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Near horizon geometries and black hole holograph

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Two quasilocal approaches to black holes are combined: Near horizon geometries (NHG) and stationary black hole holographs (BHH). Necessary and sufficient conditions on BHH data for the emergence of NHGs as resulting vacuum solutions to Einstein's equations are found.

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I. INTRODUCTION

In this paper we combine results of two topics of the quasilocal theory of black holes (BH). The first one is the theory of near horizon geometries (NHG) of extremal BHs [1–3]. They are exact solutions to Einstein's equations obtained by a naturally defined limit of neighborhoods of extremal (degenerate) Killing horizons. The first examples were derived from the extremal Reissner-Nordström solution and from the extremal Kerr. A larger family of examples (defined modulo an equation that has to be solved, though) is set by the Kundt's class of solutions to Einstein's equations [4–10]. The second topic is the recent stationary black hole holograph (BHH) [11,12]. This approach relies on the characteristic Cauchy problem for the electrovacuum Einstein's equations. If two transversal null surfaces are nonexpanding, then they become components of a bifurcated Killing horizon. The motivation for the current paper is an observation that the NHGs also admit bifurcated Killing horizons. That makes them a special case of the BHHs. In the current paper we present a solution to the inverse problem; namely, we find conditions on the BHH data that are necessary and sufficient for the corresponding hologram spacetime to be a NHG. Our result may be considered as the first step in using the BHH construction in a quest for an interesting generalization of the idea of NHG. For simplicity, we will restrict our work here to 4D spacetimes and the vacuum Einstein's equations.

A BHH data subset (S, g, ω) is a compact 2-manifold S (a BHH space) endowed with a metric tensor (a BHH metric tensor),

$$g = g_{AB} dx^A dx^B, \tag{1}$$

and a 1-form (a BHH 1-form),

$$\omega = \omega_A dx^A, \tag{2}$$

where $(x^A) = (x^1, x^2)$ is a local coordinate system at S. Note that this is a geometric version of the original definition [11,12]. The corresponding hologram is a 4D spacetime in which the 2-space S becomes the intersection between two nonexpanding null surfaces (nonexpanding horizons [13–15]), while g becomes the metric tensor induced in S. The 1-form ω becomes the pullback to S of the rotation 1-form potential of one of the horizons, and, respectively, minus the rotation 1-form potential of the other one pulled back to S. The spacetime geometry is determined via the characteristic Cauchy problem for vacuum Einstein's equations in the causal future and in the past of the intersection S. The BHH theorem states that in this spacetime the nonexpanding horizons set a bifurcated Killing horizon. Among all the black hole spacetimes obtained in this way, there are also all the NHGs. Indeed, it is known, that each NHG contains a bifurcated Killing horizon [4,16]. We will find below necessary and sufficient conditions on (S, g, ω) for the corresponding hologram to be a NHG.

II. THE BLACK HOLE HOLOGRAPH

Given a BHH data set (S, g, ω), the hologram spacetime manifold M has the product topology

$$M \sim S \times \mathbb{R} \times \mathbb{R}. \tag{3}$$

The coordinates (x^A) defined on S, as well as coordinates u and v defined on the first and the second factor $\mathbb{R}$, respectively, are naturally extended to the Cartesian product. The surfaces

$$N_1 \text{ such that } u = 0 \text{ and } N_2 \text{ such that } v = 0, \tag{4}$$

respectively, are assumed to be null and nonexpanding with respect to the resulting spacetime geometry, while S is identified with the surface $u = v = 0$ in M. According to the standard characteristic Cauchy problem for vacuum Einstein's equations, in the smooth case the spacetime geometry is determined (up to remaining diffeomorphisms) in the wedges $u \geq 0$, $v \geq 0$ and $u \leq 0$, $v \leq 0$, in some neighborhood of $S = N_1 \cap N_2$, provided that the following

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conditions hold at the surfaces N_1 and N_2, and at S, respectively:

