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Homogeneous solutions of minimal massive 3D gravity

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In this paper, we systematically construct simply transitive homogeneous spacetime solutions of the three-dimensional minimal massive gravity (MMG) model. In addition to those that have analogs in topologically massive gravity, such as warped AdS and pp waves, there are several solutions genuine to MMG. Among them, there is a stationary Lifshitz metric with the dynamical exponent $z = -1$ and an anisotropic Lifshitz solution where all coordinates scale differently. Moreover, we identify a homogeneous Kundt-type solution at the chiral point of the theory. We also show that in a particular limit of the physical parameters in which the Cotton tensor drops out from the MMG field equation, homogeneous solutions exist only at the merger point in the parameter space if they are not conformally flat.

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I. INTRODUCTION

Minimal massive gravity (MMG) is a pure three-dimensional gravity model proposed in Ref. [1], which attracted much attention during the last three years. It is an extension of another widely studied theory known as "topologically massive gravity" (TMG) [2] with a particular curvature-squared term in the field equation. It is "minimal" in the sense that there is only one propagating spin-2 mode in the bulk like TMG. However, unlike TMG, it avoids the bulk-boundary unitarity clash, since in a certain range of its parameters it is possible to have the central charge of the dual CFT and the energy of the bulk graviton be positive simultaneously [1,3] (however, see Ref. [4]), which makes it a potentially useful toy model for understanding quantum gravity in four dimensions.

In this paper, we make a systematic investigation of homogeneous spacetime solutions of MMG with Lorentzian signature and obtain a large number of new ones. We will focus on simply transitive Lie groups where any two points can be related by an isometry, and the stability (isotropy) group of any point is trivial. Since a homogeneous (pseudo) Riemannian manifold M has the form of a quotient G/H, where G is its group of isometries, which is a Lie group, and H is a closed subgroup of G; this means that we take H to be just the identity. In this case, M and G can be identified, and considering left action of such a Lie group on itself, one can construct its left-invariant metric up to automorphisms using left-invariant one-forms. This results in a metric with constant coefficients, and all curvature calculations become algebraic. In three dimensions, the classification of Lie algebras was done by Bianchi [5], and one can systematically check whether corresponding metrics are solutions of a particular three-dimensional model. This method was successfully applied to TMG with a vanishing cosmological constant in Refs. [6] and [7], and more recently for nonzero cosmological constant in Ref. [8]. This was also carried out in Refs. [9,10] (see also Refs. [11,12]) for another extension of TMG called "new (or general) massive gravity" (NMG) [13].

Since MMG is closely related to the (cosmological) TMG, we will follow the analysis in Ref. [8] closely, which will make identification of most of the solutions we obtain straightforward. In Ref. [14], it was shown that solutions of TMG which have Segre-Petrov types N and D are also solutions of MMG after a redefinition of parameters. We find that, as should be expected, MMG inherits such homogeneous solutions from TMG which include warped (A)dS and $(\mathrm{A})\mathrm{dS}_2 \times \mathrm{S}^1$ solutions obtained in Ref. [15] and pp-wave spacetimes [16]. Some of the remaining solutions turn out to be solutions of TMG as well only if the cosmological constant is zero, which implies that their scalar curvatures vanish, which is not the case in MMG. Finally, we show that there are several homogeneous solutions that are genuine to MMG, one of which is a stationary Lifshitz spacetime [Eq. (86)] with a dynamical exponent $z = -1$ and an anisotropic Lifshitz solution [Eq. (93)] where all coordinates scale differently.

In Ref. [17], it was proven that three-dimensional constant scalar-invariant (CSI) Lorentzian spacetimes are locally either homogeneous or Kundt. For MMG, the latter is studied in Ref. [18]. Thus, the current work fills an important gap in the construction of all CSI solutions of

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TABLE I. Comparison of nontrivial homogeneous solutions of MMG and TMG.

Groups	Metric	MMG	TMG	Description	Type
		Homogeneous Solutions			
$SL(2;\mathbb{R})$	111-type	(13)	✓ $(R=0)$	Triaxially deformed AdS	$I_{\mathbb{R}}$
		(17)	✓	Spacelike warped AdS	D
		(19)	✓	Timelike warped AdS	D
	12-type	(24)	✓ $(R=0)$	Kundt	II
		(26)	✓	Null warped AdS	N
	3-type	(29), chiral pt.	✗	Kundt	III
	$1z\bar{z}$-type	(31)	✓ $(R=0)$	Generic	$I_{\mathbb{R}}$
$SU(2)$		(39)	✓ $(R=0)$	Triaxially deformed sphere	$I_{\mathbb{C}}$
		(41)	✓	Stretched/squashed sphere	D
A_∞		(46)	✓	Warped flat	D
A_0	B_2-type	(54), chiral pt.	✓	Logarithmic pp wave	N
	B_4-type	(59)	✗	Generic	II
ISO$(2;\theta)$	B_1-type	(70), $(\theta=\pi/2)$	✗	Generic	$I_{\mathbb{R}}$
	B_2-type	(76), $(\theta\neq 0)$	✗	Generic	II
ISO$(1,1;\theta)$	B_1-type	(84), $(\theta=\pi/4)$	✓	Space/time-like warped AdS	D
		(86), $(\theta=\pi/2)$	✗	Stationary Lifshitz	$I_{\mathbb{R}}$
		(89)	✓	pp wave	N
		(91)	✓	pp wave	N
	B_2-type	(93), $(\theta\neq 0)$	✗	Generalized Lifshitz	II
		(95), $(\theta=\pi/4)$	✓	Warped flat	D

MMG. We also show that two of the Kundt solutions found in Ref. [18] are also homogeneous, and one of them appears at the so-called chiral point of the theory that has no TMG limit.

In Refs. [16] and [19], some solutions to the MMG field equation were obtained where there is no contribution from the Chern-Simons term. Remarkably, for this case we find that homogeneous solutions exist only at a particular point in the parameter space, called the merger point, if they are not conformally flat.

The paper is organized as follows: In the next section, we introduce the MMG model and explain our method in more detail. Then in subsequent sections, we go through all possible homogeneous metrics of three-dimensional Lie algebras one by one. For each solution, we provide a coordinate representation of its metric and in most of the cases are able to identify the corresponding spacetime. In Sec. IX, we summarize our results in Table I, which includes Segre-Petrov types as was proposed in Ref. [20], and we make a comparison with TMG and indicate some future directions.

