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Shift of symmetries of naive and staggered fermions in QCD-like lattice theories

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We study the global symmetries of naive and staggered lattice Dirac operators in QCD-like theories in any dimension larger than two. In particular we investigate how the chosen number of lattice sites in each direction affects the global symmetries of the Dirac operator. These symmetries are important since they determine not only the infrared spectrum of the Dirac operator but also the symmetry breaking pattern and, thus, the lightest pseudoscalar mesons. We perform the symmetry analysis and discuss the possible zero modes and the degree of degeneracy of the lattice Dirac operators. Moreover we explicitly identify a "reduced" lattice Dirac operator which is the naive Dirac operator apart from the degeneracy. We verify our predictions by comparing Monte Carlo simulations of QCD-like theories in the strong coupling limit with the corresponding random matrix theories.

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I. INTRODUCTION

QCD-like theories describing the strong interaction between quarks and gluons are highly involved due to their nonlinear field equations. This statement also applies to QCD-like theories beyond the standard model like theories with technicolor [1] or supersymmetry [2]. Therefore quite often only numerical simulations remain as a tool to study the theory as a whole. Here, one encounters two crucial modifications of the continuum theory to overcome certain problems.

First, the time axis is Wick rotated to circumvent the sign problem resulting from a real time. Thus, the Yang-Mills action is replaced by a Euclidean Wilson gauge action or a Symanzik improved version of it. Furthermore, the Dirac operator becomes a hypercubic lattice Dirac operator which is a finite-dimensional matrix on a lattice of finite volume V. There are several lattice discretizations of the QCD-Dirac operator, the most prominent being the Dirac operators of staggered fermions [3], of Wilson fermions [4], of twisted mass fermions [5], of overlap fermions [6] and of domain wall fermions [7].

The simplest version of a lattice Dirac operator is the naive Dirac operator on a cubic lattice with periodic and antiperiodic boundary conditions. The Dirac operator of staggered fermions (without rooting) is a particular version of the naive Dirac operator. It is the nondegenerate part of the naive Dirac operator on a cubic lattice were each direction consists of an even number of lattice sites.

The second problem to be solved in lattice QCD concerns the continuum limit. It is well known that the global symmetries of staggered fermions in four dimensions do not necessarily agree with those of the continuum theory for QCD. For example the theories with two colors and the fermions in the fundamental representations or with arbitrary colors in the adjoint representation have this problem; see Refs. [8,9]. This problem was also found in three [10,11] and in two dimensions [12]. In particular in two dimensions the reason for the change of the global symmetries was recently analyzed in detail in Ref. [12]. In the present work we aim at a generalization of this discussion to arbitrary space-time dimensions.

The global symmetries are manifest in the lowest eigenvalues of the Dirac operator [13,14]. In the phase of spontaneous breaking of chiral symmetry the spectral gap is closed and chiral perturbation theory applies. The order parameter is the chiral condensate Σ which is given by the Banks-Casher formula [15] in terms of the level density ρ of the Dirac operator,

$$\Sigma = |\langle\bar{\psi}\psi\rangle| = \lim_{a\to 0}\lim_{m\to 0}\lim_{V\to\infty}\frac{\pi}{V}\int\frac{2m\rho(\lambda)d\lambda}{m^2+\lambda^2}. \tag{1}$$

The order of the limits is crucial. The Banks-Casher formula (1) is only true for Dirac operators exhibiting a chiral symmetry. In three dimensions one has to consider the following condensate:

$$\Sigma_{\mathrm{non}\chi} = |\langle\bar{\psi}\tau_3\psi\rangle| = \lim_{a\to 0}\lim_{m\to 0}\lim_{V\to\infty}\frac{\pi}{V}\int\frac{2m\rho(\lambda)d\lambda}{m^2+\lambda^2}. \tag{2}$$

Note that the expression in terms of the level density is the same in both equations because we assumed for the latter an even number of dynamical quarks. Then, half of the quarks contribute a chiral condensate weighted with a plus sign and the other half contribute one with a minus sign.

From the expressions (1) and (2) we recognize that only those eigenvalues of the Dirac operator of order $\mathcal{O}(1/V)$ are important for the spontaneous symmetry breaking. The

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corresponding modes are intimately related with those pseudoscalar mesons whose Compton wavelength is larger than the size of the lattice. Then the kinetic modes factorize from the zero-momentum modes [16,17] and can be integrated out. The remaining effective Lagrangian for the zero-momentum modes is shared with Gaussian random matrix models [13,14]. This theoretical prediction was numerically verified several times; for example for the microscopic level density

$$\rho_{\rm micro}(x) = \lim_{V\to\infty} \frac{1}{\Sigma V} \rho\left(\frac{x}{\Sigma V}\right), \tag{3}$$

(see Ref. [8]).

For a long time it has been known that QCD-like theories may yield different patterns of spontaneous symmetry breaking and, hence, different kinds of Goldstone bosons; see e.g. see Refs. [14,18,19]. In Ref. [20] the symmetry breaking patterns were derived for an arbitrary dimension larger than two. For QCD-like theories in one and two dimensions the Mermin-Wagner-Coleman theorem [21,22] forbids a spontaneous breaking of global symmetries, though two-dimensional theories seem to be at the borderline; see Ref. [12]. The authors of Ref. [20] found that the Bott periodicity [23] satisfied by the γ matrices carries over to the symmetry breaking pattern and, thus, to the effective Lagrangian for the pseudoscalar mesons. A similar classification was also found for topological insulators [24]. All in all there are ten symmetry breaking patterns associated to QCD-like Dirac operators. They correspond to the Cartan classification of Hermitian random matrices [25,26] and each of these ten random matrix ensembles yields the effective Lagrangian of the pseudoscalar mesons at lowest order.

Combining the two observations that the partition of the lattice and the space-time dimension affect the global symmetries and, hence, the symmetry-breaking pattern, one may ask what their combined impact is for an arbitrary QCD-like theory. Answering this question is the main task of the present work. For this purpose we pursue similar ideas as in Ref. [12].

We first review the symmetry analysis of the continuum theory [20], in Sec. II. In this section we also recall the properties of Clifford algebras built up by the γ matrices and the different kinds of antiunitary operators which are at the heart of the classification of global symmetries. In Sec. III we very briefly review what the naive discretization explicitly looks like and what the corresponding symmetry operations on a lattice are. In particular we discuss the artificial symmetry operations which arise when one or more directions have an even partition of lattice sites. Those additional symmetry operators anticommute with the lattice Dirac operator and build a Clifford algebra themselves. They are the origin for the change of symmetries which is analyzed in Sec. IV. In Sec. V we discuss the symmetry-breaking patterns and the number of zero modes and in Sec. VI we derive an explicit representation of the non-degenerate part of the lattice Dirac operator. This "reduced" Dirac operator is the staggered Dirac operator when each direction of the lattice contains an even number of sites. We also perform lattice simulations in the strong coupling limit and compare it with random matrix ensembles predicted by our analysis. Those comparisons are shown in Figs. 1 and 2. The random matrix results employed in these comparisons are recalled in the Appendix. In Sec. VII we summarize our results and discuss some important implications. The Tables I, II and III provide an overview of the change of symmetries when varying the space-time dimension, the number of directions with an even partition and the representation of the gauge group. We want to emphasize that we deal with all QCD-like gauge theories on an equal footing and do not restrict ourselves to the gauge groups $\mathrm{SU}(N_{\rm c})$.

In contrast to standard literature we do not apply Einstein's summation convention throughout the present work because of computational reasons.

II. DIRAC OPERATOR OF QCD-LIKE THEORIES IN EUCLIDEAN CONTINUUM

We consider the Euclidean massless Dirac operator of a QCD-like gauge theory in d dimensions, i.e.

$$\mathcal{D} = \sum_{\mu=1}^{d} (\partial_\mu \mathbb{1}_{d_r} + \imath A_\mu(x))\gamma_\mu \tag{4}$$

where $\imath$ is the imaginary unit. The vector fields $\imath A_\mu(x) \in r(\mathfrak{g})$, where x is a point in space-time, are elements in an irreducible representation r of a finite-dimensional compact Lie algebra $\mathfrak{g}$; in particular $A_\mu^\dagger = A_\mu$ is Hermitian. The dimension of this representation is denoted by d_r and should not be confused with the space-time dimension d. In QCD the Lie algebra is $\mathfrak{g} = \mathrm{su}(3)$. The dimension d_r is emphasised by the identity matrix $\mathbb{1}_{d_r}$ multiplied with the partial derivative ∂_μ in the μth direction. Since we consider Euclidean space-time the generalized γ matrices are given by the Clifford algebra [27]

$$[\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu}\mathbb{1}_{2^{\lfloor d/2 \rfloor}}, \qquad \mathrm{tr}\,\gamma_\mu = 0 \quad \text{and} \quad \gamma_\mu^\dagger = \gamma_\mu \tag{5}$$

where $[.,.]_+$ is the anticommutator and γ_μ are Hermitian $2^{\lfloor d/2 \rfloor} \times 2^{\lfloor d/2 \rfloor}$ matrices. The function $\lfloor d/2 \rfloor$ is the floor function yielding the largest integer equal to or smaller than $d/2$.

The Clifford algebra above generates the fundamental representation of the Lie algebra of the unitary group $\mathrm{U}(2^{\lfloor d/2 \rfloor})$ via the multiplication of algebra elements and scalars as well as the addition of algebra elements. This fact will become helpful later on.

TABLE I. Classification of QCD-like continuum theories with respect to the spontaneous breaking of their flavor symmetry group like chiral symmetries; see Ref. [20]. In the first line of each entry the symmetries of the continuum Dirac operator $\mathcal{D} = -\mathcal{D}^\dagger$ regarding the antiunitary operator $\mathcal{C}$ and the matrix $\gamma^{(5)}$, if applicable, are shown. In the second line we show the symmetry-breaking pattern but without taking into account the anomalous symmetry breaking of the axial symmetry group. The space-time dimension d has to be larger than two for these patterns while $d = 2$ does not necessarily exhibit a spontaneous symmetry breaking because of the Mermin-Wagner-Coleman theorem. However it was observed in numerical simulations that some QCD-like theories might show a symmetry breaking in two dimensions; see Ref. [12]. In the third line of the entries we recall the symmetry class of the Cartan scheme [23,24] and the corresponding random matrix model exhibiting the same spectral statistics for the lowest eigenvalues as the corresponding Dirac operator; see Table III. The indices n and ν determine the dimension of the random matrix and play the roles of the volume and the topological charge.