(i) The pullback of the spacetime metric to each of the surfaces N_1 and N_2, respectively, is the following degenerate metric:

$$g_{AB}dx^A dx^B. \tag{5}$$

(ii) The vectors $\ell = \partial_u$, $n = \partial_v$ are future oriented and satisfy

$$\nabla_\ell \ell|_{N_1} = 0, \qquad \nabla_n n|_{N_2} = 0. \tag{6}$$

(iii) The pullback to S of the 1-form $-n_\mu \nabla_\nu \ell^\mu$ is

$$-n_\mu \nabla_A \ell^\mu|_S = \omega_A. \tag{7}$$

The BHH theorem [11,12] states that the spacetime metric tensor determined by the data admits a Killing vector K that, using the remaining diffeomorphisms, can be given the form

$$K = u\partial_u - v\partial_v. \tag{8}$$

Therefore, the surfaces N_1 and N_2 form a (nonextremal) bifurcated Killing horizon while the pullback of the 1-form $\omega_\nu = -n_\mu \nabla_\nu \ell^\mu$ to N_2 is its rotation 1-form potential. In this sense, the construction works as a stationary BH holograph: given any two-dimensional data (S, g, ω), it produces four-dimensional spacetime in the domain of dependence of the bifurcate Killing horizon, $N_1 \cup N_2$.

III. NHG FROM BHH

Suppose now, that a BHH data set (S, g, ω) satisfies the following equation

$$\omega_{(A;B)} + \omega_A \omega_B - \frac{1}{2} R_{AB} = 0 \tag{9}$$

where by ";" we denote the torsion-free covariant derivative defined on S by the metric g, and R_{AB} is the Ricci tensor of g. This equation is soluble [4,17,18] only when S is either a topological 2-sphere,

$$S = S_2,$$

or a 2-torus,

$$S = S_1 \times S_1.$$

In the latter case, the only solution is

$$\omega_A = 0 = R_{AB};$$

therefore, we will be assuming henceforth that the manifold S is a 2-sphere S_2. Whenever Eq. (9) holds, the hologram metric tensor can be written down explicitly. Indeed, the metric tensor

$$ds^2 = -2du\left(dv - 2v\omega - \frac{1}{2}v^2[\omega_{A;}{}^A + 2\omega_A\omega^A]du\right) + g_{AB}dx^A dx^B \tag{10}$$

is an exact solution of the vacuum Einstein equations [4] that matches the hologram data (5)–(7). Owing to uniqueness (mod diffeomorphisms) in BHH [12], this is the corresponding BH hologram. Such geometries are called near horizon. A remarkable property of this BH hologram (10) is the emergence of a second Killing vector field, namely

$$L = \partial_u. \tag{11}$$

The surface N_2 is extremal Killing horizon of the Killing vector field L, still being a component of the bifurcated nonextremal horizon of the Killing vector field K. Therefore, our first conclusion is that all BHH data (S, g, ω) such that the Eq. (9) is satisfied define a NHG with the extremal Killing horizon N_2.

IV. FLIPPED NHG BHH DATA

The gauge freedom we have in setting up the initial data for BHH [12] yields some ambiguity in identifying NHGs. For instance, condition (9) is not necessary as other BHH data may also define a NHG as the hologram spacetime. For example, the following transformation in the space of the holographic data:

$$(S, g, \omega) \mapsto (S, g, -\omega) \tag{12}$$

corresponds to switching of the factors in $S \times \mathbb{R} \times \mathbb{R}$, namely

$$(x^A, u, v) \mapsto (x^A, v, u), \tag{13}$$

because on $S = N_1 \cap N_2$

$$-(\partial_v)_\mu \nabla_A (\partial_u)^\mu = (\partial_u)_\mu \nabla_A (\partial_v)^\mu$$

holds. Hence, all data (S, g, ω) which satisfies the switched Eq. (9), which is

$$\omega_{(A;B)} - \omega_A \omega_B + \frac{1}{2} R_{AB} = 0, \tag{14}$$

also define a NHG, this time with the extremal horizon N_1.

