New Kaluza-Klein instantons and the decay of AdS vacua

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(Received 17 April 2017; published 19 July 2017)

We construct a generalization of Witten's Kaluza-Klein instanton, where a higher-dimensional sphere (rather than a circle as in Witten's instanton) collapses to zero size and the geometry terminates at a bubble of nothing, in a low energy effective theory of M theory. We use the solution to exhibit the instability of nonsupersymmetric AdS_5 vacua in M theory compactified on positive Kähler-Einstein spaces, providing further evidence for the recent conjecture that any nonsupersymmetric anti–de Sitter vacuum supported by fluxes must be unstable.

DOI: 10.1103/PhysRevD.96.026016

I. INTRODUCTION

Stability is an important criterion for the consistency of Kaluza-Klein vacua. Due to nontrivial topologies of their internal spaces, the standard positive energy theorem [1–3] does not necessarily apply. In fact, it was shown by Witten [4] that the original Kaluza-Klein theory [5,6] on a product of the four-dimensional Minkowski spacetime and a circle is unstable against a semiclassical decay process, unless protected by boundary conditions on fermions. The instanton that mediates the decay is the analytical continuation of a five-dimensional Schwarzschild solution,

$$ds^2 = \frac{dr^2}{1-\frac{R^2}{r^2}} + r^2 d\Omega_3^2 + \left(1-\frac{R^2}{r^2}\right)d\phi^2,$$

where $d\Omega_3^2$ is the metrics on the unit 3-sphere, and ϕ is the coordinate on the Kaluza-Klein circle. Smoothness of the solution at $r = R$ requires ϕ to be periodic with period $2\pi R$, and the Kaluza-Klein radius at $r = \infty$ is R. As we move toward small r, the circle collapses and becomes zero size at $r = R$, where the geometry terminates. Another analytic continuation of the polar angle on the 3-sphere turns this into a Lorentzian signature solution, where the "bubble of nothing" expands with velocity that asymptotes to the speed of light.

Witten's instanton has also played a role in the stability of anti–de Sitter (AdS) vacua. There have been several proposals for nonsupersymmetric AdS geometries. Among them is $\mathrm{AdS}_5 \times S^5/\Gamma$, where Γ is a discrete subgroup of the $SU(4)$ rotational symmetry of S^5 [7]. Supersymmetry is completely broken if Γ does not fit within an $SU(3)$ subgroup of the $SU(4)$ symmetry. It turns out that, if Γ has a fixed point on S^5 or if the radius of S^5 is not large enough, the perturbative spectrum on AdS_5 contains closed string tachyons that violate the Breitenloner-Freedman bound [8,9]. When Γ has no fixed point and S^5 is sufficiently large, the instability modes are lifted and the configuration becomes perturbatively stable. However, in this case, S^5/Γ is not simply connected, and there is a Witten-type instanton where a homotopically nontrivial cycle on S^5/Γ collapses to zero size at a bubble of nothing [10]. This eliminates $\mathrm{AdS}_5 \times S^5/\Gamma$ as a candidate for a stable nonsupersymmetric AdS geometry.

In this paper, we present instanton solutions where a higher-dimensional sphere rather than a circle collapses, in a low energy effective theory of M theory. Such generalizations of Witten's instanton were attempted earlier, for example in [11], where it was found that fluxes needed to cancel the intrinsic curvature on the sphere to prevent it from collapsing. Motivated by the recent conjecture [12] (see also [13–15]) that any nonsupersymmetric AdS vacuum supported by fluxes must be unstable, we found examples in nonsupersymmetric setups which avoid the difficulty in the earlier attempt.

We will focus on AdS_5 times positive Kähler-Einstein spaces, which break supersymmetry [16]. $\mathrm{AdS}_5 \times \mathbb{C}\mathrm{P}^3$ is the only example of this type known to be stable against linearized supergravity perturbations [17]. Since its internal space is simply connected, we do not expect it to have a Witten-type instanton, where an S^1 collapses to zero size at a bubble. On the other hand, $\mathbb{C}\mathrm{P}^3$ can be realized as S^2 fibration over S^4, and it is possible for its S^2 fibers to collapse. Indeed, we find such an instanton solution with finite action.

Our solution avoids the difficulty with fluxes encountered in [11] as follows. The AdS_5 geometry in question is supported by the 4-form flux in M theory, with nonzero components both on the S^2 fibers and on the S^4 base of the internal space. As we move toward the center of AdS_5, the

4-form flux reorients itself. By the time we reach the bubble of nothing, the flux has no components in the S^2 fiber direction. Thus, the S^2 can collapse at the bubble without violating the flux conservation.

$\mathrm{AdS}_5 \times \mathbb{CP}^3$ is only marginally stable with a normalizable mode at the Breitenlohner-Freedman bound. Thus, this vacuum is also in danger of becoming unstable by higher derivative corrections to the 11-dimensional supergravity. It is interesting to point out that in our instanton solution the normalizable mode at the Breitenlohner-Freedman bound is turned on and is responsible for triggering the collapse of the S^2 fiber.

If there is a bubble of nothing instanton in AdS, it causes an instability that can be detected instantaneously on the boundary of AdS [10,18]. This is because any observer in AdS can receive signals from any point on a Cauchy surface within a finite amount of time, and an observer at the boundary in particular has access to an infinite volume space near the boundary within an infinitesimal amount of time. Therefore, our new instanton solution in the perturbatively stable nonsupersymmetric AdS_5 configuration offers further evidence for the conjecture of [12].

