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Eigenvalue dynamics for multimatrix models

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By performing explicit computations of correlation functions, we find evidence that there is a sector of the two matrix model defined by the $SU(2)$ sector of $\mathcal{N}=4$ super Yang-Mills theory that can be reduced to eigenvalue dynamics. There is an interesting generalization of the usual Van der Monde determinant that plays a role. The observables we study are the Bogomol'nyi-Prasad-Sommerfield operators of the $SU(2)$ sector and include traces of products of both matrices, which are genuine multimatrix observables. These operators are associated with supergravity solutions of string theory.

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I. MOTIVATION

The large N expansion continues to be a promising approach toward the strong coupling dynamics of quantum field theories. For example, 't Hooft's proposal that the large N expansions of Yang-Mills theories are equivalent to the usual perturbation expansion in terms of topologies of world sheets in string theory [1] has been realized concretely in the AdS/CFT correspondence [2]. Besides the usual planar limit where classical operator dimensions are held fixed as we take $N\to\infty$, there are nonplanar large N limits of the theory [3] defined by considering operators with a bare dimension that is allowed to scale with N as we take $N\to\infty$. These limits are also relevant for the AdS/CFT correspondence. Indeed, operators with a dimension that scales as N include operators relevant for the description of giant graviton branes [4–6] while operators with a dimension of order N^2 include operators that correspond to new geometries in supergravity [7–9]. These convincing motivations have motivated sustained study of large N field theory. Despite this, carrying out the large N expansion for most matrix models is still beyond our current capabilities.

One class of models for which the large N expansion can be computed is the singlet sector of matrix quantum mechanics of a single Hermitian matrix [10]. We can also consider a complex matrix model as long as we restrict ourselves to potentials that are analytic in Z (summed with the dagger of this which needs to be added to get a real potential) and observables constructed out of traces of a product of Z s or out of a product of $Z^\dagger$s [11]. In these situations we can reduce the problem to eigenvalue dynamics. This is a huge reduction in degrees of freedom since we have reduced from $O(N^2)$ degrees of freedom, associated with the matrix itself, to $O(N)$ eigenvalue degrees of freedom. Studying saddle points of the original matrix action does not reproduce the large N values of observables. This is a consequence of the large number of degrees of freedom: we expect fluctuations to be suppressed by $1/N^2$ so that if N^2 variables in total are fluctuating, then we can have fluctuations of size $1/N^2\times N^2\sim 1$ which are not suppressed as $N\to\infty$. In terms of eigenvalues there are only N variables fluctuating so that fluctuations are bounded by $N\times 1/N^2\sim 1/N$ which vanishes as $N\to\infty$. Thus, classical eigenvalue dynamics captures the large N limit. For example, one can formulate the physics of the planar limit by using the density of eigenvalues as a dynamical variable. The resulting collective field theory defines a field theory that explicitly has $1/N$ as the loop expansion parameter [12,13]. It has found application both in the context of the $c=1$ string [14–16] and in descriptions of the Lin-Lunin-Maldecena (LLM) geometries [17].

Standard arguments show that eigenvalue dynamics corresponds to a familiar system: noninteracting fermions in an external potential [10]. This makes the description extremely convenient because the fermion dynamics is rather simple. This eigenvalue dynamics is also a natural description of the large N but nonplanar limits discussed above. Giant graviton branes which have expanded into the AdS_5 of the spacetime correspond to highly excited fermions or, equivalently, to single highly excited eigenvalues: the giant graviton is an eigenvalue [5,9]. Giant graviton branes which have expanded into the S^5 of the spacetime correspond to holes in the Fermi sea, and hence to collective excitations of the eigenvalues where many eigenvalues are excited [9]. Half-BPS geometries also have a natural interpretation in terms of the eigenvalue dynamics: every fermion state can be identified with a particular supergravity geometry [8,9]. The map between the two descriptions was discovered by Lin, Lunin, and Maldacena in [7]. The fermion state can be specified by stating which states in phase space are occupied by a fermion, so we can divide phase space up into occupied and unoccupied states. By requiring regularity of the corresponding supergravity solution exactly the same structure arises: the complete set of regular solutions are specified by boundary conditions obtained by dividing a certain plane into black (identified with occupied states in the fermion phase space) and white (unoccupied states) regions. See [7] for the details.

Our main goal in this paper is to ask if a similar eigenvalue description can be constructed for a two matrix model. Further, if such a construction exists, does it have a natural AdS/CFT interpretation? Work with a similar motivation but focusing on a different set of questions has appeared in [18–22]. We will consider the dynamics of two complex matrices, corresponding to the $SU(2)$ sector of $\mathcal{N}=4$ super Yang-Mills theory. Further, we consider the theory on $R\times S^3$ and expand all fields in spherical harmonics of the S^3. We will consider only the lowest s-wave components of these expansions so that the matrices are constant on the S^3. The reduction to the s-wave will be motivated below. In this way we find a matrix model quantum mechanics of two complex matrices. Expectation values are computed as follows:

$$\langle\cdots\rangle=\int[dZdZ^\dagger dYdY^\dagger]e^{-S}\cdots. \tag{1.1}$$

At first sight it appears that any attempts to reduce (1.1) to an eigenvalue description are doomed to fail: the integral in (1.1) runs over two independent complex matrices Z and Y which will almost never be simultaneously diagonalizable. However, perhaps there is a class of questions, generalizing the singlet sector of a single Hermitian matrix model, that can be studied using eigenvalue dynamics. To explore this possibility, let us review the arguments that lead to eigenvalue dynamics for a single complex matrix Z. We can use the Schur decomposition [11,23,24],

$$Z=U^\dagger DU \tag{1.2}$$

with U a unitary matrix and D an upper triangular matrix, to explicitly change variables. Since we only consider observables that depend on the eigenvalues (the diagonal elements of D) we can integrate U and the off diagonal elements of D out of the model, leaving only the eigenvalues. The result of the integrations over U and the off diagonal elements of D is a nontrivial Jacobian. Denoting the eigenvalues of Z by z_i, those of $Z^\dagger$ are given by complex conjugation, $\bar{z}_i$. The resulting Jacobian is [11]

$$J=\Delta(z)\Delta(\bar{z}), \tag{1.3}$$

where

$$\Delta(z)=\begin{vmatrix}1&1&\cdots&1\\ z_1&z_2&\cdots&z_N\\ \vdots&\vdots&\vdots&\vdots\\ z_1^{N-1}&z_2^{N-1}&\cdots&z_N^{N-1}\end{vmatrix}=\prod_{j>k}^{N}(z_j-z_k) \tag{1.4}$$

is the usual Van der Monde determinant. A standard argument now maps this into noninteracting fermion dynamics [10]. Trying to apply a very direct change of variables argument to the two matrix model problem appears difficult. There is, however, an approach which both agrees with the above noninteracting fermion dynamics and can be generalized to the two matrix model. The idea is to construct a basis of operators that diagonalizes the inner product of the free theory. The construction of an orthogonal basis, given by the Schur polynomials, was achieved in [8]. Each Schur polynomial $\chi_R(Z)$ is labeled by a Young diagram R with no more than N rows. In [8] the exact (to all orders in $1/N$) two point function of Schur polynomials was constructed. The result is

$$\langle\chi_R(Z)\chi_S(Z^\dagger)\rangle=f_R\delta_{RS}, \tag{1.5}$$

where all spacetime dependence in the correlator has been suppressed. This dependence is trivial as it is completely determined by conformal invariance. The notation f_R denotes the product of the factors of Young diagram R. Remarkably there is an immediate and direct connection to noninteracting fermions: the fermion wave function can be written as

$$\psi_R(\{z_i,\bar{z}_i\})=\chi_R(Z)\Delta(z)e^{-\frac{1}{2}\sum_i z_i\bar{z}_i}. \tag{1.6}$$

This relation can be understood as a combination of the state operator correspondence (we associate a Schur polynomial operator on R^4 with a wave function on $R\times S^3$) and the reduction to eigenvalues [which is responsible for the $\Delta(z)$ factor] [9]. In this map the number of boxes in each row of R determines the amount by which each fermion is excited. In this way, each row in the Young diagram corresponds to a fermion and hence to an eigenvalue. Having one very long row corresponds to exciting a single fermion by a large amount, which corresponds to a single large (highly excited) eigenvalue. In the dual AdS gravity, a single long row is a giant graviton brane that has expanded in the AdS_5 spacetime. Having one very long column corresponds to exciting many fermions by a single quantum, which corresponds to many eigenvalues excited by a small amount. In the dual anti–de Sitter (AdS) gravity, a single long column is a giant graviton brane that has expanded in the S^5 space.

The first questions we should tackle when approaching the two matrix problem should involve operators built using many Z fields and only a few Y fields. In this case at least a rough outline of the one matrix physics should be visible, and experience with the one matrix model will prove to be valuable.