V. A GENERAL CASE OF NHG FROM BHH

In virtue of Eq. (2.5) of [12], a gauge freedom, represented by the coordinate transformation

$$(x^A, u, v) \mapsto (x^A, e^{-\lambda}u, e^{\lambda}v), \tag{15}$$

where $\lambda\colon S \to \mathbb{R}$ is a sufficiently regular but otherwise arbitrary function on S, has been left over. This, in particular, means that if BHH data (S, g, ω) satisfies Eq. (9), i.e., it gives rise to a NHG with an extremal Killing horizon structure at the null surface N_2, then the data yielded by the transformation

$$(S, g, \omega) \mapsto (S, g, \omega + d\lambda) \tag{16}$$

will determine a gauge equivalent solution. Note that this is also in accordance with the statement that BHH provides us uniqueness of solutions only "up to diffeomorphisms."

Accordingly, the most general form of condition (9), guaranteeing that the BHH data $(S, g, \omega + d\lambda)$ yields, up to diffeomorphisms, the same NHG as (S, g, ω) does, reads as

$$\omega_{(A;B)} + \lambda_{;AB} + (\omega_A + \lambda_{,A})(\omega_B + \lambda_{,B}) - \frac{1}{2}R_{AB} = 0. \tag{17}$$

This nonlinear equation on the unknown function λ can be written as a linear equation on the nowhere vanishing function

$$f \coloneqq e^{\lambda}.$$

Indeed, that substitution turns Eq. (17) into (below, 'D' ≡ '$;$')

$$\left(D_A D_B + \omega_A D_B + \omega_B D_A + D_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}R_{AB}\right)f = 0. \tag{18}$$

In conclusion, a BHH data set (S, g, ω), as described in Sec. II, gives rise to a NHG with an extremal Killing horizon structure at the null surface N_2 if and only if there exists on S a real nonvanishing function f that satisfies Eq. (18). In the next section an equivalent necessary and sufficient condition is given that explicitly constrains g and ω.

VI. NECESSARY AND SUFFICIENT CONDITIONS ON BHH TO GIVE RISE TO NHG WITH RESPECT TO N_2

As explained below, for a given BHH data set (S, g, ω), a necessary and sufficient condition guaranteeing the existence of a real nonvanishing solution f to Eq. (18) is equivalent to the existence of a real (modulo a constant phase factor) and nowhere vanishing function $\tilde{f}$ that has the explicit form (25), and that also satisfies Eq. (23).

In doing so, note first that for an arbitrary choice of a nowhere vanishing real valued function h on S_2, the replacement $(g, f, \omega) \mapsto (g', f', \omega')$, with

$$\omega' = \omega - d\ln h, \qquad f' = hf, \qquad g' = g, \tag{19}$$

is a symmetry transformation of Eq. (18). As a special case of Eq. (19) one gets Eq. (17) from Eq. (18) by choosing

$$h = \frac{1}{f}.$$

Recall also that, given the metric g on S_2, the 1-form field ω, as any of the 1-form fields on a 2-sphere, can be uniquely given in terms of its scalar potentials $\mathcal{U}$ and $\mathcal{B}$ and the Hodge $\star$, namely

$$\omega = \star d\mathcal{U} + d\ln\mathcal{B}. \tag{20}$$

Here $\mathcal{U}$ and $\mathcal{B}$ are determined by the relations

$$\Delta\mathcal{U} = \star d\omega \quad \text{and} \quad \Delta\ln\mathcal{B} = \star d\star\omega, \tag{21}$$

where Δ is the Laplace operator defined on S_2 by the metric g. Note that the equations in (21) can always be uniquely solved on S_2 provided that a sufficiently regular ω is given.