II. MINIMAL MASSIVE GRAVITY

In this section, we will give a brief introduction to MMG [1] and describe our method for constructing its homogeneous solutions. The theory is defined by the field equation

$$G_{\mu\nu} + a g_{\mu\nu} + b C_{\mu\nu} + c J_{\mu\nu} = 0, \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor, and the Cotton tensor $C_{\mu\nu}$, which is symmetric, traceless and covariantly conserved, is related to the Schouten tensor $S_{\sigma\nu}$ as

$$C^{\mu}{}_{\nu} \equiv \frac{1}{\sqrt{-g}}\varepsilon^{\mu\rho\sigma}\nabla_{\rho}S_{\sigma\nu}, \qquad S_{\sigma\nu} \equiv R_{\sigma\nu} - \frac{1}{4}R g_{\sigma\nu}, \tag{2}$$

with $\varepsilon_{012} = +1$. The J tensor is given as

$$J^{\mu\nu} \equiv R^{\mu\rho}R^{\nu}{}_{\rho} - \frac{3}{4}R^{\mu\nu}R - \frac{1}{2}g^{\mu\nu}\left(R^{\rho\sigma}R_{\rho\sigma} - \frac{5}{8}R^2\right). \tag{3}$$

It is not covariantly conserved, but instead one finds [1]

$$\sqrt{-g}\nabla_{\mu}J^{\mu\nu} = \varepsilon^{\nu\rho\sigma}S_{\rho}{}^{\tau}C_{\sigma\tau}, \tag{4}$$

which is not automatically zero. It follows that the MMG field equation (1) cannot be derived from an action that contains only the metric field [1]. However, for any solution of the field equation (1), one can show that the right-hand side of Eq. (4) vanishes, which establishes the consistency of the model in a novel way. Moreover, it is still possible to couple matter [15] and calculate charges of its solutions [21].

Finally, the coefficients a, b and c in terms of physical parameters are

$$a = \frac{\bar{\Lambda}_0}{\bar{\sigma}}, \qquad b = \frac{1}{\mu\bar{\sigma}}, \qquad c = \frac{\gamma}{\mu^2\bar{\sigma}}. \tag{5}$$

When $\gamma = 0$ (i.e., $c = 0$), the model reduces to the (cosmological) TMG model [2], where such solutions were studied before [6–10].

There are two special points in the parameter space of the MMG theory [1]. The first is called the "chiral point," at which one of the central charges vanishes, and is given by

$$\bar{\sigma} + \frac{\gamma}{2}\left(\bar{\sigma}^2 - \frac{\gamma\bar{\Lambda}_0}{\mu^2}\right) \pm \sqrt{\bar{\sigma}^2 - \frac{\gamma\bar{\Lambda}_0}{\mu^2}} = 0 \quad \text{or}$$
$$1 + \frac{c}{2b^2}(1 - ac) \pm \sqrt{1 - ac} = 0. \tag{6}$$

The second one is called the "merger point," where two possible values of the cosmological constant coincide:

$$\bar{\Lambda}_0 = \frac{\mu^2\bar{\sigma}^2}{\gamma} \quad \text{or} \quad ac = 1. \tag{7}$$

In order to find homogeneous solutions of MMG, we will follow the method of Ref. [8] that was successfully used for the (cosmological) TMG [2] model, which can be summarized as follows: First, a Lie algebra basis is fixed for each three-dimensional Lie algebra $\mathfrak{g}$ which induces left-invariant Maurer-Cartan one-forms. A left-invariant metric for the Lie group at the identity is identified by a non-degenerate metric on the Lie algebra up to an automorphism group of this Lie algebra. Starting from an arbitrary left-invariant metric on the algebra, it is put into a simple form using automorphisms. The metric is expressed in terms of left-invariant one-forms with constant coefficients, which implies that all curvature calculations, and hence the MMG field equation (1), become algebraic. This method is different but equivalent to the one used in [6,7,9,10], where instead of fixing the Lie algebra basis, an orthonormal frame is chosen [22]. Then, $SO(1,2)$ Lorentz transformations are used to simplify the structure constants. We prefer the strategy of Ref. [8], since it enables us to compare our solutions with those of (cosmological) TMG [2] directly. Moreover, geometric identification of common solutions becomes trivial.

Instead of solving algebraic equations for the constants in the metric $\{u, v, w, \ldots\}$ in terms of the parameters $\{a, b, c\}$ of the MMG theory (5), it is more convenient and illuminating to display the parameters in the theory in terms of the parameters of the metric. This reduces to solving a system of linear equations

$$A \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = V \tag{8}$$

for $\{a, b, c\}$ where A is a matrix is of the dimension $k \times 3$ and V is a $k \times 1$ vector with $k = 3$, 4 or 6. The number k is determined by the number of independent components of the field equation (1). The rank of the matrix A can be at most 3. When the rank of A is 3, the linear equation (8) has a unique solution, provided that the solution exists. If the solution exists, the cases when the rank of A is less than 3 should be considered separately, as in such situations new solutions may arise. If A is of the dimension 3×3, then computing the determinant of A is enough to determine when the rank of A is less than 3. When A is not a square matrix and for the cases when Eq. (8) does not have a general solution, a more careful analysis is required. For example, it may happen that for a particular relation among the parameters of the metric, the system becomes consistent.

To identify distinct Lie algebras, one must determine sets of structure constants which cannot be related by linear transformations. For three-dimensional Lie algebras, this classification was done by Bianchi [5] but usually presented in a more modern approach described in Ref. [23] (see also Ref. [24]). Besides the Abelian $\mathbb{R}^3$ and the two familiar algebras $\mathfrak{sl}_2$ and $\mathfrak{su}_2$, we also have the Lie algebras $\mathfrak{a}_\infty$ and $\mathfrak{a}_0$, and two continuous families of Lie algebras: $\mathfrak{iso}(1,1;\theta)$ and $\mathfrak{iso}(2;\theta)$ where the parameter θ varies in $(0, \frac{\pi}{2}]$. In the first, θ values $\{0, \frac{\pi}{4}\}$ are special and should be considered separately, which in total leads to nine Bianchi classes. Finally, $\mathfrak{iso}(1,1;0)$ and $\mathfrak{iso}(2;0)$ are isomorphic to each other.

We now begin constructing the homogeneous spacetime solutions of MMG going through the above list of algebras. We assume that metrics are Lorentzian with mostly plus signatures and follow the conventions and terminology of Ref. [8], to which we refer for details.

III. SOLUTIONS ON $SL(2,\mathbb{R})$

For the Lie algebra $\mathfrak{sl}_2$ of $SL(2,\mathbb{R})$ a basis $\{\tau_0, \tau_1, \tau_2\}$ can be fixed with

$$[\tau_0, \tau_1] = \tau_2, \qquad [\tau_2, \tau_1] = \tau_0, \qquad [\tau_2, \tau_0] = \tau_1. \tag{9}$$

Let θ^a be a dual basis of τ_a. Elements of $SL(2,\mathbb{R})$ can be parametrized by a group representative as (see for example Ref. [25])

$$\mathcal{V}(x) = e^{t(\tau_0+\tau_2)} e^{\sigma\tau_1} e^{\zeta\tau_2}. \tag{10}$$

It follows that the Maurer-Cartan one-forms are

$$\mathcal{V}^{-1} d\mathcal{V} = (e^\sigma \cosh\zeta dt - \sinh\zeta d\sigma)\tau_0 + (\cosh\zeta d\sigma - e^\sigma \sinh\zeta dt)\tau_1 + (d\zeta + e^\sigma dt)\tau_2. \tag{11}$$

There are four classes of left-invariant metrics on $SL(2,\mathbb{R})$ that are given below (see Ref. [8]).