For the case of two matrices we can again construct a basis of operators that diagonalize the free field two point function. These operators $\chi_{R,(r,s)ab}(Z,Y)$ are a generalization of the Schur polynomials, called restricted Schur polynomials [25–27]. They are labeled by three Young diagrams (R, r, s) and two multiplicity labels (a, b). For an operator constructed using n Z s and m Y s, $R \vdash n+m$, $r \vdash n$, and $s \vdash m$. The multiplicity labels distinguish between different copies of the (r, s) irreducible representation of $S_n \times S_m$ that arise when we restrict the irreducible representation R of S_{n+m} to the $S_n \times S_m$ subgroup. The two point function is

$$\langle \chi_{R,(r,s)ab}(Z,Y)\chi_{T,(t,u)cd}(Z^\dagger, Y^\dagger)\rangle = f_R \frac{\mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s}\delta_{RT}\delta_{rt}\delta_{su}\delta_{ac}\delta_{bd}, \quad (1.7)$$

where f_R was defined after (1.5) and hooks_a denotes the product of the hook lengths associated with Young diagram a. These operators do not have a definite dimension. However, they only mix weakly under the action of the dilatation operator, and they form a convenient basis in which to study the spectrum of anomalous dimensions [28]. This action has been diagonalized in a limit in which R has order 1 rows (or columns), $m \ll n$, and n is of order N. Operators of a definite dimension are labeled by graphs composed of nodes that are traversed by oriented edges [29,30]. There is one node for each row, so that each node corresponds to an eigenvalue. The directed edges start and end on the nodes. There is one edge for each Y field, and the number of oriented edges ending on a node must equal the number of oriented edges emanating from a node. See Fig. 1 for an example of a graph labeling an operator. This picture, derived in the Yang-Mills theory, has an immediate and compelling interpretation in the dual gravity: each node corresponds to a giant graviton brane, and the directed edges are open string excitations of these branes. The constraint that the number of edges ending on a node equals the number of edges emanating from the node is simply encoding the Gauss law on the brane world volume, which is topologically an S^3. For this reason the graphs labeling the operators are called Gauss graphs. If we are to obtain a system of noninteracting eigenvalues, we should only consider Gauss graphs that have no directed edges stretching between nodes. See Fig. 2 for an example. In fact, these

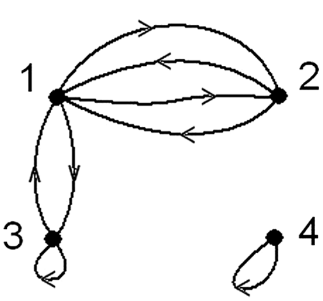

FIG. 1. An example of a graph labeling an operator with a definite scaling dimension. Each node corresponds to an eigenvalue. Edges connect the different nodes so that the eigenvalues are interacting.

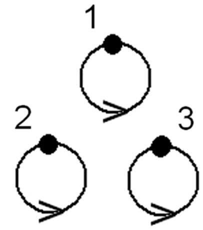

FIG. 2. An example of a graph labeling a BPS operator. Each node corresponds to an eigenvalue. There are no edges connecting the different nodes so that these eigenvalues are not interacting.

all correspond to BPS operators. We thus arrive at a very concrete proposal:

If there is a free fermion description arising from the eigenvalue dynamics of the two matrix model, it will describe the BPS operators of the $SU(2)$ sector.

The BPS operators are associated with supergravity solutions of string theory. Indeed, the only one-particle states saturating the BPS bound in gravity are associated with massless particles and lie in the supergravity multiplet. Thus, eigenvalue dynamics will reproduce the supergravity dynamics of the gravity dual.

The BPS operators are all constructed from the s wave of the spherical harmonic expansion on S^3 [9]. This is our motivation for only considering operators constructed using the s wave of the fields Y and Z. One further comment is that it is usually not consistent to simply restrict to a subset of the dynamical degrees of freedom. Indeed, this is possible only if the subset of degrees of freedom dynamically decouples from the rest of the theory. In the case that we are considering this is guaranteed to be the case, in the large N limit, because the Chan-Paton indices of the directed edges are frozen at large N [29].

We should mention that eigenvalue dynamics as dual to supergravity has also been advocated by Berenstein and his collaborators [31–37]. See also [38–41] for related studies. Using a combination of numerical and physical arguments, which are rather different from the route we have followed, compelling evidence for this proposal has already been found. The basic idea is that at strong coupling the commutator squared term in the action forces the Higgs fields to commute and hence, at strong coupling, the Higgs fields of the theory should be simultaneously diagonalizable. In this case, an eigenvalue description is possible. Notice that our argument is a weak coupling large N argument, based on diagonalization of the one loop dilatation operator, that comes to precisely the same conclusion. In this article we will make some exact analytic statements that agree with and, in our opinion, refine some of the physical picture of the above studies. For example, we will start to make precise statements about what eigenvalue dynamics does and does not correctly reproduce.

II. EIGENVALUE DYNAMICS FOR $AdS_5 \times S^5$

To motivate our proposal for eigenvalue dynamics, we will review the $\frac{1}{2}$-BPS sector stressing the logic that we will subsequently use. The way in which a direct change of variables is used to derive the eigenvalue dynamics can be motivated by considering correlation functions of arbitrary observables—that are functions only of the eigenvalues. Because we are considering BPS operators, correlators computed in the free field theory agree with the same computations at strong coupling [42], so that we now work in the free field theory. Performing the change of variables we find

$$\begin{aligned}\langle \cdots \rangle &= \int [dZdZ^\dagger] e^{-\mathrm{Tr}ZZ^\dagger} \cdots \\ &= \int \prod_{i=1}^N dz_i d\bar{z}_i e^{-\sum_k z_k \bar{z}_k} \Delta(z)\Delta(\bar{z}) \cdots \\ &= \int \prod_{i=1}^N dz_i d\bar{z}_i |\psi_{\mathrm{gs}}(\{z_i,\bar{z}_i\})|^2 \cdots,\end{aligned}$$

where the ground-state wave function is given by

$$\psi_{\mathrm{gs}}(\{z_i,\bar{z}_i\}) = \Delta(z) e^{-\frac{1}{2}\sum_i z_i \bar{z}_i}. \qquad (2.1)$$

We will shortly qualify the adjective "ground state." Under the state-operator correspondence, this wave function is the state corresponding to the identity operator. The above transformation is equivalent to the identification

$$[dZ] e^{-\frac{1}{2}\mathrm{Tr}(ZZ^\dagger)} \leftrightarrow \prod_{i=1}^N dz_i \psi_{\mathrm{gs}}(\{z_i,\bar{z}_i\}). \qquad (2.2)$$

The role of each of the elements of the wave function is now clear:

(1) Under the state operator correspondence, dimensions of operators map to energies of states. The dimensions of BPS operators are not corrected; i.e. they take their free field values. This implies an evenly spaced spectrum and hence a harmonic oscillator wave function. This explains the $e^{-\frac{1}{2}\sum_i z_i \bar{z}_i}$ factor. It also suggests that the wave function will be a polynomial times this Gaussian factor.

(2) There is a gauge symmetry $Z \to UZU^\dagger$ that is able to permute the eigenvalues. Consequently we are discussing identical particles. Two matrices drawn at random from the complex Gaussian ensemble will not have degenerate eigenvalues, so we choose the particles to be fermions. This matches the fact that the wave function is a Slater determinant.

The wave function (2.1) satisfies these properties. Further, if we require that the wave function is a polynomial in the eigenvalues z_i times the exponential $e^{-\frac{1}{2}\sum_i z_i \bar{z}_i}$, then (2.1) is the state of lowest energy (we did not write down a Hamiltonian, but any other wave function has more nodes and hence a higher energy) so it deserves to be called the ground state. The wave function (2.1) is the state corresponding to the $AdS_5 \times S^5$ spacetime in the $\frac{1}{2}$-BPS sector.

The above discussion can be generalized to write down a wave function corresponding to the $AdS_5 \times S^5$ spacetime in the $SU(2)$ sector. Equation (2.2) is generalized to

$$[dZdY] e^{-\frac{1}{2}\mathrm{Tr}(ZZ^\dagger)-\frac{1}{2}\mathrm{Tr}(YY^\dagger)} \to \prod_{i=1}^N dz_i dy_i \Psi_{\mathrm{gs}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\}). \qquad (2.3)$$

The wave function must obey the following properties:

(1) Our wave functions again describe states that correspond to BPS operators. The dimensions of the BPS operators take their free field values, implying an evenly spaced spectrum and hence a harmonic oscillator wave function. This suggests the wave function is a polynomial times the Gaussian factor $e^{-\frac{1}{2}\sum_i z_i \bar{z}_i - \frac{1}{2}\sum_i y_i \bar{y}_i}$ factor.

(2) There is a gauge symmetry $Z \to UZU^\dagger$ and $Y \to UYU^\dagger$ that is able to permute the eigenvalues. Consequently we are discussing N identical particles. Matrices drawn at random will not have degenerate eigenvalues, so we choose the particles to be fermions. Thus we expect the wave function is a Slater determinant.

We are working within the AdS/CFT correspondence. Our main goal is to understand how geometry in the dual gravity theory emerges. We expect a smooth geometry with small curvature emerges in the strongly coupled limit of the CFT. Correlators of operators belonging to the BPS sector of $\mathcal{N}=4$ SYM take their free field values even in the strong coupling limit [42]. Thus, although we study the free field theory, our intuition should come from the dual gravity. In the free field theory the eigenvalue density is expected to have a $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry [as in (3.5)]. This follows simply by integrating over the noneigenvalue degrees of freedom in the Gaussian two matrix model. The strong coupling answer, where we again integrate over the noneigenvalue degrees of freedom of the two matrices, but now in the strong coupling limit, will not match this free matrix model. It will match the dual gravity. In the $AdS_5 \times S^5$ geometry we have an $SO(6)$ isometry of the S^5, which acts in the dual field theory as $SO(6)$ rotations of the six adjoint scalars of $\mathcal{N}=4$ SYM [see (6.1)]. These are $\mathcal{R}$ symmetry rotations. When we restrict to the eigenvalues of Z and Y, we reduce this to an $SO(4)$ symmetry. Since the geometry should emerge from the eigenvalues [31], this symmetry should manifest in the single eigenvalue probability density. This leads us to the last property we impose on our theory:

(3) The probability density associated with a single particle $\rho_{gs}(z_1,\bar{z}_1,y_1,\bar{y}_1)$ must have an $SO(4)$ symmetry; i.e. it should be a function of $|z_i|^2+|y_i|^2$.

