PHYSICAL REVIEW D **96,** 036008 (2017)

Conformal invariant vacuum nonlinear electrodynamics

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(Received 14 June 2017; published 23 August 2017)

In this paper, a general case of conformal invariant vacuum nonlinear electrodynamics is studied. We analyze the consistency of this electrodynamics model with fundamental principles such as causality, unitarity, and the Ellis-Hawking dominant energy condition. Certain features of the electromagnetic waves in this model are investigated.

DOI: 10.1103/PhysRevD.96.036008

I. INTRODUCTION

Vacuum nonlinear electrodynamics (NED) effects are the object of strong interest in contemporary field theory, due to the approaching opportunities for their experimental verification. The main hopes are associated with the development of intense laser facilities such as ELI [1] and XCELS [2]. Furthermore, an especially inspiring recent report [3] about vacuum birefringence observation through the detection of polarized optical radiation coming from isolated pulsars provides strong expectations of NED investigation advances in further astrophysical missions such as XIPE [4] and IXPE [5].

Vacuum nonlinear electrodynamics models arise from diverse assumptions. For instance, the Born-Infeld [6] model solves the problem of infinite energy of a pointlike charge by bounding the field strength in the charge center. Heisenberg-Euler [7] theory considers quantum radiative corrections caused by electron-positron vacuum polarization in a strong electromagnetic field and predicts birefringence for electromagnetic waves in a vacuum. The Podolsky model [8] takes into account higher derivatives of the field strength tensor. There are various modifications and extensions for these theories [9–11] which partially or fully inherit properties of Born-Infeld and Heisenberg-Euler electrodynamics. However, due to the lack of reliable experimental data in favor of one of them, a set of completely new NED Lagrangians was proposed. Among them, empirical models [12–15], inspired by astrophysics and cosmology, take a special place. The development of these models in papers [12–14] opened up an unusual theoretical view on Universe acceleration due to nonlinear electromagnetic processes. Moreover, charged regular black holes as a new class of compact astrophysical objects were predicted in Refs. [16,17].

Because of the variety within the choices of NED models, there is a need to clarify the selection rules for them. The main common criterion in the development of all NED models listed above is correspondence to Maxwell electrodynamics in the weak-field limit. At the same time, most of the models do not possess the symmetries inherent to Maxwell theory. It is well known [18] that Maxwell electrodynamics is invariant under a 15-parametric Lie group, comprised of a Poincaré group (ten parameters), coordinate scaling (one parameter), and a conformal group (four parameters). In addition, this theory possesses such properties as dual invariance and zero trace of the stress-energy tensor, and it certainly satisfies fundamental principles, such as causality, unitarity, and the Hawking-Ellis dominant energy condition. The compliance with Maxwell electrodynamics symmetries may be a powerful selection tool for the NED models.

In this paper, we consider a general case of conformal invariant NED, possessing all the symmetries listed above. The birefringence and optical nonreciprocity conditions are defined.

The paper is organized as follows: In Sec. II, we introduce the conformal invariant NED Largangian and verify the fundamental principles. In Sec. III, we construct the dominant energy condition for nonlinear electrodynamics in tetrad representation and check this condition for the proposed model. Section IV is devoted to peculiarity in wave propagation and vacuum refraction indexes. In Sec. V, we summarize our results.

II. CONFORMAL INVARIANT NONLINEAR ELECTRODYNAMICS: SYMMETRIES AND FUNDAMENTALS

Let us consider a general case of Lorentz-invariant action for vacuum nonlinear electrodynamics, whose Lagrangian in the space-time with the metric tensor g_{ik} depends on two invariants J_2, J_4 of the field strength tensor:

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$$S = \frac{1}{c} \int \sqrt{-g} \mathcal{L}(J_2, J_4) d^4x, \tag{1}$$

where $J_2 = F_{ik}F^{ki}$ and $J_4 = F_{ik}F^{kl}F_{lm}F^{mi}$, and $g = \det||g_{ik}||$. It is easy to derive a symmetric stress-energy tensor for the action (1):

$$\begin{aligned} T_{ik} &= \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{ik}} \\ &= 4\left[\frac{\partial \mathcal{L}}{\partial J_2} + J_2 \frac{\partial \mathcal{L}}{\partial J_4}\right] F^{(2)}_{ik} \\ &\quad + \left[(2J_4 - J_2^2)\frac{\partial \mathcal{L}}{\partial J_4} - \mathcal{L}\right] g_{ik}, \end{aligned} \tag{2}$$

with the trace

$$T = T^i_i = 4\left[\frac{\partial \mathcal{L}}{\partial J_2} J_2 + 2J_4 \frac{\partial \mathcal{L}}{\partial J_4} - \mathcal{L}\right], \tag{3}$$

where the second power of the electromagnetic field tensor is introduced: $F^{(2)}_{ik} = F_{im}F^{m\cdot}_{\cdot k}$.