A. 111-type metric

The 111-type metric is of the form

$$g = u\theta^0\theta^0 + v\theta^1\theta^1 + w\theta^2\theta^2, \tag{12}$$

where $uw < 0$ and $v > 0$ for the Lorentzian, mostly plus signature. The case $-u = v = w$ corresponds to the round AdS_3 written as Hopf fibration over AdS_2 spacetime in the Poincaré coordinates. The general line element is

$$ds^2 = e^{2\sigma}(u\cosh^2\zeta + v\sinh^2\zeta)dt^2 + (u\sinh^2\zeta + v\cosh^2\zeta)d\sigma^2 - 2(u+v)e^{\sigma}\cosh\zeta\sinh\zeta dt d\sigma + w(d\zeta + e^{\sigma}dt)^2. \tag{13}$$

The coefficients a, b, c and the scalar curvature R in terms of u, v, and w are

$$a = \frac{1}{Q}\cdot\frac{[(u+v+w)^2 - 4vw]^3}{8uvw},$$
$$b = -\frac{1}{Q}\cdot 8\sqrt{-uvw}[u^2 - (v-w)^2](u+v+w),$$
$$c = \frac{1}{Q}\cdot 8uvw[(u+v+w)^2 - 4vw],$$
$$R = -\frac{(u+v+w)^2 - 4vw}{2uvw} = -\frac{cQ}{16(uvw)^2} = -\frac{(aQ)^{1/3}}{(uvw)^{2/3}}, \tag{14}$$

where

$$Q = [(u+v+w)^2 - 4vw]^2 + 8[u^2 - (v+w)^2][u^2 - (v-w)^2]. \tag{15}$$

This solution in general represents a triaxially deformed AdS spacetime. Note that when $c = 0$, i.e., for TMG, $R = 0$. In this case, the cosmological constant a vanishes too. Also, it can be shown that when $R = 0$, we have $Q \neq 0$. Therefore, for this solution Q is nonzero, since Q and R cannot vanish at the same time.

The matrix A in Eq. (8) is a 3×3 matrix with the determinant

$$\det A = -Q\cdot\frac{(u+v)(v-w)(u+w)}{8(-uvw)^{5/2}}. \tag{16}$$

Thus, the cases $Q = 0$, $u = -v$ (or equivalently $u = -w$) and $v = w$ should be considered separately. We have checked that the case $Q = 0$ does not give rise to any solution to Eq. (1).

$u = -v$—In this case, the spacetime metric (13) becomes

$$ds^2 = v[-e^{2\sigma}dt^2 + d\sigma^2] + w(d\zeta + e^{\sigma}dt)^2 \equiv g_{(2)} + w(d\zeta + \chi)^2, \tag{17}$$

where $w > 0$. This solution was found before in Ref. [15] and is called the spacelike warped[1] AdS. Note that $d\chi = \mathrm{vol}_{g_{(2)}}$. The coefficients a and b in terms of u, v, and w are

$$a = \frac{16v^2(w-4v) + c(4v-7w)(4v-3w)}{192v^4},$$
$$b = \frac{8v^2 + c(4v-3w)}{12\sqrt{v^2w}}. \tag{18}$$

The curvature scalar is as in Eq. (14).

$v = w$—In this case, $u < 0$, and the spacetime metric (13) becomes

$$ds^2 = u(e^{\sigma}\cosh\zeta dt - \sinh\zeta d\sigma)^2 + w[(\cosh\zeta d\sigma - e^{\sigma}\sinh\zeta dt)^2 + (d\zeta + e^{\sigma}dt)^2]. \tag{19}$$

This metric was identified as timelike warped AdS in Ref. [8], and the coefficients a and b are

$$a = \frac{-16w^2(u+4w) + c(3u+4w)(7u+4w)}{192w^4},$$
$$b = \frac{8w^2 + c(3u+4w)}{12w\sqrt{-u}}. \tag{20}$$

Again, the curvature scalar is given in Eq. (14).

B. 12-type metric

The 12-type metric is of the form

$$g = v(-\theta^0\theta^0 + \theta^1\theta^1) + w\theta^2\theta^2 + z(\theta^0 + \theta^1)^2, \tag{21}$$

with $z \neq 0$, $v > 0$ and $w > 0$. Here, z can be scaled to ± 1. Notice that it is a z-deformation of the spacelike warped AdS metric (17).

The coefficients a, b, c and the scalar curvature R in terms of v and w are

$$a = \frac{1}{Q}\cdot\frac{(w-4v)^3}{8v^2},$$
$$b = \frac{1}{Q}\cdot 8(2v-w)\sqrt{v^2w},$$
$$c = \frac{1}{Q}\cdot 8v^2(w-4v),$$
$$R = \frac{w-4v}{2v^2}, \tag{22}$$

where $Q = -(w-4v)^2 + 8(4v^2 - w^2)$. Note that in the TMG limit, i.e., $c = 0$, both the scalar curvature and cosmological constant a vanish. Adapting the coordinate transformations given in Ref. [8] to our case as

$$t = \frac{1}{2x} + \frac{y}{2l^2}, \qquad e^{\sigma} = 2x, \qquad \zeta = \frac{\rho}{kl} + \ln x, \tag{23}$$

we obtain

[1]Constants v and w are related to the warping parameter ν in Ref. [15] as $v = \frac{l^2}{(\nu^2+3)}$, $w = \frac{4l^2\nu^2}{(\nu^2+3)^2}$. The limit $\nu \to 1$ in Eq. (17) corresponds to the AdS metric, where $-u = v = w$.

$$ds^2 = d\rho^2 + 2dydx + \left(\frac{R}{2} + \frac{3k^2}{4l^2}\right)x^2dy^2 + \frac{2k}{l}xd\rho dy + \frac{z}{l^4}e^{-\frac{2}{kl}\rho}dy^2, \tag{24}$$

where $v = l^2$ and $w = k^2l^2$ with $k > 0$. This solution is Kundt type and corresponds to a special case found in Ref. [18], namely its equation (58) with some particular choices.

A in Eq. (8) is a 4×3 matrix. Only $Q = 0$ and $v = w$ cases should be considered separately, and the first does not provide any solution.