The single particle probability density referred to in point 3 above is given, for any state $\Psi(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})$ as usual, by

$$\rho(z_1,\bar{z}_1,y_1,\bar{y}_1)=\int\prod_{i=2}^{N}dz_i d\bar{z}_i dy_i d\bar{y}_i|\Psi(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2. \tag{2.4}$$

There is a good reason why the single particle probability density is an interesting quantity to look at: at short distances the eigenvalues feel a repulsion from the Slater determinant, which vanishes when two eigenvalues are equal. At long distances the confining harmonic oscillator potential dominates, ensuring the eigenvalues are clumped together in some finite region and do not wander off to infinity. In the end we expect that at large N the locus where the eigenvalues lie defines a specific surface, generalizing the idea of a density of eigenvalues for the single matrix model. This large N surface is captured by $\rho(z_1,\bar{z}_1,y_1,\bar{y}_1)$. We will make this connection more explicit in a later section.

There appears to be a unique wave function singled out by the above requirements. It is given by

$$\Psi_{gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})=\mathcal{N}\Delta(z,y)e^{-\frac{1}{2}\sum_k z_k\bar{z}_k-\frac{1}{2}\sum_k y_k\bar{y}_k}, \tag{2.5}$$

where

$$\Delta(z,y)=\begin{vmatrix} y_1^{N-1} & y_2^{N-1} & \cdots & y_N^{N-1} \\ z_1y_1^{N-2} & z_2y_2^{N-2} & \cdots & z_Ny_N^{N-2} \\ \vdots & \vdots & ⋮⋮⋮ & \vdots \\ z_1^{N-2}y_1 & z_2^{N-2}y_2 & \cdots & z_N^{N-2}y_N \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} = \prod_{j>k}^{N}(z_jy_k-y_jz_k) \tag{2.6}$$

generalizes the usual Van der Monde determinant and $\mathcal{N}$ is fixed by normalizing the wave function. Normalizing the wave function in the state picture corresponds to choosing a normalization in the original matrix model so that the expectation value of 1 is 1. In the next section we will discuss the proposal (2.5) with a special emphasis on the symmetries realized by this wave function. As we will review, a wave function given as a product of Van der Monde determinants is also a natural guess. We will argue that (2.5) realizes more symmetries than a product of Van der Monde determinants does. We will then use the wave function to compute correlators. Surprisingly, for a large class of correlators the wave function (2.5) gives the exact answer.

III. SYMMETRIES OF THE $\mathrm{AdS}_5\times\mathrm{S}^5$ WAVE FUNCTION

The original two (complex) matrix model enjoys an $SO(4)\simeq SU(2)_L\times SU(2)_R$ symmetry. Indeed, the generators

$$\begin{aligned} J_3^R&=Z_{ij}\frac{\partial}{\partial Z_{ij}}-Z_{ij}^\dagger\frac{\partial}{\partial Z_{ij}^\dagger}+Y_{ij}\frac{\partial}{\partial Y_{ij}}-Y_{ij}^\dagger\frac{\partial}{\partial Y_{ij}^\dagger},\\ J_+^R&=Y_{ij}\frac{\partial}{\partial Z_{ij}^\dagger}-Z_{ij}\frac{\partial}{\partial Y\dagger_{ij}},J_-^R=Z_{ij}^\dagger\frac{\partial}{\partial Y_{ij}}-Y_{ij}^\dagger\frac{\partial}{\partial Z_{ij}},\\ J_3^L&=Z_{ij}\frac{\partial}{\partial Z_{ij}}-Z_{ij}^\dagger\frac{\partial}{\partial Z_{ij}^\dagger}-Y_{ij}\frac{\partial}{\partial Y_{ij}}+Y_{ij}^\dagger\frac{\partial}{\partial Y_{ij}^\dagger},\\ J_+^L&=Y_{ij}^\dagger\frac{\partial}{\partial Z_{ij}^\dagger}-Z_{ij}\frac{\partial}{\partial Y_{ij}},\qquad J_-^L=Z_{ij}^\dagger\frac{\partial}{\partial Y_{ij}^\dagger}-Y_{ij}\frac{\partial}{\partial Z_{ij}} \end{aligned} \tag{3.1}$$

annihilate $\mathrm{Tr}(ZZ^\dagger)+\mathrm{Tr}(YY^\dagger)$. The above $SO(4)$ symmetry can also be realized at the level of the eigenvalues. In this case, the generators are

$$\begin{aligned} J_3^R&=z_i\frac{\partial}{\partial z_i}-\bar{z}_i\frac{\partial}{\partial\bar{z}_i}+y_i\frac{\partial}{\partial y_i}-\bar{y}_i\frac{\partial}{\partial\bar{y}_i},\\ J_+^R&=y_i\frac{\partial}{\partial\bar{z}_i}-z_i\frac{\partial}{\partial\bar{y}_i},\qquad J_-^R=\bar{z}_i\frac{\partial}{\partial y_i}-\bar{y}_i\frac{\partial}{\partial z_i},\\ J_3^L&=z_i\frac{\partial}{\partial z_i}-\bar{z}_i\frac{\partial}{\partial\bar{z}_i}-y_i\frac{\partial}{\partial y_i}+\bar{y}_i\frac{\partial}{\partial\bar{y}_i},\\ J_+^L&=\bar{y}_i\frac{\partial}{\partial\bar{z}_i}-z_i\frac{\partial}{\partial y_i},\qquad J_-^L=\bar{z}_i\frac{\partial}{\partial\bar{y}_i}-y_i\frac{\partial}{\partial z_i}. \end{aligned} \tag{3.2}$$

It is simple to verify that

$$\begin{aligned} J_3^L\Psi_{gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})&=J_+^L\Psi_{gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})\\ &=J_-^L\Psi_{gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})=0 \end{aligned} \tag{3.3}$$

so that the wave function is manifestly invariant under $SU(2)_L$. Further, since

$$J_3^R\Psi_{gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})=N(N-1)\Psi_{gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\}), \tag{3.4}$$

it transforms covariantly under $U(1)\subset SU(2)_R$ generated by J_3^R. Thus, in summary, out of the original $SO(4)$ symmetry, the wave function is invariant under $SU(2)_L$ and covariant under a $U(1)\subset SU(2)_R$. Since we will restrict to the subset of BPS operators that are holomorphic in Y and Z, this is the biggest symmetry we should expect.

A few comments are in order. If the interaction is switched off, the system is invariant under separate $U(N)$ actions on Z and Y. Thus, in this case, the model has a $U(N)\times U(N)$ symmetry. If we restrict ourselves to correlators of operators that never have Ys and Zs in the same trace, the wave function

$$\Psi_{\rm VdM}=\mathcal{N}\Delta(z)\Delta(y)e^{-\frac{1}{2}\sum_j(z_j\bar{z}_j+y_j\bar{y}_j)} \tag{3.5}$$

will reproduce the exact values for all correlators. Notice that this wave function is covariant under $U(1)_L\times U(1)_R\subset SU(2)_L\times SU(2)_R$ generated by J_3^L and J_3^R; i.e. it has less symmetry than (2.5). Further, if we consider correlators of operators that include products of Z and Y matrices, the symmetry is broken to $U(N)$. The integration over the noneigenvalue degrees of freedom is nontrivial, but the result will again be a polynomial in the eigenvalues. The precise form of the polynomial will depend on the choice of operators in the correlator, and we will not get a simple rule for translating a specific operator. In the next section we will show that using (2.5), we will in fact obtain a simple rule for translating a specific operator into the eigenvalue language and the translation will not depend on the choice of the other operators in the correlator. For these reasons, we do not discuss $\Psi_{\rm VdM}$ further.

To end this section we consider the location of the zeros of (2.5). For each eigenvalue we have a vector with coordinates (z_i,y_i) on $\mathbb{C}^2$. Physically we expect that the wave function must vanish whenever $n>1$ eigenvalues coincide, leading to an enhanced symmetry of the joint eigenvalue configuration [31]. The wave function vanishes whenever the vectors associated with two distinct eigenvalues are parallel, i.e. whenever $(z_i,y_i)=\lambda(z_j,y_j)$. If $\lambda\neq 1$, the eigenvalues are not coincident, there is no enhanced symmetry of the joint eigenvalue configuration, and physically there is no reason why such an eigenvalue configuration should be weighted with zero. Thus, there are more zeros than what we expect. Clearly then (2.5) will get various things wrong, but given that it realizes more symmetries than $\Psi_{\rm VdM}$, it may be good enough for some computations. We will confirm this in the next section by showing that this wave function reproduces the correct exact answer for a large class of matrix model correlators.

Finally, note that it is useful to think of the wave function as a function of two points in $\mathbb{C}P^1\times\mathbb{C}^*$, with (z_i,y_i) simultaneously the coordinates of a point and the affine coordinates of the projective sphere base. With this interpretation, the singularities are associated with points coinciding in the base which is physically more sensible.