d	Real representation	Complex representation	Quaternion representation
$8m$	$[\gamma^{(5)},\mathcal{D}]_+ = [\mathcal{C},\mathcal{D}]_- = [\mathcal{C},\gamma^{(5)}]_- = 0$, $\mathcal{C}^2 = \mathbb{1}$ $\mathrm{U}(2N_\mathrm{f}) \to \mathrm{USp}(2N_\mathrm{f})$ $\mathrm{B\vert DI}, \chi\mathrm{GOE}_\nu(n), \nu \in \mathbb{Z}$	$[\gamma^{(5)},\mathcal{D}]_+ = 0$ $\mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f})$ $\mathrm{AIII}, \chi\mathrm{GUE}_\nu(n), \nu \in \mathbb{Z}$	$[\gamma^{(5)},\mathcal{D}]_+ = [\mathcal{C},\mathcal{D}]_- = [\mathcal{C},\gamma^{(5)}]_- = 0$, $\mathcal{C}^2 = -\mathbb{1}$ $\mathrm{U}(2N_\mathrm{f}) \to \mathrm{O}(2N_\mathrm{f})$ $\mathrm{CII}, \chi\mathrm{GSE}_\nu(n), \nu \in \mathbb{Z}$
$8m+1$	$[\mathcal{C},\mathcal{D}]_- = 0, \mathcal{C}^2 = \mathbb{1}$ $\mathrm{O}(2N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f})$ $\mathrm{B\vert D}, \mathrm{GAOE}_\nu(n), \nu \in \mathbb{Z}_2$	no further symmetries $\mathrm{U}(2N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f})$ $\mathrm{A}, \mathrm{GUE}(n)$	$[\mathcal{C},\mathcal{D}]_- = 0, \mathcal{C}^2 = -\mathbb{1}$ $\mathrm{USp}(2N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f})$ $\mathrm{C}, \mathrm{GASE}(n)$
$8m+2$	$[\gamma^{(5)},\mathcal{D}]_+ = [\mathcal{C},\mathcal{D}]_+ = [\mathcal{C},\gamma^{(5)}]_+ = 0$, $\mathcal{C}^2 = -\mathbb{1}$ $\mathrm{O}(2N_\mathrm{f}) \times \mathrm{O}(2N_\mathrm{f}) \to \mathrm{O}(2N_\mathrm{f})$ $\mathrm{B\vert DIII}, \mathrm{GBSE}_\nu(n), \nu \in \mathbb{Z}_2$	$[\gamma^{(5)},\mathcal{D}]_+ = 0$ $\mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f})$ $\mathrm{AIII}, \chi\mathrm{GUE}_\nu(n), \nu \in \mathbb{Z}$	$[\gamma^{(5)},\mathcal{D}]_+ = [\mathcal{C},\mathcal{D}]_+ = [\mathcal{C},\gamma^{(5)}]_+ = 0$, $\mathcal{C}^2 = \mathbb{1}$ $\mathrm{USp}(2N_\mathrm{f}) \times \mathrm{USp}(2N_\mathrm{f}) \to \mathrm{USp}(2N_\mathrm{f})$ $\mathrm{CI}, \mathrm{GBOE}(n)$
$8m+3$	$[\mathcal{C},\mathcal{D}]_+ = 0, \mathcal{C}^2 = -\mathbb{1}$ $\mathrm{O}(2N_\mathrm{f}) \to \mathrm{O}(N_\mathrm{f}) \times \mathrm{O}(N_\mathrm{f})$ $\mathrm{AII}, \mathrm{GSE}(n)$	no further symmetries $\mathrm{U}(2N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f})$ $\mathrm{A}, \mathrm{GUE}(n)$	$[\mathcal{C},\mathcal{D}]_+ = 0, \mathcal{C}^2 = \mathbb{1}$ $\mathrm{USp}(4N_\mathrm{f}) \to \mathrm{USp}(2N_\mathrm{f}) \times \mathrm{USp}(2N_\mathrm{f})$ $\mathrm{AI}, \mathrm{GOE}(n)$
$8m+4$	$[\gamma^{(5)},\mathcal{D}]_+ = [\mathcal{C},\mathcal{D}]_- = [\mathcal{C},\gamma^{(5)}]_- = 0$, $\mathcal{C}^2 = -\mathbb{1}$ $\mathrm{U}(2N_\mathrm{f}) \to \mathrm{O}(2N_\mathrm{f})$ $\mathrm{CII}, \chi\mathrm{GSE}_\nu(n), \nu \in \mathbb{Z}$	$[\gamma^{(5)},\mathcal{D}]_+ = 0$ $\mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f})$ $\mathrm{AIII}, \chi\mathrm{GUE}_\nu(n), \nu \in \mathbb{Z}$	$[\gamma^{(5)},\mathcal{D}]_+ = [\mathcal{C},\mathcal{D}]_- = [\mathcal{C},\gamma^{(5)}]_- = 0$, $\mathcal{C}^2 = \mathbb{1}$ $\mathrm{U}(2N_\mathrm{f}) \to \mathrm{USp}(2N_\mathrm{f})$ $\mathrm{B\vert DI}, \chi\mathrm{GOE}_\nu(n), \nu \in \mathbb{Z}$
$8m+5$	$[\mathcal{C},\mathcal{D}]_- = 0, \mathcal{C}^2 = -\mathbb{1}$ $\mathrm{USp}(2N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f})$ $\mathrm{C}, \mathrm{GASE}(n)$	no further symmetries $\mathrm{U}(2N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f})$ $\mathrm{A}, \mathrm{GUE}(n)$	$[\mathcal{C},\mathcal{D}]_- = 0, \mathcal{C}^2 = \mathbb{1}$ $\mathrm{O}(2N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f})$ $\mathrm{B\vert D}, \mathrm{GAOE}_\nu(n), \nu \in \mathbb{Z}_2$
$8m+6$	$[\gamma^{(5)},\mathcal{D}]_+ = [\mathcal{C},\mathcal{D}]_+ = [\mathcal{C},\gamma^{(5)}]_+ = 0$, $\mathcal{C}^2 = \mathbb{1}$ $\mathrm{USp}(2N_\mathrm{f}) \times \mathrm{USp}(2N_\mathrm{f}) \to \mathrm{USp}(2N_\mathrm{f})$ $\mathrm{CI}, \mathrm{GBOE}(n)$	$[\gamma^{(5)},\mathcal{D}]_+ = 0$ $\mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f})$ $\mathrm{AIII}, \chi\mathrm{GUE}_\nu(n), \nu \in \mathbb{Z}$	$[\gamma^{(5)},\mathcal{D}]_+ = [\mathcal{C},\mathcal{D}]_+ = [\mathcal{C},\gamma^{(5)}]_+ = 0$, $\mathcal{C}^2 = -\mathbb{1}$ $\mathrm{O}(2N_\mathrm{f}) \times \mathrm{O}(2N_\mathrm{f}) \to \mathrm{O}(2N_\mathrm{f})$ $\mathrm{B\vert DIII}, \mathrm{GBSE}_\nu(n), \nu \in \mathbb{Z}_2$
$8m+7$	$[\mathcal{C},\mathcal{D}]_+ = 0, \mathcal{C}^2 = \mathbb{1}$ $\mathrm{USp}(4N_\mathrm{f}) \to \mathrm{USp}(2N_\mathrm{f}) \times \mathrm{USp}(2N_\mathrm{f})$ $\mathrm{AI}, \mathrm{GOE}(n)$	no further symmetries $\mathrm{U}(2N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f})$ $\mathrm{A}, \mathrm{GUE}(n)$	$[\mathcal{C},\mathcal{D}]_+ = 0, \mathcal{C}^2 = -\mathbb{1}$ $\mathrm{O}(2N_\mathrm{f}) \to \mathrm{O}(N_\mathrm{f}) \times \mathrm{O}(N_\mathrm{f})$ $\mathrm{AII}, \mathrm{GSE}(n)$

We now recall some well-known facts about the Clifford algebra with d generators; see also Ref. [27]. The generalized γ matrices can be represented in terms of the three Pauli matrices σ_i, $i = 1, 2, 3$. However we will employ a basis independent representation of the Dirac matrices. We recall that for odd dimension d (odd number of generators of the Clifford algebra) there is no chiral symmetry while for even dimensions the matrix

$$\gamma^{(5)} = \imath^{-d(d-1)/2} \prod_{j=1}^{d} \gamma_j = \imath^{-d(d-1)/2} \gamma_1 \gamma_2 \cdots \gamma_d \qquad (6)$$

anticommutes with all generators such that there is a chiral symmetry. Moreover, the generators of the Clifford algebra always satisfy an antiunitary symmetry

$$[C, \imath^{d(d-1)/2}\gamma_\nu]_- = 0 \quad \text{and} \quad [C, \imath^{d(d-1)/2}\gamma^{(5)}]_- = 0 \qquad (7)$$

where $[.,.]_-$ is the commutator. The operator C can be explicitly written as

$$C = K\chi = \begin{cases} K\imath^m \prod_{j=1}^{2m}\gamma_{2j}, & d = 4m, 4m+1, \\ K\imath^{m+1}\prod_{j=1}^{2m+2}\gamma_{2j-1}, & d = 4m+2, 4m+3, \end{cases} \qquad (8)$$

where K is the complex conjugation operator and $\gamma_{d+1} = \gamma^{(5)}$ for $d \in 2\mathbb{N}$. It satisfies

$$C^2 = (-1)^{(d+2)(d+1)d(d-1)/8}\mathbb{1}_{2^{\lfloor d/2\rfloor}} \qquad (9)$$

which is the origin for the Bott periodicity [23] of Clifford algebras. The matrix χ [see Eq. (8)] is a unitary, Hermitian matrix which is either symmetric or antisymmetric depending on the sign of C^2.

The antiunitary symmetry (7) has to be combined with the one which might be satisfied by the vector fields A_μ. Note that the partial derivatives are anti-Hermitian and real, i.e. $\partial_\mu = -\partial_\mu^\dagger = K\partial_\mu K$. Also the vector fields $\imath A_\mu$ are anti-Hermitian, i.e. $(\imath A_\mu)^\dagger = -\imath A_\mu$. Additionally they might satisfy an antiunitary symmetry,

$$[K\zeta, \imath A_\mu]_- = 0 \quad \text{and} \quad (K\zeta)^2 = \zeta^*\zeta = \pm\mathbb{1} \qquad (10)$$

where ζ is a unitary matrix. If there is no such symmetry then the representation r is called complex. Examples are the fundamental representations of the gauge groups $\mathrm{SU}(N_c \geq 3)$ and $\mathrm{U}(N_c \geq 2)$. When there is an antiunitary operator $K\zeta$, the representation r is called real or quaternion when $(K\zeta)^2 = \mathbb{1}$ or $(K\zeta)^2 = -\mathbb{1}$, respectively. Examples of real representations are the fundamental representation of $\mathrm{SO}(2N_c + 1 \geq 5)$ and $\mathrm{SO}(2N_c \geq 8)$ and the adjoint representation of any compact Lie algebra. The fundamental representation of $\mathrm{SU}(N_c = 2) = \mathrm{USp}(2N_c = 2)$ and in general the unitary symplectic group $\mathrm{USp}(2N_c)$ are examples of quaternion representations.

The symmetry discussion above for the continuum Dirac operator is summarized in Table I and was already performed in Ref. [20]. For this purpose, we defined the charge conjugation operator

$$\mathcal{C} = K\zeta\chi \qquad (11)$$

which only exists for real and quaternion representations of the chosen gauge group. In Table I we also point out the symmetry-breaking pattern, the symmetry class via the Cartan classification scheme [25,26], and the random matrix theory describing the infrared energy spectrum of the Dirac operator. Note that all ten symmetry classes of Hermitian operators can be found as it is the case for topological insulators [24].

