Higher dimensional higher derivative ϕ^4 theory

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(Received 27 May 2017; published 26 July 2017)

We construct several towers of scalar quantum field theories with an $O(N)$ symmetry which have higher derivative kinetic terms. The Lagrangians in each tower are connected by lying in the same universality class at the d-dimensional Wilson-Fisher fixed point. Moreover the universal theory is studied using the large N expansion and we determine d-dimensional critical exponents to $O(1/N^2)$. We show that these new universality classes emerge naturally as solutions to the linear relation of the dimensions of the fields deduced from the underlying force-matter interaction of the universal critical theory. To substantiate the equivalence of the Lagrangians in each tower we renormalize each to several loop orders and show that the renormalization group functions are consistent with the large N critical exponents. While we focus on the first two new towers of theories and renormalize the respective Lagrangians to 16 and 18 dimensions there are an infinite number of such towers. We also briefly discuss the conformal windows and the extension of the ideas to theories with spin-$\frac{1}{2}$ and spin-1 fields as well as the idea of lower dimension completeness.

I. INTRODUCTION

The Wilson-Fisher fixed point of scalar quantum field theories has provided a remarkable basis for tackling a variety of different problems in physics [1–4]. The most widely known example is the use of the Wilson-Fisher fixed point underlying the Ising model as well as the superfluid phase transition, dilute polymer solutions, and the Heisenberg ferromagnet. Information about the properties of their phase transitions can be accessed by the continuum scalar quantum field theory with a quartic interaction. When endowed with an $O(N)$ symmetry the $N = 1$ case corresponds to the Ising model whereas the ferromagnet is described by the value of $N = 3$. Equally dilute polymer solutions and superfluidity correspond to the cases of $N = 0$ and 2 respectively. The connection with the Heisenberg magnet is perhaps remarkable, for instance, in that information on the phase transition in three dimensions can be obtained by renormalizing the $O(N)$ ϕ^4 scalar theory in *four* dimensions. Central to this connection is the Wilson-Fisher fixed point and the underlying critical renormalization group equation [1–4], which is a core property in d dimensions. Moreover underlying the transition is a universal quantum field theory which is central to the theories at the Wilson-Fisher fixed point. Put another way in the approach to three dimensions through the ϵ expansion, where the spacetime dimension is $d = 4 - 2\epsilon$, only the quartic operator present in the $O(N)$ scalar theory is relevant. In practical terms to obtain accurate information on the phase transition properties, one would have to know the renormalization group functions of $O(N)$ ϕ^4 theory to a large loop order in four dimensions where it is renormalizable. This has been achieved over many years in the advance to five loop

renormalization group functions in the modified minimal subtraction ($\overline{\text{MS}}$) scheme in [5–10]. More recently these calculations have been extended to *six* loops in [11–14]. From a bigger perspective our understanding of the universality property of this particular Wilson-Fisher fixed point has been extended above and beyond four dimensions in recent years.

Similarly the connection to the $O(N)$ nonlinear sigma model, which is renormalizable in two dimensions, is already well established and the extension to six dimensions was achieved in [15–17] based on the early work of [18,19] and eight dimensions in [20]. The connections were constructed by explicit perturbative renormalization of the respective higher dimensional $O(N)$ symmetric scalar theories and, in addition, knowledge of the critical exponents of the universal theory in arbitrary spacetime dimensions. The latter is possible through an expansion akin to conventional perturbation theory but where the expansion parameter is dimensionless similar to the perturbative coupling constant in the critical dimension of a theory. Specifically this is the parameter $1/N$ when an $O(N)$ symmetry is present. When N is large then $1/N$ is small and this can be regarded as a perturbative parameter which unlike a coupling constant is dimensionless in any spacetime dimension. What is also remarkable is that the core critical exponents have been determined at a fixed point, such as the Wilson-Fisher one, to *three* orders in $1/N$ [21–24]. Also these exponents, which are renormalization group invariants and known as exact functions of d at each order in $1/N$, have a correspondence with the renormalization group functions of the critical spacetime dimension of each of the theories in the same universality class.

Therefore evaluating the exponents in the ϵ expansion near each critical dimension provides information on each set of renormalization group functions of the theory in that critical dimension. Equally knowledge of the latter at a particular loop order establishes the connection of the six and higher dimensional quantum field theories with the same universality class as $O(N)$ ϕ^4 theory and the $O(N)$ nonlinear sigma model. From a Lagrangian point of view what is central in the universal theory is a core interaction which in essence has a force-matter structure of the form $\sigma\phi^i\phi^i$. Here ϕ^i is the $O(N)$ scalar field and σ is a second scalar field representing the force mediating particle. This interaction drives the Wilson-Fisher fixed point dynamics and is the only interaction in the universal theory used for the computation of the d-dependent critical exponents in the $1/N$ expansion. Although each Lagrangian in the critical spacetime dimension will have this same core interaction, they differ in the other interactions. By performing a dimensional analysis of the Lagrangians of the higher dimensional theories, one can abstract the essence of those extra interactions. They will only involve σ self-interactions which will differ from dimension to dimension. We will term these as spectator interactions in contrast to core, due to the sense that they only participate or are activated in one specific dimension. Their absence in the Lagrangian at its critical dimension would render the theory nonrenormalizable and hence not connected to the universal theory. This resulting tower of theories in the $O(N)$ scalar theory universality tree is now well established from high order perturbative and $1/N$ computations. For example, see [15–17,20].

One of the more recent applications of the $1/N$ formalism of [21–23] is its connection with d-dimensional conformal field theories. Earlier work in this direction was provided in [25–28] for example. In this approach it is hoped that the ideas of two dimensional conformal field theories can be developed in dimensions greater than two and the $1/N$ tool assists this activity. Equally a less well-explored aspect of scalar and other conformal field theories is that where the kinetic term of the scalar field involves higher derivatives. One recent study of tying these two threads together was in [29–31]. Although [29–31] focused on the free $\square^2$ conformal field theories and its symmetry algebra, this has to be appreciated first before the extension to interacting theories. Moreover that work was motivated by connections with AdS/CFT ideas. It is therefore the purpose of the article to contribute to this debate by examining scalar field theories using the $1/N$ expansion formalism of [21,22] in a new way. We will show how one can construct new towers of interacting theories based around distinct universal theories and connected by a Wilson-Fisher fixed point. As the initial distinction is at the level of the ϕ^i field kinetic term, there will in principle be a countably infinite number of new universality classes but we will concentrate on the first

few of these in detail. As the starting point of our description of the Wilson-Fisher fixed point is $O(N)$ ϕ^4 scalar theory, we will term these new towers of theories as higher derivative ϕ^4 theories. Part of our construction will involve computing several large N critical exponents for the first two new towers explicitly using the methods of [21,22]. Then to establish the connection with the respective field theories in specific dimensions we will construct the Lagrangians and renormalize them to several loop orders. In some cases this will be to three loops and the tower will extend to 16 and 18 spacetime dimensions respectively.

Having these explicit connections with higher derivative theories can serve at the very least as an alternative forum or laboratory to test new ideas in connections with conformal field theories as there is an underlying Wilson-Fisher fixed point. Our study will not be the first in this area. For instance, higher derivative nonlinear sigma models were considered in [32,33] where various one loop β-functions were computed. More recently, similar ideas have been put forward in [34,35]. Although these provide valuable insights, our analysis has the benefit of cross-connecting theories dimensionally. Our main focus throughout will be on scalar theories but we will not restrict the discussion solely to these. Theories which have similar $1/N$ expansions in the formalism of [21,22] will be discussed in part. The key two are the $O(N)$ Gross-Neveu model [36] and the non-Abelian Thirring model [37,38]. These two theories which are renormalizable in two dimensions serve as the base Lagrangian in the large N formalism and are in the same universality class as four dimensional Gross-Neveu-Yukawa [39] and quantum chromodynamics (QCD) respectively. In each case the matter field is fermionic but this is no obstacle to having a parallel higher derivative kinetic term which we will discuss here. Indeed an early construction of a renormalizable extension of a fermion field involving higher derivatives was noted in [40]. A final issue which we will touch on as well, since it will be a manifestly new feature of the higher derivative ϕ^4 theories, is that of what we will call lower dimension completeness. The extension of ϕ^4 theory to six and higher dimensions at the standard Wilson-Fisher fixed point is called ultraviolet completion. As the higher derivative $O(N)$ ϕ^4 theories will have a critical dimension greater than four, a natural question to ask is what would be the lower dimensional theory in the higher derivative universality class. In some sense this is in the opposite direction to the ultraviolet completion in terms of dimensions.