IV. CORRELATORS

In this section we will provide detailed tests of this wave function by computing correlators with the wave function and comparing them to the exact results from the matrix model. The comparison is accomplished by using the equation

$$\begin{aligned}&\int[dYdZdY^\dagger dZ^\dagger]e^{-{\rm Tr}(ZZ^\dagger)-{\rm Tr}(YY^\dagger)}\cdots\\&\quad=\int\prod_{i=1}^N dz_id\bar{z}_idy_id\bar{y}_i|\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\cdots\end{aligned} \tag{4.1}$$

to compute correlators of observables (denoted by $\cdots$ above) that depend only on the eigenvalues. We have already argued above that we expect that the observables that are correctly computed using eigenvalue dynamics are the BPS operators of the CFT. As a first example, consider correlators of traces $O_J={\rm Tr}(Z^J)$. These can be computed exactly in the matrix model, using a variety of different techniques—see for example [11,23,43]. The result is

$$\langle{\rm Tr}(Z^J){\rm Tr}(Z^{\dagger J})\rangle=\frac{1}{J+1}\left[\frac{(J+N)!}{(N-1)!}-\frac{N!}{(N-J-1)!}\right] \tag{4.2}$$

if $J<N$ and

$$\langle{\rm Tr}(Z^J){\rm Tr}(Z^{\dagger J})\rangle=\frac{1}{J+1}\frac{(J+N)!}{(N-1)!} \tag{4.3}$$

if $J\geq N$. These expressions could easily be expanded to generate the $1/N$ expansion if we wanted to do that. We would now like to consider the eigenvalue computation. It is useful to write the wave function as

$$\begin{aligned}\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})&=\frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1a_2\cdots a_n}\frac{z^0_{a_1}y^{N-1}_{a_1}}{\sqrt{0!(N-1)!}}\\&\quad\cdots\frac{z^{k-1}_{a_k}y^{N-k}_{a_k}}{\sqrt{(k-1)!(N-k)!}}\cdots\\&\quad\cdots\frac{z^{N-1}_{a_N}y^0_{a_N}}{\sqrt{(N-1)!0!}}e^{-\frac{1}{2}\sum_q z_q\bar{z}_q-\frac{1}{2}\sum_q y_q\bar{y}_q}.\end{aligned} \tag{4.4}$$

The gauge invariant observable in this case is given by

$${\rm Tr}(Z^J){\rm Tr}(Z^{\dagger J})=\sum_{i=1}^N z_i^J\sum_{j=1}^N\bar{z}_j^J. \tag{4.5}$$

It is now straightforward to find

$$\begin{aligned}&\int\prod_{i=1}^N dz_id\bar{z}_idy_id\bar{y}_i|\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\sum_{i=1}^N z_i^J\sum_{j=1}^N\bar{z}_j^J\\&\quad=\frac{1}{J+1}\frac{(J+N)!}{(N-1)!}.\end{aligned} \tag{4.6}$$

When evaluating the above integral, only the terms with $i = j$ contribute. From this result we see that we have not reproduced traces with $J < N$ correctly—we do not even get the leading large N behavior right. We have, however, correctly reproduced the exact answer (to all orders in $1/N$) of the two point function for all single traces of dimension N or greater. For $J > N$ there are trace relations of the form

$$\mathrm{Tr}(Z^J) = \sum_{i,j,\ldots,k} c_{ij\cdots k}\mathrm{Tr}(Z^i)\mathrm{Tr}(Z^j)\cdots\mathrm{Tr}(Z^k), \tag{4.7}$$

$i, j, \ldots, k \leq N$ and $i + j + \cdots + k = J$. The fact that we reproduce two point correlators of traces with $J > N$ exactly implies that we also start to reproduce sums of products of traces of less than N fields. This suggests that the important thing is not the trace structure of the operator, but rather the dimension of the state.

The fact that we only reproduce observables that have a large enough dimension is not too surprising. Indeed, supergravity cannot be expected to correctly describe the backreaction of a single graviton or a single string. To produce a state in the CFT dual to a geometry that is different from the AdS vacuum one needs to allow a number of giant gravitons (eigenvalues) to condense. The eigenvalue dynamics is correctly reproducing the two point function of traces when their energy is greater than that required to blow up into a giant graviton.

With a very simple extension of the above argument we can argue that we also correctly reproduce the correlator $\langle\mathrm{Tr}(Y^J)\mathrm{Tr}(Y^{\dagger J})\rangle$ with $J \geq N$. A much more interesting class of observables to consider are mixed traces, which contain both Y and Z fields. To build BPS operators using both Y and Z fields we need to construct symmetrized traces. A very convenient way to perform this construction is as follows:

$$\mathcal{O}_{J,K} = \frac{J!}{(J+K)!}\mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K \mathrm{Tr}(Z^{J+K}). \tag{4.8}$$

The normalization up front is just the inverse of the number of terms that appear. With this normalization, the translation between the matrix model observable and an eigenvalue observable is

$$\mathcal{O}_{J,K} \leftrightarrow \sum_i z_i^J y_i^K. \tag{4.9}$$

Since we could not find this computation in the literature, we will now explain how to evaluate the matrix model two point function exactly, in the free field theory limit. Since the dimension of BPS operators are not corrected, this answer is in fact exact. To start, perform the contraction over the $Y, Y^\dagger$ fields

$$\begin{aligned}\langle\mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle &= \left(\frac{J!}{(J+K)!}\right)^2\left\langle\mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K\mathrm{Tr}(Z^{J+K})\mathrm{Tr}\left(Y^\dagger\frac{\partial}{\partial Z^\dagger}\right)^K\mathrm{Tr}(Z^{\dagger J+K})\right\rangle \\ &= \left(\frac{J!}{(J+K)!}\right)^2 K!\left\langle\mathrm{Tr}\left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}\right)^K\mathrm{Tr}(Z^{J+K})\mathrm{Tr}(Z^{\dagger J+K})\right\rangle.\end{aligned} \tag{4.10}$$

Given the form of the matrix model two point function

$$\langle Z_{ij}Z^\dagger_{kl}\rangle = \delta_{il}\delta_{jk}, \tag{4.11}$$

we know that we can write any free field theory correlator as

$$\langle\cdots\rangle = e^{\mathrm{Tr}(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger})}\cdots|_{Z=Z^\dagger=0}. \tag{4.12}$$

Using this identity we now find

$$\begin{aligned}\langle\mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle &= \left(\frac{J!}{(J+K)!}\right)^2 K!\frac{(J+K)!}{J!} \\ &\quad\times\langle\mathrm{Tr}(Z^{J+K})\mathrm{Tr}(Z^{\dagger J+K})\rangle.\end{aligned} \tag{4.13}$$

Thus, the result of the matrix model computation is

$$\begin{aligned}\langle\mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle &= \frac{J!K!}{(J+K+1)!} \\ &\quad\times\left[\frac{(J+K+N)!}{(N-1)!} - \frac{N!}{(N-J-K-1)!}\right]\end{aligned} \tag{4.14}$$

if $J + K < N$ and

$$\langle\mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \frac{J!K!}{(J+K+1)!}\frac{(J+K+N)!}{(N-1)!} \tag{4.15}$$

if $J + K \geq N$. Notice that for these two matrix observables we again get a change in the form of the correlator as the dimension of the trace exceeds N.

Next, consider the eigenvalue computation. We need to perform the integral

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \int \prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2 \sum_{k=1}^N z_k^J y_k^K \sum_{j=1}^N \bar{z}_j^J \bar{y}_j^K. \tag{4.16}$$

After some straightforward manipulations we have

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \pi^{-2N} \int \prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i \frac{|z_1|^0 |y_1|^{2N-2}}{0!(N-1)!}\cdots\frac{|z_k|^{2k-2}|y_k|^{2N-2k}}{(k-1)!(N-k)!}\cdots \times \frac{|z_N|^{2N-2}|y_N|^0}{(N-1)!0!}\times e^{-\sum_q z_q\bar{z}_q - \sum_q y_q\bar{y}_q}\sum_{k,j=1}^N z_k^J y_k^K \bar{z}_j^J \bar{y}_j^K. \tag{4.17}$$

Only terms with $k = j$ contribute so that

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \sum_{k=1}^N \frac{(N-k+K)!}{(N-k)!}\frac{(J+k-1)!}{(k-1)!} = \frac{K!J!}{(K+J+1)!}\frac{(J+K+N)!}{(N-1)!}. \tag{4.18}$$

Thus, we again correctly reproduce the exact (to all orders in $1/N$) answer for the two point function of single trace operators of dimension N or greater. Inspecting (3.1) we notice that we have obtained $\mathcal{O}_{J,K}$ from $\mathcal{O}_{J+K}$ by applying J_-^L, that is, by applying an $SU(2)_L$ rotation. Since both the original matrix description and the eigenvalue description enjoy $SU(2)_L$ symmetry, the agreement of the $\langle \mathcal{O}^\dagger_{J,K}\mathcal{O}_{J,K}\rangle$ correlator is not independent of the agreement of the $\langle \mathcal{O}^\dagger_{J+K}\mathcal{O}_{J+K}\rangle$ correlator.