$v = w$—Here the coefficients a and b are equal to

$$a = -\frac{16w + c}{64w^2}, \qquad b = \frac{8w + c}{12\sqrt{w}}. \tag{25}$$

In this case, the coefficient of the third term in the metric (24) above vanishes, since $k = 1$. By defining a new coordinate $\theta = xe^{\rho/l}$, it becomes

$$ds^2 = d\rho^2 + 2e^{-\frac{\rho}{l}}dyd\theta + \frac{z}{l^4}e^{-\frac{2\rho}{l}}dy^2, \tag{26}$$

which corresponds to the null warped AdS (Schrödinger) spacetime that was obtained before in Ref. [15].

C. 3-type metric

The 3-type metric is of the form

$$g = v(-\theta^0\theta^0 + \theta^1\theta^1 + \theta^2\theta^2) + z(\theta^0\theta^2 + \theta^1\theta^2), \tag{27}$$

with $z \neq 0$ and $v > 0$. Note that it is a z-deformation of the metric (12) with $-u = v = w$ (i.e., round AdS). The constant z can be scaled to ± 1.

The coefficients a, b, c and the scalar curvature R in terms of v are equal to

$$a = -\frac{9}{40v}, \quad b = \frac{8\sqrt{v}}{15}, \quad c = -\frac{8v}{5}, \quad R = -\frac{3}{2v}. \tag{28}$$

Note that the z-deformation has no effect on the scalar curvature, which is the same as that of round AdS_3. Moreover, the solution is attained at the chiral point; i.e., the coefficients satisfy the equality (6) with the plus sign.

Using the coordinate transformations given above (23) with $k = 1$, we obtain

$$ds^2 = d\rho^2 + 2dydx + \left(\frac{2x}{l} + \frac{z}{l^3}e^{-\rho/l}\right)dyd\rho + \frac{z}{l^4}e^{-\rho/l}xdy^2, \tag{29}$$

where $v = l^2$. This is a particular case of a Kundt solution given in equation (47) of Ref. [18].

D. $1z\bar{z}$—type metric

The $1z\bar{z}$-type metric is of the form

$$g = v(-\theta^0\theta^0 + \theta^1\theta^1) + w\theta^2\theta^2 + 2z\theta^0\theta^1, \tag{30}$$

with $vz \neq 0$ and $w > 0$. Like Eq. (21), it is a deformation of the spacelike warped AdS metric (17). When $v = w$, then this solution is a deformation of round AdS. The line element is

$$ds^2 = [v + z\sinh 2\zeta](-e^{2\sigma}dt^2 + d\sigma^2) + w(d\zeta + e^{\sigma}dt)^2 + 2ze^{\sigma}\cosh 2\zeta d\sigma dt - 2z\sinh 2\zeta d\sigma^2, \tag{31}$$

which was identified with the type (b) solution of Ref. [7] in Ref. [8].

The coefficients a, b, c and the scalar curvature R are equal to

$$\begin{aligned} a &= \frac{1}{Q}\cdot\frac{(4vw - w^2 + 4z^2)^3}{8w(v^2 + z^2)}, \\ b &= \frac{1}{Q}\cdot 8(w - 2v)(w^2 + 4z^2)\sqrt{w(v^2 + z^2)}, \\ c &= \frac{1}{Q}\cdot 8w(4vw - w^2 + 4z^2)(v^2 + z^2), \\ R &= -\frac{4vw - w^2 + 4z^2}{2w(v^2 + z^2)}, \end{aligned} \tag{32}$$

where $Q = (4vw - w^2 + 4z^2)^2 + 8(w^2 - 4v^2)(w^2 + 4z^2)$. Notice that unlike TMG, for which $c = a = 0$, in MMG the scalar curvature can be nonvanishing.

Here the matrix A defined in Eq. (8) is a 4×3 matrix. The only special case that should be considered separately is when $Q = 0$, which does not produce any solution to Eq. (1).

IV. SOLUTIONS ON $SU(2)$

We fix a basis $\{\tau_1, \tau_2, \tau_3\}$ and its dual basis θ^a for the Lie algebra $\mathfrak{su}_2$ with

$$[\tau_1, \tau_2] = \tau_3, \qquad [\tau_2, \tau_3] = \tau_1, \qquad [\tau_3, \tau_1] = \tau_2. \tag{33}$$

An element of $SU(2)$ can be parametrized by (see Ref. [25])

$$\mathcal{V} = e^{\phi\tau_3}e^{\xi\tau_2}e^{\psi\tau_3}. \tag{34}$$

The Maurer-Cartan one-forms are

$$\begin{aligned} \mathcal{V}^{-1}d\mathcal{V} = &(\sin\psi d\xi - \cos\psi\sin\xi d\phi)\tau_1 \\ &+ (\cos\psi d\xi + \sin\psi\sin\xi d\phi)\tau_2 \\ &+ (d\psi + \cos\xi d\phi)\tau_3. \end{aligned} \tag{35}$$

A left-invariant metric g on $SU(2)$ can be written as

$$g = u\theta^1\theta^1 + v\theta^2\theta^2 + w\theta^3\theta^3, \tag{36}$$

with $uvw < 0$, not all negative. The coefficients a, b, c and the scalar curvature R are

$$\begin{aligned} a &= \frac{1}{Q}\cdot\frac{[(u-v-w)^2-4vw]^3}{8uvw}, \\ b &= \frac{1}{Q}\cdot 8\sqrt{-uvw}[u^2-(v-w)^2](u-v-w), \\ c &= \frac{1}{Q}\cdot 8uvw[(u-v-w)^2-4vw], \\ R &= -\frac{(u-v-w)^2-4vw}{2uvw}, \end{aligned} \tag{37}$$

where

$$Q = [(u-v-w)^2-4vw]^2 + 8[u^2-(v+w)^2][u^2-(v-w)^2]. \tag{38}$$

The line element is given by

$$ds^2 = (u-v)(\sin\psi d\xi - \cos\psi \sin\xi d\phi)^2 + v(d\xi^2 + \sin^2\xi d\phi^2) + w(d\psi + \cos\xi d\phi)^2, \tag{39}$$

which corresponds to a triaxially deformed sphere, and in MMG the scalar curvature given in Eq. (37) is nonvanishing, unlike TMG.

Moreover, A in equation (8) is a 3×3 matrix with the determinant

$$\det A = \frac{Q}{8(uvw)^3}\cdot\sqrt{-uvw}(u-v)(u-w)(v-w). \tag{40}$$

Thus, the cases $Q=0$ and $u=v$ (which is enough due to the symmetry) should be considered separately. Again, the case $Q=0$ does not give any solution.

$u=v$—Note that in this case, $w<0$. The coefficients a and b are

$$\begin{aligned} a &= \frac{16v^2(4v-w)+c(4v-3w)(4v-7w)}{192v^4}, \\ b &= -\frac{8v^2+c(3w-4v)}{12\sqrt{-v^2w}}. \end{aligned} \tag{41}$$

In this case, the metric (39) simplifies to a Hopf fibration over S^2. Depending on whether $|w|>1$ or $|w|<1$, we have stretched or squashed warpings, respectively.