III. NAIVE AND STAGGERED LATTICE DIRAC OPERATOR

On the lattice the Hilbert space $\mathcal{H} = \mathbb{C}^{d_r} \otimes \hat{V} \otimes \mathbb{C}^{\lfloor d/2\rfloor}$ is finite dimensional. It consists of three parts. The first part is the color space $\mathbb{C}^{d_r}$ where the gauge group acts. The second one is the cubic lattice $\hat{V} = \bigotimes_{j=1}^{d} \mathbb{C}^{L_j}$ with the volume $V = \prod_{j=1}^{d} L_j$ and $L_\mu \in \mathbb{N}$ being the number of lattice sites in the direction μ. The third part is the spinor space $\mathbb{C}^{\lfloor d/2\rfloor}$. Thus the dimension of the Hilbert space is in total $d_{\mathcal{H}} = 2^{\lfloor d/2\rfloor} d_r V$.

In the naive discretization the covariant derivative is replaced by the difference of the translation operator T_μ and its Hermitian adjoint, i.e.

$$\partial_\mu \mathbb{1}_{d_r} + \imath A_\mu(x) \to T_\mu - T_\mu^\dagger. \qquad (12)$$

The translation operator T_μ acts as follows on a state $|\psi(x)\rangle \in \mathcal{H}$ at a fixed lattice site $x = (x_1, \ldots, x_d) \in \hat{V}$:

$$T_\mu|\psi(x)\rangle = (-1)^{\delta_{\mu d}\delta_{x_d L_d}} U_\mu(x)|\psi(x + e_\mu)\rangle, \qquad (13)$$

where e_μ is the d-dimensional vector with a 1 at the position μ and otherwise zero and $x_j \in \mathbb{Z}_{L_j}$. The matrices $U_\mu(x)$ are elements of the representation $r(\mathcal{G})$ where $\mathcal{G}$ is the gauge group. This representation satisfies the same antiunitary symmetry as the Lie algebra $\mathfrak{g}$ if it exists. The sign in Eq. (13) reflects the boundary conditions which are periodic in the spatial directions $\mu = 1, \ldots, d-1$ and antiperiodic for the temporal direction $\mu = d$. We want to emphasize that the results of the symmetry analysis will be independent of the periodic or antiperiodic boundary conditions. Hence the sign only plays a minor role. The naive lattice Dirac operator is given by

$$D = \sum_{\mu=1}^{d}(T_\mu - T_\mu^\dagger)\gamma_\mu. \qquad (14)$$

As already found for two dimensions (see Ref. [12]), depending on the number of lattice sites in a fixed direction μ there might be an operator that anticommutes with the lattice Dirac operator (14). Suppose the number L_μ is even. Then we can consider the operator

$$\Gamma_\mu|\psi(x)\rangle = (-1)^{x_\mu}|\psi(x)\rangle \qquad (15)$$

which assigns to each even lattice site (according to the parity in the direction μ) a "+1" and to each odd lattice site a "−1." Obviously, the operator Γ_μ is diagonal and consists of equally many eigenvalues ± 1 and only acts on the lattice part $\hat{V}$ of the Hilbert space. This artificial operator satisfies the following commutation relations with the transfer matrices:

$$[\Gamma_\mu, T_\mu]_+ = 0 \quad \text{and} \quad [\Gamma_\mu, T_\nu]_- \overset{\mu\neq\nu}{=} 0. \tag{16}$$

Combining this artificial operator with the γ matrix γ_μ, i.e.

$$\Gamma_\mu^{(5)} = \Gamma_\mu \gamma_\mu \tag{17}$$

(note that we do *not* sum over μ), we obtain the anti-commutation relation

$$[\Gamma_\mu^{(5)}, D]_+ = \sum_{\nu=1}^{d} [\Gamma_\mu \gamma_\mu, (T_\nu - T_\nu^\dagger)\gamma_\nu]_+ = 0. \tag{18}$$

The lattice may not only have one direction with an even partition. For example for staggered fermions all L_μ are even. For each direction with an even partition we construct an operator $\Gamma_\mu^{(5)}$ and each of them satisfies Eq. (18). Suppose $N_\mathrm{L} \leq d$ directions have an even partition and we choose these directions as $\mu = 1, \ldots, N_\mathrm{L}$ without loss of generality. We emphasize again that the different boundary conditions for spatial and temporal directions play a minor role. Additionally we define the operator $\Gamma_{N_\mathrm{L}+1}^{(5)} = \gamma^{(5)}$ depending on whether the space-time dimension d is even. The operators $\{\Gamma_j^{(5)}\}_{j=1,\ldots,N_\mathrm{Cl}}$ also build a Clifford algebra of $N_\mathrm{Cl} = N_\mathrm{L} + [d+1]_2$ generators, i.e.

$$[\Gamma_i^{(5)}, \Gamma_j^{(5)}]_+ = 2\delta_{ij} \mathbb{1}_{d_\mathcal{H}}, \qquad \mathrm{tr}\Gamma_j^{(5)} = 0 \quad \text{and} \quad (\Gamma_j^{(5)})^\dagger = \Gamma_j^{(5)}. \tag{19}$$

The function $[d+1]_2$ is 1 if d is even and vanishes for odd d. All operators $\{\Gamma_j^{(5)}\}_{j=1,\ldots,N_\mathrm{Cl}}$ anticommute with the Dirac operator; see Eq. (18).

In the case where there is a charge conjugation operator

$$C_\mathrm{lat} = K\zeta\chi \quad \text{and} \quad C_\mathrm{lat}^2 = (-1)^{(d+2)(d+1)d(d-1)/8} \mathrm{sign}[(K\zeta)^2] \mathbb{1}_{d_\mathcal{H}}, \tag{20}$$

namely for real and quaternion representations r, with

$$[C_\mathrm{lat}, \imath^{d(d-1)/2} D]_- = 0, \tag{21}$$

we have furthermore the relation

$$[C_\mathrm{lat}, \imath^{d(d-1)/2} \Gamma_j^{(5)}]_- = 0 \tag{22}$$

for all $j = 1, \ldots, N_\mathrm{Cl}$.

IV. SYMMETRY ANALYSIS

The relations (18), (19), (21), and (22) together with the definitions (17) and (20) completely determine the degeneracy of the eigenvalues and the symmetry class of the lattice Dirac operator D including the symmetry-breaking pattern. This has to be done for even and odd N_Cl, separately. We do this by pursuing the same ideas as in the discussion for the two-dimensional lattice QCD Dirac operator [12].

A. Even number N_Cl of Clifford generators

When the number N_Cl of generators of the Clifford algebra anticommuting with the Dirac operator is even one can construct a $N_\mathrm{Cl}+1$ Hermitian, unitary operator $\hat{\Gamma}^{(5)} = \imath^{N_\mathrm{Cl}(N_\mathrm{Cl}-1)/2} \prod_{j=1}^{N_\mathrm{Cl}} \Gamma_j^{(5)}$ which anticommutes with all generators but commutes with the naive Dirac operator D. Multiplying D with $\hat{\Gamma}^{(5)}$ one obtains the commutation relations

$$[D\hat{\Gamma}^{(5)}, \Gamma_j^{(5)}]_- = 0 \quad \text{for all } j = 1, \ldots, N_\mathrm{Cl}. \tag{23}$$

Since the combinations $\Gamma_i^{(5)} \pm \imath\Gamma_j^{(5)}$ are nilpotent the set $\{\Gamma_j^{(5)}\}_{j=1,\ldots,N_\mathrm{Cl}}$ generates a direct sum of fundamental representations of the Lie algebra of the unitary group $\mathrm{U}(2^{N_\mathrm{Cl}/2})$ by multiplication of elements and of scalars and by addition. The fundamental representation of the Clifford algebra generated by N_Cl Hermitian elements is unique up to unitary transformations because N_Cl is even. Thus there is a unitary matrix $U \in \mathrm{U}(d_\mathcal{H})$ with

$$U\Gamma_j^{(5)} U^\dagger = \mathbb{1}_{d_\mathcal{H}/2^{N_\mathrm{Cl}/2}} \otimes \gamma_j' \quad \text{for all } j = 1, \ldots, N_\mathrm{Cl}, \tag{24}$$

with $\{\gamma_j'\}_{j=1,\ldots,N_\mathrm{Cl}}$ in the fundamental representation of the Clifford algebra in $2^{N_\mathrm{Cl}/2}$ dimensions. The tensor notation shall emphasize the splitting of the Hilbert space into two terms. We will also employ it in the following as a bookkeeping tool.

Equation (24) also implies

$$U\hat{\Gamma}^{(5)} U^\dagger = \mathbb{1}_{d_\mathcal{H}/2^{N_\mathrm{Cl}/2}} \otimes \gamma'^{(5)} = \imath^{N_\mathrm{Cl}(N_\mathrm{Cl}-1)/2} \mathbb{1}_{d_\mathcal{H}/2^{N_\mathrm{Cl}/2}} \otimes \prod_{j=1}^{N_\mathrm{Cl}} \gamma_j'. \tag{25}$$

With the aid of Schur's lemma [28], the commutation relation (23) yields $UD\hat{\Gamma}^{(5)}U^\dagger = D_\mathrm{red} \otimes \mathbb{1}_{2^{N_\mathrm{Cl}/2}}$ or equivalently

$$UDU^\dagger = D_\mathrm{red} \otimes \gamma'^{(5)} \tag{26}$$

with a reduced lattice Dirac operator $D_\mathrm{red} = -D_\mathrm{red}^\dagger$ only acting on a Hilbert space of dimension $d_\mathcal{H}/2^{N_\mathrm{Cl}/2}$.

TABLE II. The symmetry-breaking patterns of the naive lattice Dirac operator for the three kinds of theories with a dimension $d > 2$ and the corresponding Cartan class [23,24] and the random matrix theory (see Table III and the Appendix), with which the Monte Carlo simulations have to be compared. Instead of the dimension d of the continuum theory one obtains the symmetry of a shifted dimension $d_{\rm red} = d - N_{\rm L}$. The staggered Dirac operator corresponds to $d_{\rm red} = 0 = 8m$ though the degeneracy $d_{\rm tri} = 2^{\lfloor (N_{\rm L}+[d+1]_2)/2 \rfloor}$ will not be present for this operator. The degeneracy is also the reason why the number of flavors is increased to $d_{\rm tri}N_{\rm f}$ compared to the continuum theory, cf. Table I. Note that the topological charge ν vanishes because no zero modes are present in the naive Dirac operator with $d > 2$.