The choice of Lagrangian follows from specific model phenomenology. As the main criterion for constructing our model, we will take the maximum correspondence to the symmetries of Maxwell's electrodynamics. Primarily, we will demand that the stress-energy tensor be traceless, $T = 0$, which leads to the following equation for the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial J_2} J_2 + 2J_4 \frac{\partial \mathcal{L}}{\partial J_4} - \mathcal{L} = 0. \tag{4}$$

The general solution to this equation can be obtained by the characteristics method and has the form

$$\mathcal{L} = J_2 W\left(\frac{J_2}{\sqrt{2J_4}}\right) = J_2 W(z), \tag{5}$$

where W is an arbitrary function of the field invariants' ratio $z = J_2/\sqrt{2J_4}$.

This Lagrangian of the vacuum nonlinear electrodynamics, besides that of the traceless stress-energy tensor, also possesses other symmetries of Maxwell electrodynamics. It is obvious that the model is invariant under Poincaré-group and coordinate scaling. In order to show invariance under the conformal group, let us consider the metric tensor transformation $g_{ik} \to \tilde{g}_{ik} = \Omega(x) g_{ik}$, where Ω is an arbitrary, positive-defined scaling factor. As was pointed out in Ref. [18], the electromagnetic field invariants under such transformations change as

$$J_2 \to \tilde{J}_2 = J_2/\Omega^2, \qquad J_4 \to \tilde{J}_4 = J_4/\Omega^4. \tag{6}$$

If we take into account that the metric tensor determinant transforms as $\sqrt{-g} \to \sqrt{-\tilde{g}} = \Omega^2\sqrt{-g}$, we will obtain, by using Eqs. (5) and (6), that the action (1) is invariant under conformal group. So the model is conformal invariant, and due to this property, we will call it conformal vacuum nonlinear electrodynamics (CNED). Moreover, the stress-energy tensor of this model is proportional to the stress-energy tensor of Maxwell theory:

$$T_{ik} = 4[W + z(1 - z^2)W']\left\{F^{(2)}_{ik} - \frac{g_{ik}}{4} J_2\right\}, \tag{7}$$

where W' denotes the derivative. Although the model function $W(z)$ is still considered to be arbitrary, its choice must fulfill fundamental principles, the primary of which are unitarity and causality conditions.

The causality principle guarantees that the group velocity for the elementary electromagnetic excitations does not exceed the speed of light in a vacuum. At the same time, the unitarity criterion provides the positive definiteness of the norm of every elementary excitation of the vacuum. To clarify the restrictions on the Lagrangian coming from these principles, one should consider electromagnetic wave propagation on the background of an electromagnetic field which is constant in time and space. The dispersion relation for these waves for nonlinear electrodynamics with an arbitrary Lagrangian was obtained in Ref. [19] and is extremely difficult to analyze. Therefore, similarly to Ref. [20], we will perform our description in a more particular case, where the background field is under the following restriction: $(\mathbf{EB}) = 0$. This means that the field is purely magnetic or purely electric in a special Lorentz frame. In this case, due to the vacuum birefringence, the electromagnetic wave splits into two orthogonal, propagating modes. The restrictions from causality and unitarity conditions, applied to each of these modes, were obtained in Ref. [20] as a set of inequalities [that reference's Eqs. (27)–(31)], which in terms of independent invariants J_2 and J_4 can be rewritten as

$$\frac{\partial \mathcal{L}}{\partial J_2} \geq 0, \qquad \frac{\partial \mathcal{L}}{\partial J_4} \geq 0, \qquad \frac{\partial \mathcal{L}}{\partial J_2} + J_2 \frac{\partial \mathcal{L}}{\partial J_4} \geq 0, \tag{8}$$