The plan of the paper is the following. In Sec. II we describe the $\mathrm{AdS}_5 \times \mathbb{CP}^3$ solution and introduce our instanton ansatz. Boundary conditions on the instanton at the bubble and at the infinity are discussed in Sec. III. It turns out that there are algebraic relations among variables in our ansatz, as shown in Sec. IV. These relations reduce the problem to a second order ordinary differential equation on a single function, which we will numerically solve in Sec. V. Finiteness of the instanton action is verified in Sec. VI. In Sec. VII, we discuss $\mathrm{AdS}_5 \times \frac{SU(3)}{U(1)\times U(1)}$. It is not known whether this geometry is stable against linearized supergravity perturbation. Regardless, we will show that it allows a bubble of nothing solution and is therefore unstable nonperturbatively. In the final section, we discuss additional features of our instanton solutions.

II. $\mathrm{AdS}_5 \times \mathbb{CP}^3$ GEOMETRY AND INSTANTON ANSATZ

For any Kähler-Einstein sixfold M_6, there exists an $\mathrm{AdS}_5 \times M_6$ solution [16] to the 11-dimensional supergravity equations of motion,

$$R_{MN} = \frac{1}{3}\left(F_M{}^{PQR}F_{NPQR} - \frac{1}{12}g_{MN}F^{PQRS}F_{PQRS}\right),$$
$$\nabla_M F^{MPQR} = -\frac{1}{576}\varepsilon^{M_1\ldots M_8 PQR}F_{M_1M_2M_3M_4}F_{M_5M_6M_7M_8}. \tag{1}$$

Such a solution can be found by setting the 4-form field strength as

$$F = c\omega \wedge \omega, \tag{2}$$

where ω is the Kähler 2-form of internal space and c is some constant, which will be related to the AdS radius. With this ansatz, the right-hand side of the second equation in (1) vanishes since F is nonzero only on M_6, and the left-hand side vanishes by the Kähler integrability condition on ω. On the other hand, the first equation in (1) gives

$$R_{\mu\nu} = -2c^2 g_{\mu\nu}, \qquad R_{mn} = 2c^2 g_{mn}, \tag{3}$$

where $\mu = 0, \ldots, 4$ are the indices in the noncompact directions and $m = 5, \ldots, 10$ are on M_6. Therefore, the noncompact directions can be chosen to be AdS_5, and M_6 must be an Einstein manifold. The configuration breaks supersymmetry [16,19] since there are no nontrivial solutions to $\delta\Psi_M = 0$ for the supersymmetry variation of the gravitini,

$$\delta\Psi_M = D_M\varepsilon = \nabla_M\varepsilon + \frac{1}{144}(\Gamma_{MNPQR}F^{NPQR} - 8\Gamma_{NPQ}F_M{}^{NPQ})\varepsilon. \tag{4}$$

As we mentioned in the Introduction, the only known perturbatively stable case is $M_6 = \mathbb{CP}^3$. This space can be realized as an S^2 fibration, and we look for an instanton solution where the fiber collapses. In the next few sections we focus on $M_6 = \mathbb{CP}^3$ and in Sec. VII we show that an instanton solution for $M_6 = \frac{SU(3)}{U(1)\times U(1)}$ can be constructed similarly.

To make the S^2 fibration explicit, we use the following set of coordinates on $\mathbb{CP}^3$ [20]:

$$\begin{aligned}
e_1 &= \sqrt{g(r)}d\mu,\\
e_i &= \frac{\sqrt{g(r)}}{2}\sin\mu\Sigma_{i-1} \quad \text{for } i = 2,3,4,\\
e_5 &= \sqrt{h(r)}(d\theta - A_1\sin\phi + A_2\cos\phi),\\
e_6 &= \sqrt{h(r)}\sin\theta(d\phi - \cot\theta(A_1\cos\phi + A_2\sin\phi) + A_3),
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\Sigma_1 &= \cos\gamma d\alpha + \sin\gamma\sin\alpha d\beta,\\
\Sigma_2 &= -\sin\gamma d\alpha + \cos\gamma\sin\alpha d\beta,\\
\Sigma_3 &= d\gamma + \cos\alpha d\beta,\\
A_i &= \cos\left(\frac{\mu}{2}\right)^2\Sigma_i.
\end{aligned} \tag{6}$$

Here the first 4-tetrad corresponds to the base S^4 and the last two correspond to the S^2 fiber. We multiplied them by the functions $g(r)$, $h(r)$ to make their sizes dynamical. We take the vierbein on Euclidean AdS space to be

$$\begin{aligned} e_7 &= dr, \\ e_k &= \sqrt{f(r)}\hat{e}_k \quad \text{for } k = 8, 9, 10, 11, \end{aligned} \tag{7}$$

where $\hat{e}_k$ is any tetrad on the S^4. The metric in this frame is

$$\begin{aligned} ds^2 = {}& g(r)\left(d\mu^2 + \frac{1}{4}\sin^2\mu\sum_{i=1}^{3}\Sigma_i^2\right) + h(r)(d\theta - A_1\sin\phi + A_2\cos\phi)^2 \\ &+ h(r)\sin^2\theta(d\phi - \cot\theta(A_1\cos\phi + A_2\sin\phi) + A_3)^2 + dr^2 + f(r)d\Omega_4^2. \end{aligned} \tag{8}$$

We used the freedom of coordinate redefinition by fixing the coefficient near dr^2 to be 1.