The article is organized as follows. We review the essential aspects of the large N method of [21–23] as well as introducing the new solutions which lead to the higher derivative $O(N)$ theories in the following section. Once these new universal large N Lagrangians have been constructed we compute various d-dimensional critical

exponents in the $1/N$ expansion in Sec. III. In order to compare the information contained within these exponents, Sec. IV is devoted to constructing the towers of ultraviolet complete Lagrangians for the two main extensions of higher derivative Lagrangians we concentrate on. The explicit renormalization of these Lagrangians is carried out in Sec. V. We also briefly discuss the status of the conformal windows in these theories there. Section VI lays the groundwork for the extension of these ideas to theories with fermions which are also accessible via the large N expansion. As the higher derivative ϕ^4 core Lagrangians are renormalizable in a dimension higher than the canonical one in four dimensions, there is the possibility of having lower dimensional theories in the same universality class. These Lagrangians are tentatively explored in Sec. VII while we provide conclusions in Sec. VIII.

II. BACKGROUND

In order to motivate the connection which a higher derivative ϕ^4 theory has with its canonical four dimensional counterpart, we review the basic Lagrangian within the large N formalism. The four dimensional renormalizable Lagrangian is

$$L^{(4)} = \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{8} g_1^2 (\phi^i \phi^i)^2 \qquad (2.1)$$

where g_1 is a dimensionless coupling in four dimensions and throughout $1 \le i \le N$. While this version is the one widely used to construct the renormalization group functions [5–10], the interaction can be rewritten in terms of an auxiliary field σ to produce the equivalent Lagrangian

$$L^{(4)} = \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{g_1}{2} \sigma \phi^i \phi^i - \frac{1}{2} \sigma^2. \qquad (2.2)$$

It is this version of the original scalar quartic theory which is the starting point for the large N construction provided by Vasil'ev *et al.*, [21–23]. In this approach the critical exponents of the theory are determined directly from the field theory as a series in powers of $1/N$ without using the perturbative renormalization group functions. Defining the full dimensions of the fields ϕ^i and σ by α and β respectively then [21,22]

$$\alpha = \mu - 1 + \frac{1}{2} \eta, \qquad \beta = 2 - \eta - \chi \qquad (2.3)$$

where $d = 2\mu$. The canonical dimensions of the fields are determined by a dimensional analysis of the Lagrangian with the proviso that the action S is dimensionless. The spacetime dimension enters via the d-dimensional measure associated with the relation between the Lagrangian and S. The other quantities, η and χ, correspond to critical exponents and represent or are a measure of the quantum

corrections. They are referred to as anomalous dimensions. The exponent associated with ϕ^i is η whereas χ is the anomalous dimension of the vertex operator $\sigma \phi^i \phi^i$.

Strictly in the Vasil'ev *et al.* definition of the canonical dimensions, the Lagrangian

$$\bar{L}^{(4)} = \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2} \bar{\sigma} \phi^i \phi^i - \frac{1}{2g_1^2} \bar{\sigma}^2 \qquad (2.4)$$

is used [21,22], where the σ field is rescaled by a power of the coupling. In this version of $L^{(4)}$ the $\bar{\sigma} \phi^i \phi^i$ interaction is central and drives the universality class across the dimensions where the $O(N)$ nonlinear sigma model, $O(N)$ ϕ^4 theory and $O(N)$ ϕ^3 theory in six dimensions [15,16], are all critically equivalent at the Wilson-Fisher fixed point together with higher dimensional extensions [20]. Therefore $L^{(4)}$ is the version which was used in [21–23] to define the critical point propagators of the two fields which are

$$\langle \phi^i(x) \phi^j(y) \rangle \sim \frac{A \delta^{ij}}{((x-y)^2)^\alpha},$$

$$\langle \sigma(x) \sigma(y) \rangle \sim \frac{B}{((x-y)^2)^\beta} \qquad (2.5)$$

in coordinate space where A and B are x and y independent amplitudes. Within computations they always appear in the combination $z = A^2 B$. It is important to recognize that these are the propagators in the asymptotic approach to the critical point which is why they do not involve equality symbols. As noted in [22] there are corrections to scaling. While the canonical exponents are determined by dimensional arguments, expressions for η and χ are deduced by solving the skeleton Dyson-Schwinger equations at criticality where the integrals are regularized analytically [21–23]. This is a crucial point as one is in effect performing a perturbative expansion in the anomalous dimension of the vertex operator. Therefore the regularization is formally introduced by the shift $\chi \to \chi + \Delta$ where Δ is the regularizing parameter [21–23]. The advantage of this is that the usual dimensional regularization of perturbation theory is not used. So the underlying Feynman integrals are evaluated as *exact* functions of d at each order in the $1/N$ expansion. The notation introduced in [21] reflects this since

$$\eta = \sum_{i=1}^{\infty} \frac{\eta_i(\mu)}{N^i} \qquad (2.6)$$

with similar expansions for other quantities such as χ and z. In other words the coefficients of the expansion parameter are functions of the spacetime dimension.

At this point it is worth making the connection of the large N formulation with the conventional perturbative approach. The exponents, such as η which has been determined to $O(1/N^3)$ in [21–23], are renormalization group invariants and thus are more fundamental than the renormalization group functions themselves. However both are related since the evaluation of a renormalization group function at a value of the coupling defined by a zero of the β-function defines a critical exponent. In other words we have relations such as

$$\eta = \gamma_\phi(g_c) \qquad (2.7)$$

where $\gamma_\phi(g)$ is the renormalization group anomalous dimension of the field ϕ^i. Here g_c is the critical coupling defined by $\beta(g_c) = 0$ and we use g to represent either a single coupling, such as g_1 in (2.1), or a vector of couplings for theories where there is more than one coupling constant which will arise later. While this outlines the essence of the critical point renormalization group relevant to large N, the final aspect which makes the connection is that the solution to $\beta(g_c) = 0$ is determined from the d-dimensional β-function. One of the solutions to this equation will be the Wilson-Fisher fixed point. In carrying out the classical dimensional analysis of the Lagrangian above to determine the canonical dimensions of the field, we omitted the analysis of the coupling constant. Ordinarily it derives from the interaction in (2.1) but in the large N context one uses the quadratic term in (2.4). In either situation one finds that the dimension of the coupling is $(d-4)$ in d dimensions. In other words the critical dimension, from the point of view of renormalizability, is four and in that dimension the coupling is dimensionless. This is reflected in the d-dimensional β-function whose first term will be proportional to $(d-4)g$ with the actual constant depending on the convention used to define the β-function. The dimensional dependence of the remaining part of the β-function depends on the renormalization scheme which is chosen. In general the coefficients of the one and higher loop terms in the series will be of the form $A_n + B_n(d-4)$ where A_n is related to the residue of the simple pole in the regularizing parameter in the coupling constant renormalization constant while B_n is connected to its finite part. In the strictly four dimensional β-function only the sequence of numbers A_n would be present and the first term of $\beta(g)$ would involve A_1. As an aside we note that this description of the structure of the d-dimensional β-function applies to all the other renormalization group functions. The finite part of the renormalization constant appears in the same way in the d-dimensional expression. The relevance of these finite parts appearing in the renormalization group functions emerges when one determines the underlying critical exponents in whatever scheme one has performed the perturbative renormalization. Solving $\beta(g_c) = 0$ in d dimensions and determining $\gamma_\phi(g_c)$, the overall expression

has to be scheme independent as the critical exponents are renormalization group invariants. Omitting the respective B_n sequences from the renormalization group functions would lead to a contradiction of this. For explicit three loop examples we refer the interested reader to [41] where the β-functions of QCD were examined in this context in the $\overline{\text{MS}}$ and the momentum subtraction (MOM) schemes of Celmaster and Gonsalves [42,43].