It is also interesting to consider multitrace correlators. We will start with the correlator between a double trace and a single trace, and we will again start with the matrix model computation

$$\langle O_{J_1,K_1}O_{J_2,K_2}O^\dagger_{J_1+J_2,K_1+K_2}\rangle = \frac{J_1!}{(J_1+K_1)!}\frac{J_2!}{(J_2+K_2)!}\frac{(J_1+J_2)!}{(J_1+K_1+J_2+K_2)!} \times \left\langle \mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^{K_1}\mathrm{Tr}(Z^{J_1+K_1})\mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^{K_2}\mathrm{Tr}(Z^{J_2+K_2})\mathrm{Tr}\left(Y^\dagger\frac{\partial}{\partial Z^\dagger}\right)^{K_1+K_2}\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\right\rangle. \tag{4.19}$$

We could easily set $K_1 = K_2 = 0$ and obtain traces involving only a single matrix. Begin by contracting all $Y, Y^\dagger$ fields to obtain

$$\langle O_{J_1,K_1}O_{J_2,K_2}O^\dagger_{J_1+J_2,K_1+K_2}\rangle = \frac{J_1!}{(J_1+K_1)!}\frac{J_2!}{(J_2+K_2)!}\frac{(J_1+J_2)!}{(J_1+K_1+J_2+K_2)!}(K_1+K_2)! \times \left\langle \frac{\partial}{\partial Z_{i_1j_1}}\cdots\frac{\partial}{\partial Z_{i_{K_1}j_{K_1}}}\mathrm{Tr}(Z^{J_1+K_1})\frac{\partial}{\partial Z_{i_{K_1+1}j_{K_1+1}}}\cdots\frac{\partial}{\partial Z_{i_{K_1+K_2}j_{K_1+K_2}}}\mathrm{Tr}(Z^{J_2+K_2}) \times \frac{\partial}{\partial Z^\dagger_{j_1i_1}}\cdots\frac{\partial}{\partial Z^\dagger_{j_{K_1+K_2}i_{K_1+K_2}}}\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\right\rangle. \tag{4.20}$$

It is now useful to integrate by parts with respect to $Z^\dagger$, using the identity

$$\left\langle \frac{\partial}{\partial Z_{ij}}f(Z)g(Z)\frac{\partial}{\partial Z^\dagger_{ji}}h(Z^\dagger)\right\rangle = n_f\langle f(Z)g(Z)h(Z^\dagger)\rangle, \tag{4.21}$$

where $f(Z)$ is of degree n_f in Z. Repeatedly using this identity, we find

$$\begin{aligned}\langle O_{J_1,K_1}O_{J_2,K_2}O^\dagger_{J_1+J_2,K_1+K_2}\rangle &= \frac{J_1!}{(J_1+K_1)!}\frac{J_2!}{(J_2+K_2)!}\frac{(J_1+J_2)!}{(J_1+K_1+J_2+K_2)!}(K_1+K_2)!\\ &\quad\times\frac{(J_1+K_1)!}{J_1!}\frac{(J_2+K_2)!}{J_2!}\langle \mathrm{Tr}(Z^{J_1+K_1})\mathrm{Tr}(Z^{J_2+K_2})\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle\\ &= \frac{(J_1+J_2)!(K_1+K_2)!}{(J_1+K_1+J_2+K_2)!}\langle \mathrm{Tr}(Z^{J_1+K_1})\mathrm{Tr}(Z^{J_2+K_2})\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle. \end{aligned}\tag{4.22}$$

This last correlator is easily computed. For example, if $J_1+K_1<N$ and $J_2+K_2<N$, we have

$$\begin{aligned}\langle O_{J_1,K_1}O_{J_2,K_2}O^\dagger_{J_1+J_2,K_1+K_2}\rangle &= \frac{(J_1+J_2)!(K_1+K_2)!}{(J_1+K_1+J_2+K_2+1)!}\left[\frac{(J_1+K_1+J_2+K_2+N)!}{(N-1)!}\right.\\ &\quad\left.+\frac{N!}{(N-J_1-K_1-J_2-K_2-1)!}-\frac{(N+J_1+K_1)!}{(N-J_2-K_2-1)!}-\frac{(N+J_2+K_2)!}{(N-J_1-K_1-1)!}\right],\end{aligned}\tag{4.23}$$

and if $J_1+K_1\geq N$ and $J_2+K_2\geq N$, we have

$$\langle O_{J_1,K_1}O_{J_2,K_2}O^\dagger_{J_1+J_2,K_1+K_2}\rangle = \frac{(J_1+J_2)!(K_1+K_2)!}{(J_1+K_1+J_2+K_2+1)!}\frac{(J_1+K_1+J_2+K_2+N)!}{(N-1)!}.\tag{4.24}$$

It is a simple exercise to check that, in terms of eigenvalues, we have

$$\begin{aligned}\langle \mathcal{O}_{J_1,K_1}\mathcal{O}_{J_2,K_2}\mathcal{O}^\dagger_{J_1+J_2,K_1+K_2}\rangle &= \int\prod_{i=1}^N dz_i d\bar z_i dy_i d\bar y_i|\Psi_{\mathrm{gs}}(\{z_i,\bar z_i,y_i,\bar y_i\})|^2\sum_{k=1}^N z_k^{J_1}y_k^{K_1}\sum_{l=1}^N z_l^{J_2}y_l^{K_2}\sum_{j=1}^N \bar z_j^{J_1+J_2}\bar y_j^{K_1+K_2}\\ &= \frac{(J_1+J_2)!(K_1+K_2)!}{(J_1+K_1+J_2+K_2+1)!}\frac{(J_1+K_1+J_2+K_2+N)!}{(N-1)!}\end{aligned}\tag{4.25}$$

so that once again we have reproduced the exact answer as long as the dimension of each trace is not less than N. The agreement that we have observed for multitrace correlators continues as follows: as long as the dimension of each trace is greater than $N-1$, the matrix model and the eigenvalue descriptions agree and both give

$$\langle O_{J_1,K_1}O_{J_2,K_2}\cdots O_{J_n,K_n}O^\dagger_{J,K}\rangle = \frac{J!K!}{(J+K+1)!}\frac{(J+K+N)!}{(N-1)!}\delta_{J_1+\cdots+J_n,J}\delta_{K_1+\cdots+K_n,K}\tag{4.26}$$

for the exact value of this correlator. We have limited ourselves to a single daggered observable in the above expression for purely technical reasons: it is only in this case that we can compute the matrix model correlator using the identity (4.21). It would be interesting to develop analytic methods that allow more general computations.

Finally, we can also test multitrace correlators with a dimension of order N^2. A particularly simple operator is the Schur polynomial labeled by a Young diagram R with N rows and M columns. For this R we have

$$\chi_R(Z) = (\det Z)^M = z_1^M z_2^M\cdots z_N^M,\tag{4.27}$$

$$\chi_R(Z^\dagger) = (\det Z^\dagger)^M = \bar z_1^M\bar z_2^M\cdots\bar z_N^M.\tag{4.28}$$

The dual LLM geometry is labeled by an annulus boundary condition that has an inner radius of $\sqrt{M}$ and an outer radius of $\sqrt{M+N}$. The two point correlator of this Schur polynomial is

$$\begin{aligned}\langle\chi_R(Z)\chi_R(Z^\dagger)\rangle &= \int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i \chi_R(Z)\chi_R(Z^\dagger)|\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\\ &= \pi^{-2N}\int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i \frac{|z_1|^{0+2M}|y_1|^{2N-2}}{0!(N-1)!}\cdots\frac{|z_k|^{2k-2+2M}|y_k|^{2N-2k}}{(k-1)!(N-k)!}\\ &= \cdots\frac{|z_N|^{2N-2+2M}|y_N|^0}{(N-1)!0!}\times e^{-\sum_q z_q\bar{z}_q-\sum_q y_q\bar{y}_q}\\ &= \prod_{i=1}^N\frac{(i-1+M)!}{(i-1)!},\end{aligned} \tag{4.29}$$

which is again the exact answer for this correlator.

After this warm-up example we will now make a few comments that are relevant for the general case. The details are much more messy, so we will not manage to make very precise statements. We have, however, included this discussion as it does provide a guide as to when eigenvalue dynamics is applicable. A Schur polynomial labeled with a Young diagram R that has row lengths r_i is given in terms of eigenvalues as (our labeling of the rows is defined by $r_1 \geq r_2 \geq \cdots \geq r_N$)

$$\chi_R(Z) = \frac{\epsilon_{a_1a_2\cdots a_N} z_{a_1}^{N-1+r_1} z_{a_2}^{N-2+r_2}\cdots z_{a_N}^{r_N}}{\epsilon_{b_1b_2\cdots b_N} z_{b_1}^{N-1} z_{b_2}^{N-2}\cdots z_{b_{N-1}}}. \tag{4.30}$$

Using this expression, we can easily write the exact two point function as follows:

$$\begin{aligned}\langle\chi_R(Z)\chi_R(Z^\dagger)\rangle &= \frac{1}{N!\pi^N}\prod_{j=0}^{N-1}\frac{1}{j!}\int\prod_{i=1}^N dz_i d\bar{z}_i \epsilon_{a_1a_2\cdots a_N} z_{a_1}^{N-1+r_1} z_{a_2}^{N-2+r_2}\cdots z_{a_N}^{r_N}\\ &\quad\times\epsilon_{b_1b_2\cdots b_N}\bar{z}_{b_1}^{N-1+r_1}\bar{z}_{b_2}^{N-2+r_2}\cdots\bar{z}_{b_N}^{r_N} e^{-\sum_k z_k\bar{z}_k}\\ &= \prod_{j=0}^{N-1}\frac{(j+r_{N-j})!}{j!} = f_R.\end{aligned} \tag{4.31}$$

Using our wave function we can compute the two point function of Schur polynomials. The result is

$$\begin{aligned}\langle\chi_R(Z)\chi_R(Z^\dagger)\rangle &= \int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i \chi_R(Z)\chi_R(Z^\dagger)|\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\\ &= \frac{1}{\pi^N}\prod_{j=0}^{N-1}\frac{1}{j!}\int\prod_{i=1}^N dz_i d\bar{z}_i |z_{a_1}|^{2N-2}|z_{a_2}|^{2N-4}\cdots|z_{a_{N-1}}|^2\\ &\quad\times\frac{\epsilon_{b_1b_2\cdots b_N} z_{b_1}^{N-1+r_1} z_{b_2}^{N-2+r_2}\cdots z_{b_N}^{r_N}}{\epsilon_{c_1c_2\cdots c_N} z_{c_1}^{N-1} z_{c_2}^{N-2}\cdots z_{c_{N-1}}}\\ &\quad\times\frac{\epsilon_{d_1d_2\cdots d_N}\bar{z}_{d_1}^{N-1+r_1}\bar{z}_{d_2}^{N-2+r_2}\cdots\bar{z}_{d_N}^{r_N}}{\epsilon_{e_1e_2\cdots e_N}\bar{z}_{e_1}^{N-1}\bar{z}_{e_2}^{N-2}\cdots\bar{z}_{e_{N-1}}} e^{-\sum_k z_k\bar{z}_k}.\end{aligned} \tag{4.32}$$