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial J_2} + J_2 \frac{\partial \mathcal{L}}{\partial J_4} \\ &\quad + 2J_2\left[\frac{\partial^2 \mathcal{L}}{\partial J_2^2} + 2J_2 \frac{\partial^2 \mathcal{L}}{\partial J_2 \partial J_4} + J_2^2 \frac{\partial^2 \mathcal{L}}{\partial J_4^2} + \frac{\partial \mathcal{L}}{\partial J_4}\right] \geq 0, \end{aligned} \tag{9}$$

$$\frac{\partial^2 \mathcal{L}}{\partial J_2^2} + 2J_2 \frac{\partial^2 \mathcal{L}}{\partial J_2 \partial J_4} + J_2^2 \frac{\partial^2 \mathcal{L}}{\partial J_4^2} + \frac{\partial \mathcal{L}}{\partial J_4} \geq 0. \tag{10}$$

This set of inequalities imposes restrictions on the Lagrangian of any arbitrary nonlinear theory; however, in the case of electrodynamics with conformal properties CNED, this set can be significantly simplified. To perform this simplification, let us derive the additional relations between the derivatives of the Lagrangian. By differentiating Eq. (4) with respect to the invariants J_2 and J_4, we obtain

$$J_2\frac{\partial^2\mathcal{L}}{\partial J_2^2}+2J_4\frac{\partial^2\mathcal{L}}{\partial J_2\partial J_4}=0, \tag{11}$$

$$2J_4\frac{\partial^2\mathcal{L}}{\partial J_4^2}+J_2\frac{\partial^2\mathcal{L}}{\partial J_2\partial J_4}+\frac{\partial\mathcal{L}}{\partial J_4}=0. \tag{12}$$

For the considered field configuration, when $(\mathbf{EB})\to 0$, the invariant J_4 tends to the value $J_4\to J_2^2/2$, and at the same time $J_2\neq 0$. In consequence, Eqs. (11) and (12) take the forms

$$\frac{\partial^2\mathcal{L}}{\partial J_2^2}+J_2\frac{\partial^2\mathcal{L}}{\partial J_2\partial J_4}=0 \tag{13}$$

and

$$J_2^2\frac{\partial^2\mathcal{L}}{\partial J_4^2}+J_2\frac{\partial^2\mathcal{L}}{\partial J_2\partial J_4}+\frac{\partial\mathcal{L}}{\partial J_4}=0. \tag{14}$$

Finally, the addition of the last two equations leads to the expression

$$\frac{\partial^2\mathcal{L}}{\partial J_2^2}+2J_2\frac{\partial^2\mathcal{L}}{\partial J_2\partial J_4}+J_2^2\frac{\partial^2\mathcal{L}}{\partial J_4^2}+\frac{\partial\mathcal{L}}{\partial J_4}=0, \tag{15}$$

through which the inequality (10) is always satisfied, and (9) transforms into the last inequality in (8). As a result, the causality and unitarity criteria for conformal invariant vacuum nonlinear electrodynamics take the form of the inequalities (8), which can be rewritten in terms of the model function W as

$$W(z)>0,\qquad W'(z)<0, \tag{16}$$

where one should take into account that $z=\pm 1$ when $(\mathbf{EB})=0$.

In addition to the general constraints, let us consider other possible restrictions on the Lagrangian coming from observational data. Often, the Hawking-Ellis dominant energy condition is considered such a restriction. It is well known that this condition is satisfied for Maxwell electrodynamics, so it is important for it to keep being satisfied for CNED.

III. DOMINANT ENERGY CONDITION FOR CONFORMAL NED

The dominant energy condition sets restrictions on the stress-energy tensor, which guarantee that for every timelike observer, energy density will be non-negative, and energy flux will be a casual vector (timelike or null). These requirements ensure dominance of the energy density over the other components in the stress-energy tensor. The dominant energy condition [21] leads to the following inequalities:

$$T_{ik}a^ia^k\geq 0,\qquad T_{ki}T^{im}a_ma^k\geq 0, \tag{17}$$

for any causal, future-pointing vector a^k.

Usually, verification of fulfillment of these inequalities is performed in coordinate representation for each particular case of the stress-energy tensor.

In this section, we will develop an alternative way based on tetrad representation. This approach seems to be more general and formal for calculations.