We have discussed the background to the d-dimensional renormalization group functions at length as it is relevant to the relation to the d-dimensional large N critical exponents of [21–23]. In our discussion we have been careful in our description. First, the role of the renormalization scheme is relevant in the d-dimensional renormalization group functions. However we will not comment any further on this here as throughout we will use the $\overline{\text{MS}}$ scheme for all our perturbative computations. We recall that in the $\overline{\text{MS}}$ scheme the renormalization constants are defined at a specific subtraction point by removing only those terms which are divergent with respect to the parameter of the regularization. This naturally leads to the second aspect of our discussion which is that at no point did we refer to a specific regularization let alone dimensional regularization which is the one widely used for explicit perturbative renormalization at higher loop order. In this regularization the critical dimension of the field theory is replaced by a complex variable d which is then written in our four dimensional case as $d = 4 - 2\epsilon$ where ϵ is the regularizing parameter of dimensional regularization. The use of d here differs in sense from that used in our earlier description of the d-dimensional renormalization group functions. For instance, if one used a different regularization to dimensional regularization, such as lattice regularization, and renormalized in a MOM scheme, for example, then the finite parts of the renormalization constant represented by B_n would appear in the corresponding d-dimensional renormalization group functions. In other words there is a clear demarcation between use of dimensional regularization and the construction of the renormalization group functions in d-dimensional spacetime. In the context of the universal theory represented by the Lagrangian used for the large N expansion, (2.4), it is the d-dimensional renormalization group functions which are used to compare with the exponents defined by the relations like (2.7). We have discussed this issue at length due to its subtle nature. Therefore, we summarize the situation by clarifying that the exponents derived in the large N expansion for the universal theory are functions of the actual spacetime dimension d itself, which is completely arbitrary, and the exponents are derived in an *analytically* regularized Lagrangian. By contrast in perturbation theory using ϵ within dimensional regularization it is regarded as being a small quantity.

The connection between the large N exponents and the perturbative renormalization group functions comes

through the definitions such as (2.7). The quantities in (2.7) are functions of d and N and can be expanded as a double Taylor series in the respective parameters ϵ and $1/N$ which are small and dimensionless. Here ϵ is not being used as a regularizing parameter in the strict sense as it would be in dimensional regularization as the exponents are the outcome of the large N formalism with the universal theory analytically regularized for all d. Instead we have set $d = D - 2\epsilon$ where D is an integer and represents the critical dimension of one of the theories in the tower connected by the universal interaction. If one computes the renormalization group functions for a specific theory with critical dimension D in that tower then expands the exponent in powers of ϵ at each order in $1/N$ for both the large N derived approximation to the exponent and the perturbative renormalization group function determined at the critical D Wilson-Fisher fixed point, one finds total agreement of both expansions. Of course in practical terms one only knows several orders in the respective expansions, but this is usually sufficient to establish consistency. As such this approach has become a standard check on explicit high loop order perturbative expansions where the theory has an $O(N)$ or similar symmetry with a *dimensionless* parameter. Equally it has been used to determine coefficients in the polynomial of N at higher loop orders which have not been evaluated perturbatively. Recent examples are the determination of the six loop $O(N)$ ϕ^4 β-function and other renormalization group functions in the $\overline{\text{MS}}$ scheme [5–14]. The $O(\epsilon^6)$ terms of the derived exponents of [11–14] were in exact agreement with the various $O(1/N^2)$ and $O(1/N^3)$ exponents of [21–24]. One of our aims here is to take the sequence of tower of theories to a new level but also provide the corresponding explicit perturbative and large N results to establish that our underlying universal theory is correct.

Before we proceed to that level of verification we need to first construct the relevant Lagrangians which populate the tower along a common Wilson-Fisher fixed point thread in d dimensions. One way to proceed is to do this for a sequence of critical dimensions by using the universal interaction as the basis for defining the canonical dimensions of the fields for a specific D and then construct the spectator part of the Lagrangian which ensures renormalizability. This will systematically build a tower in much the same way as the ultraviolet completion of $O(N)$ ϕ^4 theory to six dimensions [15,16] and higher [17]. However, we have chosen to begin at another point which is within the universal theory itself but at the critical point. For the established tower (2.4) we noted that the canonical dimensions of the fields were determined by dimensionally analyzing the kinetic term for ϕ^i and the universal interaction to produce the full dimensions of the fields which are α and β and are given by (2.3). They satisfy

$$2\alpha + \beta = d - \chi. \tag{2.8}$$

However, this is not the only way to consider the dimensional analysis within the universal theory. Instead of using the kinetic term for ϕ^i and (2.8) to find (2.3) we can write down a sequence of solutions to (2.8) which includes (2.3) as a specific case. This solution is

$$\alpha = \mu - n + \frac{1}{2}\eta, \qquad \beta = 2n - \eta - \chi \tag{2.9}$$

where n is any positive integer. Viewed this way means that one opens up new threads of towers since the kinetic term for ϕ^i will involve higher derivatives. Although unlike the scalar theories considered in [29–31], for example, these will be interacting theories with a conformal symmetry at a fixed point. More specifically the first few Lagrangians representing the universal theories will be

$$\bar{L}^{(8)} = \frac{1}{2}(\Box\phi^i)^2 + \frac{1}{2}\bar{\sigma}\phi^i\phi^i - \frac{1}{2g_1^2}\bar{\sigma}^2 \tag{2.10}$$

and

$$\bar{L}^{(12)} = \frac{1}{2}(\Box\partial_\mu\phi^i)^2 + \frac{1}{2}\bar{\sigma}\phi^i\phi^i - \frac{1}{2g_1^2}\bar{\sigma}^2 \tag{2.11}$$

for $n = 2$ and 3 in the formulation with the coupling constant adjustment used for the large N expansion. Eliminating the auxiliary σ field produces

$$L^{(8)} = \frac{1}{2}(\Box\phi^i)^2 + \frac{g_1^2}{8}(\phi^i\phi^i)^2 \tag{2.12}$$

and

$$L^{(12)} = \frac{1}{2}(\Box\partial_\mu\phi^i)^2 + \frac{g_1^2}{8}(\phi^i\phi^i)^2 \tag{2.13}$$

which are the higher derivative extensions of scalar $O(N)$ ϕ^4 theory. We have dropped the overline on the Lagrangian since the $\bar{\sigma}$ field is absent. Our notation is that the Lagrangian $L^{(D)}$ is perturbatively renormalizable in D dimensions with critical dimension $4n$ in terms of the solution (2.9). So the critical dimension for the $n = 2$ thread is 8 and for $n = 3$ it is 12. This explains our notation for the usual quartic $O(N)$ scalar theory of (2.1). As an aside one can in principle have a solution with $n = 0$ which would have the formal Lagrangian

$$L^{(0)} = \frac{1}{2}\phi^i\phi^i + \frac{g_1^2}{8}(\phi^i\phi^i)^2. \tag{2.14}$$

However, this is in effect trivial as there are no spacetime variables to provide arguments for the fields. At best one is counting graphs with a quartic interaction.

III. LARGE N EXPONENTS

Having introduced new sets of quartic scalar theories with a critical dimension $D = 4n$ and an $O(N)$ symmetry, it is possible to determine the various critical exponents of each theory in the same way that the $n = 1$ exponents are known [21–23]. Indeed it turns out that the leading order exponents for the fields as well as that for η_2 can be immediately deduced from [21,22] for any positive value of n. This is because in the construction of Vasil'ev *et al.* the $O(1/N^2)$ diagrams contributing to η_2 were computed as functions of α and β using only (2.8) in the derivation before (2.9) was substituted in the final step to find the values of the exponents for the $n = 1$ case of interest. One aspect of the evaluation of the $O(1/N^2)$ graphs in [21,22] was the use of what is termed uniqueness or conformal integration. This was originally introduced in [44] in strictly three dimensions and then extended by others to d dimensions. In the coordinate space formulation of [21,22] the rule in brief is that if the sum of the exponents of the lines joining a 3-point vertex is equal to the spacetime dimension then the integral over the vertex location can be performed. In the large N context for $n = 1$ this was exploited in [21,22]. However, since the canonical dimensions of our fields for our higher n solutions satisfy the same uniqueness condition independently of n, then the use of uniqueness for general α and β in

the derivation of the $O(1/N^2)$ exponents can be simply used for $n = 2$ and 3. Therefore, we have revisited [22] and determined η_1, η_2, and χ_1 for these specific values as examples. These will be needed for later in order to allow us to check off explicit perturbative results in the respective tower of theories which have each of $L^{(8)}$ and $L^{(12)}$ as the seed Lagrangians. It is worth contrasting the use of uniqueness here with another aspect of the conformal integration rule. This is that there is not one condition for a coordinate space vertex to be integrable. If the exponents at a vertex sum to the spacetime dimension plus a positive integer then the vertex can be integrated. See [45], for example, for lectures on this construction. Although, the resulting expression may be cumbersome. While it is possible to consider theories based on the one step from uniqueness criterion, it is not our main focus here.