When the integration over the angles θ_i associated with $z_i = r_i e^{i\theta_i}$ are performed, a nonzero result is obtained only if powers of the z_i match the powers of the $\bar{z}_i$. The difference between the above expression and the exact answer is simply that in the eigenvalue expression these powers are separately set to be equal in the measure and in the product of Schur polynomials—there are two matchings, while in the exact answer the power of z_i arising from the product of the measure and the product of Schur polynomials is matched to the power of $\bar{z}_i$ from the product of the measure and the product of Schur polynomials—there is a single matching happening. Thus, the eigenvalue computation may miss some terms that are present in the exact answer.[1] For Young diagrams with a few corners and $O(N^2)$ boxes (the annulus above is a good example) the eigenvalues

[1]This is the reason why (4.6) captures only one of the terms present in the two point function for $J < N$.

clump into groupings, with each grouping collecting eigenvalues of a similar size corresponding to rows with a similar row length [41]. This happens because the product of the Gaussian falloff $e^{-z\bar{z}}$ and a polynomial of fixed degree $|z^2|^n$ is sharply peaked at $|z| = n$. Thus, for example, if $r_i \approx M_1$ for $i = 1, 2, \ldots, \frac{N}{2}$ and $r_i \approx M_2$ for $i = 1 + \frac{N}{2}, 2 + \frac{N}{2}, \ldots, N$ with M_1 and M_2 well separated $[M_1 - M_2 \geq O(N)]$, under the integral we can replace

$$\frac{\epsilon_{b_1 b_2 \cdots b_N} z_{b_1}^{N-1+r_1} z_{b_2}^{N-2+r_2} \cdots z_{b_N}^{r_N}}{\epsilon_{c_1 c_2 \cdots c_N} z_{c_1}^{N-1} z_{c_2}^{N-2} \cdots z_{c_{N-1}}} \to \prod_{i=1}^{\frac{N}{2}} z_{a_i}^{M_1} z_{a_{i+\frac{N}{2}}}^{M_2}. \tag{4.33}$$

After making a replacement of this type, we recover the exact answer. This replacement is not exact—we need to appeal to large N to justify it. It would be very interesting to explore this point further and to quantify in general (if possible) what the corrections to the above replacement are. For Young diagrams with many corners, row lengths are not well separated and there is no similar grouping that occurs, so that the eigenvalue description will not agree with the exact result, even at large N. A good example of a geometry with many corners is the superstar [44]. The corresponding LLM boundary condition is a number of very thin concentric annuli, so that we effectively obtain a gray disk, signaling a singular supergravity geometry. It is then perhaps not surprising that the eigenvalue dynamics does not correctly reproduce this two point correlator.

Having discussed the two point function of Schur polynomials in detail, the product rule

$$\chi_R(Z)\chi_S(Z) = \sum_T f_{RST}\chi_T(Z) \tag{4.34}$$

with f_{RST} a Littlewood-Richardson coefficient, implies that there is no need to consider correlation functions of products of Schur polynomials.

V. OTHER BACKGROUNDS

In the $\frac{1}{2}$ BPS sector there is a wave function corresponding to every LLM geometry. The (not normalized) wave function has already been given in (1.6). In this section we consider the problem of writing eigenvalue wave functions that correspond to geometries other than $AdS_5 \times S^5$. The simplest geometry we can consider is the annulus geometry considered in the previous section, where we argued that the eigenvalue dynamics reproduces the exact correlator of the Schur polynomials dual to this geometry. Our proposal for the state that corresponds to this LLM spacetime is

$$\begin{aligned}\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\}) = &\frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1 a_2 \cdots a_n}\frac{z_{a_1}^M y_{a_1}^{N-1}}{\sqrt{M!(N-1)!}} \\ &\cdots\frac{z_{a_k}^{k-1+M} y_{a_k}^{N-k}}{\sqrt{(k-1+M)!(N-k)!}} \\ &\cdots\frac{z_{a_N}^{N-1+M} y_{a_N}^{0}}{\sqrt{(N-1+M)!0!}} e^{-\frac{1}{2}\sum_q z_q\bar{z}_q - \frac{1}{2}\sum_q y_q\bar{y}_q}.\end{aligned} \tag{5.1}$$

This is simply obtained by multiplying the ground state wave function by the relevant Schur polynomial and normalizing the resulting state. The connection between matrix model correlators and expectation values computed using the above wave function is the following[2]:

$$\begin{aligned}\langle \cdots \rangle_{\rm LLM} &= \frac{\langle \cdots \chi_R(Z)\chi_R(Z^\dagger)\rangle}{\langle \chi_R(Z)\chi_R(Z^\dagger)\rangle} \\ &= \int \prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2 \cdots.\end{aligned} \tag{5.2}$$

We can use this wave function to compute correlators that we are interested in. Traces involving only Zs, for example, lead to

$$\begin{aligned}\langle \mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J})\rangle_{\rm LLM} &= \int \prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2 \sum_{k=1}^N z_k^J \sum_{l=1}^N \bar{z}_l^J \\ &= \sum_{k=0}^{N-1}\frac{(J+k+M)!}{(k+M)!} \\ &= \frac{1}{J+1}\left[\frac{(J+M+N)!}{(M+N-1)!} - \frac{(J+M)!}{(M-1)!}\right],\end{aligned} \tag{5.3}$$

which agrees with the exact result, as long as $J > N-1$. Thus, in this background, eigenvalue dynamics is correctly reproducing the same set of correlators as in the original $AdS_5 \times S^5$ background. Traces involving only Y fields are also correctly reproduced

[2]The new normalization for matrix model correlators is needed to ensure that the identity operator has expectation value 1. This matches the normalization adopted in the eigenvalue description.

$$\langle \mathrm{Tr}(Y^J)\mathrm{Tr}(Y^{\dagger J})\rangle_{\mathrm{LLM}} = \int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\mathrm{LLM}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2 \sum_{k=1}^{N} y_k^J \sum_{l=1}^{N} \bar{y}_l^J = \frac{1}{J+1}\frac{(J+N)!}{(N-1)!}, \tag{5.4}$$

where $J \geq N$. Notice that these results are again exact; i.e. we reproduce the matrix model correlators to all orders in $1/N$. Finally, let us consider the most interesting case of traces involving both matrices. The LLM wave function we have proposed does not reproduce the exact matrix model computation. The matrix model computation gives

$$\begin{aligned}
\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle_{\mathrm{LLM}} &= \left(\frac{J!}{(J+K)!}\right)^2 \left\langle \mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K \mathrm{Tr}(Z^{J+K})\mathrm{Tr}\left(Y^{\dagger}\frac{\partial}{\partial Z^{\dagger}}\right)^K \mathrm{Tr}(Z^{\dagger J+K})\right\rangle_{\mathrm{LLM}} \\
&= \left(\frac{J!}{(J+K)!}\right)^2 K! \left\langle \mathrm{Tr}\left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^{\dagger}}\right)^K \mathrm{Tr}(Z^{J+K})\mathrm{Tr}(Z^{\dagger J+K})\right\rangle_{\mathrm{LLM}} \\
&= \left(\frac{J!}{(J+K)!}\right)^2 K! \frac{(J+K)!}{J!} \langle \mathrm{Tr}(Z^{J+K})\mathrm{Tr}(Z^{\dagger J+K})\rangle_{\mathrm{LLM}} \\
&= \frac{J!K!}{(J+K+1)!}\left[\frac{(J+K+M+N)!}{(M+N-1)!} - \frac{(J+K+M)!}{(M-1)!}\right]
\end{aligned} \tag{5.5}$$

if $J+K \geq N$. Next, consider the eigenvalue computation. We need to perform the integral

$$\begin{aligned}
\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle_{\mathrm{LLM,eigen}} &= \int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\mathrm{LLM}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2 \sum_{k=1}^{N} z_k^J y_k^K \sum_{j=1}^{N} \bar{z}_j^J \bar{y}_j^K \\
&= \sum_{k=1}^{N} \frac{(N-k+K)!}{(N-k)!}\frac{(J+M+k-1)!}{(M+k-1)!}.
\end{aligned} \tag{5.6}$$

It is not completely trivial to compare (5.5) and (5.6), but it is already clear that they do not reproduce exactly the same answer. To simplify the discussion, let us consider the case that $M = O(\sqrt{N})$. In this case, in the large N limit, we can drop the second term in (5.5) to obtain

$$\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle_{\mathrm{LLM}} = \frac{J!K!}{(J+K+1)!}\frac{(J+K+M+N)!}{(M+N-1)!}(1+\cdots), \tag{5.7}$$

where $\cdots$ stand for terms that vanish as $N\to\infty$. In the sum appearing in (5.6), change variables from k to $k'-M$ and again appeal to large N to write

$$\begin{aligned}
\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle_{\mathrm{LLM,eigen}} &= \sum_{k'=M+1}^{M+N} \frac{(N+M-k'+K)!}{(N+M-k')!}\frac{(J+k'-1)!}{(k'-1)!} \\
&= \sum_{k'=1}^{M+N} \frac{(N+M-k'+K)!}{(N+M-k')!}\frac{(J+k'-1)!}{(k'-1)!}(1+\cdots) \\
&= \frac{J!K!}{(J+K+1)!}\frac{(J+K+M+N)!}{(M+N-1)!}(1+\cdots).
\end{aligned} \tag{5.8}$$

In the last two lines above $\cdots$ again stands for terms that vanish as $N\to\infty$. Thus, we find agreement between (5.5) and (5.6). It is again convincing to see genuine multimatrix observables reproduced by the eigenvalue dynamics. Notice that in this case the agreement is not exact, but rather is realized to the large N limit. This is what we expect for the generic situation—the $\mathrm{AdS}_5\times\mathrm{S}^5$ case is highly symmetric and the fact that eigenvalue dynamics reproduces so many observables exactly is a consequence of this symmetry. We only expect eigenvalue dynamics to reproduce classical gravity, which should emerge from the CFT at $N=\infty$.