The next step is to write an ansatz for the 4-form field, utilizing the $SU(3)$-structure of the squashed $\mathbb{C}\mathrm{P}^3$ given by the 2-form J and 3-form Ω as [21–24]

$$\begin{aligned} J &= -\sin\theta\cos\phi(e^{12} + e^{34}) - \sin\theta\sin\phi(e^{13} + e^{42}) - \cos\theta(e^{14} + e^{23}) + e^{56}, \\ \mathrm{Re}\Omega &= \cos\theta\cos\phi(e^{126} + e^{346}) + \cos\theta\sin\phi(e^{136} + e^{426}) + \sin\phi(e^{125} + e^{345}) \\ &\quad - \cos\phi(e^{135} + e^{425}) - \sin\theta(e^{146} + e^{236}), \\ \mathrm{Im}\Omega &= -\cos\theta\cos\phi(e^{125} + e^{345}) - \cos\theta\sin\phi(e^{135} + e^{425}) + \sin\phi(e^{126} + e^{346}) \\ &\quad - \cos\phi(e^{136} + e^{426}) + \sin\theta(e^{145} + e^{235}). \end{aligned} \tag{9}$$

Here $e^{12} = e^1 \wedge e^2$ etc. These forms satisfy

$$\begin{aligned} d_6 J &= \frac{3}{2}W_1\mathrm{Im}\Omega, \\ d_6\mathrm{Im}\Omega &= 0, \\ d_6\mathrm{Re}\Omega &= W_1 J\wedge J + W_2\wedge J, \end{aligned} \tag{10}$$

where d_6 is an external derivative of $\mathbb{C}\mathrm{P}^3$, and W_1 and W_2 are torsion classes of the $SU(3)$-structure given by

$$\begin{aligned} W_1 &= \frac{2}{3}\frac{g(r) + h(r)}{g(r)\sqrt{h(r)}}, \\ W_2 &= \frac{2h(r) - g(r)}{g(r)\sqrt{h(r)}}\left(\frac{2}{3}J - 2e^{56}\right). \end{aligned} \tag{11}$$

A general manifold with $SU(3)$ structure has more terms in the relations (10), but in the case of our interest (squashed $\mathbb{C}\mathrm{P}^3$) other torsion classes vanish.

Note that this $SU(3)$-structure is different from the usual Fubini-Study Kähler structure of $\mathbb{C}\mathrm{P}^3$. We use ω to denote the Fubini-Study Kähler 2-form to distinguish it from J. These two $SU(3)$-structures are associated to different realizations of $\mathbb{C}\mathrm{P}^3$ as coset spaces. The first is $\mathbb{C}\mathrm{P}^3 = \frac{SU(4)}{U(3)}$, which is a symmetric space and the complex structure of $SU(4)$ gives the Fubini-Study structure. The second is $\mathbb{C}\mathrm{P}^3 = \frac{Sp(2)}{S(U(2)\times U(1))}$, which is not manifestly symmetric but homogeneous. Therefore, we can use the latter even after we change the relative sizes of the base S^4 and the fiber S^2. This is the reason why we use the second structure to build an ansatz for the 4-form field.

Left-invariant 2-forms and 3-forms are spanned by J, W_2 and $\mathrm{Re}\Omega, \mathrm{Im}\Omega$ respectively. Therefore, the most general ansatz for the 4-form respecting the symmetries is

$$\begin{aligned} F_4 = {}& \xi_1(r)J\wedge J + \xi_2(r)J\wedge e^{56} + d(\xi_3(r)\mathrm{Im}\Omega) \\ &+ d(\xi_4(r)\mathrm{Re}\Omega) + \xi_5(r)e^{8,9,10,11}. \end{aligned} \tag{12}$$

With the ansatz for the metric (8) and the 4-form field strength (12), we are ready to impose the 11-dimensional supergravity equations of motion (1). The second equation in (1), namely the Maxwell equation for the 4-form, can be solved by

$$\begin{aligned} &\xi_1(r) = \frac{C_1}{g(r)^2}, \qquad \xi_2(r) = -\frac{2C_1}{g(r)^2} + \frac{C_2}{g(r)h(r)}, \\ &\xi_3(r) = 0, \qquad \xi_4(r) = -\frac{3\sqrt{2}\xi(r)}{g(r)h(r)^{1/2}}, \qquad \xi_5(r) = 0, \end{aligned} \tag{13}$$

where the function $\xi(r)$ satisfies the differential equation,

$$\xi'' + \frac{2f'\xi'}{f} - \frac{4h(\xi - \frac{3}{2})}{g^2} - \frac{2\xi}{h} = 0. \tag{14}$$

From now on, we will set the dimensionful constant in (2) to be $c = \sqrt{2}$. In order for the 4-form (12) to converge to (2) as $r \to \infty$, one must impose $C_1 = 9\sqrt{2}$, $C_2 = 0$ and $\xi(\infty) = 1$. Thus, the 4-form can be expressed as

$$F_4 = \frac{9\sqrt{2}}{g(r)^2} J \wedge J - \frac{18\sqrt{2}}{g(r)^2} J \wedge e^{56} - d\left(\frac{3\sqrt{2}\xi(r)}{g(r)h(r)^{1/2}} \mathrm{Re}\Omega\right). \tag{15}$$