In order to develop this approach, let us choose two isotropic, future-pointing, real four-vectors l^k and n^k. We take into account that any causal, future-pointing vector a^k can be represented as a superposition of two noncollinear isotropic vectors also pointing into the future $a^k=\alpha l^k+\beta n^k$, where α,β are real constants. Without loss of generality, one can take that $l^kn_k=1$. At the same time, the causality a^k sets a restriction on coefficients, $\alpha\beta\geq 0$.

Now we supplement vectors l^k and n^k by two other complex, isotropic vectors m^k and $\bar{m}^k$, so that altogether these vectors form a tetrad with the following scalar products:

$$\begin{aligned}&l_kl^k=n_kn^k=m_km^k=n_km^k=l_km^k=0,\\&l_kn^k=-\bar{m}_km^k=1,\end{aligned} \tag{18}$$

where the bar denotes complex conjugation. The tetrad represents a basis, which can be used for electromagnetic field tensor decomposition:

$$\begin{aligned}F_{ik}=&f_1(l_in_k-l_kn_i)+f_2(l_im_k-l_km_i)\\&+\bar{f}_2(l_i\bar{m}_k-l_k\bar{m}_i)+f_3(n_im_k-n_km_i)\\&+\bar{f}_3(n_i\bar{m}_k-n_k\bar{m}_i)+if_4(m_i\bar{m}_k-m_k\bar{m}_i),\end{aligned} \tag{19}$$

where f_1, f_4 are real and f_2, f_3 are complex functions of coordinates, whose values assign the electromagnetic field configuration in the given space-time. By using the representation (19) and the scalar products (18), it is easy to derive the auxiliary relations:

$$\begin{aligned}&J_2=F_{ik}F^{ki}=2[f_1^2-f_4^2+2(\bar{f}_2f_3+\bar{f}_3f_2)],\\&J_4=F_{ik}F^{kl}F_{lm}F^{mi}=\frac{J_2^2}{2}+4[f_1f_4+i(f_2\bar{f}_3-f_3\bar{f}_2)]^2,\\&F_{ik}^{(2)}l^in^k=f_1^2+\bar{f}_2f_3+\bar{f}_3f_2.\end{aligned} \tag{20}$$

Substitution of the stress-energy tensor for arbitrary vacuum nonlinear electrodynamics (2), auxiliary relations (20), and a^k decomposition to the dominant energy condition (17) gives the representation of this condition in tetrad formalism:

$$T_{ik}a^i a^k = 8\left[\frac{\partial\mathcal{L}}{\partial J_2}+J_2\frac{\partial\mathcal{L}}{\partial J_4}\right]\left\{\alpha^2|f_3|^2+\beta^2|f_2|^2+\frac{\alpha\beta(f_1^2+f_4^2)}{2}\right\} + 2\alpha\beta T\geq 0,$$

$$T_{ik}T^{\mathrm{km}}a_m a^i = 16T\left[\frac{\partial\mathcal{L}}{\partial J_2}+J_2\frac{\partial\mathcal{L}}{\partial J_4}\right]\left\{\alpha^2|f_3|^2+\beta^2|f_2|^2+\frac{\alpha\beta(f_1^2+f_4^2)}{2}\right\} + 2\alpha\beta\left\{T^2+(4J_4-J_2^2)\left[\frac{\partial\mathcal{L}}{\partial J_2}+J_2\frac{\partial\mathcal{L}}{\partial J_4}\right]^2\right\}\geq 0, \tag{21}$$

where T is the stress-energy tensor trace. This form of the Hawking-Ellis dominant energy condition is suitable for arbitrary Lorentz-invariant nonlinear vacuum electrodynamics. Its verification proves that for the chosen Lagrangian, for any arbitrary field configuration represented by functions f_i and any constants α and β, the inequalities take place [Eq. (21)].

In the particular case of conformal vacuum nonlinear electrodynamics, the verification of the dominant energy condition becomes greatly simplified, since $T = 0$. Taking into account the non-negative definiteness of $4J_4 - J_2^2 = 4(\mathbf{E}^2-\mathbf{B}^2)^2+16(\mathbf{EB})^2\geq 0$ for any field configuration and the restriction on coefficients $\alpha\beta>0$, one can obtain that both inequalities (21) will be satisfied when

$$\frac{\partial\mathcal{L}}{\partial J_2}+J_2\frac{\partial\mathcal{L}}{\partial J_4}\geq 0, \tag{22}$$

which coincides with the last inequality in the earlier requirements (8) coming from causality and unitarity criteria. Therefore, the fulfilment of these criteria for CNED implies the dominant energy condition.