Much of our intuition came from thinking about the Gauss graph operators constructed in [29,30]. It is natural to ask if we can write down wave functions dual to the Gauss graph operators. The simplest possibility is to consider a Gauss graph operator obtained by exciting a single eigenvalue by J levels, and then attaching a total of K Y strings to it. The extreme simplicity of this case follows because we can write the (normalized) Gauss graph operator in terms of a familiar Schur polynomial as

$$\hat{O} = \sqrt{\frac{J!}{K!(J+K)!}\frac{(N-1)!}{(N+J+K-1)!}}\mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K \chi_{(J+K)}(Z), \tag{5.9}$$

where we have used the notation (n) to denote a Young diagram with a single row of n boxes. Consider the correlator

$$\langle \hat{O}\mathrm{Tr}(Y^\dagger)^K \mathrm{Tr}(Z^{\dagger J})\rangle = \left\langle \mathrm{Tr}\left(\frac{\partial}{\partial Y}\right)^K \hat{O}\mathrm{Tr}(Z^{\dagger J})\right\rangle = \sqrt{\frac{J!K!}{(J+K)!}\frac{(N+J+K-1)!}{(N-1)!}}. \tag{5.10}$$

This answer is exact, in the free field theory. In what limit should we compare this answer to eigenvalue dynamics? Our intuition is coming from the $\frac{1}{2}$- BPS sector where we know that rows of Schur polynomials correspond to eigenvalues and we know exactly how to write the corresponding wave function. If we only want small perturbations of this picture, we should keep $K \ll J$. In this case we should simplify

$$\frac{J!}{(J+K)!} \to \frac{1}{J^K},$$
$$\frac{(N+J+K-1)!}{(N-1)!} = \frac{(N+J+K-1)!}{(N+J-1)!}\frac{(N+J-1)!}{(N-1)!} \to (N+J-1)^K \frac{(N+J-1)!}{(N-1)!}. \tag{5.11}$$

How should we scale J as we take $N \to \infty$? The Schur polynomials are a sum over all possible matrix trace structures. We want these sums to be dominated by traces with a large number of matrices (N or more) in each trace. To accomplish this we will scale $J = O(N^{1+\epsilon})$ with $\epsilon > 0$. In this case, at large N, we can replace

$$\frac{1}{J^K}(N+J-1)^K \to 1 \tag{5.12}$$

and hence, the result that should be reproduced by the eigenvalue dynamics is given by

$$\langle \hat{O}\mathrm{Tr}(Y^\dagger)^K \mathrm{Tr}(Z^{\dagger J})\rangle = \sqrt{K!\frac{(N+J-1)!}{(N-1)!}}. \tag{5.13}$$

In the eigenvalue computation, we will use the wave function of the ground state and the wave function of the Gauss graph operator $[\Psi_{\mathrm{GG}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})]$ to compute the amplitude

$$\int \prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i \Psi^*_{\mathrm{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})\left(\sum_i \bar{y}_i\right)^K \times \sum_j \bar{z}_j^J \Psi_{\mathrm{GG}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}). \tag{5.14}$$

We expect the amplitude (5.14) to reproduce (5.13). Our proposal for the wave function corresponding to the above Gauss graph operator is

$$\Psi_{GG}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = \frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1 a_2 \cdots a_n}\frac{z^0_{a_1} y^{N-1}_{a_1}}{\sqrt{0!(N-1)!}}\cdots\frac{z^{k-1}_{a_k} y^{N-k}_{a_k}}{\sqrt{(k-1)!(N-k)!}}\cdots \cdots\frac{z^{N-2}_{a_{N-1}} y_{a_{N-1}}}{\sqrt{(N-2)!1!}}\frac{z^{J+N-1}_{a_N} y^K_{a_N}}{\sqrt{(J+N-1)!K!}} e^{-\frac{1}{2}\sum_q z_q \bar{z}_q - \frac{1}{2}\sum_q y_q \bar{y}_q}. \tag{5.15}$$

The eigenvalue with the largest power of z (i.e. z_{a_N}) was the fermion at the very top of the Fermi sea. It has been excited by J powers of z and K powers of y. It is now trivial to verify that (5.14) does indeed reproduce (5.13).

Finally, the state with three eigenvalues excited by $J_1 > J_2 > J_3$ and with $K_1 > K_2 > K_3$ strings attached to each eigenvalue is given by

$$\Psi_{GG}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = \frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1 a_2 \cdots a_n}\frac{z^0_{a_1} y^{N-1}_{a_1}}{\sqrt{0!(N-1)!}}\cdots\frac{z^{k-1}_{a_k} y^{N-k}_{a_k}}{\sqrt{(k-1)!(N-k)!}}\cdots \cdots\frac{z^{N-4}_{a_{N-3}} y^3_{a_{N-3}}}{\sqrt{(N-4)!3!}}\frac{z^{J_3+N-3}_{a_{N-2}} y^{2+K_3}_{a_{N-2}}}{\sqrt{(J_3+N-3)!(2+K_3)!}}\frac{z^{J_2+N-2}_{a_{N-1}} y^{K_2+1}_{a_{N-1}}}{\sqrt{(J_2+N-2)!(K_2+1)!}} \times \frac{z^{J_1+N-1}_{a_N} y^{K_1}_{a_N}}{\sqrt{(J_1+N-1)!K_1!}} e^{-\frac{1}{2}\sum_q z_q \bar{z}_q - \frac{1}{2}\sum_q y_q \bar{y}_q}. \tag{5.16}$$

The generalization to any Gauss graph operator is now clear.

VI. CONNECTION TO SUPERGRAVITY

In this section we would like to explore the possibility that the eigenvalue dynamics of the $SU(2)$ sector has a natural interpretation in supergravity. The relevant supergravity solutions have been considered in [45–48].

There are six adjoint scalars in the $\mathcal{N}=4$ super Yang-Mills theory that can be assembled into the following three complex combinations:

$$Z=\phi^1+i\phi^2, \qquad Y=\phi^3+i\phi^4, \qquad X=\phi^5+i\phi^6. \tag{6.1}$$

The operators we consider are constructed using only Z and Y so that they are invariant under the $U(1)$ which rotates ϕ^5 and ϕ^6. Further, since our operators are BPS, they are built only from the s-wave spherical harmonic components of Y and Z, so that they are invariant under the $SO(4)$ symmetry which acts on the S^3 of the $R\times S^3$ spacetime on which the CFT is defined. Local supersymmetric geometries with $SO(4)\times U(1)$ isometries have the form [45,48]

$$ds^2_{10}=-h^{-2}(dt+\omega)^2+h^2\left[\frac{2}{Z+\frac{1}{2}}\partial_a\bar{\partial}_bKdz^ad\bar{z}^b+dy^2\right] + y(e^Gd\Omega_3^2+e^{-G}d\psi^2), \tag{6.2}$$

$$d\omega=\frac{i}{y}(\partial_a\bar{\partial}_b\partial_yKdz^ad\bar{z}^b-\partial_aZdz^ady+\bar{\partial}_aZd\bar{z}^ady). \tag{6.3}$$

Here z^1 and z^2 are a pair of complex coordinates and K is a Kahler potential which may depend on y, z^a, and $\bar{z}^a$. y^2 is the product of warp factors for S^3 and S^1. Thus we must be careful and impose the correct boundary conditions at the $y=0$ hypersurface if we are to avoid singularities. The $y=0$ hypersurface includes the four-dimensional space with coordinates given by the z^a. These boundary conditions require that when the S^3 contracts to zero, we need $Z=-\frac{1}{2}$, and when the ψ circle collapses, we need $Z=\frac{1}{2}$ [45,48]. There is a surface separating these two regions and, hence, defining the supergravity solution. So far the discussion given closely matches what is found for the $\frac{1}{2}$-BPS supergravity solutions. In that case the $y=0$ hypersurface includes a two-dimensional space which is similarly divided into two regions, giving the black droplets on a white plane. The edges of the droplets are completely arbitrary, which is an important difference from the case we are considering. The surface defining local supersymmetric geometries with $SO(4)\times U(1)$ isometries is not completely arbitrary—it too has to satisfy some additional constraints as spelled out in [48]. It is natural to ask if the surface defining the supergravity solution is visible in the eigenvalue dynamics?

To answer this question we will now review how the surface defining the local supersymmetric geometries with $SO(4)\times U(1)$ isometries corresponding to the $\frac{1}{2}$-BPS LLM geometries is constructed. According to [48], the boundary condition for these geometries have walls between the two boundary conditions determined by the equation[3]

$$z^2\bar{z}^2=e^{-2\hat{D}(z^1,\bar{z}^1)}, \tag{6.4}$$

where $\hat{D}(z^1,\bar{z}^1)$ is determined by expanding the function D as follows (it is the y coordinate that we set to zero to get the LLM plane):

$$D=\log(y)+\hat{D}(z,\bar{z})+O(y). \tag{6.5}$$

The function D is determined by the equations

$$y\partial_yD=\frac{1}{2}-Z, \qquad V=-i(dz\partial_z-d\bar{z}\partial_{\bar{z}})D, \tag{6.6}$$

where $Z(y,z^1,\bar{z}^1)$ is the function obeying Laplace's equation that determines the LLM solution and $V(y,z^1,\bar{z}^1)$ is the one form appearing in the combination $(dt+V)^2$ in the LLM metric.