$d_{\rm red} = d - N_{\rm L}$	Real representation	Complex representation	Quaternion representation
$8m$	$\mathrm{U}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{USp}(2d_{\rm tri}N_{\rm f})$ DI, χGOE$_0(n)$	$\mathrm{U}(d_{\rm tri}N_{\rm f}) \times \mathrm{U}(d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f})$ AIII, χGUE$_0(n)$	$\mathrm{U}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{O}(2d_{\rm tri}N_{\rm f})$ CII, χGSE$_0(n)$
$8m + 1$	$\mathrm{O}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f})$ D, GAOE$_0(n)$	$\mathrm{U}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f}) \times \mathrm{U}(d_{\rm tri}N_{\rm f})$ A, GUE(n)	$\mathrm{USp}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f})$ C, GASE(n)
$8m + 2$	$\mathrm{O}(2d_{\rm tri}N_{\rm f}) \times \mathrm{O}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{O}(2d_{\rm tri}N_{\rm f})$ B|DIII, GBSE$_0(n)$	$\mathrm{U}(d_{\rm tri}N_{\rm f}) \times \mathrm{U}(d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f})$ AIII, χGUE$_0(n)$	$\mathrm{USp}(2d_{\rm tri}N_{\rm f}) \times \mathrm{USp}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{USp}(2d_{\rm tri}N_{\rm f})$ CI, GBOE(n)
$8m + 3$	$\mathrm{O}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{O}(d_{\rm tri}N_{\rm f}) \times \mathrm{O}(d_{\rm tri}N_{\rm f})$ AII, GSE(n)	$\mathrm{U}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f}) \times \mathrm{U}(d_{\rm tri}N_{\rm f})$ A, GUE(n)	$\mathrm{USp}(4d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{USp}(2d_{\rm tri}N_{\rm f}) \times \mathrm{USp}(2d_{\rm tri}N_{\rm f})$ AI, GOE(n)
$8m + 4$	$\mathrm{U}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{O}(2d_{\rm tri}N_{\rm f})$ CII, χGSE$_0(n)$	$\mathrm{U}(d_{\rm tri}N_{\rm f}) \times \mathrm{U}(d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f})$ AIII, χGUE$_0(n)$	$\mathrm{U}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{USp}(2d_{\rm tri}N_{\rm f})$ DI, χGOE$_0(n)$
$8m + 5$	$\mathrm{USp}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f})$ C, GASE(n)	$\mathrm{U}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f}) \times \mathrm{U}(d_{\rm tri}N_{\rm f})$ A, GUE(n)	$\mathrm{O}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f})$ D, GAOE$_0(n)$
$8m + 6$	$\mathrm{USp}(2d_{\rm tri}N_{\rm f}) \times \mathrm{USp}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{USp}(2d_{\rm tri}N_{\rm f})$ CI, GBOE(n)	$\mathrm{U}(d_{\rm tri}N_{\rm f}) \times \mathrm{U}(d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f})$ AIII, χGUE$_0(n)$	$\mathrm{O}(2d_{\rm tri}N_{\rm f}) \times \mathrm{O}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{O}(2d_{\rm tri}N_{\rm f})$ B|DIII, GBSE$_0(n)$
$8m + 7$	$\mathrm{USp}(4d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{USp}(2d_{\rm tri}N_{\rm f}) \times \mathrm{USp}(2d_{\rm tri}N_{\rm f})$ AI, GOE(n)	$\mathrm{U}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{U}(d_{\rm tri}N_{\rm f}) \times \mathrm{U}(d_{\rm tri}N_{\rm f})$ A, GUE(n)	$\mathrm{O}(2d_{\rm tri}N_{\rm f})$ $\downarrow$ $\mathrm{O}(d_{\rm tri}N_{\rm f}) \times \mathrm{O}(d_{\rm tri}N_{\rm f})$ AII, GSE(n)

In Sec. VI we write $D_{\rm red}$ more explicitly by choosing a basis in the spinor space.

In the case that there is no additional antiunitary symmetry (complex representation of the gauge group) the eigenvalues of D are $2^{N_{\rm Cl}/2-1}$-fold degenerate. Moreover all eigenvalues come in "chiral" pairs $(\lambda, -\lambda)$ apart from the case $N_{\rm Cl} = 0$. These "chiral" pairs are reminiscent of the chirality of the original lattice Dirac operator D. The Dirac operator will have the global symmetries of the three-dimensional continuum theory.

In the case of a real or quaternion representation, we also have to consider the transformation of the charge conjugation operator $C_{\rm lat}$. We define $C'_{\rm lat}=UC_{\rm lat}U^\dagger=K\zeta'\otimes\chi'$ with unitary matrices $\zeta'\in{\rm U}(d_{\mathcal H}/2^{N_{\rm Cl}/2})$ and $\chi'\in{\rm U}(2^{N_{\rm Cl}/2})$. The commutation relations (22) and (21) become

$$[C'_{\rm lat},\imath^{d(d-1)/2}\mathbb{1}_{d_{\mathcal H}/2^{N_{\rm Cl}/2}}\otimes\gamma'_j]_-=0$$
$$\Rightarrow[C'_{\rm lat},\imath^{N_{\rm Cl}(N_{\rm Cl}-1)/2}\mathbb{1}_{d_{\mathcal H}/2^{N_{\rm Cl}/2}}\otimes\gamma'^{(5)}]_-=0 \quad (27)$$

and

$$\pm[C'_{\rm lat},\imath^{d(d-1)/2}D_{\rm red}\otimes\gamma'^{(5)}]_- = [C_{\rm red},\imath^{d_{\rm red}(d_{\rm red}-1)/2}D_{\rm red}]_-\otimes\imath^{N_{\rm Cl}(N_{\rm Cl}-1)/2}\chi'\gamma'^{(5)}=0 \quad (28)$$

with $C_{\rm red}=K\zeta'$ and $d_{\rm red}=d-N_{\rm L}$, respectively. To derive Eq. (28) we need

$$\imath^{d(d-1)/2}=\pm\imath^{d_{\rm red}(d_{\rm red}-1)/2+N_{\rm Cl}(N_{\rm Cl}-1)/2} \quad (29)$$

which is true because $N_{\rm L}=N_{\rm Cl}-[d+1]_2$ with $[d+1]_2=0$, 1 and $d[d+1]_2$ is always even.

What remains to be calculated is the square $C^2_{\rm red}$. For this purpose we notice that $(C'_{\rm lat})^2=C^2_{\rm lat}$. Furthermore the unitary matrix χ' can be chosen as

$$\chi'=(\gamma'^{(5)})^{d_{\rm red}(d_{\rm red}-1)/2}\times\begin{cases}\imath^m\prod_{j=1}^{2m}\gamma'_{2j}, & N_{\rm Cl}=4m,\\ \imath^{m+1}\prod_{j=1}^{2m+2}\gamma'_{2j-1}, & N_{\rm Cl}=4m+2,\end{cases} \quad (30)$$

because of the commutation relations

$$[K\chi',\imath^{d(d-1)/2}\gamma'_j]_-=0\quad\text{for all } j=1,\ldots,N_{\rm Cl}. \quad (31)$$

All other choices of χ' are unitarily equivalent. The commutation relations (31) directly follow from the first line of Eq. (27), since the first component of the tensor product is trivial. Employing the relation $C^2_{\rm lat}=(C'_{\rm lat})^2=C^2_{\rm red}\otimes(K\chi')^2$ we obtain

$$C^2_{\rm red}=(-1)^{(d_{\rm red}+2)(d_{\rm red}+1)d_{\rm red}(d_{\rm red}-1)/8}\times{\rm sign}[(K\zeta)^2]\mathbb{1}_{d_{\mathcal H}/2^{N_{\rm Cl}/2}}. \quad (32)$$

To simplify the sign we have used the identity

$$(-1)^{(d+2)(d+1)d(d-1)/8+(N_{\rm Cl}+2)(N_{\rm Cl}+1)N_{\rm Cl}(N_{\rm Cl}-1)/8} = (-1)^{(d_{\rm red}+2)(d_{\rm red}+1)d_{\rm red}(d_{\rm red}-1)/8+d_{\rm red}(d_{\rm red}-1)N_{\rm Cl}(N_{\rm Cl}-1)/4} \quad (33)$$

which is even true for odd $N_{\rm Cl}$ as can be checked by choosing $d,N_{\rm L}=1,\ldots,8$ because of the periodicity in 8.

The symmetries of the Dirac operator D experience a shift in the Bott periodic Table I due to the lattice directions with even partition. Moreover, the degeneracy of the eigenvalues of the lattice Dirac operator D is either $2^{N_{\rm Cl}/2-1}$-fold or $2^{N_{\rm Cl}/2}$-fold depending on whether $D_{\rm red}$ exhibits Kramers degeneracy or "chiral pairs" of eigenvalues or not. We emphasize that the whole Dirac operator D always exhibits "chiral pairs" of eigenvalues.

Summarizing the discussion for an even number $N_{\rm Cl}$ of Clifford generators anticommuting with the Dirac operator, the symmetries of D are shifted via the Bott periodic Table I from the original symmetries of the continuum theory in d dimensions to a theory sharing the symmetries of the same continuum theory but only in $d_{\rm red}=d-N_{\rm L}$ dimensions. This is true for all representations. Thus, for any representation the shift of the symmetry class is always to an odd-dimensional continuum theory because $N_{\rm Cl}=N_{\rm L}+[d+1]_2$ is even. This will be different for the case of odd $N_{\rm Cl}$. The symmetry-breaking patterns are summarized in Table II.

B. Odd number $N_{\rm Cl}$ of Clifford generators

For an odd number $N_{\rm Cl}$ of Clifford generators the corresponding fundamental representation of the Clifford algebra is not unique; rather, there are two. Indeed we have in our case that the product of all generators $\prod_{j=1}^{N_{\rm Cl}}\Gamma_j^{(5)}$ is not proportional to the identity as it would be for the case when it is a multiple of one of the two inequivalent fundamental representations. The product is

$$\tilde\Gamma=\imath^{N_{\rm Cl}(N_{\rm Cl}-1)/2}\prod_{j=1}^{N_{\rm Cl}}\Gamma_j^{(5)}=\Gamma^{(5)}\tilde\gamma = \imath^{N_{\rm Cl}(N_{\rm Cl}-1)/2}\Gamma^{(5)}\begin{cases}\left(\prod_{j=1}^{N_{\rm L}}\gamma_j\right)\gamma^{(5)}, & d\in2\mathbb{N},\\ \prod_{j=1}^{N_{\rm L}}\gamma_j, & d\in2\mathbb{N}+1,\end{cases} \quad (34)$$

where $\Gamma^{(5)}=\prod_{j=1}^{N_{\rm L}}\Gamma_j$ is a diagonal matrix with an equal number of eigenvalues ±1. Also the matrix $\tilde\Gamma$ has the same number of eigenvalues with ±1 which follows from $\tilde\Gamma=\tilde\Gamma^\dagger=\tilde\Gamma^{-1}$ and ${\rm tr}\,\tilde\Gamma=({\rm tr}\,\Gamma^{(5)})({\rm tr}\,\tilde\gamma)=0$. Thus we have the same number of both fundamental representations of the Clifford algebra with $N_{\rm Cl}$ elements. Then the matrices $\{\imath\tilde\Gamma\Gamma_j^{(5)}\}_{j=1,\ldots,N_{\rm Cl}}$ build a multiple of only one of the two fundamental representations. In particular we can find a unitary matrix $U\in{\rm U}(d_{\mathcal H})$ with

$$U\Gamma_j^{(5)}U^\dagger=\gamma^{(5)}_{\rm red}\otimes\gamma'_j\quad\text{for all } j=1,\ldots,N_{\rm Cl},$$
$$U\tilde\Gamma U^\dagger=\gamma^{(5)}_{\rm red}\otimes\mathbb{1}_{2^{(N_{\rm Cl}-1)/2}}, \quad (35)$$

where $\gamma'_{N_{\rm Cl}}=\imath^{N_{\rm Cl}(N_{\rm Cl}-1)/2}\prod_{j=1}^{N_{\rm Cl}-1}\gamma'_j$ and $\gamma^{(5)}_{\rm red}$ are diagonal matrices with eigenvalues ±1 each with multiplicity

$d_{\mathcal{H}}/2^{(N_{\rm Cl}+1)/2}$. Again the tensor notation shall help to separate the Hilbert space into a space where the naive Dirac operator acts trivially and a reduced Hilbert space.