After the discussion of general properties and fundamental restrictions, we proceed to observational manifestations of conformal nonlinear electrodynamics.

IV. VACUUM REFRACTION INDEXES IN THE CONFORMAL NED

Vacuum birefringence is one of the vivid effects inherent for many models of nonlinear electrodynamics, and its experimental research is the most promising way to test a status of the specific model.

When the electromagnetic wave propagates in a strong external field in a nonlinear vacuum, it splits into two normal modes as in a birefringent optical crystal.

To obtain vacuum refraction indexes for these modes in conformal nonlinear electrodynamics, let us consider electromagnetic wave excitation $\tilde{f}_{ik}=f_{ik}\exp\{-iS\}$, with the amplitude f_{ik} and eikonal $S(\mathbf{r},t)$, propagating on the background of a constant and homogenous field F_{ik}. We suppose that the excitation is weak compared to the external field, $|f_{ik}|\ll|F_{ik}|$.

The general form of dispersion law for such excitations in the case of an arbitrary Lorentz-invariant nonlinear electrodynamics was obtained in early works [19,22]. The dispersion law for CNED with the Lagrangian (5) in pseudo-Riemmanian space-time with the metric tensor g_{ik} takes the form

$$\left[G^{ik}_{(1)}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k}\right]\times\left[G^{mn}_{(2)}\frac{\partial S}{\partial x^m}\frac{\partial S}{\partial x^n}\right]=0, \tag{23}$$

where metric tensors of effective space-time are introduced:

$$G^{ik}_{(1)}=Ug^{ik}+V\frac{F^{ik}_{(2)}}{\sqrt{2J_4}},\qquad G^{mn}_{(2)}=g^{mn}, \tag{24}$$

and for brevity we use the notations for coefficients

$$U = z^2(z^2-1)(z^2-2)W''+z(3z^4-7z^2+3)W'-W,$$
$$V=-2(z^2-2)\{z(z^2-1)W''+(3z^2-2)W'\}. \tag{25}$$

As it follows from (24), electromagnetic excitations, as in an optic crystal, split into two modes, each of which propagates in effective space-time with the metric tensor dependent on the wave polarization.

The speed of one of the modes does not depend on the external field. This wave, similarly with the crystal optics, we will call the ordinary wave. The refraction index for it is always equal to unity, $n_2 = 1$. The propagation speed and refraction index for another mode will substantially depend on the external field. To obtain this dependance, we suppose that background space-time is pseudo-Euclidian, $g^{ik}=\mathrm{diag}\{+1,-1,-1,-1\}$, and we represent the wave eikonal in the form $S=\omega t-(\mathbf{kr})$. We also assume that the wave vector $\mathbf{k}$ is real and coupled to the refraction index n by a relation inherent to homogenous waves in continuous media, $\mathbf{k}=\omega n\mathbf{q}/c$, where $\mathbf{q}$ is the unity vector.

Using the auxiliary expression

$$F^{ik}_{(2)}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k}=\frac{\omega^2}{c^2}\mathbf{E}^2-\frac{2\omega}{c}(\mathbf{k}[\mathbf{EB}])+[\mathbf{kB}]^2-(\mathbf{kE})^2 =\frac{\omega^2}{c^2}\{\mathbf{E}^2+n^2([\mathbf{qB}]^2-(\mathbf{qE})^2)-2n(\mathbf{q}[\mathbf{EB}])\}, \tag{26}$$

one can represent the dispersion law (23) in the form

$$n^2\left\{1+\frac{V((\mathbf{qE})^2-[\mathbf{qB}]^2)}{2U\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+2(\mathbf{EB})^2}}\right\} +\frac{nV(\mathbf{q}[\mathbf{EB}])}{U\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+2(\mathbf{EB})^2}} =1+\frac{V\mathbf{E}^2}{2U\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+2(\mathbf{EB})^2}}. \tag{27}$$

