) or with both of them (ultracold limit). We discussed the necessary conditions for each of these limits in Sec. II B.

We also discussed the thermodynamics of the near EVH limit. Indeed, there are two types of fluctuations around the EVH black hole: those which remain in the EVH surface and those which make the solution parameters out of EVH surface (normal to the EVH surface in the parameter space). Allowing for the fluctuations normal to the EVH surface, we find near horizon near-EVH geometries. Depending on the sing of λ_3, we found Kerr-dS_3, BTZ or flat space cosmology factor in the near horizon near-EVH geometries. We also studied the thermodynamic behavior of these near horizon geometries. There is a one-to-one relation between these behaviors and the thermodynamics of the black hole itself. Since λ is the parameter of the theory which is fixed and a_i's are solution parameters which can be varied, it is interesting to study the phase transition between three possible three-dimensional geometries via changing a_i's in the context of original EVH black hole thermodynamics. On the other hand, we limited the variations to be "normal" to the EVH surface and kept λ_3 and G_3 constant. It is possible to take λ_3 as a variable and study the thermodynamics of near horizon near-EVH geometries for generic variations which could be parallel or normal to the EVH surface. In this case one may study the thermodynamics of the near horizon geometry in the context of extended phase space thermodynamics e.g. [60,61].

Presence of the AdS_3 factor in the near horizon geometry enables us to describe the physics of the black hole in the vicinity of its horizon by a two-dimensional CFT in the context of AdS_3/CFT_2. In particular, the entropy of near horizon BTZ black hole is obtained via Cardy formula (see Appendix C). When the three-dimensional part of the near horizon metric is locally dS_3 or flat, one may apply dS/CFT

[7]In asymptotically AdS spacetimes, that subspace is restricted by the strong energy condition to be only AdS_3 [26,27].

[50–52] and flat space holography [53,54] proposals to study the near horizon geometry. Especially for the proposed dS_3/CFT_2 in the Nariai limit, it is instructive to compare the result with what is obtained via Kerr/CFT in the Nariai limit of Kerr-dS [62][8]

In the second example, we considered another type of asymptotically de Sitter solution, the cosmological soliton. Using the symplectic phase space method [38], we computed its thermodynamic quantities and integrable charges. Our result for the mass is independent of the radius of the codimension-two surface on which the conserved charge is computed. Therefore, our result for the mass is different from the masses, $M_{\rm in}$ and $M_{\rm out}$ given in [57].[9] The discrepancy can be a consequence of using the extended phase space thermodynamics that they have considered.

An interesting observation is that the entropy of the cosmological soliton is proportional to its temperature. Thus, the extremal limit of this solution is already the EVH limit. In this sense, this is not a standard Nariai limit.

Unlike most other EVH black holes, the horizon of the cosmological soliton is smooth everywhere and free of curvature singularity. This is a counter example to the lore that EVH black holes are naked singularities.

In the near horizon of EVH cosmological soliton, we found only dS_3 subspace in contrast to the EVH Kerr-dS for which AdS_3 and three-dimensional flat space is also possible. This dS_3 factor of the near horizon turns into Kerr-dS_3 in the near EVH limit. We also discussed thermodynamics of this type of geometries and its relation to soliton thermodynamics. After reduction on the $(d-3)$-dimensional space, we get an asymptotically dS_3 space. Studying the dual CFT_2 (if it exists) is another interesting question which should be answered.

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APPENDIX A: NEAR HORIZON EXTREMAL GEOMETRY OF KERR-DS SOLUTION

The near horizon of the extremal limit of more general Kerr-NUT-(A)ds spacetimes in even dimensions has been study recently in [43]. Here, we briefly note this limit for (2.2). Since the surface gravity of this black hole is proportional to $V'(r_h)$ (2.8), the extremal limit simply is given by $V'(r_h)=0$. We are interested in the near horizon limit so we can expand the metric around r_h. In particular, the metric time-time component g_{tt} is proportional to $(V(r)-2m)$ and it vanishes at r_h. Therefore, the near horizon expansion, g_{tt} should start from $V''(r_h)$. The same argument also works for g_{rr}^{-1}. Changing the coordinates as

$$\varphi^i \to \phi^i = \varphi^i - \Omega_H^i t, \qquad r - r_h = \gamma\rho, \qquad t = \frac{\tau}{X\gamma},$$
$$X = \frac{|V''(r_h)|}{4m}(1-\lambda r_h^2) \tag{A1}$$

and taking the $\gamma \to 0$ limit, we find the near horizon extremal geometry,

$$ds^2 = \frac{4mF}{V''}\left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2}\right) + h_{ij}(r_h)(d\phi^i - \rho k^i d\tau)$$
$$\times (d\phi^j - \rho k^j d\tau) + k_{ij}(r_h) d\mu^i d\mu^j, \tag{A2}$$

where h_{ij} and k_{ij} can be read from (2.2) as

$$h_{ij}(r) = \frac{(r^2+a_i^2)}{(1+\lambda a_i^2)}\mu_i^2\delta_{ij} + \frac{2m}{VF}\frac{a_i\mu_i^2}{(1+\lambda a_i^2)}\frac{a_j\mu_j^2}{(1+\lambda a_j^2)},$$
$$k_{ij}(r) = \frac{(r^2+a_i^2)}{(1+\lambda a_i^2)}\delta_{ij} + \frac{\lambda}{W(1-\lambda r^2)}\frac{(r^2+a_i^2)\mu_i}{(1+\lambda a_i^2)}\frac{(r^2+a_j^2)\mu_j}{(1+\lambda a_j^2)}, \tag{A3}$$