In the next step we consider the Clifford algebra of $N_{\rm Cl}-1$ elements $\{\imath 1_{d_{\mathcal{H}}/2^{(N_{\rm Cl}-1)/2}}\otimes\gamma'_{N_{\rm Cl}}\gamma'_j\}_{j=1,\ldots,N_{\rm Cl}-1}$ which commutes with the lattice Dirac operator $UDU^\dagger$, i.e.

$$[UDU^\dagger,\imath 1_{d_{\mathcal{H}}/2^{(N_{\rm Cl}-1)/2}}\otimes\gamma'_{N_{\rm Cl}}\gamma'_j]_-=0 \tag{36}$$

because $[D,\Gamma^{(5)}_{N_{\rm Cl}}\Gamma^{(5)}_j]_-=0$ for all $j=1,\ldots,N_{\rm Cl}$. Schur's lemma [28] [now the commutation of D with a fundamental representation of $\mathrm{U}(2^{(N_{\rm Cl}-1)/2})$] tells us that the Dirac operator has the form

$$UDU^\dagger=D_{\rm red}\otimes 1_{2^{(N_{\rm Cl}-1)/2}}\quad\text{with}$$
$$D_{\rm red}=-D^\dagger_{\rm red}\quad\text{and}\quad[D_{\rm red},\gamma^{(5)}_{\rm red}]_+=0. \tag{37}$$

The last equality follows from the anticommutation relation of $[D,\tilde\Gamma]_+=0$. An explicit form of $D_{\rm red}$ is given in Sec. VI.

For complex representations the discussion above implies that all eigenvalues are $2^{(N_{\rm Cl}-1)/2}$-fold degenerate. Furthermore the reduced Dirac operator $D_{\rm red}$ has a chiral form with no further symmetries. Thus, the lattice theory should share the symmetries of the continuum theory in four dimensions.

When we consider a real or a quaternion representation $r(\mathfrak{g})$ of a gauge group, we have to evaluate the implications for the antiunitary symmetries. We consider the antiunitary operator in the new basis $C'_{\rm lat}=UC_{\rm lat}\tilde\Gamma^{N_{\rm Cl}(N_{\rm Cl}-1)/2}U^\dagger=K\zeta'\otimes\chi'$ which is equivalent to the original antiunitary operator $C_{\rm lat}$. The factor $\tilde\Gamma^{N_{\rm Cl}(N_{\rm Cl}-1)/2}$ is introduced in the new antiunitary operator because of the following anticommutation relation with the reduced Dirac operator:

$$\begin{aligned}&U[C_{\rm lat}\tilde\Gamma^{N_{\rm Cl}(N_{\rm Cl}-1)/2},\imath^{d(d-1)/2+N_{\rm Cl}(N_{\rm Cl}-1)/2}D]_-U^\dagger\\&\quad=[C'_{\rm lat},\imath^{d(d-1)/2+N_{\rm Cl}(N_{\rm Cl}-1)/2}D_{\rm red}\otimes 1_{2^{(N_{\rm Cl}-1)/2}}]_-\\&\quad=\pm[C_{\rm red},\imath^{d_{\rm red}(d_{\rm red}-1)/2}D_{\rm red}]_-\otimes\chi'=0,\end{aligned} \tag{38}$$

due to the anticommutation relation $[\tilde\Gamma,D]_+=0$ and Eq. (21). Again we have chosen the notation $C_{\rm red}=K\zeta'$ and $d_{\rm red}=d-N_{\rm L}$.

Now we need to calculate $C^2_{\rm red}$. For this purpose we need the commutation relations between $C'_{\rm lat}$ and $U\Gamma^{(5)}_jU^\dagger$ which read

$$\begin{aligned}&[C'_{\rm lat},\imath^{d(d-1)/2}\gamma^{(5)}_{\rm red}\otimes\gamma'_j]_-\\&\quad=U[C_{\rm lat}\tilde\Gamma^{N_{\rm Cl}(N_{\rm Cl}-1)/2},\imath^{d(d-1)/2}\Gamma^{(5)}_j]_-U^\dagger=0,\end{aligned} \tag{39}$$

for all $j=1,\ldots,N_{\rm Cl}$. The commutation relation with $U\tilde\Gamma U^\dagger$ is

$$\begin{aligned}&[C'_{\rm lat},\imath^{d_{\rm red}(d_{\rm red}-1)/2}\gamma^{(5)}_{\rm red}\otimes 1_{2^{(N_{\rm Cl}-1)/2}}]_-\\&\quad=\pm U\left[C_{\rm lat}\tilde\Gamma^{N_{\rm Cl}(N_{\rm Cl}-1)/2},\imath^{d(d-1)/2}\prod_{j=1}^{N_{\rm Cl}}\Gamma^{(5)}_j\right]_-U^\dagger=0\end{aligned} \tag{40}$$

up to an overall sign. Combining Eqs. (39) and (40) yields

$$[C'_{\rm lat},\imath^{N_{\rm Cl}(N_{\rm Cl}-1)/2}1_{d_{\mathcal{H}}/2^{(N_{\rm Cl}-1)/2}}\otimes\gamma'_j]_-=0. \tag{41}$$

Thus we can choose

$$\chi'=\begin{cases}\imath^m\prod_{j=1}^{2m}\gamma'_{2j}, & N_{\rm Cl}=4m+1,\\ \imath^{m+1}\prod_{j=1}^{2m+2}\gamma'_{2j-1}, & N_{\rm Cl}=4m+3\end{cases} \tag{42}$$

because all other choices would be unitarily equivalent.

In the last step we employ the relation $(C_{\rm lat}\tilde\Gamma^{N_{\rm Cl}(N_{\rm Cl}-1)/2})^2=C^2_{\rm red}\otimes(K\chi')^2$ which yields us the sign

$$\begin{aligned}C^2_{\rm red}=&(-1)^{(d_{\rm red}+2)(d_{\rm red}+1)d_{\rm red}(d_{\rm red}-1)/8}\\&\times\mathrm{sign}[(K\zeta)^2]1_{d_{\mathcal{H}}/2^{(N_{\rm Cl}-1)/2}}\end{aligned} \tag{43}$$

because $\mathrm{sign}[(K\chi')^2]=(-1)^{(N_{\rm Cl}+2)(N_{\rm Cl}+1)N_{\rm Cl}(N_{\rm Cl}-1)/8}$ and $\mathrm{sign}[(C_{\rm lat}\tilde\Gamma)^2]=(-1)^{d_{\rm red}(d_{\rm red}-1)/2}\mathrm{sign}[C^2_{\rm lat}]$. Moreover we used the identity (33) which is also true for odd $N_{\rm Cl}$.

Combining Eqs. (37), (38) and (43) we can summarize that the eigenvalues of D are either $2^{(N_{\rm Cl}-1)/2}$ degenerate or $2^{(N_{\rm Cl}+1)/2}$ degenerate if Kramers degeneracy applies. Moreover the symmetries of $D_{\rm red}$ or equivalently of D are those of the continuum theory at even dimension $d_{\rm red}=d-N_{\rm L}$ and not of dimension d. Hence, also for an odd number of Clifford generators $N_{\rm Cl}$ anticommuting with the lattice Dirac operator D the symmetries are shifted along the Bott periodic Table I. Interestingly, the shift is exactly the same as for even $N_{\rm Cl}$.

V. SYMMETRY-BREAKING PATTERN AND ZERO MODES

In the previous sections we have seen that the lattice Dirac operator D may drastically degenerate when some or even all (case of staggered fermions [3]) lattice directions exhibit an even partition of lattice sites. The reduced lattice Dirac operator $D_{\rm red}$ acts on a Hilbert space of dimension $d_{\mathcal{H}}/d_{\rm tri}$ with $d_{\rm tri}=2^{\lfloor N_{\rm Cl}/2\rfloor}$ whose value depends on whether $N_{\rm Cl}$ is even or not. The characteristic polynomial of the lattice Dirac operator with a quark mass m is then

$$\begin{aligned}\det(D+m1_{d_{\mathcal{H}}})=&\det(D_{\rm red}+m1_{d_{\mathcal{H}}/d_{\rm tri}})^{d_{\rm tri}/2}\\&\times\det(-D_{\rm red}+m1_{d_{\mathcal{H}}/d_{\rm tri}})^{d_{\rm tri}/2}\end{aligned} \tag{44}$$

for an even number $N_{\rm Cl}$ of Clifford elements anticommuting with D and

$$\det(D + m\mathbb{1}_{d_\mathcal{H}}) = \det(D_{\rm red} + m\mathbb{1}_{d_\mathcal{H}/d_{\rm tri}})^{d_{\rm tri}} \tag{45}$$

for odd $N_{\rm Cl}$. Thus the number of physical flavors is enhanced by $d_{\rm tri}$. In particular the symmetry-breaking patterns are those of the continuum theory of dimension $d_{\rm red} = d - N_{\rm L}$ with $d_{\rm tri}N_{\rm f}$ flavors. This is also shown in Table II.

The zero modes of the naive lattice Dirac operator are also enhanced by the factor $d_{\rm tri}$. However exact zero modes are only present when D or equivalently $D_{\rm red}$ is in the symmetry class B and BIII because the off-diagonal operators are always of square form (we always have the same number of vectors with positive and negative chirality). The symmetry class B means that $D_{\rm red}$ is a real antisymmetric matrix of odd dimension with no additional symmetries such that it has one exact zero mode. This zero mode is currently interpreted as a Majorana fermion in condensed matter physics; see Ref. [29]. The class BIII implies that $D_{\rm red}$ has a chiral structure whose off-diagonal block is antisymmetric and odd dimensional such that $D_{\rm red}$ has two zero modes: one has positive chirality and the other one has negative chirality.

We recognize two things. First a QCD-like theory with a complex gauge group representation will never yield a naive Dirac operator with zero modes. Hence we can restrict the discussion about the zero modes to real and quaternion representations. Second, whether there is a zero mode strongly depends on the dimension $d_\mathcal{H}/d_{\rm tri}$ of the reduced Hilbert space as well as on $d_{\rm red} = d - N_{\rm L}$, the effective dimension reflected by the symmetries of $D_{\rm red}$. Therefore we have to go through the four cases $d_{\rm red} = 8\mathbb{N}_0 + j$ with $j = 1, 3, 5, 7$ where these zero modes may appear.