The solution for n can be expressed from this equation in a general form; however, it will be cumbersome and hard to analyze, so we will consider it in several partial cases, the simplest and most important of which corresponds to the wave propagation in the purely magnetic field $\mathbf{B} \neq 0$, $\mathbf{E} = 0$. For this field configuration, $z = z_1 = -1$ and also $U = W_1' - W_1$, $V = 2W_1'$, where the derivatives of the function W should be taken at the point z_1, as noted by the index. The dispersion law (27) in this case will take a simpler form:

$$n^2 = \left[1 - \frac{W_1'}{W_1' - W_1}\sin^2\theta\right]^{-1}\Bigg|_{z=-1}, \qquad (28)$$

where $\theta = (\widehat{\mathbf{qB}})$ is the angle between the wave propagation direction and the magnetic field. The distinctive feature of this expression is that the vacuum refraction index does not depend on the magnetic field strength, and for each specific model function, W is completely determined by the wave propagation direction.

Let us consider another specific case, in which the wave propagates in a pure electric field $\mathbf{E} \neq 0$, $\mathbf{B} = 0$, for which $z = z_2 = 1$, and also $U = -W_2' - W_2$ and $V = 2W_2'$. It is easy to derive the refraction index from the dispersion law (27):

$$n^2 = \left[1 + \frac{W_2'}{W_2}\sin^2\psi\right]^{-1}\Bigg|_{z=1}, \qquad (29)$$

where $\psi = (\widehat{\mathbf{qE}})$ is the angle between the wave propagation direction and the electric field strength.

Finally, we describe the case in which $\mathbf{E} \neq 0$, $\mathbf{B} \neq 0$, but at the same time $\mathbf{E} \perp \mathbf{B}$. For clarity, we will consider that in the chosen reference frame, the electric field value exceeds the magnetic field $|\mathbf{E}| > |\mathbf{B}|$. As earlier, for this field configuration $z = 1$. Since the dispersion law (27) contains the terms odd under reflection of $\mathbf{q}$, this leads to a difference of the refraction indexes for the waves propagating in mutually opposite directions. The difference of refraction indexes for opposite directions can be easily found from the dispersion law, taking into account Eq. (25):

$$n_+ - n_- = \frac{2W'(\mathbf{q}[\mathbf{EB}])}{W(\mathbf{E}^2 - \mathbf{B}^2) + W'([\mathbf{qE}]^2 - (\mathbf{qB})^2)}. \qquad (30)$$

This effect is called optical nonreciprocity and can be measured in experiments with ring lasers [23,24].

V. DISCUSSION

We found that in conformal-invariant nonlinear electrodynamics with the traceless stress-energy tensor, weak electromagnetic waves propagate on the background of a homogeneous pure electric or pure magnetic field similarly to the waves in an optic crystal. So the wave splits into two modes, and the refraction index for the ordinary mode is equal to unity as in Maxwell vacuum. It is also follows from the causality and unitarity conditions (16) that the refraction index for the extraordinary mode is greater than or equal to unity, depending on the wave propagation direction with the respect to the field vector. Advances in optical measurements make it possible to perform the experimental research of vacuum nonlinear electrodynamics effects in the laboratory. Such projects as ELI and PVLAS search for vacuum birefringence in strong magnetic fields. The main efforts of these projects are aimed at vacuum refraction index measurement and comparison with the Heisenberg-Euler theory prediction:

$$n_1 = 1 + \frac{4\alpha^2 B^2}{90\pi m^4}\sin^2\theta, \qquad n_2 = 1 + \frac{7\alpha^2 B^2}{90\pi m^4}\sin^2\theta, \qquad (31)$$

where m is the electron mass, α is the fine structure constant, B is the induction of the background magnetic field, and θ is the angle between $\mathbf{k}$ and the field vector $\mathbf{B}$. In contradiction to Eq. (31), in the conformal-invariant nonlinear electrodynamics with the Lagrangian (5), the refraction index for the first normal mode is always equal to unity, and for the mode with orthogonal polarization to the first one, the refraction index will depend on the angle θ in a more complex way than predicted by Heisenberg-Euler theory. It also should be noted that the indexes (28) do not depend on the field strength, and the birefringence is caused by a dimensionless coupling constant, which is a part of the W function as a parameter. Therefore, when carrying out experiments to search for the vacuum birefringence effects in a magnetic field, it is necessary not to limit measurements by choosing the angle $\theta = \pi/2$, which provides the maximum value of the effect, but it is also required to find the refraction index dependence on the angle θ for both normal modes.

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