V. SOLUTIONS ON A_∞

The Lie algebra $\mathfrak{a}_\infty$ of A_∞ is spanned by r, x, and y and has only one nontrivial bracket

$$[r,x] = -y. \tag{42}$$

We denote the dual basis as $\{\tilde{r},\tilde{x},\tilde{y}\}$. The Baker-Campbell-Hausdorff formula allows us to write a representative as (see Ref. [25])

$$\mathcal{V} = e^{sr}e^{tx}e^{\rho y}. \tag{43}$$

The Maurer-Cartan one-forms are

$$\mathcal{V}^{-1}d\mathcal{V} = (ds)r + (dt)x + (d\rho - tds)y. \tag{44}$$

By the automorphism group, a left-invariant metric can be fixed as [8]

$$g = u\tilde{r}\,\tilde{r} + v\tilde{x}\,\tilde{x} \pm \tilde{y}\,\tilde{y}, \tag{45}$$

where $uv \neq 0$, and u or v can be scaled to ± 1. The line element reads as

$$ds^2 = uds^2 + vdt^2 \pm (d\rho - tds)^2, \tag{46}$$

which is a Hopf fibration over a flat space. We have

$$a = \frac{16|uv| + 21c}{192(uv)^2}, \quad b = \pm\frac{8|uv| - 3c}{12\sqrt{|uv|}}, \quad R = \frac{1}{2|uv|}.$$

VI. SOLUTIONS ON A_0

The Lie algebra $\mathfrak{a}_0$ of A_0, spanned by r, x, and y, has nonvanishing brackets

$$[r,x] = x, \qquad [r,y] = x + y. \tag{47}$$

We denote the dual basis as $\{\tilde{r},\tilde{x},\tilde{y}\}$. Again, by the Baker-Campbell-Hausdorff formula we can choose the representative

$$\mathcal{V} = e^{\xi x + \rho y}e^{\alpha r}. \tag{48}$$

Then the Maurer-Cartan one-forms are

$$\mathcal{V}^{-1}d\mathcal{V} = (e^{-\alpha}d\xi - \alpha e^{-\alpha}d\rho)x + (e^{-\alpha}d\rho)y + (d\alpha)r. \tag{49}$$

The following four types of metrics are available [8].

A. B_1-type metric

The metric is given by

$$B_1 = z\tilde{r}^2 \pm \tilde{x}^2 + v\tilde{y}^2. \tag{50}$$

There is no solution for a, b, c.

B. B_2-type metric

The metric is given by

$$B_2 = z\tilde{r}^2 \pm 2\tilde{x}\,\tilde{y}, \tag{51}$$

with $z > 0$. The MMG field equation (1) is satisfied if

$$a = -\frac{4z + c}{4z^2}, \qquad b = \mp \frac{2z + c}{2\sqrt{z}}, \qquad R = -\frac{6}{z}. \tag{52}$$

Note that the solution is attained at the chiral point (6).

Under the coordinate transformations

$$\alpha \to \log(w), \qquad \rho \to -lx^+, \qquad \xi \to lx^-, \tag{53}$$

where $z = l^2$, the metric (51) becomes

$$ds^2 = \mp \frac{l^2}{w^2}[2\log(w)(dx^+)^2 + 2dx^+dx^- \mp dw^2], \tag{54}$$

which is the logarithmic pp-wave solution found in Ref. [16].

C. B_3-type metric

The metric is given by

$$B_3 = z\tilde{r}^2 + \tilde{r}\,\tilde{x} + v\tilde{y}^2. \tag{55}$$

In this case, $a = 0$ is the necessary and sufficient condition to solve Eq. (1). The Ricci, Cotton, and J-tensors are identically zero. The metric of this Ricci flat spacetime is

$$ds^2 = zd\alpha^2 + e^{-\alpha}(d\xi d\alpha - \alpha d\rho d\alpha) + ve^{-2\alpha}d\rho^2. \tag{56}$$

As we discuss in Sec. IX, it must be maximally symmetric based on a result of Ref. [26], and hence should locally be Minkowski spacetime.

D. B_4-type metric

The metric is given by

$$B_4 = z\tilde{r}^2 + \tilde{r}\,\tilde{y} + u\tilde{x}^2, \tag{57}$$

where $u > 0$. The coefficients a, b, c, and the scalar curvature R are found to be

$$a = -\frac{u}{18}, \quad b = -\frac{4}{9\sqrt{u}}, \quad c = -\frac{2}{9u}, \quad R = 2u. \tag{58}$$

The line element is

$$ds^2 = zd\alpha^2 + e^{-\alpha}d\rho d\alpha + ue^{-2\alpha}(d\xi - \alpha d\rho)^2. \tag{59}$$

Unfortunately, we could not determine to which spacetime geometry this metric corresponds. Higher-order-curvature scalars are as follows:

$$R_{\mu\nu}R^{\mu\nu} = 12u^2, \qquad R_{\mu\rho}R^{\rho\nu}R_\nu{}^\mu = 8u^3. \tag{60}$$

VII. SOLUTIONS ON ISO(2; θ)

Let the Lie algebra basis and the dual basis of $\mathfrak{iso}(2;\theta)$ be $\{l, m_1, m_2\}$ and $\{\tilde{l}, \tilde{m}_1, \tilde{m}_2\}$, respectively. The nonvanishing brackets are

$$[l, m_1] = 2\cos\theta m_1 + 2\sin\theta m_2,$$
$$[l, m_2] = 2\cos\theta m_2 - 2\sin\theta m_1, \tag{61}$$

where $\theta \in [0, \pi/2]$. We choose the group representative

$$\mathcal{V} = e^{xm_1 + ym_2}e^{\rho l}. \tag{62}$$

Then the Maurer-Cartan one-forms are

$$\begin{aligned}\mathcal{V}^{-1}d\mathcal{V} = {}& (d\rho)l \\ & + e^{-2\rho\cos\theta}[\cos(2\rho\sin\theta)dx + \sin(2\rho\sin\theta)dy]m_1 \\ & + e^{-2\rho\cos\theta}[-\sin(2\rho\sin\theta)dx + \cos(2\rho\sin\theta)dy]m_2.\end{aligned} \tag{63}$$

There are two types of metrics, as given below [8]. The case $\theta = 0$ should be analyzed separately.

A. B_1-type metric

The metric is given by

$$B_1 = u\tilde{l}\,\tilde{l} + v\tilde{m}_1\,\tilde{m}_1 + w\tilde{m}_2\,\tilde{m}_2, \tag{64}$$

where $uvw < 0$, not all negative. The coefficient v or w can be rescaled freely.

There is no general solution for a, b, and c. The scalar curvature is given by

$$R = -\frac{2[12vw\cos^2\theta + (v - w)^2\sin^2\theta]}{uvw}. \tag{65}$$

The matrix A in Eq. (8) is 4×3, and the cases $\theta = 0$, $\theta = \frac{\pi}{2}$, and $v = w$ should be considered separately.