Consider an annulus that has an outer edge at radius $M+N$ and an inner edge at a radius M. This solution has (these solutions were constructed in the original LLM paper [7])

$$Z(y,z^1,\bar{z}^1)=-\frac{1}{2}\left(\frac{|z^1|^2+y^2-M}{\sqrt{(|z^1|^2+y^2+M)^2-4|z^1|^2M}} + \frac{|z^1|^2+y^2-M-N}{\sqrt{(|z^1|^2+y^2+M+N)^2-4|z^1|^2(M+N)}}\right),$$

$$V(y,z^1,\bar{z}^1)=\frac{d\phi}{2}\left(\frac{|z^1|^2+y^2+M}{\sqrt{(|z^1|^2+y^2+M)^2-4|z^1|^2M-}} + \frac{|z^1|^2+y^2+M+N}{\sqrt{(|z^1|^2+y^2+M+N)^2-4|z^1|^2(M+N)}}\right).$$

Evaluating at $y=0$, the second of (6.6) says

$$V=-i(dz\partial_z-d\bar{z}\partial_{\bar{z}})\hat{D}. \tag{6.7}$$

Setting $z^1=re^{-i\phi}$ and assuming that $\hat{D}$ depends only on r we find

$$r\frac{\partial\hat{D}}{\partial r}=-\frac{M+N}{r^2-M-N}+\frac{M}{r^2-M}, \tag{6.8}$$

which is solved by

[3]This next equation is (6.35) of [48]. We will relate z^1 and z^2 to z_i (the eigenvalues of Z) and y_i (the eigenvalues of Y) when we make the correspondence to eigenvalues.

$$\hat{D} = \frac{1}{2}\log\frac{|z^1\bar{z}^1 - M|}{|z^1\bar{z}^1 - M - N|}. \tag{6.9}$$

Thus, the wall between the two boundary conditions is given by

$$|z^2|^2 = \frac{M + N - z^1\bar{z}^1}{z^1\bar{z}^1 - M}. \tag{6.10}$$

The same analysis applied to the $AdS_5 \times S^5$ solution gives

$$|z^1|^2 + |z^2|^2 = N. \tag{6.11}$$

For the pair of geometries described above, we know the wave function in the eigenvalue description. We will now return to the eigenvalue description and see how these surfaces are related to the eigenvalue wave functions.

At large N, since fluctuations are controlled by $1/N^2$, we expect a definite eigenvalue distribution. These eigenvalues will trace out a surface specified by the support of the single fermion probability density

$$\rho(z_1, \bar{z}_1, y_1, \bar{y}_1) = \int \prod_{i=2}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})|^2. \tag{6.12}$$

Denote the points lying on this surface using coordinates z, y.

Using the wave function $\Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$ corresponding to the $AdS_5 \times S^5$ spacetime, the probability density for a single eigenvalue is

$$\rho(z, \bar{z}, y, \bar{y}) = \frac{1}{N\pi^2}\sum_{i=0}^{N-1}\frac{(z\bar{z})^i}{i!}\frac{(y\bar{y})^{N-i-1}}{(N-i-1)!}e^{-z\bar{z}-y\bar{y}} = \frac{(z\bar{z} + y\bar{y})^{N-1}}{N\pi^2(N-1)!}e^{-z\bar{z}-y\bar{y}}, \tag{6.13}$$

which is maximized at

$$z\bar{z} + y\bar{y} = N - 1. \tag{6.14}$$

Thus, if we identify the points z, y with the supergravity coordinates z^1, z^2 as follows:

$$z^2 = y, \qquad z^1 = z, \tag{6.15}$$

we find

$$|z^1|^2 + |z^2|^2 = N \tag{6.16}$$

at large N, so that the eigenvalues condense on the surface that defines the wall between the two boundary conditions.

Let us now compute the positions of our eigenvalues, using $\Psi_{\rm LLM}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$. The probability density for a single eigenvalue is easily obtained by computing the following integral:

$$\rho(z_1, \bar{z}_1, y_1, \bar{y}_1) = \int \prod_{i=2}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm LLM}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})|^2 = \frac{1}{N\pi^2}\sum_{i=0}^{N-1}\frac{(z_1\bar{z}_1)^{M+i}}{(M+i)!}\frac{(y_1\bar{y}_1)^{N-i-1}}{(N-i-1)!}e^{-z_1\bar{z}_1-y_1\bar{y}_1}. \tag{6.17}$$

Following the analysis we performed above, we find that the probability density is maximized when the following relations are satisfied:

$$\sum_{i=0}^{N-1}\left[\frac{(y\bar{y})^{N-i-1}}{(N-i-1)!}\left(\frac{(z\bar{z})^{M+i-1}}{(M+i-1)!} - \frac{(z\bar{z})^{M+i}}{(M+i)!}\right)\right] = 0, \tag{6.18}$$

$$\sum_{i=0}^{N-1}\left[\frac{(z\bar{z})^{M+i}}{(M+i)!}\left(\frac{(y\bar{y})^{N-i-2}}{(N-i-2)!} - \frac{(y\bar{y})^{N-i-1}}{(N-i-1)!}\right)\right] = 0. \tag{6.19}$$

The above holds only if each term in each sum is zero. We find

$$z\bar{z} = M + i, \qquad y\bar{y} = N - i - 1, i = 1, \qquad 2, \ldots, N - 1. \tag{6.20}$$

Thus, if we identify the points z, y and the supergravity coordinate z^1, z^2 as follows:

$$z^2 = \frac{y}{\sqrt{|z|^2 - M}}, \qquad z^1 = z, \tag{6.21}$$

we find that (6.10) gives

$$\frac{|y|^2}{i} = \frac{M + N - |z|^2}{|z|^2 - M} \tag{6.22}$$

in complete agreement with where our wave function is localized. This again shows that the eigenvalues are collecting on the surface that defines the wall between the two boundary conditions. Although these examples are rather simple, they teach us something important: the map between the eigenvalues and the supergravity coordinates depends on the specific geometry we consider.

The fact that eigenvalues condense on the surface that defines the wall between the two boundary conditions is something that was already anticipated by Berenstein and Cotta in [33]. The proposal of [33] identifies the support of the eigenvalue distribution with the degeneration locus of the three sphere in the full ten-dimensional metric. Our results appear to be in perfect accord with this proposal.

VII. OUTLOOK

There are a number of definite conclusions resulting from our study. One of our key results is that we have found substantial evidence for the proposal that there is a sector of the two matrix model that is described (sometimes exactly) by eigenvalue dynamics. This is rather nontrivial since, as we have already noted, it is simply not true that the two matrices can be simultaneously diagonalized. The fact that we have reproduced correlators of operators that involve products of both matrices in a single trace is convincing evidence that we are reproducing genuine two matrix observables. The observables we can reproduce correspond to BPS operators. In the dual gravity these operators map to supergravity states corresponding to classical geometries. The local supersymmetric geometries with $SO(4)\times U(1)$ isometries are determined by a surface that defines the boundary conditions needed to obtain a nonsingular supergravity solution. At large N where we expect classical geometry, the eigenvalues condense on this surface. In this way the supergravity boundary conditions appear to match the large N eigenvalue description perfectly.

The eigenvalue dynamics appears to provide some sort of a coarse grained description. Correlators of operators dual to states with a very small energy are not reproduced correctly: for example, the energy of states dual to single traces has to be above some threshold (N) before they are correctly reproduced. For complicated operators with a detailed multitrace structure we would thus expect to get the gross features correct, but we may miss certain finer details—see the discussion after (4.32). Developing this point of view, perhaps using the ideas outlined in [38], may provide a deeper understanding of the eigenvalue wave functions.

The eigenvalue description we have developed here is explicit enough that we could formulate the dynamics in terms of the density of eigenvalues. This would provide a field theory that has $1/N$ appearing explicitly as a coupling. It would be very interesting to work out, for example, what the generalization of the Das-Jevicki Hamiltonian [49] is.

The picture of eigenvalue dynamics that we are finding here is almost identical to the proposal discussed by Berenstein and his collaborators [31–37], developed using numerical methods and clever heuristic arguments. The idea of these works is that the eigenvalues represent microscopic degrees of freedom. At large N one can move to collective degrees of freedom that represent the ten-dimensional geometry of the dual gravitational description. This is indeed what we are seeing. They have also considered cases with reduced supersymmetry and orbifold geometries [50–52]. These are natural examples to consider using the ideas and methods we have developed in this article. Developing other examples of eigenvalue dynamics will allow us to further test the proposals for wave functions and the large N distributions of eigenvalues that we have put forward in this article.

An important question that should be tackled is to ask how one could derive (and not guess) the wave functions we have described. Progress with this question is likely to give some insights into how it is even possible to have a consistent eigenvalue dynamics. One would like to know when an eigenvalue description is relevant and to what classes of observables it is applicable.

Another important question is to consider the extension to more matrices, including gauge and fermion degrees of freedom. The Gauss graph labeling of operators continues to work when we include gauge fields and fermions [53,54], so that our argument goes through without modification and we again expect that eigenvalue dynamics in these more general settings will be an effective approach to compute these more general correlators of BPS operators. Another important extension is to consider the eigenvalue dynamics, perturbed by off diagonal elements, which should allow one to start including stringy degrees of freedom. Can this be done in a controlled systematic fashion? In this context, the studies carried out in [55–57] will be relevant.

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