The next step is to express the first equation in (1), namely the Einstein equations, in our ansatz as ξ'^2

$$\begin{aligned}
-\frac{g''}{2g} - \frac{f'g'}{fg} - \frac{g'h'}{2gh} - \frac{g'^2}{2g^2} - \frac{\xi'^2}{24g^2h} - \frac{\xi^2}{12g^2h^2} - \frac{h}{g^2} - \frac{2(\xi-\frac{3}{2})^2}{3g^4} + \frac{3}{g} = 0,\\
-\frac{h''}{2h} - \frac{f'h'}{fh} - \frac{g'h'}{gh} - \frac{\xi'^2}{24g^2h} - \frac{\xi^2}{3g^2h^2} + \frac{h}{g^2} + \frac{(\xi-\frac{3}{2})^2}{3g^4} + \frac{1}{h} = 0,\\
-\frac{f''}{2f} - \frac{f'g'}{fg} - \frac{f'h'}{2fh} - \frac{f'^2}{2f^2} + \frac{\xi'^2}{12g^2h} + \frac{\xi^2}{6g^2h^2} + \frac{(\xi-\frac{3}{2})^2}{3g^4} + \frac{3}{f} = 0,\\
\frac{8f'g'}{fg} + \frac{4f'h'}{fh} + \frac{3f'^2}{f^2} + \frac{h'^2}{2h^2} + \frac{4g'h'}{gh} + \frac{3g'^2}{g^2} - \frac{\xi'^2}{4g^2h} + \frac{\xi^2}{2g^2h^2} + \frac{2h}{g^2} + \frac{(\xi-\frac{3}{2})^2}{g^4} - \frac{12}{g} - \frac{2}{h} - \frac{12}{f} = 0,\\
\xi'' + \frac{2f'\xi'}{f} - \frac{4h(\xi-\frac{3}{2})}{g^2} - \frac{2\xi}{h} = 0.
\end{aligned} \tag{16}$$

For our reference below, we added the Maxwell equation for the 4-form (14) in the end. There are four independent functions, four Einstein equations and one Maxwell equation. Due to the Bianchi identities, only three out of four Einstein equations are independent.

As a consistency check, we can easily verify that the Euclidean version of $\mathrm{AdS}_5 \times \mathbb{C}\mathrm{P}^3$,

$$f(r) = \sinh^2 r, \qquad h(r) = \frac{1}{2}, \qquad g(r) = \frac{1}{2}, \qquad \xi(r) = 1, \tag{17}$$

solves these equations. There is another simple solution,

$$f(r) = \frac{4}{3}\left(\frac{2}{3}\right)^{2/3} \sinh\left(\frac{1}{2}\left(\frac{3}{2}\right)^{5/6}\sqrt{2}r\right)^2, \quad h(r) = \left(\frac{2}{3}\right)^{2/3},$$
$$g(r) = \frac{1}{2^{1/3}3^{2/3}} \quad \xi(r) = \frac{4}{3}, \tag{18}$$

which is a stretched $\mathbb{C}\mathrm{P}^3$ solution [25]. One can see that $h(r) = 2g(r)$; i.e., $\mathbb{C}\mathrm{P}^3$ is stretched along its fiber.

III. BOUNDARY CONDITIONS

In this section, we will study boundary conditions to instanton solutions at the infinity of AdS and at the bubble of nothing.

For $r \to \infty$, the solution should approach the vacuum $\mathrm{AdS}_5 \times \mathbb{C}\mathrm{P}^3$, and we can linearize (16). In this set of equations, three are second order differential equations for g, h, ξ and one is first order for f (modulo the redundancy by the Bianchi identities). We should also note that there is translational invariance in r, which is the residual symmetry in our gauge (8). Therefore, there are six linearly independent modes, and they are $e^{2(\pm\sqrt{7}-1)r}$, $e^{2(\pm\sqrt{10}-1)r}$, e^{-2r}, and $r \cdot e^{-2r}$. Among them, three are normalizable and three are non-normalizable. Note that e^{-2r} and $r \cdot e^{-2r}$ are at the Breitenlohner-Freedman bound. Conformal invariance on the boundary requires the $r \cdot e^{-2r}$ mode to vanish [26]. This condition also guarantees that the instanton action is finite, as we will see in Sec. VI. For now, we only set the two diverging modes, $e^{2(\sqrt{7}-1)r}$ and $e^{2(\sqrt{10}-1)r}$, to vanish at $r = \infty$. We will keep the $r \cdot e^{-2r}$ mode to be adjustable in the next couple of sections and demand it to vanish in Sec. VI.

Let us turn our attention to boundary conditions at the bubble of nothing. In order for the S^2 fiber to shrink to zero size, the 4-form flux should not have components on the S^2; otherwise the flux conservation would prevent it from collapsing. Thus, $\xi(r)$ in (15) must be chosen in such a way that F_4 is proportional to the volume form of the base S^4. For this purpose, it is convenient to rewrite (15) as

$$\frac{1}{3\sqrt{2}} F_4 = \frac{4(\frac{3}{2} - \xi(r))}{g(r)^2} e^{1234} - \frac{2\xi(r)}{g(r)h(r)} J \wedge e^{56} + \frac{\xi'(r)}{g(r)\sqrt{h(r)}} \mathrm{Re}\Omega \wedge dr. \tag{19}$$

Note that the second and third terms in the right-hand side have $h(r)$ and $\sqrt{h(r)}$ in the denominators, which should vanish at the bubble. However, e^{56} and $\mathrm{Re}\Omega$ also go to zero since they have the factors $h(r)$ and $\sqrt{h(r)}$ respectively. Therefore these second and third terms vanish if we set

$\xi = 0$ and $\xi' = 0$, and F_4 becomes proportional to the volume form e^{1234} on the base.