For $L^{(8)}$ the exponents which determine the leading order behavior of scaling behavior of the respective fields are

$$\eta_1^{(8)} = -\frac{6[\mu - 1][\mu - 4]\Gamma(2\mu - 3)}{\Gamma^2(\mu + 2)\Gamma(\mu)\Gamma(-1 - \mu)},$$

$$\chi_1^{(8)} = -\frac{\mu[\mu + 1][4\mu^2 - 30\mu + 47]}{9[\mu - 3][\mu - 4]}\eta_1^{(8)} \qquad (3.1)$$

where we use the notation $\eta_i^{(D)}$ for exponents so as to be clear that they are derived from $L^{(D)}$. Extending the formalism of [21,22] to the next order we find

$$\eta_2^{(8)} = \left[-\frac{[2\mu^4 - 13\mu^3 - 2\mu^2 + 85\mu - 108]}{9[\mu - 3][\mu - 4]}[B(3 - \mu) - B(\mu - 1)] + [4\mu^{10} - 72\mu^9 + 433\mu^8 - 697\mu^7 - 3085\mu^6 + 15845\mu^5 \right.$$
$$\left. - 26504\mu^4 + 11816\mu^3 + 15436\mu^2 - 16416\mu + 2592]/[18[\mu + 1][\mu - 1][\mu - 2][\mu - 3]^2[\mu - 4]^2\mu] \right]\eta_1^{(8)2} \qquad (3.2)$$

where

$$B(z) = \psi(z) + \psi(\mu - z) \qquad (3.3)$$

and $\psi(z) = \frac{d \ln \Gamma(z)}{dz}$. Compared to the same exponents for the $n = 1$ case these expressions are more involved. This is because in the derivation the arguments of the Γ- and ψ-functions will involve n. This is more apparent in the $n = 3$ case since we have

$$\eta_1^{(12)} = -\frac{80[\mu - 1][\mu - 2][\mu - 6]\Gamma(2\mu - 5)}{\Gamma^2(\mu + 3)\Gamma(\mu)\Gamma(-2 - \mu)},$$

$$\chi_1^{(12)} = -\frac{\mu[\mu + 1][\mu + 2][4\mu^4 - 92\mu^3 + 767\mu^2 - 2722\mu + 3453]}{150[\mu - 4][\mu - 5]^2[\mu - 6]}\eta_1^{(12)} \qquad (3.4)$$

at leading order in $1/N$ and

$$\eta_2^{(12)} = \left[-\frac{[4\mu^7 - 80\mu^6 + 499\mu^5 - 890\mu^4 + 2791\mu^3 - 39980\mu^2 + 153756\mu - 180000]}{300[\mu-4][\mu-5]^2[\mu-6]}[B(6-\mu) - B(\mu-3)] \right.$$
$$- [24\mu^{14} - 648\mu^{13} + 6813\mu^{12} - 38367\mu^{11} + 197774\mu^{10} - 1486770\mu^9 + 9018717\mu^8 - 29983215\mu^7$$
$$+ 37756752\mu^6 + 56662008\mu^5 - 238972400\mu^4 + 227278992\mu^3 + 47272320\mu^2$$
$$\left. - 150912000\mu + 34560000]/[400[\mu+2][\mu+1][\mu-1][\mu-2][\mu-3][\mu-4]^2[\mu-5]^2[\mu-6]^2\mu] \right] \eta_1^{(12)2} \qquad (3.5)$$

at next order. We have included these exponents together with the renormalization group functions which are discussed later in an electronic format in an attached data file [46]. The situation for $n = 0$ is different in that the leading order large N equations of [21,22] involving α and β do not have a solution for this value of n. The first solution of substance is when $n = 1$.

Having derived these from the large N form of $L^{(8)}$ and $L^{(12)}$, they remain to be checked against explicit perturbative computations of the underlying renormalization group functions. For both theories we have computed the anomalous dimensions of ϕ^i together with the mass of ϕ^i and the β-function to the first two terms. This means two loops for the last two quantities but three loops for the field anomalous dimension. In a quartic theory there is no one loop contribution to the wave function renormalization and we require a nontrivial check on our large N exponents. The β-function is key to determining the location of the Wilson-Fisher fixed points in the respective ϵ expansions around $d = D - 2\epsilon$. To determine the renormalization group functions for the purely quartic theories, we used conventional perturbation theory. However this requires a modification to the treatment of the canonical four dimensional quartic scalar theory. Specifically the propagators of $L^{(8)}$ and $L^{(12)}$ are $1/(k^2)^2$ and $1/(k^2)^3$ respectively. For the renormalization of the 2-point function we used the integration by parts algorithm of Laporta [47] to reduce the Feynman graphs to a few basic master integrals which can then be evaluated by direct methods. To three loops the majority of these are simple integrals built out of basic bubbles which are easy to compute. Two three loop master integrals do not fall into this class but we have determined their ϵ expansions with respect to the critical dimension of each theory by applying the formalism developed by Tarasov in [48,49]. This allows one to connect the value of such master integrals in a dimension d say with that in dimension $(d + 2)$ as well as a linear combination of integrals with fewer propagators. Since the two three loop masters are known in four dimensions [50] then applying the Tarasov method in hand with the Laporta algorithm recursively allowed us to construct the required 2-point three loop masters in eight and twelve dimensions. For the β-function we used the same general approach but employed the vacuum bubble expansion method of [51,52]. The corresponding two loop master vacuum

bubble integrals were similarly deduced using Tarasov's method where the four dimensional two loop master of [53] was used as the base for extending to eight and twelve dimensions. It is worth addressing a concern which is that with the higher order propagator in each of these dimensions there may be infrared divergences. This is not the case. A propagator such as $1/(k^2)^2$ would be infrared problematic in four dimensions but not in six or higher dimensions. Equally integrals which involve $1/(k^2)^4$ would not be infrared safe in eight dimensions but in twelve dimensions the poles in ϵ emerging from an integral will represent purely ultraviolet divergences. In general a propagator of the form $1/(k^2)^\alpha$ is infrared safe in $d > 2\alpha$ dimensions. Finally, for the renormalization of (2.12) and (2.13) as well as the other theories we examined here we used QGRAF [54] to generate the Feynman diagrams in electronic form. This is necessary as we have designed an automatic Feynman graph routine which is written in the symbolic manipulation language FORM and TFORM [55,56]. The necessary integration by parts relations to obtain the master integrals were generated via the early and recent versions of the REDUZE package [57,58]. These relations were encoded in a FORM module to allow us simply to import the Feynman rules module corresponding to whichever theory we are interested in.