First we consider $d_{\rm red} \in 8\mathbb{N}_0 + 1$ and a real representation of the gauge group or $d_{\rm red} \in 8\mathbb{N}_0 + 5$ and a quaternion representation. Then we find a symmetry class B when $N_{\rm Cl}$ is even, because this symmetry class does not satisfy a chiral symmetry, and when $d_\mathcal{H}/d_{\rm tri} = 2^{\lfloor d/2\rfloor} d_r V/2^{N_{\rm Cl}/2}$ is odd where $d_r = 2^{c_1}c_2$, with $c_1 \in \mathbb{N}_0$ and c_2 is an odd integer, is the dimension of the representation. We have $N_{\rm L} = N_{\rm Cl} - [d+1]_2 \le d$ and $V = 2^{N_{\rm L}+b_1}b_2 \in 2^{N_{\rm L}}\mathbb{N}$ with $b_1 \in \mathbb{N}_0$ and b_2 is an odd integer. Then the equation

$$\begin{aligned} 0 &= \left\lfloor \frac{d}{2} \right\rfloor + N_{\rm L} + c_1 + b_1 - \frac{N_{\rm L} + [d+1]_2}{2} \\ &= \frac{d + N_{\rm L} - 1}{2} + c_1 + b_1 \end{aligned} \tag{46}$$

has to be satisfied to find a zero mode. However this will only be the case for $d = d_{\rm red} = 1$ and $N_{\rm L} = b_1 = c_1 = 0$. Since we excluded the lattice theory in one dimension because there is no spontaneous symmetry breaking, we conclude that the symmetry class B never shows up for the naive discretization. In other words, when D and, thus, $D_{\rm red}$ are real antisymmetric matrices they will always be of even dimension (Cartan class D) regardless of what QCD-like lattice gauge theory one considers and which effective dimension $d_{\rm red} = d - N_{\rm L}$ we consider.

In the third and fourth cases we consider $d_{\rm red} \in 8\mathbb{N}_0 + 2$ and a real representation of the gauge group or $d_{\rm red} \in 8\mathbb{N}_0 + 6$ and a quaternion representation. Then, $N_{\rm Cl}$ has to be odd because the symmetry class BIII exhibits a chiral structure. For a zero mode the off-diagonal block has to be odd dimensional, i.e. $d_\mathcal{H}/(2d_{\rm tri}) = 2^{\lfloor d/2\rfloor} d_r V/2^{(N_{\rm Cl}+1)/2}$ has to be odd. The additional division by 2 comes from the chiral structure of the matrix $D_{\rm red}$. We have to solve the equation

$$\begin{aligned} 0 &= \left\lfloor \frac{d}{2} \right\rfloor + N_{\rm L} + c_1 + b_1 - \frac{N_{\rm L} + [d+1]_2 + 1}{2} \\ &= \frac{d + N_{\rm L} - 2}{2} + c_1 + b_1. \end{aligned} \tag{47}$$

This is only satisfied when $d = d_{\rm red} = 2$ and $N_{\rm L} = b_1 = c_1 = 0$ because we exclude the one-dimensional case. Indeed this case was found in the simulations performed in Ref. [12]. Apart from this particular case again no generic zero modes will be found for the naive lattice Dirac operator.

Let us summarize the discussion about the zero modes. Excluding one- and two-dimensional theories, all naive lattice Dirac operators will never show generic zero modes independently of the representation of the gauge group, of the space-time dimension and of the partition of the lattice.

We summarize the results above in Table II.

VI. EXPLICIT REPRESENTATION OF THE DIRAC OPERATOR

At last we want to derive an explicit representation of the reduced lattice Dirac operator $D_{\rm red}$ which is the staggered Dirac operator for $N_{\rm L} = d$. To achieve this we define the $d_\mathcal{H} \times d_\mathcal{H}$ unitary matrices

$$\begin{aligned} V_j^{(1)} &= \frac{1}{2}(\mathbb{1}_{d_\mathcal{H}} + \Gamma_j + \gamma_j - \Gamma_j\gamma_j), \\ V_j^{(2)} &= \frac{1}{2}(\mathbb{1}_{d_\mathcal{H}} + \Gamma_j - \gamma_j + \Gamma_j\gamma_j), \\ V_j^{(3)} &= \frac{1}{2}(\mathbb{1}_{d_\mathcal{H}} - \Gamma_j + \gamma_j + \Gamma_j\gamma_j) \end{aligned} \tag{48}$$

for $j = 1, \ldots, N_{\rm L}$. These matrices are Hermitian, $V_j^{(l)} = V_j^{(l)\dagger}$, and, thus, self-inverse, $V_j^{(l)}V_j^{(l)\dagger} = \mathbb{1}_{d_\mathcal{H}}$. Moreover they satisfy

$$V_j^{(1)}V_j^{(2)} = \Gamma_j, V_j^{(1)}V_j^{(3)} = \gamma_j \quad \text{and} \quad V_j^{(2)}V_j^{(3)} = \Gamma_j\gamma_j. \tag{49}$$

The product of the matrices $V_j^{(1)}$, i.e.

$$\tilde{U} = V_1^{(1)} V_2^{(1)} \cdots V_{N_L}^{(1)}, \tag{50}$$

will serve as the change of basis we are looking for to identify $D_{\rm red}$. We want to emphasize that $\tilde{U}$ is not necessarily equal to the unitary matrix U from the previous subsections. In particular the charge conjugation operator $C_{\rm lat}$ will be different from $C'_{\rm lat}$ after conjugation with $\tilde{U}$ though it will be equivalent. Our particular choice for $\tilde{U}$ is that the transformed Dirac operator $\tilde{U}^\dagger D \tilde{U}$ will have a simple form where $D_{\rm red}$ can be readily read off.

For the transformation of the Dirac operator we need the commutation relations

$$\begin{aligned}(T_i - T_i^\dagger)\gamma_i V_i^{(1)} &= V_i^{(3)}(T_i - T_i^\dagger)\gamma_i,\\ (T_i - T_i^\dagger)\gamma_i V_j^{(1)} &\overset{i\neq j}{=} V_j^{(2)}(T_i - T_i^\dagger)\gamma_i,\\ \gamma_i V_j^{(2)} &\overset{i\neq j}{=} V_j^{(1)}\gamma_i.\end{aligned} \tag{51}$$

Again we want to recall that we do not use Einstein's summation convention. Combining the commutation relations (51) with Eqs. (14), (49), and (50) the Dirac operator in the new basis is

$$\begin{aligned}\tilde{U}^\dagger D \tilde{U} &= \sum_{\mu=1}^{d} V_{N_L}^{(1)} \cdots V_1^{(1)} V_1^{(2)} \cdots V_{\mu-1}^{(2)} V_\mu^{(3)} V_{\mu+1}^{(2)} \cdots V_{N_L}^{(2)}\\ &\quad \times (T_\mu - T_\mu^\dagger)\gamma_\mu\\ &= \sum_{\mu=1}^{N_L}\left(\prod_{j=1}^{\mu-1}\Gamma_j\right)(T_\mu - T_\mu^\dagger)\\ &\quad + \sum_{\mu=N_L+1}^{d}\left(\prod_{j=1}^{N_L}\Gamma_j\right)(T_\mu - T_\mu^\dagger)\gamma_\mu.\end{aligned} \tag{52}$$

We notice that the covariant derivatives in the first N_L directions act trivially in the spinor space. Hence, we may choose a basis of the generalized γ matrices for $\mu > N_L$ in the following way:

$$\gamma_\mu = \tilde{\gamma}_\mu \otimes \mathbb{1}_{2^{(N_{Cl}-1)/2}} \tag{53}$$

for odd N_{Cl} and

$$\gamma_\mu = \tilde{\gamma}_\mu \otimes \sigma_3^{\otimes N_{Cl}/2} \tag{54}$$

for even N_{Cl}. The matrix $\sigma_3^{\otimes j}$ is the tensor product of the third Pauli matrix taken j times. The matrices $\tilde{\gamma}_\mu$ build the generalized γ matrices in $d_{\rm red} = d - N_L$ dimensions. The remaining γ matrices γ_μ with $\mu \leq N_L$ are then of the form $\sigma_3^{\otimes J} \otimes \gamma'_\mu$ for odd N_{Cl} and $\mathbb{1}_{2^{\otimes J}} \otimes \gamma'_\mu$ for even N_{Cl} with $J = \lfloor d/2 \rfloor - \lfloor N_{Cl}/2 \rfloor$, but this is not important anymore. Equation (52) is enough to read off the reduced lattice Dirac operator which is

$$D_{\rm red} = \sum_{\mu=1}^{N_L} D_\mu^{(\rm red)} + \sum_{\mu=N_L+1}^{d} D_\mu^{(\rm red)}\gamma_\mu \tag{55}$$

with the new covariant derivatives

$$\begin{aligned}D_\mu^{(\rm red)}|\psi(x)\rangle &= (-1)^{\sum_{j=1}^{\mu-1} x_j}((-1)^{\delta_{\mu d}\delta_{x_d L_d}} U_\mu(x)|\psi(x+e_\mu)\rangle\\ &\quad - (-1)^{\delta_{\mu d}\delta_{x_d L_1}} U_\mu^\dagger(x)|\psi(x-e_\mu)\rangle)\end{aligned} \tag{56}$$

for $\mu \leq N_L$ and

$$\begin{aligned}D_\mu^{(\rm red)}|\psi(x)\rangle &= (-1)^{\sum_{j=1}^{N_L} x_j}((-1)^{\delta_{\mu d}\delta_{x_d L_d}} U_\mu(x)|\psi(x+e_\mu)\rangle\\ &\quad - (-1)^{\delta_{\mu d}\delta_{x_d L_1}} U_\mu^\dagger(x)|\psi(x-e_\mu)\rangle)\end{aligned} \tag{57}$$

for $\mu > N_L$. We emphasize that the only difference between Eqs. (56) and (57) is the overall sign.

For $N_L = d$ the Dirac operator (55) automatically reduces to the staggered Dirac operator [3]. In the case that $-U(x)$ is in the considered representation of the gauge group when $U(x)$ is, the signs can be absorbed. For example this is the case for the fundamental representation of SU(2).

For the formulas (55), (56) and (57) we assumed that the temporal direction has an odd partition of lattice sites. In the case that the temporal direction has an even number of lattice sites we switch in the formulas the directions $\mu = 1$ and $\mu = d$.

In Figs. 1 and 2 we compare Monte Carlo simulations of quenched QCD lattice Dirac operators in the naive discretization and in the strong coupling limit ($\beta \to \infty$, where group elements are drawn from the Haar measure) with random matrix theory results. We have chosen several lattices, dimensions and representations of the gauge groups SU(2) and SU(3). The number of configurations generated is 10^5 for the three- and four-dimensional lattices. For the five-dimensional lattices the number of configurations ranges from 10^3–10^4 such that the statistical error will be only a few percent. The agreement with the analytical random matrix results shown in the Appendix is quite good according to the very small sizes of the lattices.