and i, j run from 1 to n. In the near horizon geometry (A2), k^i is given by

$$k^i = \left.\frac{d\Omega^i}{dr}\right|_{r=r_h}, \tag{A4}$$

such that

$$\Omega^i = \left(\frac{W}{1+g}\frac{2m}{VF}\right)\frac{a_i}{(r^2+a_i^2)};$$
$$g \equiv \frac{2m}{VF}\sum_{i=1}^n \frac{a_i^2\mu_i^2}{(1+\lambda a_i^2)(r^2+a_i^2)}. \tag{A5}$$

(The horizon angular velocity Ω_H^i, can be read from the above expressions for Ω^i on the horizon.) Simple calculation shows that

$$V''(r_h) = \frac{8mr_h^2}{(1-\lambda r_h^2)^2}\left(\frac{1-2\lambda r_h^2}{r_h^4} - \sum_{i=1}^n \frac{(1-\lambda r_h^2)^2}{(r_h^2+a_i^2)^2}\right). \tag{A6}$$

[8] In [62], the authors only consider four-dimensional Kerr-dS but its generalization would be straightforward.

[9] We remind the reader that "in" and "out" refer to inside and outside the cosmological horizon in [57].

APPENDIX B: KERR-DS METRIC IN THE KERR-SCHILD FORM

The Kerr-Schild form of the Kerr-dS metric is given by [34]

$$ds^2 = d\bar{s}^2 + \frac{2m}{VF}(k_\mu dx^\mu)^2, \tag{B1}$$

in which the de Sitter metric $d\bar{s}^2$ and the null one-form k_μ are as follows:

$$\begin{aligned} d\bar{s}^2 &= -W(1-\lambda r^2)d\bar{\tau}^2 + Fdr^2 + \sum_{i=1}^{n}\frac{\mu_i^2}{\Xi_i}(r^2+a_i^2)d\bar{\varphi}_i^2 \\ &\quad + \sum_{i=1}^{n+\alpha}(r^2+a_i^2)\frac{d\mu_i^2}{\Xi_i} \\ &\quad + \frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{n+\alpha}(r^2+a_i^2)\frac{\mu_i d\mu_i}{\Xi_i}\right)^2, \\ k_\mu dx^\mu &= Wd\bar{\tau} + Fdr - \sum_{i=1}^{n}\frac{a_i\mu_i^2}{\Xi_i}d\bar{\varphi}_i. \end{aligned} \tag{B2}$$

where, the functions V, F, W and Ξ are defined in Eqs. (2.4) and (2.3). Using the coordinate transformation [34]

$$\begin{aligned} d\bar{\tau} &= dt + \frac{2mdr}{(1-\lambda r^2)(V-2m)}, \\ d\bar{\varphi}_i &= d\varphi_i + \frac{2ma_i dr}{(r^2+a_i^2)(V-2m)}, \end{aligned} \tag{B3}$$

one can get the metric (2.2).

APPENDIX C: BLACK HOLE ENTROPY FROM THE CARDY FORMULA ($\lambda_3 < 0$)

For $\lambda_3 < 0$, there is a BTZ black hole in the near horizon of near EVH black hole. The standard AdS_3/CFT_2 shows Virasoro operators L_0 and $\bar{L}_0$ of CFT_2 are given in terms of mass m_{BTZ} and angular momentum J_{BTZ}

$$\begin{aligned} L_0 - \frac{c}{24} &= \frac{1}{2}(\ell_3 m_{BTZ} + J_{BTZ}), \\ \bar{L}_0 + \frac{c}{24} &= \frac{1}{2}(\ell_3 m_{BTZ} - J_{BTZ}), \end{aligned} \tag{C1}$$

where c is the central charge of the corresponding Virasoro algebra. Following the seminal work of Brown-Henneaux [63] and taking into account the pinching periodicity of $\psi \in [0, 2\pi\epsilon]$ one can show

$$c = \frac{3\ell_3 3}{2G_3}\epsilon = \frac{3\ell_3 3\mathcal{A}_{d-2}}{4\pi G_d}\prod_{i=2}^{n}\frac{a_i^2}{\Xi_i}\epsilon. \tag{C2}$$

To keep c finite, we should scale G_d with ϵ [17]. Indeed, one can define the EVH limit via the following triple limits [11]

$$A_H, \quad \kappa_H, \quad G_d \to 0, \quad \frac{A}{\kappa}, \quad \frac{A}{G_d} \ \text{finite}, \tag{C3}$$

Now, we can obtain the finite entropy of the BTZ black hole via Cardy formula

$$\begin{aligned} S &= 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{c}{6}\left(\bar{L}_0 - \frac{c}{24}\right)} \\ &= \frac{\pi\rho_0}{2G_3} = \frac{\mathcal{A}_{d-2}}{4G_d}\left(\prod_{i=2}^{n}\frac{a_i^2}{\Xi_i}\right)\rho_0, \end{aligned} \tag{C4}$$

which is exactly the entropy of near horizon BTZ and the corresponding near EVH black hole.

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