Suppose the S^2 fiber becomes zero size at $r = r_0$. This means we set $h(r_0) = 0$. We also require $\xi(r_0) = \xi'(r_0) = 0$ due to our analysis in the previous paragraph. Combining these boundary conditions with the equations of motion (16), we find

$$\begin{aligned} \xi(r) &= \xi_0 (r - r_0)^2 + O((r - r_0)^4), \\ f(r) &= f_0 + O((r - r_0)^2), \\ g(r) &= g_0 + O((r - r_0)^2), \\ h(r) &= (r - r_0)^2 + O((r - r_0)^4). \end{aligned} \tag{20}$$

In order for the geometry to terminate smoothly, we need $h(r) = (r - r_0)^2 + \cdots$ with the coefficient 1 in the leading term. This condition turns out to be implied by the Einstein equations. This is in contrast to the case of Witten's instanton, for which an analogue of $h(r) = (r - r_0)^2 + \cdots$ has to be imposed as an additional boundary condition.

Thus, we find that there are three parameters f_0, g_0 and ξ_0 at the bubble. As we will see below, they can be fixed by demanding the three non-normalizable modes, $e^{2(\sqrt{7}-1)r}$, $e^{2(\sqrt{10}-1)r}$, and $r \cdot e^{-2r}$, to vanish at infinity. The location r_0 of the bubble is fixed by demanding $f(r)/\sinh^2 r \to 1$ for $r \to \infty$.

IV. ALGEBRAIC RELATIONS

Interestingly, both the equations of motion and the boundary conditions defined in the last two sections are compatible with two simple algebraic relations between the three functions $g(r)$, $h(r)$, $\xi(r)$. In fact, if we set,

$$\begin{aligned} g(r) &= G(h(r)), \\ \xi(r) &= S(h(r)), \end{aligned} \tag{21}$$

and substitute them into the equations of motion (16), we find a couple of equations that are independent of $f(r)$:

$$\begin{aligned} &\frac{3}{G} - \frac{h}{G^2} - \frac{S^2}{12h^2G^2} - \frac{2(S-\frac{3}{2})^2}{3G^4} \\ &\quad + \dot{G}\left(\frac{S^2}{3G^3h} - \frac{1}{G} - \frac{h^2}{G^3} - \frac{h(S-\frac{3}{2})^2}{3G^5}\right) \\ &\quad = -h'(r)^2\left(\frac{\dot{G}^2}{2G^2} - \frac{\ddot{G}}{2G} + \frac{\dot{G}\dot{S}^2}{24G^3} - \frac{\dot{G}}{2hG} - \frac{\dot{S}^2}{24hG^2}\right) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &-\frac{4h(S-\frac{3}{2})}{G^2} - \frac{2S}{h} + \dot{S}\left(2 + \frac{2h^2}{G^2} - \frac{2S^2}{3hG^2} + \frac{2h(S-\frac{3}{2})^2}{3G^4}\right) \\ &\quad = -h'(r)^2\left(\ddot{S} - \frac{2\dot{G}\dot{S}}{G} - \frac{\dot{S}^3}{12G^2}\right), \end{aligned} \tag{23}$$

where $\dot{} = d/dh$. Demanding that these equations hold independently of $h'(r)$, we obtain four differential equations on $G(h)$ and $S(h)$ with respect to h. Remarkably, these four equations can be solved algebraically by imposing the simple relations,

$$\begin{aligned} \xi &= 3 - \sqrt{2g}(3g + h), \\ 1 &= \sqrt{2g}(h + g). \end{aligned} \tag{24}$$

These algebraic relations are also consistent with the boundary conditions at $r = r_0$ and ∞: setting $h(r) = (r - r_0)^2$ gives $g(r) = 2^{-1/3}$ and $\xi(r) = 2^{4/3}(r - r_0)^2$ as expected at the bubble, and $h = 1/2$ at $r = \infty$ gives $g = 1/2$ and $\xi = 1$ as required for $AdS_5 \times \mathbb{C}P^3$. We found these relations experimentally, and it would be interesting to find their deeper origins. In the following, we will use them to numerically integrate the rest of the equations of motion.

V. NUMERICAL SOLUTION

With use of the algebraic relations (24), the equations of motion (16) collapse to the following three equations for the two functions $f(r)$ and $g(r)$,

$$\begin{aligned} -\frac{g''}{2g} - \frac{g'^2}{4g^2} - \frac{f'g'}{fg} - \frac{1}{6g^4}(1 - 5\sqrt{2}g^{3/2} + 12g^3) &= 0, \\ -\frac{f''}{2f} - \frac{f'^2}{2f^2} - \frac{3f'g'}{4fg}\frac{1-(2g)^{3/2}}{1-\sqrt{2}g^{3/2}} + \frac{3g'^2}{2g^2}\frac{\sqrt{2}g^{3/2}}{1-\sqrt{2}g^{3/2}} + \frac{1}{12g^4}(1 - 8\sqrt{2}g^{3/2} + 48g^3) + \frac{3}{f} &= 0, \\ \frac{3f'^2}{f^2} + \frac{6f'g'}{fg}\frac{1-(2g)^{3/2}}{1-\sqrt{2}g^{3/2}} + \frac{g'^2}{8g^2}\frac{(9 - 96\sqrt{2}g^{3/2} + 192g^3)}{(1-\sqrt{2}g^{3/2})^2} + \frac{1}{4g^4}\frac{1-5\sqrt{2}g^{3/2}}{1-\sqrt{2}g^{3/2}} - \frac{12}{f} &= 0. \end{aligned} \tag{25}$$

Only two of these three equations are independent. Eliminating $f(r)$, one finds the following equation for $g(r)$,

$$\begin{aligned}&(1-\sqrt{2}g^{3/2})^2 g^6(4g'''g'-5g''^2)-\frac{3}{4}(1+14\sqrt{2}g^{3/2}-42g^3)g^4g'^4\\&-(5-16\sqrt{2}g^{3/2}+22g^3)g^5g'^2g''-2(1-3\sqrt{2}g^{3/2})(1-2\sqrt{2}g^{3/2})(1-\sqrt{2}g^{3/2})^2g^3g''\\&-(1-\sqrt{2}g^{3/2})^2(4-9\sqrt{2}g^{3/2}-12g^3)g^2g'^2\\&-\frac{1}{9}(1-3\sqrt{2}g^{3/2})^2(1-2\sqrt{2}g^{3/2})^2(1-\sqrt{2}g^{3/2})^2=0.\end{aligned} \tag{26}$$

This is a third order differential equation for $g(r)$. Since it does not depend on r explicitly, one can lower its order by 1. The numerical integration of this equation shows the desired behavior; i.e., $g(r)$ goes from $2^{-1/3}$ to $1/2$ as r goes from r_0 to ∞.