Consequently we find the first two terms of the renormalization group functions are

$$\gamma_\phi^{(8)}(g_1) = -\frac{[N+2]}{4320}g_1^4 + \frac{[N+2][N+8]}{9331200}g_1^6 + O(g_1^8),$$
$$\gamma_m^{(8)}(g_1) = -\frac{[N+2]}{36}g_1^2 - \frac{7[N+2]}{12960}g_1^4 + O(g_1^6),$$
$$\beta^{(8)}(g_1) = \frac{[N+8]}{36}g_1^4 + \frac{[41N+202]}{19440}g_1^6 + O(g_1^8) \qquad (3.6)$$

for (2.12) and

$$\gamma_\phi^{(12)}(g_1) = -\frac{[N+2]}{4354560}g_1^4 - \frac{293[N+2][N+8]}{2633637888000}g_1^6 + O(g_1^8),$$
$$\gamma_m^{(12)}(g_1) = -\frac{[N+2]}{720}g_1^2 - \frac{29[N+2]}{3402000}g_1^4 + O(g_1^6),$$
$$\beta^{(12)}(g_1) = \frac{[N+8]}{720}g_1^4 + \frac{[255N+1112]}{9072000}g_1^6 + O(g_1^8) \qquad (3.7)$$

for the twelve dimensional case. Structurally these are similar to their four dimensional counterparts from the point of view of the factor $(N + 2)$ and $(N + 8)$. More interestingly, neither is asymptotically free in parallel with the $D = 4$ case. However the more intriguing aspect of the results in this section is that when one computes the respective critical exponents from (3.6) and (3.7) near their critical dimensions, the coefficients in the ϵ expansion are in one-to-one agreement with the same powers of ϵ and $1/N$ of η and χ. As the σ field couples to the operator $\phi^i \phi^i$ its anomalous dimension in the large N formalism is $\eta + \chi$ and the renormalization group function this corresponds to is $\gamma_m^{(D)}(g_1)$. This agreement between the large N exponents derived from the universal theory residing at the respective eight and twelve dimensional Wilson-Fisher fixed points and their explicit high order perturbative renormalization group functions is in keeping with the situation for $n = 1$. Moreover, it has in effect opened up more avenues to explore aspects of conformal theories in higher dimensions. In principal one can examine the generalization

$$L^{(4n)} = \frac{1}{2}(\partial_{\mu_1}...\partial_{\mu_n}\phi^i)(\partial^{\mu_1}...\partial^{\mu_n}\phi^i) + \frac{1}{8}g_1^2(\phi^i\phi^i)^2 \quad (3.8)$$

explicitly.

We have chosen to stop at twelve dimensions for the practical and not conceptual reason that there is a limitation to the size of the REDUZE database we constructed using the Laporta algorithm. There is no technical obstruction to proceeding further in principle. Instead the direction we take is to examine (3.1), (3.2), (3.4), and (3.5) in dimensions beyond the critical dimensions of each of the Lagrangians we have considered in order to make connection with the tower of theories which we aim to construct and verify that they lie in the same Wilson-Fisher universality class. For instance, if one expands (3.1) and (3.2) in $d = 10 - 2\epsilon$ then unlike in eight dimensions both η_1 and η_2 begin with $O(\epsilon)$ terms. So unlike the connection of these exponents with a quartic eight dimensional scalar theory where the exponents begin with $O(\epsilon^2)$, the ultraviolet completion of $L^{(8)}$ ought to be a cubic theory. This is because a cubic theory will have a one loop self-energy graph unlike a purely quartic theory. Equally expanding the exponents around dimensions 12, 14 and higher dimensions, the leading order terms are always $O(\epsilon)$ indicating the subsequent completions involve a basic cubic interaction. This is, of course, the seed interaction of (2.10). The position with (3.4) and (3.5) is completely similar in terms of the leading order terms in ϵ.

IV. ULTRAVIOLET COMPLETE LAGRANGIANS

In order to construct the ultraviolet completions or tower of theories based respectively on $L^{(8)}$ and $L^{(12)}$, it is instructive to review the position with the conventional Wilson-Fisher universality class. This is also to allow us to compare structures with the other towers we will construct. The key is the use of the canonical dimension of the two basic fields and the spacetime dimension the Lagrangian is to be completed in. In the large N expansion the canonical dimensions are necessarily dimension dependent as the universal theory is spacetime transcendent. For the theories which are renormalizable in a fixed (even) dimension, one has to use the canonical dimensions for that specific dimension. So when $n = 1$, σ has dimension 2 and ϕ^i has dimension 1, 2, 3, and 4 in the even dimensions between four and ten. One consequence is that in each of these dimensions the $\sigma\phi^i\phi^i$ operator dimension is preserved and moreover no new ϕ^i-σ interactions can be included. Instead in order to ensure renormalizability in each dimension, extra pure σ (spectator) interactions have to be added which can include derivative interactions. Given this reasoning we find the following higher dimensional extensions of $L^{(4)}$, (2.2),

$$L^{(4,6)} = \frac{1}{2}(\partial_\mu\phi^i)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3,$$

$$L_\phi^{(4,8)} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}(\Box\sigma)^2 + \frac{1}{2}g_1\sigma\phi^i\phi^i + \frac{1}{6}g_2\sigma^2\Box\sigma$$
$$+ \frac{1}{24}g_3^2\sigma^4,$$

$$L_\phi^{(4,10)} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}(\Box\partial^\mu\sigma)(\Box\partial_\mu\sigma) + \frac{1}{2}g_1\sigma\phi^i\phi^i$$
$$+ \frac{1}{6}g_2\sigma^2\Box^2\sigma + \frac{1}{2}g_3\sigma(\Box\sigma)^2 + \frac{1}{24}g_4^2\sigma^3\Box\sigma$$
$$+ \frac{1}{120}g_5^3\sigma^5. \quad (4.1)$$

Our notation of $L^{(d_1,d_2)}$ for these Lagrangians is to indicate the dimension of the base quartic theory, which is d_1, and the particular critical dimension, d_2, where it is renormalizable. This is to avoid confusion with the same constructions for $L^{(8)}$ and $L^{(12)}$. Given the structure of these Lagrangians we could equally well have labeled or classified them by the number of derivatives in the 2-point terms. However, we chose the former syntax as we wish to indicate the tower aspect of the construction. For each of these theories bar $L^{(4,10)}$ the respective renormalization group functions evaluated at the Wilson-Fisher fixed point and compared with the corresponding critical exponents of [21–24] are in full agreement to the orders they have been computed in perturbation theory. For $L^{(4,6)}$ this is to four loops [15–17] based on the pioneering work of [18,19]. The verification of $L^{(4,8)}$ was carried out in [20] to two loop order. That for $L^{(4,10)}$ will be considered here to add confidence in the overall vision of an underlying universal theory in d dimensions.

Again when the $n = 1$ construction is viewed in this light, it is straightforward to write down the Lagrangians in the tower with $L^{(8)}$ as the base theory. In this instance σ now has dimension 4 and ϕ^i has dimension 1, 2, 3, and 4 in eight, ten, twelve, and fourteen dimensions. The seed interaction again is the only one between ϕ^i and σ and ensuring the renormalizability is achieved by the spectator interactions which are purely σ dependent. Consequently we have

$$L^{(8,10)} = \frac{1}{2}(\Box\phi^i)^2 + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{g_1}{2}\sigma\phi^i\phi^i,$$

$$L^{(8,12)} = \frac{1}{2}(\Box\phi^i)^2 + \frac{1}{2}(\Box\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3,$$

$$L^{(8,14)} = \frac{1}{2}(\Box\phi^i)^2 + \frac{1}{2}(\Box\partial^\mu\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^2\Box\sigma,$$

$$L^{(8,16)} = \frac{1}{2}(\Box\phi^i)^2 + \frac{1}{2}(\Box^2\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^2\Box^2\sigma$$
$$+ \frac{g_3}{2}\sigma(\Box\sigma)^2 + \frac{g_4^2}{24}\sigma^4 \tag{4.2}$$

where only independent derivative interactions are included. Repeating the exercise for $L^{(12)}$ is similarly straightforward but in this case σ has canonical dimension 6. Up to eighteen dimensions we have

$$L^{(12,14)} = \frac{1}{2}(\Box\partial_\mu\phi^i)^2 + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{g_1}{2}\sigma\phi^i\phi^i,$$

$$L^{(12,16)} = \frac{1}{2}(\Box\partial_\mu\phi^i)^2 + \frac{1}{2}(\Box\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i,$$

$$L^{(12,18)} = \frac{1}{2}(\Box\partial_\mu\phi^i)^2 + \frac{1}{2}(\Box\partial_\mu\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3. \tag{4.3}$$

One of the reasons why we have included a range of Lagrangians built from the various base Lagrangians is to compare and contrast structural similarities. For instance the spectator Lagrangians of $L^{(4,6)}$, $L^{(8,12)}$, and $L^{(12,18)}$ are formally the same although the canonical dimension of the σ field is not the same in each case. This will generalize to the sequence $L^{(4n,6n)}$ but in the dimensions between $4n$ and $6n$ there are no spectator interactions only a change in the σ 2-point term. A similar situation arises for the derivative cubic and higher σ interactions as one proceeds up each tower. It is worth stressing at this stage that we have merely constructed sequences of higher dimensional renormalizable interacting Lagrangians founded on a quartic scalar theory with a higher derivative kinetic term. We now need to make the connection with our large N exponents in order to extend the Wilson-Fisher threads in this new context.