We employed two quantities to compare the lattice data with random matrix theory results. Most symmetry classes (namely seven of the ten) exhibit a nontrivial microscopic level density about the origin; see Eq. (A5). This quantity has specific characteristics like the level repulsion from the origin as well as between the levels themselves. Combining these characteristics with the generic degeneracy of the eigenvalues and the number of the zero modes uniquely determines the symmetry class. We applied a χ^2 fitting for this quantity for the microscopic level density which

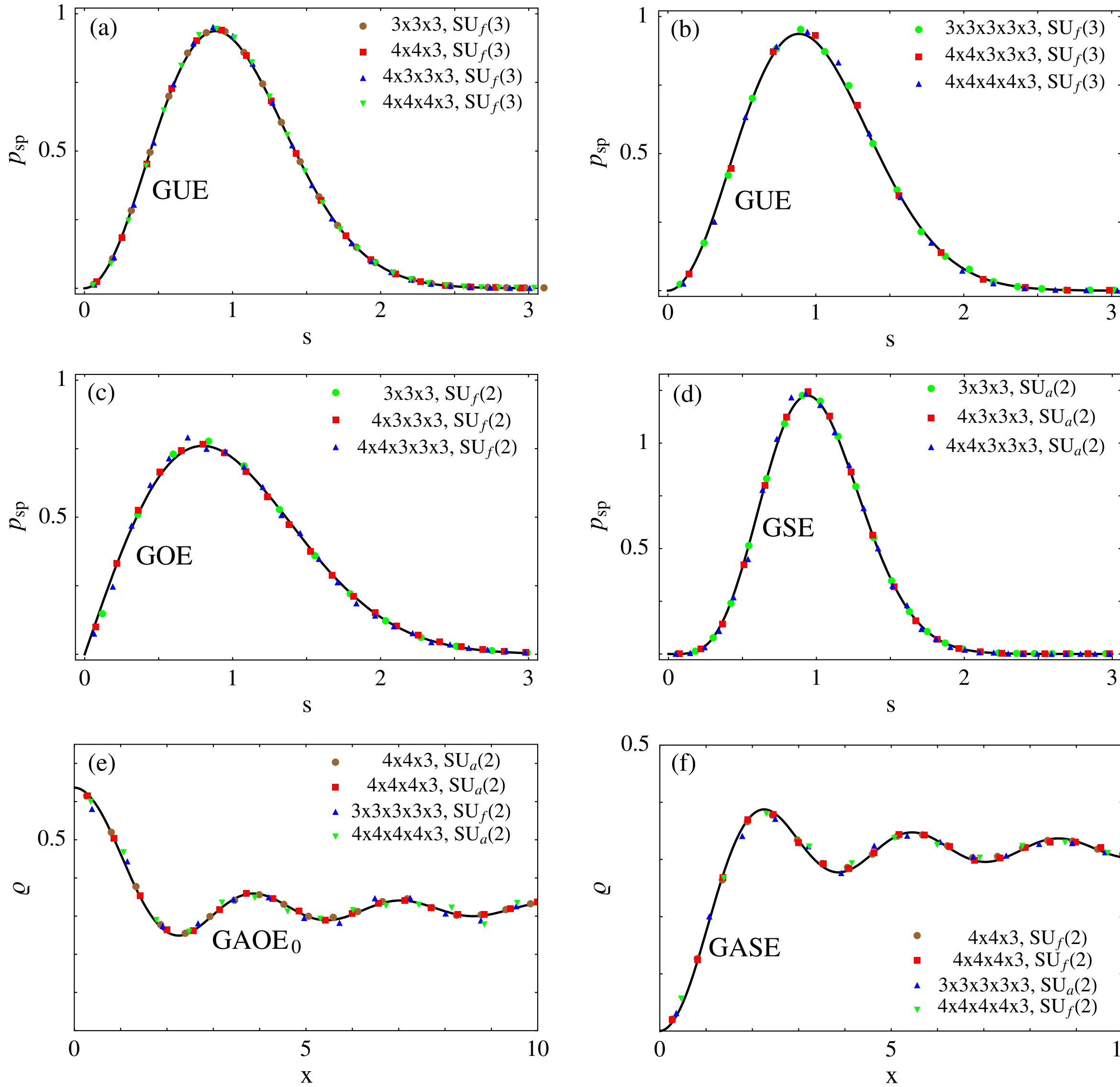

FIG. 1. Comparison of the random matrix theory predictions (black smooth curves) with lattice simulations of three-, four- and five-dimensional quenched naive Dirac operators (colored symbols) in the strong coupling limit. The abbreviations GUE, GOE, … refer to the corresponding random matrix ensembles listed in Table III and the abbreviations $\mathrm{SU}_f(N_c)$ and $\mathrm{SU}_a(N_c)$ stand for the fundamental and adjoint representations of the group $\mathrm{SU}(N_c)$. In the first plots we show the level spacing distribution $p_{\mathrm{sp}}(s)$ which is given by Wigner's surmise (A6). The lattice data for these plots were normalized to unity for the zeroth and first moments. For these plots we took into account 19 eigenvalues (= 18 level spacings) per configuration of the Dirac operator about the origin. The last two plots show the microscopic level density $\varrho(x)$; see Eq. (A5). For those plots the lattice data was rescaled via a χ^2 fitting. We simulated 10^5 configurations for the three- and four-dimensional lattices. For the five-dimensional lattices we generated below 10^4 configurations such that the statistical variance will be below five percent.

comprises three to four eigenvalues that are closest to the origin. Since we consider only very few eigenvalues the macroscopic (or global) level density will have no curvature for this short distance such that we have a trivial (only rescaling) unfolding of the spectrum.

The other three symmetry classes, abbreviated by GOE, GUE and GSE, have a flat microscopic level density about the origin. Thus we have chosen for these ensembles the level spacing distribution as an observable to determine which symmetry class the lattice data exhibits. We compared the lattice data with the Wigner surmise (A6) which is suitable enough for our aim. For this purpose we have chosen about 20 eigenvalues per configuration about the origin such that we do not have to unfold the spectrum because the macroscopic level density will be flat in this regime to a good approximation. We normalized the resulting probability density to the mean level spacing equal to 1.

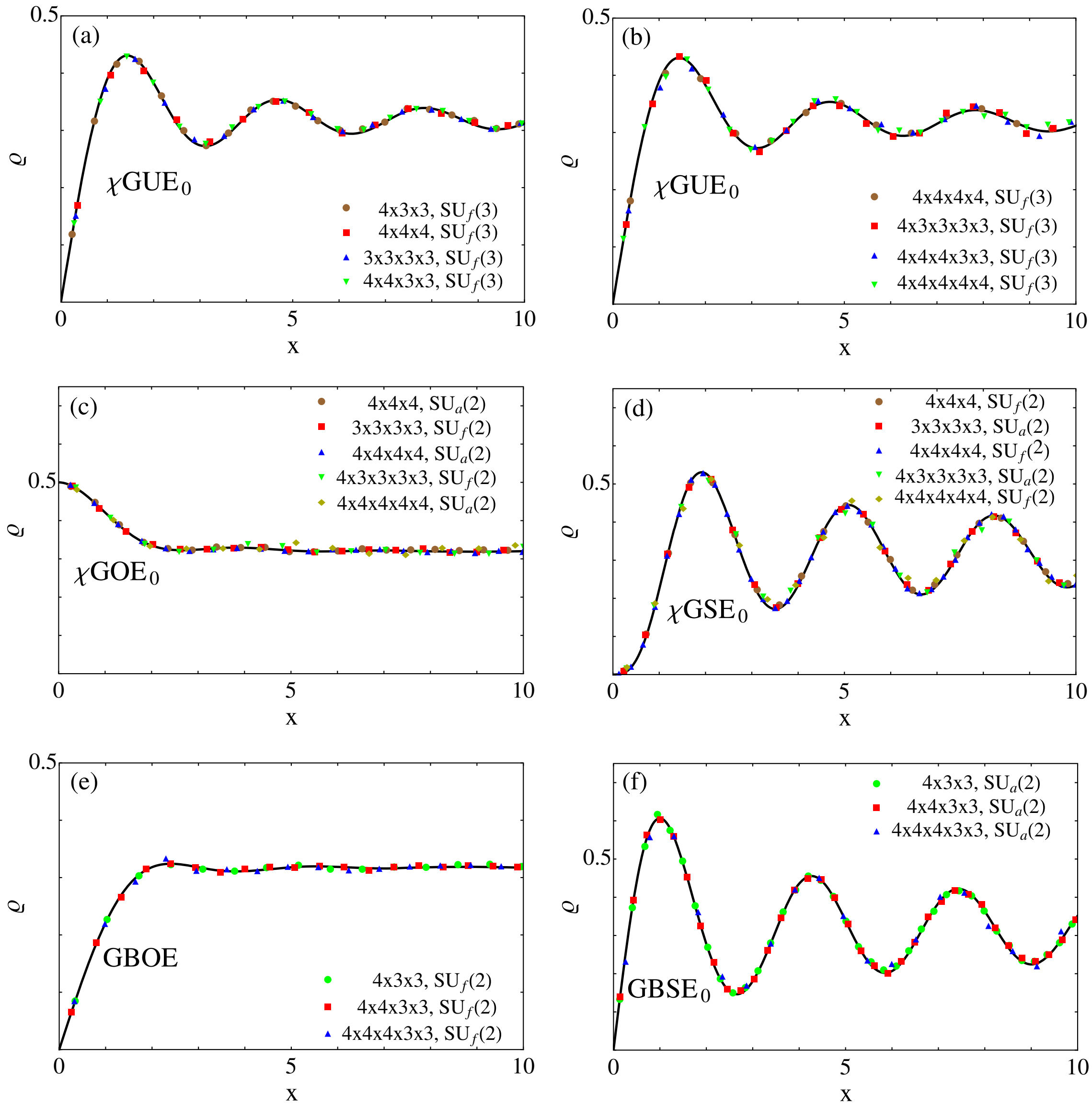

FIG. 2. Continuation of the list of comparisons of Fig. 1 between random matrix theory predictions (black smooth curves) and lattice simulations of three-, four- and five-dimensional quenched naive Dirac operators (colored symbols) in the strong coupling limit. In all six plots we consider the microscopic level density $\varrho(x)$ only [see Eq. (A5)]. As in Fig. 1 we applied a χ^2 fitting on the lattice data which consist of 10^5 configurations for the three- and four-dimensional lattices and the number of configurations varies (< 15000) for the five-dimensional lattices.

VII. CONCLUSIONS

One well-known consequence of the naive discretization of the Dirac operator is the change of its spectral properties; see Refs. [8–12]. We analyzed this change for an arbitrary QCD-like gauge theory and at arbitrary space-time dimension $d \geq 2$. When one or more directions have an even number of lattice sites the degeneracy increases exponentially. This also results in an increase of the number of flavors (neglecting the doublers still comprised in the continuum limit) by $N_\mathrm{f} \to d_\mathrm{tri} N_\mathrm{f}$ where $d_\mathrm{tri} = 2^{\lfloor (N_\mathrm{L}+[d+1]_2)/2 \rfloor}$ and N_L is the number of directions with even parity.

For $d > 2$ the Dirac operator has no zero modes such that the topological charge ν will be zero. Hence all zero modes can only appear when taking the continuum limit. For example Follana *et al.* [30] have analyzed how these zero modes show up for staggered lattice configurations which converge to configurations with nontrivial topological charge. Whether a mode is a "would-be" zero mode or not was determined by measuring the chirality of the individual modes. We have not considered this particular issue here.