We would like to comment on two important features of our numerical solutions:

(1) The equation for $g(r)$ is singular at the bubble, and the coefficient of g''' vanishes with $g(r_0)=2^{-1/3}$. Therefore, instead of numerically integrating the equation with the initial value of $g(r_0)=2^{-1/3}$, we computed the first few terms in the Taylor expansion analytically and matched them to a numerical solution.

(2) In this section, we are not requiring the $r\cdot e^{-2r}$ mode to vanish at $r\to\infty$. Thus, we are left with one free parameter f_0, which is the size of the bubble.

We presented the typical behavior of the solution in Fig. 1, where we set $f_0=1$. The horizontal axis is $r-r_0$, where r_0 is fixed by demanding $f(r)/\sinh^2 r\to 1$ as $r\to\infty$. One can see that the solution exhibits the desired behavior: $h(r)\to 1/2$, $g(r)\to 1/2$, $\xi(r)\to 0$ and $f'(r)/f(r)\to 2$ as $r\to\infty$.

VI. INSTANTON ACTION

We have shown that there is a family of solutions parametrized by the size f_0 of the bubble, which approach $AdS_5\times\mathbb{C}P^3$ at infinity. However, there is one more non-normalizable mode $r\cdot e^{-2r}$ we need to fix. In this section, we show that one can turn off this mode by adjusting f_0. This also makes the instanton action finite.

For a general value of the parameter f_0, the solution at infinity behaves as

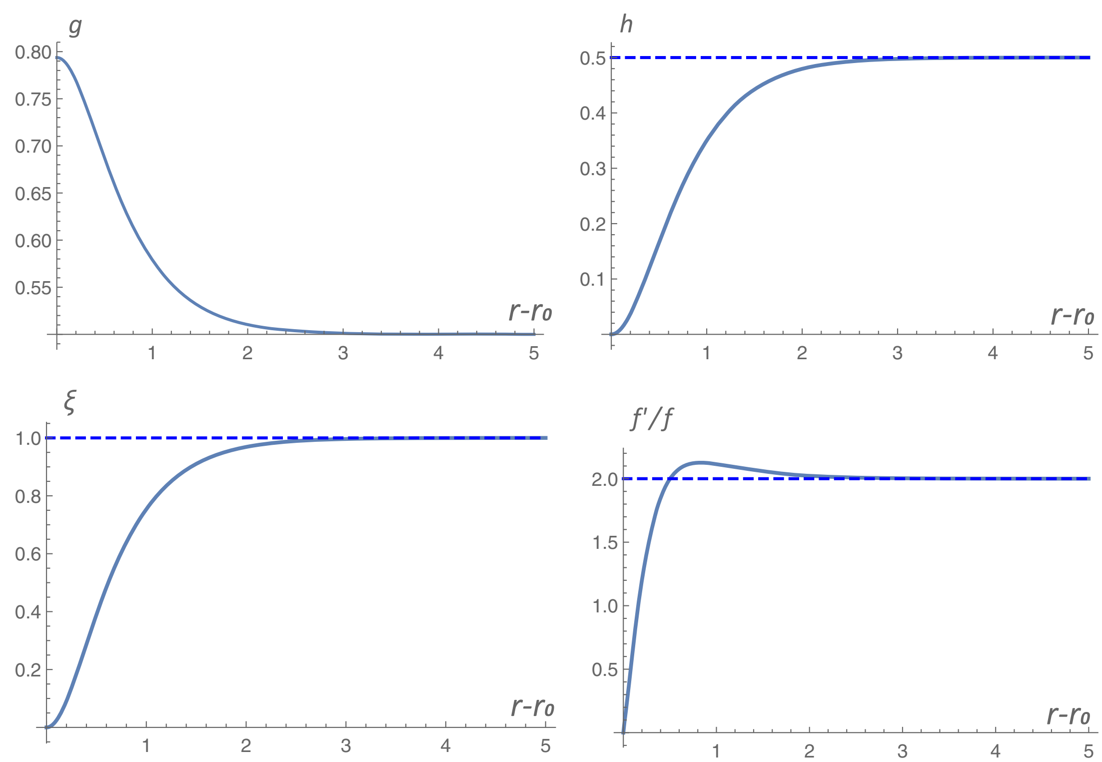

FIG. 1. Numerical solution to Eq. (26).