V. PERTURBATIVE RESULTS

We devote this section to computing the renormalization group functions to as high a loop order as is calculationally

viable for the Lagrangians we have constructed in Sec. V. To be consistent with other work on the $L^{(4)}$ thread we will use the same convention and notation as that used in [17]. In addition we will use the same underlying computational technology described in [17]. For instance, an efficient algorithm was used to easily access the renormalization of 3-point vertices. This exploited the fact that a propagator $1/(k^2)^\alpha$ is infrared safe in $d > 2\alpha$ dimensions to allow us to simply make the replacement

$$\frac{1}{(k^2)^p} \mapsto \frac{1}{(k^2)^p} + \frac{\xi g_i}{(k^2)^{p+1}} \tag{5.1}$$

for either a ϕ^i or σ propagator where g_i is the appropriate coupling constant and integer p. The parameter ξ is used to limit the expansion as one only requires a single insertion on a propagator to generate vertex graphs from the 2-point functions. As noted in [17] this approach is only applicable in the Lagrangians with 3-point vertices. Those Lagrangians with quartic and higher spectator interactions require a more direct approach such as that used for $L^{(4,8)}$ in [20]. For theories with such interactions we will use the same method. First, the renormalization of 2-point Green's functions will proceed as just described but the propagator shift is not included as it would omit graphs with quartic and higher interactions. In this case the 3-point vertex functions are renormalized by considering the Green's function at either a completely symmetric point or at a completely off-shell point. The former is appropriate to use when there is either nonderivative 3-point interactions or a single 3-point vertex. The off-shell configuration is used when there is more than one 3-point interaction and they involve derivative couplings as in the case of $L^{(4,10)}$. Equally the 4-point functions are treated by evaluating the vertex function at the completely symmetric point. Further background and calculational detail for both these instances can be found in [20]. Finally for the higher n-point functions the vacuum bubble method already outlined for the base ϕ^4 theories was used. In terms of loop orders the theories in the higher dimensions were renormalized mostly to two loops but to three loops for a few cases above the critical dimension of the base Lagrangian. This is because there are practical limitations in the construction of the databases we used to apply the Laporta and Tarasov algorithms [47–49]. The increase in the powers of the propagators means that to build the three loop 2-point masters beyond twelve dimensions, which requires a significant amount of integration by parts even for non-tensor integrals, was not viable. However, we take the point of view that it will be evident even with two loop renormalization group functions that the connection between all the theories will be established.

First, we record our results for the theories along the thread based on $L^{(8)}$. We have

$$\gamma_\phi^{(8,10)}(g_1) = \frac{g_1^2}{120} + [194N - 567]\frac{g_1^4}{864000} + [-37786N^2 - 259420N + 648000\zeta_3 + 505299]\frac{g_1^6}{21772800000} + O(g_1^8),$$

$$\gamma_\sigma^{(8,10)}(g_1) = -\frac{Ng_1^2}{60} + \frac{167Ng_1^4}{216000} + [259847N - 648000\zeta_3 + 256266]\frac{Ng_1^6}{10886400000} + O(g_1^8),$$

$$\beta^{(8,10)}(g_1) = [-N + 6]\frac{g_1^3}{240} + [-197N - 297]\frac{g_1^5}{288000} + [-859789N^2 + 25272000\zeta_3 N - 38231814N - 38232000\zeta_3$$

$$+ 43101039]\frac{g_1^7}{43545600000} + O(g_1^9) \tag{5.2}$$

for $L^{(8,10)}$. Our three loop results for $L^{(8,12)}$ are

$$\gamma_\phi^{(8,12)}(g_1, g_2) = \frac{g_1^2}{280} + [-1587Ng_1^2 - 9334g_1^2 - 6160g_1g_2 - 1587g_2^2]\frac{g_1^2}{197568000} + [-13130181N^2g_1^4 + 175046616Ng_1^4$$

$$+ 8890560000\zeta_3 g_1^4 - 9803169176g_1^4 + 425268522Ng_1^3g_2 + 712313280g_1^3g_2 + 76042962Ng_1^2g_2^2$$

$$+ 8890560000\zeta_3 g_1^2g_2^2 - 10046446142g_1^2g_2^2 - 150402798g_1g_2^3 + 209985345g_2^4]$$

$$\times \frac{g_1^2}{1254635827200000} + O(g_i^8),$$

$$\gamma_\sigma^{(8,12)}(g_1, g_2) = [Ng_1^2 + g_2^2]\frac{1}{560} + [-6254Ng_1^4 - 6160Ng_1^3g_2 - 1587Ng_1^2g_2^2 - 4667g_2^4]\frac{1}{197568000}$$

$$+ [930040938N^2g_1^6 + 8890560000\zeta_3 Ng_1^6 - 9419379728Ng_1^6 + 194624640N^2g_1^5g_2 + 491748768Ng_1^5g_2$$

$$- 34230843N^2g_1^4g_2^2 + 22226400000\zeta_3 Ng_1^4g_2^2 - 26093250026Ng_1^4g_2^2 + 1287984600Ng_1^3g_2^3$$

$$+ 417101400Ng_1^2g_2^4 + 4445280000\zeta_3 g_2^6 - 4603622893g_2^6]\frac{1}{2509271654400000} + O(g_i^8),$$

$$\beta_1^{(8,12)}(g_1, g_2) = [3Ng_1^2 + 40g_1^2 + 28g_1g_2 + 3g_2^2]\frac{g_1}{3360} + [12042Ng_1^4 - 53464g_1^4 - 133308Ng_1^3g_2 + 490392g_1^3g_2$$

$$- 14283Ng_1^2g_2^2 + 66956g_1^2g_2^2 + 57960g_1g_2^3 - 42003g_2^4]\frac{g_1}{3556224000} + [-14938245342N^2g_1^6$$

$$- 617004864000\zeta_3 Ng_1^6 + 3109795833456Ng_1^6 + 2311545600000\zeta_3 g_1^6 + 1737998549536g_1^6$$

$$- 2875390812N^2g_1^5g_2 + 1867017600000\zeta_3 Ng_1^5g_2 - 2203538203104Ng_1^5g_2 + 547658496000\zeta_3 g_1^5g_2$$

$$- 423865546832g_1^5g_2 - 308077587N^2g_1^4g_2^2 + 200037600000\zeta_3 Ng_1^4g_2^2 - 178146641322Ng_1^4g_2^2$$

$$+ 2261758464000\zeta_3 g_1^4g_2^2 + 6401866158256g_1^4g_2^2 + 8781904656Ng_1^3g_2^3 + 497871360000\zeta_3 g_1^3g_2^3$$

$$- 586662087088g_1^3g_2^3 + 3753912600Ng_1^2g_2^4 + 49787136000\zeta_3 g_1^2g_2^4 + 2136750334680g_1^2g_2^4$$

$$+ 373403520000\zeta_3 g_1g_2^5 - 374235507660g_1g_2^5 + 40007520000\zeta_3 g_2^6 - 41432606037g_2^6]$$

$$\times \frac{g_1}{451166889779200000} + O(g_i^9),$$

$$\beta_2^{(8,12)}(g_1, g_2) = [28Ng_1^3 + 9Ng_1^2g_2 + 37g_2^3]\frac{1}{3360} + [173880Ng_1^5 + 32826Ng_1^4g_2 + 241164Ng_1^3g_2^2$$

$$- 159651Ng_1^2g_2^3 + 96073g_2^5]\frac{1}{3556224000} + [121231189296N^2g_1^7 + 1194891264000\zeta_3 Ng_1^7$$

$$- 1408785519120Ng_1^7 - 4290992658N^2g_1^6g_2 + 1434936384000\zeta_3 Ng_1^6g_2 + 7192728107344Ng_1^6g_2$$

$$- 90566288568N^2g_1^5g_2^2 + 1344252672000\zeta_3 Ng_1^5g_2^2 - 1313378645040Ng_1^5g_2^2$$