$\theta = 0$—In this case, due to the enlargement of the automorphism group, the metric (64) becomes

$$B_1 = |z|(\pm\tilde{l}\,\tilde{l} \pm \tilde{m}_1\,\tilde{m}_1 \pm \tilde{m}_2\,\tilde{m}_2), \tag{66}$$

which in spacetime coordinates takes the form

$$ds^2 = |z|(\pm d\rho^2 + e^{-4\rho}[\pm dx^2 + \pm dy^2]), \tag{67}$$

which is either de Sitter for $(-, +, +)$ or AdS for $(+, +, -)$ signs with $R = \pm 24/|z|$. The Cotton tensor (2) vanishes identically, and we have the relation

$$a = -\frac{4(\pm|z| - c)}{z^2}. \quad (68)$$

$\theta = \frac{\pi}{2}$—The coefficients a, b, c in terms of u, v, and w are equal to

$$a = \frac{1}{Q} \cdot \frac{(v-w)^4}{2uvw}, \qquad b = \frac{1}{Q} \cdot 4\sqrt{-uvw}(v+w),$$
$$c = \frac{1}{Q} \cdot 2uvw, \quad (69)$$

where $Q = (v-w)^2 + 8(v+w)^2 \neq 0$. In this case, the line element is

$$ds^2 = ud\rho^2 + (v-w)[\cos(2\rho)dx + \sin(2\rho)dy]^2 + w(dx^2 + dy^2), \quad (70)$$

which is not familiar to us. Higher-order-curvature invariants are as follows:

$$R_{\mu\nu}R^{\mu\nu} = \frac{4(v-w)^2(3v^2 + 2vw + 3w^2)}{(uvw)^2},$$
$$R_{\mu\rho}R^{\rho\nu}R^{\mu}_{\nu} = -\frac{8(v-w)^6}{(uvw)^3}. \quad (71)$$

A solution of this type also exists [11,12] in NMG [13].

$v = w$—In this case, the Cotton tensor (2) vanishes identically, and

$$a = -\frac{4\cos^2\theta(u + c \cdot \cos^2\theta)}{u^2}. \quad (72)$$

The line element is simply

$$ds^2 = ud\rho^2 + ve^{-4\rho\cos\theta}[dx^2 + dy^2], \quad (73)$$

which is, for $u < 0$ and $v > 0$, either de Sitter if $\theta \neq \pi/2$ or Minkowski if $\theta = \pi/2$.

B. B_2-type metric

The metric is given by

$$B_2 = u\tilde{l}\,\tilde{l} + \tilde{l}\,\tilde{m}_1 + w\tilde{m}_2\,\tilde{m}_2, \quad (74)$$

with $u > 0$ and $w \neq 0$.

When $\theta = 0$, the metric is Minkowski, and the field equation (1) is solved only if $a = 0$. For $\theta \neq 0$, the coefficients a, b, c and the scalar curvature R are equal to

$$a = -\frac{2}{9}w\sin^2\theta, \qquad b = \frac{2}{9\sqrt{w}\sin\theta},$$
$$c = -\frac{1}{18w\sin^2\theta}, \qquad R = 8w\sin^2\theta. \quad (75)$$

The line element is

$$ds^2 = ud\rho^2 + e^{-2\rho\cos\theta}[\cos(2\rho\sin\theta)dxd\rho + \sin(2\rho\sin\theta)dyd\rho] + we^{-4\rho\cos\theta}[\sin(2\rho\sin\theta)dx - \cos(2\rho\sin\theta)dy]^2. \quad (76)$$

We could not recognize the spacetime to which it corresponds. Higher-order-curvature scalars are

$$R_{\mu\nu}R^{\mu\nu} = 192w^2\sin^4\theta,$$
$$R_{\mu\rho}R^{\rho\nu}R_{\nu}{}^{\mu} = 512w^3\sin^6\theta. \quad (77)$$

VIII. SOLUTIONS ON ISO(1,1;θ)

The basis $\{l, m_1, m_2\}$ of $\mathfrak{iso}(1,1;\theta)$ has the brackets

$$[l, m_1] = 2\cos\theta m_1 + 2\sin\theta m_2,$$
$$[l, m_2] = 2\cos\theta m_2 + 2\sin\theta m_1, \quad (78)$$

and the dual basis is $\{\tilde{l}, \tilde{m}_1, \tilde{m}_2\}$. We choose the group representative as

$$\mathcal{V} = e^{xm_1 + ym_2}e^{\rho l}, \quad (79)$$

and the Maurer-Cartan one-forms are

$$\mathcal{V}^{-1}d\mathcal{V} = (d\rho)l + e^{-2\rho\cos\theta}[\cosh(2\rho\sin\theta)dx - \sinh(2\rho\sin\theta)dy]m_1 + e^{-2\rho\cos\theta}[\cosh(2\rho\sin\theta)dy - \sinh(2\rho\sin\theta)dx]m_2, \quad (80)$$

with $\theta \in [0, \pi/2]$. When $\theta = 0$, the Lie algebras of ISO(1, 1; 0) and ISO(2; 0) coincide. Hence, the $\theta = 0$ case is already covered in Secs. VII A and VII B. From the automorphism group, two types of metrics can be fixed as given below [8].

A. B_1-type metric

The B_1-type metric is given by

$$B_1 = \delta\tilde{l}\,\tilde{l} + u(\tilde{m}_1 + \tilde{m}_2)^2 + v(\tilde{m}_1 - \tilde{m}_2)^2 + 2w(\tilde{m}_1\,\tilde{m}_1 - \tilde{m}_2\,\tilde{m}_2), \quad (81)$$

with $w^2 > uv$ and $\delta > 0$. Two of the parameters (u, v, w) can be set to ± 1 whenever they are nonzero.

The matrix A in Eq. (8) is 6×3, and there is no general solution for a, b, c. The scalar curvature is

$$R = -\frac{8[3(uv - w^2)\cos^2\theta + uv\sin^2\theta]}{\delta(uv - w^2)}. \quad (82)$$

However, the cases $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$, and $uv = 0$ should be considered separately.

$\theta = \frac{\pi}{4}$—For $w \neq 0$, the coefficients a and b are found as

$$a = \frac{2\delta(w^2 - uv)(4uv - 3w^2) + c(4uv - w^2)(4uv + 3w^2)}{3\delta^2(w^2 - uv)^2},$$
$$b = \frac{-\delta(w^2 - uv) + c(4uv - w^2)}{3w\sqrt{2\delta(w^2 - uv)}}. \quad (83)$$

The metric (81) simplifies to

$$ds^2 = \delta d\rho^2 + ue^{-4\sqrt{2}\rho}(dx + dy)^2 + v(dx - dy)^2 + 2we^{-2\sqrt{2}\rho}(dx^2 - dy^2), \quad (84)$$

which was identified as timelike or spacelike warped AdS in Ref. [8] depending on signs.