Moreover the antiunitary symmetries and chiral symmetries experience a shift along the symmetry classification of the continuum symmetries; see Table I as well as

Ref. [20]. This shift is explicitly given by $d \to d_{\rm red} = d - N_{\rm L}$. Thus the symmetries of the staggered Dirac operator [3] are always those of the corresponding continuum theories at dimension $d = 8$ regardless of what original dimension was chosen as long as the dimension is $d \geq 2$. In particular the symmetry-breaking patterns will be those at $d = 8$.

Additionally we derived an explicit form of the nondegenerate part of the Dirac operator. It reduces to the staggered Dirac operator [3] when all directions have an even partition. We performed Monte Carlo simulations with this reduced Dirac operator in the strong coupling limit and in the three-, four- and five-dimensional quenched theory. The spectral statistics of the lowest eigenvalues were compared with random matrix theory results. The good agreement of the numerics with the analytical results confirm our predictions. This agreement is at least as good as that found in four dimensions for the staggered fermions [8] despite the fact that the lattices we simulated are very small.

Our results may yield a basis to understand the continuum limit of staggered and naive fermions. In particular it is still an unsolved problem whether the global symmetries change to those of the correct continuum theory. Here we have to say that the weak coupling limit studied in Ref. [9] nurtures some doubt because on the smallest scales the global symmetries of the lattice Dirac operator always show up. When assuming, nonetheless, that the continuum limit exists, one can study the infrared spectrum of the Dirac operator and, hence, the lightest pseudoscalar mesons with random matrix theory. First attempts in this direction were already done in four [31] and three [11] dimensions.

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APPENDIX: CLASSES OF HERMITIAN RANDOM MATRICES

There are ten Hermitian symmetric matrix spaces [26,32]. Five symmetry classes satisfy a chiral symmetry and the other five do not. Eight of the ten classes have an antiunitary symmetry with respect to the complex conjugation. The corresponding antiunitary operator squares to $+\mathbb{1}$ for four classes and to $-\mathbb{1}$ for the other four classes. The ten classes are listed in Table III with the corresponding Cartan classification label; see Refs. [25,26].

The probability density can be chosen Gaussian for all ten classes, i.e.

$$P(H) \propto \exp\left[-\frac{{\rm tr} H^2}{2\sigma^2}\right], \tag{A1}$$

with variance σ which may vary from one class to another. The variance can be read off from the joint probability densities (A2) and (A3). The random matrix $\imath H$ exhibits the spectral statistics of the lowest eigenvalues of QCD-like Dirac operators satisfying the same unitary and antiunitary symmetries as $\imath H$.

The joint probability density of the eigenvalues λ of the random matrix H for the classes A (Hermitian matrices), AI (real symmetric matrices) and AII (Hermitian self-dual matrices) can be summarized in one formula

$$p(\lambda) \propto |\Delta_n(\lambda)|^{\beta_{\rm D}} \prod_{j=1}^{n} \exp\left[-\frac{\beta_{\rm D}}{2n}\lambda_j^2\right], \tag{A2}$$

while the other seven Hermitian matrix ensembles follow the formula

$$p(\lambda) \propto |\Delta_n(\lambda^2)|^{\beta_{\rm D}} \prod_{j=1}^{n} \lambda_j^{\alpha_{\rm D}} \exp\left[-\frac{\beta_{\rm D}}{2n}\lambda_j^2\right]. \tag{A3}$$

In the latter case the eigenvalues of the matrices come in "chiral pairs" $(\lambda_j, -\lambda_j)$ though not all of these matrices satisfy a chiral symmetry, e.g. the imaginary antisymmetric matrices are not chiral but their eigenvalues appear in "chiral pairs." We recall the Vandermonde determinant

$$\Delta_n(\lambda) = \prod_{1\leq a<b\leq n} (\lambda_b - \lambda_a) = \det[\lambda_a^{b-1}]_{a,b=1,\ldots,n}. \tag{A4}$$

The index $\beta_{\rm D}$ is the Dyson index which determines the strength of the level repulsion between the eigenvalues. The parameter $\alpha_{\rm D}$ is related to the topological charge and is the origin of the level repulsion from the origin.

One important spectral quantity is the microscopic level density. It is a constant for those three ensembles which do not exhibit "chiral pairs" of eigenvalues, namely real symmetric, Hermitian and Hermitian self-dual matrices. For the other seven ensembles the quenched microscopic level density is nontrivial and has the form [32]

$$\begin{aligned}
\rho_{\nu_{\rm D}}^{(\beta_{\rm D}=1)}(x) &= \frac{|x|}{2}(J_{\nu_{\rm D}}^2(x) - J_{\nu_{\rm D}+1}(x)J_{\nu_{\rm D}-1}(x)) \\
&\quad + \frac{1}{2}J_{\nu_{\rm D}}(|x|)\left(1 - \int_0^{|x|} J_{\nu_{\rm D}}(x')dx'\right), \\
\rho_{\nu_{\rm D}}^{(\beta_{\rm D}=2)}(x) &= \frac{|x|}{2}(J_{\nu_{\rm D}}^2(x) - J_{\nu_{\rm D}+1}(x)J_{\nu_{\rm D}-1}(x)), \\
\rho_{\nu_{\rm D}}^{(\beta_{\rm D}=4)}(x) &= |x|(J_{2\nu_{\rm D}}^2(2x) - J_{2\nu_{\rm D}+1}(2x)J_{2\nu_{\rm D}-1}(2x)) \\
&\quad - J_{2\nu_{\rm D}}(2|x|)\left(\frac{1}{2} - \int_{|x|}^{\infty} J_{2\nu_{\rm D}}(2x')dx'\right).
\end{aligned} \tag{A5}$$

TABLE III. The ten Gaussian random matrix models corresponding to the ten symmetries of the Cartan classification scheme [25,26]. We want to emphasize that not all of the abbreviations for these ensembles are standard. The last five classes are the chiral models while the first five exhibit no chirality but may have "chiral pairs" of eigenvalues $(\lambda, -\lambda)$; see the seventh column. Shown are their explicit matrix representations (fourth column), the Dyson index β_D (fifth column) and the exponent of the level repulsion from the origin α_D (sixth column) as well as the number of generic zero modes. Please note that some ensembles only differ in their spectral properties via subtleties like the number of the zero modes. The indices β_D, α_D and ν_D are needed for the analytical random matrix results (A2), (A3), (A5) and (A6).

RMT	Abbreviation for Gaussian ensemble	Cartan class	Random matrix H	β_D	α_D	ν_D	chiral pair	generic zeros
Hermitian matrices Lie algebra of U(n)	GUE(n)	A	$H = H^\dagger \in \mathbb{C}^{n\times n}, n \in \mathbb{N}$	2	0	0	No	0
real symmetric matrices	GOE(n)	AI	$H = H^T = H^* \in \mathbb{R}^{n\times n}, n \in \mathbb{N}$	1	0	0	No	0
Hermitian self-dual matrices	GSE(n)	AII	$H = \theta_2 H^T \theta_2 = \theta_2 H^* \theta_2$ $H \in \mathbb{C}^{2n\times 2n}, n \in \mathbb{N}$	4	0	0	No	0
imaginary antisymmetric matrices, Lie algebra of O(n)	GAOE$_\nu(n)$	B|D	$H = -H^T = -H^*$ $H \in \imath\mathbb{R}^{(2n+\nu)\times(2n+\nu)}$, $n \in \mathbb{N}, \nu = 0, 1$	2	$2\nu = 0$, 2	$\frac{\nu-1}{2} = \pm\frac{1}{2}$	Yes	$\nu = 0, 1$
Hermitiananti-self-dual matrices, Lie algebra of USp$(2n)$	GASE(n)	C	$H = -\theta_2 H^T \theta_2 = -\theta_2 H^* \theta_2$ $H \in \mathbb{C}^{2n\times 2n}, n \in \mathbb{N}$	2	2	$\frac{1}{2}$	Yes	0
chiral Hermitian matrices	χGUE$_\nu(n)$	AIII	$H = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix}$, $W \in \mathbb{C}^{n\times(n+\nu)}, n, \nu \in \mathbb{N}$	2	$2\nu + 1$	ν	Yes	ν
chiral real symmetric matrices	χGOE$_\nu(n)$	B|DI	$H = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix}$, $W = W^* \in \mathbb{R}^{n\times(n+\nu)}, n, \nu \in \mathbb{N}$	1	ν	ν	Yes	ν
chiral Hermitian self-dual matrices	χGSE$_\nu(n)$	CII	$H = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix}$, $W = \theta_2 W^* \theta_2 \in \mathbb{C}^{2n\times 2(n+\nu)}, n, \nu \in \mathbb{N}$	4	$4\nu + 3$	ν	Yes	2ν
symmetric Bogolyubov-de Gennes matrices	GBOE(n)	CI	$H = \begin{bmatrix} 0 & W^\dagger \\ W & 0 \end{bmatrix}$, $W = W^T \in \mathbb{C}^{n\times n}, n \in \mathbb{N}$	1	1	1	Yes	0
antisymmetric Bogolyubov-de Gennes matrices	GBSE$_\nu(n)$	B|DIII	$H = \begin{bmatrix} 0 & W^\dagger \\ W & 0 \end{bmatrix}$, $W = -W^T \in \mathbb{C}^{(2n+\nu)\times(2n+\nu)}$, $n \in \mathbb{N}, \nu = 0, 1$	4	$4\nu + 1$	$\frac{\nu-1}{2} = \pm\frac{1}{2}$	Yes	$2\nu = 0$, 2

The function $J_\nu(x) = \int_{-\pi}^{\pi} \exp[\imath x \sin(\varphi) - \imath\nu\varphi] d\varphi/(2\pi)$ is the Bessel function of the first kind. The densities are normalized such that $\lim_{|x|\to\infty} \rho(x) = 1/\pi$. The index ν_D is related to α_D and the topological charge ν and can be read off from Table III.

The densities (A5) together with the degree of degeneracy of the eigenvalues and the number of the generic zero modes are ideal for deciding to which of the seven symmetry classes a specific spectrum belongs. But what about the classes of real symmetric, Hermitian and Hermitian self-dual matrices? For these three classes another quantity is needed which is the level spacing distribution. It describes the distribution of the spacing between adjacent eigenvalues. This distribution is very well described by Wigner's surmise [33]

$$p_{\rm sp}(s) = 2\frac{(\Gamma[(\beta_{\rm D}+2)/2])^{\beta_{\rm D}+1}}{(\Gamma[(\beta_{\rm D}+1)/2])^{\beta_{\rm D}+2}} s^{\beta_{\rm D}} \times \exp\left[-\left(\frac{\Gamma[(\beta_{\rm D}+2)/2]}{\Gamma[(\beta_{\rm D}+1)/2]}\right)^2 s^2\right] \tag{A6}$$

with the gamma function $\Gamma(x)$. A better approximation of the level spacing distribution is via a Padé expansion which converges rapidly to the true level spacing distribution; see Ref. [34]. Though the distribution (A6) is not the exact result of the level spacing distribution for $n \to \infty$ it is a good approximation. Its root-mean-square deviation to the correct expression is much less than one per mill.

In the Monte Carlo simulations shown in Figs. 1 and 2, we make use of Eqs. (A5) and (A6). They are the analytical curves we compare with the numerics. In this way we confirm our predictions.

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