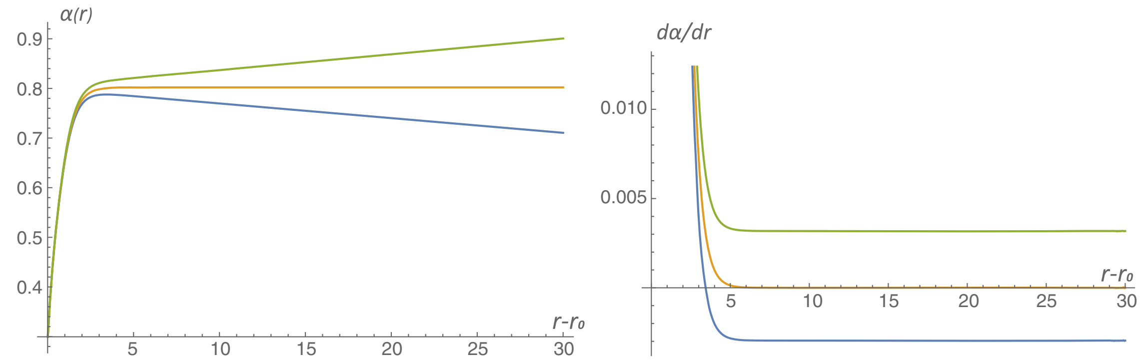

FIG. 2. The graphs of $\alpha(r)$ and its derivative. The horizontal axes are $r - r_0$. The values of f_0 from top to bottom are 0.63, 0.62, 0.61 respectively.

$$g(r) = \frac{1}{2} + (a(f_0) + b(f_0)r)e^{-2r} + \cdots. \quad (27)$$

Numerically, $b(f_0)$ vanishes at $f_0 = 0.6203025\ldots$ To show that $b(f_0)$ can be set exactly equal to zero by adjusting f_0, we present Fig. 2 for $\alpha(r) = (g(r) - \frac{1}{2})e^{2r}$. Since $\alpha(r) \sim a(f_0) + b(f_0)r$ for $r \to \infty$, we see that $b(f_0)$ changes its sign near 0.62. Therefore, there must be f_0 near 0.62 such that $b(f_0) = 0$.

The action for 11-dimensional supergravity takes the form

$$S = \int_{M_{11}} \sqrt{\det G}\left(\frac{1}{4}R - \frac{1}{48}F_{MNPQ}F^{MNPQ} + \cdots\right), \quad (28)$$

where we have ignored the Chern-Simons terms and fermions, which are irrelevant to our discussion. Using the supergravity equations (1), the 4-form kinetic term is related to the Einstein term, $F_{MNPQ}F^{MNPQ} = 36R$. Therefore, the instanton action reduces to

$$S = -\frac{1}{2}\int_{M_{11}} \sqrt{\det G}R. \quad (29)$$

It is straightforward to see that, with the $r \cdot e^{-2r}$ mode removed, the instanton action is finite and positive after subtracting the value for the vacuum AdS.

We conclude that $AdS_5 \times \mathbb{C}P^3$ is unstable due to the finite action instanton.

VII. INSTABILITY OF $AdS_5 \times \frac{SU(3)}{U(1)\times U(1)}$

In this section, we will show that the $AdS_5 \times \frac{SU(3)}{U(1)\times U(1)}$ model has an instanton that mediates its decay. It is not known whether this solution is perturbatively stable or not. In either case, the existence of the bubble of nothing solution shown here means that it is unstable.

To construct the solution, let us first review the geometry of $\frac{SU(3)}{U(1)\times U(1)}$. It can be viewed as a flag manifold $\mathbb{F}(1,2,3)$ or a twistor space over $\mathbb{C}P^2$. It admits Kähler structure with an Einstein metric and therefore is a solution of the supergravity equations of motion. It is also an S^2 fibration over the $\mathbb{C}P^2$ base. The last fact is best understood from the coset point of view. One can choose the $SU(2)$ subgroup in $SU(3)$ and decompose $SU(3) = \frac{SU(3)}{SU(2)} \times SU(2)$. The former term in the product is homogeneous space S^5 and the latter is S^3. Therefore, $SU(3)$ is an S^3 fibration over S^5. The two $U(1)$ subgroups in the denominator of the coset $\frac{SU(3)}{U(1)\times U(1)}$ turn each sphere into a complex projective space resulting in $S^2 \hookrightarrow \frac{SU(3)}{U(1)\times U(1)} \to \mathbb{C}P^2$ fibration.

It worth mentioning that both $\mathbb{C}P^3$ and $\frac{SU(3)}{U(1)\times U(1)}$ are twistor spaces of S^4 and $\mathbb{C}P^3$ [22]. This fact seems to be the main reason of the similarity between the collapsing solutions of the models. Choosing the vielbein as [20,27]

$$\begin{aligned}
e_1 &= \sqrt{2g(r)}d\mu,\\
e_2 &= \sqrt{\frac{g(r)}{2}}\sin\mu\Sigma_1,\\
e_3 &= \sqrt{\frac{g(r)}{2}}\sin\mu\Sigma_2,\\
e_4 &= \sqrt{\frac{g(r)}{2}}\sin\mu\cos\mu\Sigma_3,\\
e_5 &= \sqrt{h(r)}(d\theta - A_1\sin\phi + A_2\cos\phi),\\
e_6 &= \sqrt{h(r)}\sin\theta(d\phi - \cot\theta(A_1\cos\phi + A_2\sin\phi) + A_3),
\end{aligned} \quad (30)$$

with

$$\begin{aligned}
\Sigma_1 &= \cos\gamma d\alpha + \sin\gamma d\beta,\\
\Sigma_2 &= -\sin\gamma d\alpha + \cos\gamma d\beta,\\
\Sigma_3 &= d\gamma + \cos\alpha d\beta,\\
A_1 &= \cos\mu\Sigma_1,\\
A_2 &= \cos\mu\Sigma_2,\\
A_3 &= \frac{1}{2}(1+\cos^2\mu)\Sigma_3,
\end{aligned} \quad (31)$$

we see that all the formulas and results become exactly the same as in the $\mathbb{C}P^3$ case. Namely, $\frac{SU(3)}{U(1)\times U(1)}$ has the $SU(3)$-structure defined by (9) and has torsion classes (11). Since the $SU(3)$-structure is the same, the ansatz for the flux will have the same solution (13). Finally and most importantly, the equations of motion of supergravity take exactly the same form (16). The last fact makes all the results of the previous sections applicable to this case. The only thing which is different is the expression of the vielbein in terms of the coordinates.