$$- 10178257905N^2g_1^4g_2^3 + 1869684768000\zeta_3 Ng_1^4g_2^3 + 4814949206106Ng_1^4g_2^3 + 746807040000\zeta_3 Ng_1^3g_2^4$$

$$- 890895567408Ng_1^3g_2^4 - 34486893072Ng_1^2g_2^5 + 941510304000\zeta_3 g_2^7 + 1196618048425g_2^7]$$

$$\times \frac{1}{451166889779200000} + O(g_i^9) \tag{5.3}$$

where ζ_z is the Riemann zeta function. However, for $L^{(8,14)}$ computational limitations meant we can only provide two loop results which are

$$
\gamma_\phi^{(8,14)}(g_1,g_2) = \frac{g_1^2}{1120} + [107964Ng_1^2 - 1533897g_1^2 + 718200g_1g_2 - 54586g_2^2]\frac{g_1^2}{768144384000} + O(g_i^6),
$$

$$
\gamma_\sigma^{(8,14)}(g_1,g_2) = [-18Ng_1^2 + 7g_2^2]\frac{1}{136080} + [13056633Ng_1^4 - 6826680Ng_1^3g_2 + 467334Ng_1^2g_2^2
$$

$$
+ 275849g_2^4]\frac{1}{23332385664000} + O(g_i^6),
$$

$$
\beta_1^{(8,14)}(g_1,g_2) = [-18Ng_1^2 + 621g_1^2 - 252g_1g_2 + 7g_2^2]\frac{g_1}{272160} + [171159480Ng_1^4 - 12056931g_1^4 - 67296096Ng_1^3g_2
$$

$$
- 377785296g_1^3g_2 + 1869336Ng_1^2g_2^2 - 7019838g_1^2g_2^2 - 59274432g_1g_2^3 + 1103396g_2^4]
$$

$$
\times \frac{g_1}{186659085312000} + O(g_i^6),
$$

$$
\beta_2^{(8,14)}(g_1,g_2) = [72Ng_1^3 - 6Ng_1^2g_2 + g_2^3]\frac{1}{30240} + [-7007148Ng_1^5 + 25365069Ng_1^4g_2 + 27512136Ng_1^3g_2^2
$$

$$
- 639018Ng_1^2g_2^3 + 2198333g_2^5]\frac{1}{15554923776000} + O(g_i^7). \tag{5.4}
$$

Equally for similar computational constraints we could only determine the full renormalization group functions for $L^{(8,16)}$ at one loop. We found

$$
\gamma_\phi^{(8,16)}(g_i) = \frac{g_1^2}{6048} + [-1468755Ng_1^2 - 55406142g_1^2 - 4477968g_1g_2 - 29420424g_1g_3
$$

$$
- 2792128g_2^2 + 2456232g_2g_3 - 2116377g_3^2]\frac{g_1^2}{1003811081011200} + O(g_i^6),
$$

$$
\gamma_\sigma^{(8,16)}(g_i) = [45Ng_1^2 + 296g_2^2 - 384g_2g_3 + 159g_3^2]\frac{1}{5987520} + [-2880219870Ng_1^4 - 1664530560Ng_1^3g_2
$$

$$
- 338294880Ng_1^3g_3 - 479937600Ng_1^2g_2^2 + 584739000Ng_1^2g_2g_3 - 243713475Ng_1^2g_3^2
$$

$$
- 1869662576g_2^4 + 691909088g_2^3g_3 - 1637070624g_2^2g_3^2 - 2769180480g_2^2g_4^2
$$

$$
+ 4241645772g_2g_3^3 + 3466964160g_2g_3g_4^2 - 1825036161g_3^4 - 1227798000g_3^2g_4^2
$$

$$
- 86444820g_4^4]\frac{1}{496886485100544000} + O(g_i^6),
$$

$$
\beta_1^{(8,16)}(g_i) = [45Ng_1^2 + 4356g_1^2 + 1584g_1g_2 + 792g_1g_3 + 296g_2^2 - 384g_2g_3 + 159g_3^2]\frac{g_1}{11975040} + O(g_i^4),
$$

$$
\beta_2^{(8,16)}(g_i) = [1782Ng_1^3 + 135Ng_1^2g_2 + 2516g_2^3 - 866g_2^2g_3 - 447g_2g_3^2 + 3168g_2g_4^2 - 330g_3^3
$$

$$
- 2376g_3g_4^2]\frac{1}{11975040} + O(g_i^5),
$$

$$
\beta_3^{(8,16)}(g_i) = [2376Ng_1^3 + 135Ng_1^2g_3 + 176g_2^3 - 696g_2^2g_3 - 1020g_2g_3^2 + 961g_3^3]\frac{1}{11975040} + O(g_i^5),
$$

$$
\beta_4^{(8,16)}(g_i) = [1782Ng_1^4 + 45Ng_1^2g_4^2 + 352g_2^4 + 704g_2^3g_3 + 528g_2^2g_3^2 + 1880g_2^2g_4^2 + 176g_2g_3^3
$$

$$
+ 1200g_2g_3g_4^2 + 22g_3^4 + 555g_3^2g_4^2 + 891g_4^4]\frac{1}{2993760} + O(g_i^6). \tag{5.5}
$$

The two loop wave function anomalous dimensions were computed to provide a nontrivial check on the one loop coupling constant renormalization.

For the $n = 3$ thread we are limited to two loops throughout but our renormalization group functions for the first three Lagrangians are

$$\gamma_\phi^{(12,14)}(g_1) = -\frac{g_1^2}{6048} + [-103653N + 179983]\frac{g_1^4}{460886630400} + O(g_1^6),$$

$$\gamma_\sigma^{(12,14)}(g_1) = -\frac{Ng_1^2}{1120} + \frac{611Ng_1^4}{4267468800} + O(g_1^6),$$

$$\beta^{(12,14)}(g_1) = [-27N + 74]\frac{g_1^3}{120960} + [-3489939N + 6027443]\frac{g_1^5}{4608866304000} + O(g_1^7),$$

$$\gamma_\phi^{(12,16)}(g_1) = -\frac{g_1^2}{15120} + [8661N + 12118]\frac{g_1^4}{1152216576000} + O(g_1^6),$$

$$\gamma_\sigma^{(12,16)}(g_1) = \frac{Ng_1^2}{10080} + \frac{47Ng_1^4}{96018048000} + O(g_1^6),$$

$$\beta^{(12,16)}(g_1) = [3N + 8]\frac{g_1^3}{120960} + [-31845N + 142192]\frac{g_1^5}{2304433152000} + O(g_1^7) \tag{5.6}$$

and

$$\gamma_\phi^{(12,18)}(g_1, g_2) = -\frac{g_1^2}{66528} + [-148285Ng_1^2 + 143716g_1^2 + 440286g_1g_2 - 148285g_2^2]$$
$$\times \frac{g_1^2}{1226880210124800} + O(g_i^6),$$

$$\gamma_\sigma^{(12,18)}(g_1, g_2) = -[Ng_1^2 + g_2^2]\frac{1}{133056} + [-76427Ng_1^4 + 440286Ng_1^3g_2 - 148285Ng_1^2g_2^2 + 71858g_2^4]$$
$$\times \frac{1}{1226880210124800} + O(g_i^6),$$

$$\beta_1^{(12,18)}(g_1, g_2) = [-5Ng_1^2 + 13g_1^2 + 33g_1g_2 - 5g_2^2]\frac{g_1}{1330560}$$
$$+ [150438125Ng_1^4 + 590701817g_1^4 + 97868100Ng_1^3g_2 + 945987075g_1^3g_2$$
$$- 14828500Ng_1^2g_2^2 + 290724292g_1^2g_2^2 + 214657575g_1g_2^3 + 7185800g_2^4]$$
$$\times \frac{g_1}{245376042024960000} + O(g_i^7),$$

$$\beta_2^{(12,18)}(g_1, g_2) = [11Ng_1^3 - 5Ng_1^2g_2 + 6g_2^3]\frac{1}{443520}$$
$$+ [214657575Ng_1^5 + 139088071Ng_1^4g_2 + 204846675Ng_1^3g_2^2 + 12091250Ng_1^2g_2^3$$
$$+ 190227857g_2^5]\frac{1}{81792014008320000} + O(g_i^7). \tag{5.7}$$