Now, by applying the symmetry transformation (19) with the choice

$$h = \mathcal{B}, \tag{22}$$

one gets from Eq. (18)

$$\left(D_A D_B + \tilde{\omega}_A D_B + \tilde{\omega}_B D_A + D_{(A}\tilde{\omega}_{B)} + \tilde{\omega}_A\tilde{\omega}_B - \frac{1}{2}R_{AB}\right)\tilde{f} = 0, \tag{23}$$

where

$$\tilde{\omega} = \star d\mathcal{U}, \qquad \tilde{f} = \mathcal{B}f. \tag{24}$$

Note that Eq. (23) consists of three partial differential equations, one per each pair of values $A, B = 1, 2$. By following the basic steps of the argument applied in the appendix of [13] and in Sec. II.2 of [19], the integrability condition of Eq. (23) can be used to verify the existence of a (possibly) complex constant $\mathcal{A}_0$ with the help of which the most general solution to Eq. (23) can be given as

$$\tilde{f} = \mathcal{A}_0(R - 2i\Delta\mathcal{U})^{-\frac{1}{3}}e^{i\mathcal{U}}, \tag{25}$$

where

$$R = g^{AB} R_{AB}$$

denotes the Ricci scalar of the metric g.

Before giving our new necessary and sufficient conditions as explicit restrictions on g and $\mathcal{U}$, note first that $\mathcal{A}_0$, without loss of generality, may be assumed to be real as if it was complex its phase factor could always be eliminated by using the freedom $\mathcal{U} \mapsto \mathcal{U} + \mathcal{U}_0$, with $\mathcal{U}_0 = \text{const}$, we have in choosing a solution to Eq. (21), which, on the other hand, leaves ω intact.

In summary, our necessary and sufficient conditions (modulo the flipping of section I 3) can then be given by the following:

Theorem: A BHH data set (S_2, g, ω) gives rise to a NHG with extremal Killing horizon N_2 if and only if the function

$$(R - 2i\Delta\mathcal{U})^{-\frac{1}{3}} e^{i\mathcal{U}}, \tag{26}$$

which can be given in terms of the scalar curvature R of g and the scalar potential $\mathcal{U}$ of ω, has the following properties:

(i) It is real (modulo a constant phase factor),
(ii) it is nonvanishing, and
(iii) it is a solution to Eq. (23).

VII. NONROTATING BHH DATA

Let us finally consider, as a special case, a nonrotating NHG represented by the BHH data (S_2, g, ω) with

$$d\omega = 0. \tag{27}$$

Note that then, in virtue of Eq. (26) and the vanishing of $\tilde{\omega} = \star d\mathcal{U}$, the function $R^{-\frac{1}{3}}$ is a solution to Eq. (23), which implies [13]

$$0 = \int_{S_2} d^2x \sqrt{\det g} \left(\Delta - \frac{1}{2} R \right) R^{-\frac{1}{3}}$$
$$= -\frac{1}{2} \int_{S_2} d^2x \sqrt{\det g} (R^{\frac{1}{3}})^2,$$

a condition requiring the vanishing of R throughout S_2. This, however, leads to a contradiction as a 2-sphere does not admit a flat metric tensor, which, in turn, verifies the nonexistence of nonrotating vacuum NHGs, in accordance with [20].

VIII. SUMMARY AND OUTLOOK

The subset of BHH data that corresponds to NHGs was identified. It was shown that, up to the flips of the horizons, a BHH data set (S_2, g, ω) gives rise to a NHG if and only if the function in Eq. (26), determined by the scalar curvature R of g and the scalar potential $\mathcal{U}$ of ω, is real (up to constant a phase factor), nowhere vanishing, and satisfies Eq. (23). A general exact solution to these constraints is not known yet.

It is important to emphasize that the found intimate interrelation between the BHH construction and the NHGs may play a significant role in attempting to give suitable generalizations of the concept of NHGs. The NHGs are exact solutions to Einstein's equations that at the same time provide the 0th order in a suitable expansion of spacetime metric about an extremal Killing horizon. In the nonextremal case a suitable generalization of the NHGs is not known yet.

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