When $w = 0$, the Cotton tensor vanishes identically, and we are at the merger point (7) with $c = -\delta/4 = 1/a = 4/R$. The metric (84) becomes $(\mathrm{A})\mathrm{dS}_2 \times \mathrm{S}^1$ that was found in Ref. [15], which is clearly not a solution of TMG, since c cannot be zero. Its absence is related to a no-go result on solutions of TMG with a hypersurface orthogonal Killing vector [27]. However, it exists in NMG [13], as was found in Ref. [28].

$\theta = \frac{\pi}{2}$—The coefficients a, b, c in terms of u, v, and w are equal to

$$a = \frac{2u^2v^2}{\delta(uv + 8w^2)(uv - w^2)}, \qquad b = -\frac{2w\sqrt{\delta(w^2 - uv)}}{uv + 8w^2},$$
$$c = \frac{\delta(uv - w^2)}{2(uv + 8w^2)}. \quad (85)$$

Its spacetime metric is

$$ds^2 = \frac{\delta}{4}\frac{dr^2}{r^2} + \frac{d\alpha^2}{r^2} - r^2 dt^2 + 2wd\alpha dt, \quad (86)$$

where we set $u = 1$, $v = -1$ and define

$$r = e^{2\rho}, \qquad \alpha = x + y, \qquad t = x - y. \quad (87)$$

Notice that the following constant rescalings $r \to \lambda r, \alpha \to \lambda\alpha, t \to \lambda^{-1}t$ leave the metric (86) invariant. When $w = 0$ [which sets $b = 0$ and satisfies the merger point condition (7)], this corresponds to the static Lifshitz spacetime with the dynamical exponent $z = -1$, and for $w \neq 0$ it is a stationary Lifshitz metric (see Ref. [29]). Note that the rotation parameter w is nonzero only when there is a contribution from the Cotton tensor.

$u = 0$—The coefficients a and b are

$$a = -\frac{4\cos^2\theta(\delta + c\cdot\cos^2\theta)}{\delta^2},$$
$$b = \frac{w(\delta + 2c\cdot\cos^2\theta)}{2\sqrt{w^2\delta}(\cos\theta - 2\sin\theta)}. \quad (88)$$

The spacetime metric (81) takes the form

$$ds^2 = \delta d\rho^2 + ve^{-4\rho(\cos\theta - \sin\theta)}(dx - dy)^2 + 2we^{-4\rho\cos\theta}(dx^2 - dy^2), \quad (89)$$

which was identified in Ref. [8] as AdS pp-wave for $\theta \neq \pi/4$ and $\theta \neq \pi/2$. When $\theta = \pi/4$, it is AdS, and when $\theta = \pi/2$, it is a flat-space pp-wave [8].

$v = 0$—The coefficients a and b are equal to

$$a = -\frac{4\cos^2\theta(\delta + c\cdot\cos^2\theta)}{\delta^2},$$
$$b = -\frac{w(\delta + 2c\cdot\cos^2\theta)}{2\sqrt{w^2\delta}(\cos\theta + 2\sin\theta)}. \quad (90)$$

The spacetime metric (81) becomes

$$ds^2 = \delta d\rho^2 + ue^{-4\rho(\cos\theta + \sin\theta)}(dx + dy)^2 + 2we^{-4\rho\cos\theta}(dx^2 - dy^2), \quad (91)$$

which again corresponds to an AdS pp-wave in general [8]. But for $\theta = \pi/4$, it is the null warped AdS (Schrödinger) spacetime, and when $\theta = \pi/2$, it is a flat-space pp-wave [8].

B. B_2-type metric

For $\theta \neq 0$, the metric is given by

$$B_2 = \delta\tilde{l}\,\tilde{l} + \tilde{l}\,\tilde{m}_1 + u(\tilde{m}_1 + \tilde{m}_2)^2 + v(\tilde{m}_1 - \tilde{m}_2)^2 + 2w(\tilde{m}_1\,\tilde{m}_1 - \tilde{m}_2\,\tilde{m}_2), \quad (92)$$

with $w^2 = uv > 0$ and $u + v \neq 2w$. For $w = 0$, both Cotton and J-tensors vanish and $a = 0$ in Eq. (1), which locally corresponds to Minkowski spacetime, as we discuss in Sec. IX. Hence, we assume $w \neq 0$, which means that $v = w$ is not allowed. One of the coefficients u or v can be set to ± 1.

Using the coordinate transformations given in Eq. (87), the line element becomes

$$ds^2 = \frac{\delta}{4}\frac{dr^2}{r^2} + ur^{-2n}d\alpha^2 + vr^{-2m}dt^2 + 2wr^{-(n+m)}d\alpha dt + \frac{1}{4}r^{-(n+1)}d\alpha dr + \frac{1}{4}r^{-(m+1)}dtdr, \quad (93)$$

where $n = (\cos\theta + \sin\theta)$ and $m = (\cos\theta - \sin\theta)$. Note that the metric is invariant under the scalings $r \to \lambda r$, $\alpha \to \lambda^n \alpha, t \to \lambda^m t$. Hence, the solution possesses a generalized (anisotropic) Lifshitz symmetry. For $\theta = \pi/2$, it becomes a stationary Lifshitz solution with dynamical exponent $z = -1$ similar to the one we found above (86).

The coefficients a, b, c and the scalar curvature R are

$$a = -\frac{32vw^2\sin^2\theta}{9(v-w)^2}, \qquad b = -\frac{|v-w|}{18w\sqrt{v}\sin\theta},$$
$$c = -\frac{(v-w)^2}{288vw^2\sin^2\theta}, \qquad R = \frac{128vw^2\sin^2\theta}{(v-w)^2}. \tag{94}$$

The only special case that should be considered separately is $\theta = \frac{\pi}{4}$.

$\theta = \frac{\pi}{4}$—The coefficients a and b are given as

$$a = \frac{32vw^2[(v-w)^2 + 168c \cdot vw^2]}{3(v-w)^4},$$
$$b = -\frac{(v-w)^2 - 48c \cdot vw^2}{12w|v-w|\sqrt{2v}}. \tag{95}$$

Its metric is (93) with $m = 0$ and $n = \sqrt{2}$, which corresponds to warped flat space [8].

IX. SUMMARY AND DISCUSSION

In this paper, we constructed homogeneous solutions of MMG, several of which are new. We summarize our results in Table I, where we include only those that are nontrivial in the sense that none of the terms in the MMG field equation (1) vanishes identically. In its last column, we give a classification of our solutions with respect to the Segre-Petrov type of their traceless Einstein tensor

$$P^a{}_b \equiv R^a{}_b - \frac{1}{3}R\delta^a{}_b, \tag{96}$$

as was proposed in Ref. [20], to which we refer for details.