One might wonder why the equations are exactly the same. It follows from the fact that we constructed the ansatz which respects all the symmetries of the model with squashed fiber. The bases of the compact manifold in both cases are Einstein manifolds and therefore their contribution to the Einstein equations will enter in a similar manner. Besides, the $SU(3)$-structure is rooted in the twistor origin of both spaces and it is used to build the ansatz for the flux. Because the spaces have the same origin, the ansatz gives the same result in both cases.

VIII. DISCUSSION

We would like to end this paper by explaining how our solution evades the issue raised in [11]. Suppose we try to collapse a d-dimensional sphere in the internal space supported by a flux, at the bubble located at $r = r_0$. The flux gives a contribution to the Einstein equations proportional to $1/h(r)^d$, where $h(r)$ is the square radius of the sphere, while the contribution from curvature is proportional to $1/(r-r_0)^2$. Taking into account that $h(r) \sim (r-r_0)^2$ for the smoothness, it was argued in [11] that the only possible way to make these two terms of the same order is to set $d = 1$, i.e., a circle. However, this does not apply to our case since the amount of flux on the sphere can vary.

It is instructive to see it explicitly in the Einstein equation (16) for $h(r)$,

$$-\frac{h''}{2h} - \frac{f'h'}{fh} - \frac{g'h'}{gh} - \frac{\xi'^2}{24g^2h} + \frac{h}{g^2} + \frac{(\xi-\frac{3}{2})^2}{3g^4} + \frac{1}{h} - \frac{\xi^2}{3g^2h^2} = 0. \tag{32}$$

One can see that the curvature contribution (first term) is of order $1/(r-r_0)^2$, while the flux (last term) is of order $\xi(r)^2/h(r)^2$ [flux for the S^4 is proportional to the fourth power of $g(r)$ as it should be]. This is consistent with the estimate of [11] for $d = 2$. However, in our solution, these two terms can balance each other since $\xi(r) \to 0$ when $h(r) \to 0$. In this way, the flux evaporates from the S^2 fiber, and the Einstein equations can be satisfied.

Another possibility to deal with flux conservation is to introduce a domain wall at the bubble to absorb the flux. This idea was used in [10] to collapse the supersymmetry breaking S^1 in the $AdS_5 \times S^5/\Gamma$ geometry of [7]. More recently, instanton solutions with S^2 collapsing have been constructed in some models in six dimensions. Blanco-Pillardo *et al.* [28] considered the Einstein gravity in six dimensions coupled to $SU(2)$ Yang-Mills gauge field and an adjoint Higgs field with a specific potential to break the gauge group to $U(1)$ and found a smooth solution of this type. Brown and Dahlen [29] considered the Einstein-Maxwell system without the Higgs and added a domain wall as a source. It would be interesting to realize such solutions in a low energy effective theory of M/string theory in a controlled approximation.

According to [17], $AdS_5 \times \mathbb{C}P^1 \times \mathbb{C}P^2$ and $AdS_5 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ are not stable perturbatively. Thus, we do not need instantons for these geometries to be consistent with the conjecture of [12]. In fact, our ansatz is not applicable to them since the configurations are too restrictive for the fluxes to slip off. In our solution, it is important that $\mathbb{C}P^3$ has the nontrivial fibration structure and not a direct product since this allows our nontrivial solution to the 4-form equations. A similar argument applies to the Freud-Rubin-type compactification (when the flux is proportional to the volume form of the AdS) and the compactifications where the flux is proportional to the volume form of the compact space (unless one can turn on another lower dimensional form).

Finally, we want to mention that it is possible that our solution is related to resolved M-theory conifold solutions with G_2-holonomy [30]. They are Ricci-flat seven-dimensional manifolds which have conic structure and its metrics reads

$$ds^2 = \frac{1}{1-\frac{1}{r^4}}dr^2 + \frac{1}{4}r^2\left(1-\frac{1}{r^4}\right)|d_A u|^2 + \frac{r^2}{2}ds^2_{M_4}, \tag{33}$$

where M_4 is either S^4 or $\mathbb{C}P^2$ and $|d_A u|^2$ is the metrics on the S^2 fiber which are exactly the same as in the present paper for $\mathbb{C}P^3$ and $\frac{SU(3)}{U(1)\times U(1)}$ respectively. Moreover, the S^2 fiber collapses at finite $r = 1$, while the M_4 radius stays finite. Unfortunately, the radii grow linearly at infinity. It may be possible to construct a desirable solution by multiplying this geometry with the flat $\mathbb{R}^4$ and by adding some flux along the conifold in order to change the behavior at infinity from the linear growth to the constant. An idea along this line may allow us to generalize our solutions further.

ACKNOWLEDGMENTS

The authors would like to thank G. Horowitz, S. Gukov, I. Klebanov, and H. Reall for discussions. This research is supported in part by the U.S. Department of Energy Grant No. DE-SC0011632. H. O. is also supported in part by the Simons Investigator Award, by the World Premier International Research Center Initiative, Ministry of Education, Culture, Sports, Science and Technology (MEXT), Japan, by the Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Scientific Research No. C-26400240, and by the JSPS Grant-in-Aid for Scientific Research on Innovative Areas No. 15H05895.

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