From Table I, we see that homogeneous solutions of MMG in comparison to TMG can be grouped into three categories, as follows:

Group 1: Solutions which are type N or D in the Segre-Petrov classification can be obtained from TMG solutions with a redefinition of constants as was shown in Ref. [14]. Corresponding solutions have the same curvature.

For type-D solutions—namely Eqs. (17), (19), (41), (46), (84), and (95)—we have

$$a_{\mathrm{MMG}} = a_{\mathrm{TMG}} + c \cdot \frac{1}{48}\left(R + \frac{4}{9b_{\mathrm{TMG}}^2}\right)\left(R + \frac{4}{3b_{\mathrm{TMG}}^2}\right), \tag{97}$$

$$b_{\mathrm{MMG}} = b_{\mathrm{TMG}} - c \cdot \frac{b_{\mathrm{TMG}}}{4}\left(R + \frac{4}{9b_{\mathrm{TMG}}^2}\right). \tag{98}$$

For type-N solutions—that is, Eqs. (26), (54), (89), and (91)—we have

$$a_{\mathrm{MMG}} = a_{\mathrm{TMG}} - c \cdot \frac{Ra_{\mathrm{TMG}}}{24}, \tag{99}$$

$$b_{\mathrm{MMG}} = b_{\mathrm{TMG}} - c \cdot \frac{Rb_{\mathrm{TMG}}}{12}. \tag{100}$$

Group 2: Solutions (13), (24), (31), and (39) exist in TMG, but only if the cosmological constant vanishes. Hence, they have $R = 0$ in TMG. But for MMG, for these solutions the cosmological constant is proportional to the MMG parameter c, and therefore $R \neq 0$ is possible.

Group 3: Solutions (29), (59), (70), (76), (86), and (93) exist only in MMG.

It is interesting to note that for all solutions in groups 2 and 3, we have $R^2 = 16a/c$. Moreover, three of the solutions in the third group—that is, Eqs. (59), (76), and (93)—appear when $ac = 1/81$ and $9b^2 = -8c$. Whether this particular point in the parameter space of MMG has any physical significance like chiral (6) and merger (7) points remains to be seen. Also, more work is required to understand spacetimes that we found in Eqs. (59), (70), and (76).

Within the third group, Lifshitz-type solutions—that is, Eqs. (86) and (93)—are especially attractive due to their possible holographic applications (for a review, see Ref. [30]). Moreover, only few exact Lifshitz solutions which are stationary are known [29]. In Eq. (86), the dynamical exponent is $z = -1$, and rotation is present only when there is a contribution from the Cotton tensor. The second one [Eq. (93)] enjoys a generalized Lifshitz symmetry where each coordinate scales differently. Such solutions are also very rare. In four dimensions, one example was found for Einstein gravity coupled to massive vectors in Ref. [31] and another one in conformal gravity in Ref. [32]. It would be interesting to study our solutions from the dual CFT perspective. Another related issue is to search for Lifshitz black holes [33].

In many of the solutions above, it is possible to set $b = 0$ by choosing other parameters appropriately, after which remarkably one always ends up at the merger point (7). Thus, these are solutions of the MMG theory without the Cotton tensor. The fundamental equation of this specific model can be obtained from MMG (1) by taking the limit $\mu \to \infty$, $\gamma \to \infty$ while keeping γ/μ^2 constant, which was considered before in Refs. [16] and [19]. For this to be consistent, one should still make sure that the Bianchi identity (4) is satisfied, i.e.,

$$V^\mu = \epsilon^{\mu\rho\sigma} S_\rho{}^\tau C_{\sigma\tau} = 0. \tag{101}$$

For our solutions, it turns out that V^μ is identically zero except for the following three cases:

(i) A_0 spacetime with a B_1-type metric (50):

$$V^\mu = \left[\mp \frac{2}{vz^3}, 0, 0\right]. \tag{102}$$

(ii) $\mathrm{ISO}(2;\theta)$ spacetime with a B_1-type metric (64):

$$V^\mu = \left[-\frac{64(v-w)^2 \cos\theta \sin^2\theta}{u^3 vw}, 0, 0\right]. \tag{103}$$

(iii) $\mathrm{ISO}(1,1;\theta)$ spacetime with a B_1-type metric (81):

$$V^\mu = \left[\frac{64uv \sin 4\theta \sin\theta}{(w^2 - uv)z^3}, 0, 0\right]. \tag{104}$$

Recall from Sec. VI A that there is no B_1-type solution for A_0, which is in agreement with V^μ being nonzero in Eq. (102). From our analysis in Secs. VII A and VIII A, one can easily see that the last two vectors become zero precisely at the solutions we found, independent of the value of b. Moreover, for all our solutions, whenever $b = 0$ is possible, the merger point condition (7) is satisfied. Exceptions appear only when the Cotton tensor identically vanishes. Hence, we reach the conclusion that when the Cotton tensor is absent in the MMG equation (1), simply transitive homogeneous solutions exist only at the merger point (7), provided that they are not conformally flat. They satisfy the Bianchi condition (101).

For our solutions for which the Cotton tensor vanishes, it is useful to recall that conformally flat spacetime solutions of MMG are locally maximally symmetric away from the merger point (7), which was proven in Ref. [26]. Indeed, Eqs. (67) and (73) are (A)dS spacetimes which are in general away from the merger point, but for a specific choice of the parameter c they also exist at the merger point. For Eq. (56), the cosmological constant vanishes ($a = 0$), and hence we conclude that it must be locally Minkowski, since the merger point condition, i.e., $ac = 1$, is impossible to satisfy. This applies also to Eq. (92) with $w = 0$. On the other hand, at the merger point, a conformally flat solution is not necessarily maximally symmetric. For example, the metric (84) with $w = 0$ corresponds to $\mathrm{(A)dS}_2 \times \mathrm{S}^1$, which was found in Ref. [15], and it exists only at the merger point.

We found that two of our solutions given in Eqs. (29) and (52) exist at the chiral point (6) of the parameter space. Only in the latter is it possible to set $b = 0$, in which case one ends up at the merger point (7), as we noted above.

In this paper, we focused on simply transitive homogeneous spacetimes. A natural generalization would be to allow a nontrivial isotropy group as was studied in Ref. [34] for TMG, and in Ref. [10] for NMG. The following metric that was discussed in both [34] and [10] has a four-dimensional isometry group with no three-dimensional simply transitive subgroup:

$$ds^2 = -dt^2 + v^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{105}$$

This is a solution of the MMG field equation (1) at the merger point (7) with

$$a = \frac{1}{c} = \frac{R}{4} = \frac{1}{2v^2}. \tag{106}$$

The Cotton tensor vanishes identically, and its Segre-Petrov type is D.

It would be interesting to repeat our investigation in models closely related to MMG [35,36]. Finally, trying to classify all stationary axisymmetric solutions of MMG as was done for TMG [37] using a method developed in Ref. [38] would be worth studying. We hope to explore these issues in the near future.

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