Finally we note the renormalization group functions for $L^{(4,10)}$ which extends the $n = 1$ thread to the next dimension are

$$\gamma_\phi^{(4,10)}(g_i) = -\frac{g_1^2}{40} + [-5301Ng_1^2 + 16758g_1^2 + 120540g_1g_2 + 302820g_1g_3 - 14114g_2^2 - 18032g_2g_3 - 15779g_3^2]$$

$$\times \frac{g_1^2}{254016000} + O(g_i^4),$$

$$\gamma_\sigma^{(4,10)}(g_i) = [-9Ng_1^2 - 86g_2^2 + 112g_2g_3 - 71g_3^2]\frac{1}{15120} + [664524Ng_1^4 + 6713280Ng_1^3g_2 + 1451520Ng_1^3g_3$$

$$- 1128852Ng_1^2g_2^2 + 1202544Ng_1^2g_2g_3 - 797022Ng_1^2g_3^2 + 4415512g_2^4 + 6451480g_2^3g_3$$

$$- 14360000g_2^2g_3^2 - 3621996g_2^2g_4^2 - 10763088g_2g_3^3 + 10666782g_2g_3g_4^2 + 8993886g_3^4$$

$$- 1885086g_3^2g_4^2 - 496125g_4^4]\frac{1}{96018048000} + O(g_i^4),$$

$$\beta_1^{(4,10)}(g_i) = [-9Ng_1^2 + 504g_1^2 + 840g_1g_2 + 420g_1g_3 - 86g_2^2 + 112g_2g_3 - 71g_3^2]\frac{g_1}{30240} + O(g_i^5),$$

$$\beta_2^{(4,10)}(g_i) = [756Ng_1^3 - 81Ng_1^2g_2 + 1298g_2^3 + 1764g_2^2g_3 - 1395g_2g_3^2 - 1134g_2g_4^2$$

$$- 308g_3^3 + 1134g_3g_4^2]\frac{1}{90720} + O(g_i^5),$$

$$\beta_3^{(4,10)}(g_i) = [756Ng_1^3 - 81Ng_1^2g_3 - 448g_2^3 - 1782g_2^2g_3 + 3024g_2g_3^2 + 1134g_2g_4^2$$

$$+ 565g_3^3 - 1134g_3g_4^2]\frac{1}{90720} + O(g_i^6),$$

$$\beta_4^{(4,10)}(g_i) = [11340Ng_1^4 - 81Ng_1^2g_4^2 + 896g_2^4 - 4256g_2^3g_3 - 7728g_2^2g_3^2 + 234g_2^2g_4^2 - 4088g_2g_3^3 + 8820g_2g_3g_4^2$$

$$+ 2268g_2g_5^3 - 700g_3^4 + 3015g_3^2g_4^2 - 2268g_3g_5^3 - 1134g_4^4]\frac{1}{6804} + O(g_i^6),$$

$$\beta_5^{(4,10)}(g_i) = [-27216Ng_1^5 - 81Ng_1^2g_5^3 - 3584g_2^5 - 8960g_2^4g_3 - 8960g_2^3g_3^2 + 10080g_2^3g_4^2$$

$$- 4480g_2^2g_3^3 + 15120g_2^2g_3g_4^2 - 10854g_2^2g_5^3 - 1120g_2g_3^4 + 7560g_2g_3^2g_4^2 - 9072g_2g_3g_5^3 - 5670g_2g_4^4$$

$$- 112g_3^5 + 1260g_3^3g_4^2 - 3159g_3^2g_5^3 - 2835g_3g_4^4 + 5670g_4^2g_5^3]\frac{1}{54432} + O(g_i^7). \tag{5.8}$$

All the renormalization group functions have been determined using dimensional regularization with the renormalization constants defined with respect to the $\overline{\text{MS}}$ scheme. It is worth noting that in the critical dimension of each Lagrangian we used, the coupling constants are dimensionless in that dimension but the standard arbitrary scale is introduced to preserve dimensionlessness of the couplings in the regularized theory.

The main reason for constructing these renormalization group functions is to verify that the critical exponents at the Wilson-Fisher fixed point are consistent with the large N critical exponents given in Sec. III for each of the underlying universal theories. In order to carry out the comparison, we follow the process given in [15,16] and first find the values of the critical coupling constants, g_{ic}, by solving

$$\beta_j^{(d_1,d_2)}(g_{ic}) = 0 \tag{5.9}$$

where g_{ic} is a power series in $1/N$. Each coefficient is itself a power series in ϵ aside from the leading order $1/N$ term

which only involves ϵ due to the structure of the N dependence at two and higher loops. Once these critical couplings are determined, the field anomalous dimensions $\gamma_\phi^{(d_1,d_2)}(g_i)$ and $\gamma_\sigma^{(d_1,d_2)}(g_i)$ are evaluated at criticality as series in $1/N$. Then the coefficients of each term in ϵ of each successive power of $1/N$ should be in total agreement with the critical exponents η and $\eta + \chi$ respectively. We have checked this correspondence holds for all of the sets of renormalization group functions for the threads $n = 2$ and 3 we have computed and the large N exponents (3.1), (3.2), (3.4), and (3.5). Such agreement should be regarded as evidence for the underlying universality of the core interaction across the dimensions in the same spirit as that of the original and well-established universality of the Wilson-Fisher fixed point of $O(N)$ ϕ^4 theory or $n = 1$ thread in the present language. Equally the agreement is a reassuring check that we have correctly performed the renormalization to several loop orders which relied on elevating the various master integrals to higher dimensions. There are already several internal checks on the

perturbative results in that the double and triple poles of the two and three loop renormalization constants are already predetermined by the lower loop results. We have ensured that these have been satisfied first before recording our two and three loop expressions.

Having established the connection with the underlying universal theory, it is worth briefly analyzing aspects of the nontrivial fixed point structure of each theory and in particular the location, if it exists, of the conformal window. This falls into two classes of analysis. In QCD there is a conformal window when the signs of the one and two loop terms of the strictly four dimensional β-function are different with the nontrivial fixed point being called the Banks-Zaks fixed point [59]. Such a class of critical points of the renormalization group equations is a feature of a single coupling theory. Several of the theories we have constructed have the same potential property and we have determined the conformal windows for these. For instance, the two loop term of $\beta^{(8,10)}(g)$ is always negative but the one loop term changes sign at $N = 6$. Hence it is straightforward to see that the conformal window is $1 \leq N < 6$. When $N = 6$ the two and three loop terms are both negative which is the reason for the strict inequality. Above $N = 6$ the nontrivial critical coupling of the so-called Banks-Zaks fixed point is real whereas it becomes purely imaginary below $N = 6$. By contrast the eight dimensional ϕ^4 theory is not asymptotically free and the two loop term of its β-function is positive. There is a parallel picture for the $n = 3$ thread as the base twelve dimensional ϕ^4 theory has two positive β-function terms but there are Banks-Zaks fixed points for the higher dimensional single coupling theories. For instance for the (12,14) theory there is a conformal window at $N = 1$ and 2 with real critical couplings for $N \geq 3$. By contrast $\beta^{(12,16)}(g)$ has a positive one loop term but the two loop term is negative for $N \geq 5$. So there is a Banks-Zaks type fixed point for this range of N.

The second class of fixed point analysis concerns theories with more than one coupling constant. To access the conformal window in this instance we have to solve a set of equations [15,16], which for the two coupling theories considered here, are

$$\beta_1(g_i) = \beta_2(g_i) = 0, \qquad \frac{\partial \beta_1}{\partial g_1}\frac{\partial \beta_2}{\partial g_2} - \frac{\partial \beta_1}{\partial g_2}\frac{\partial \beta_2}{\partial g_1} = 0. \quad (5.10)$$

The first two equations determine the critical couplings, and the final one, which is the vanishing of the Hessian, provides the condition where there is a change in the stability property of a fixed point. Moreover, as in [15,16] we can determine the window as a perturbative series in ϵ which, in principle, provides insight into other dimensions. For $\beta_i^{(8,12)}(g_1, g_2)$ we found four solutions to (5.10) for the critical value of N which are

$$N_{(A)}^{(8,12)} = 1.015123 - 0.024469\epsilon - 0.324484\epsilon^2 + O(\epsilon^3),$$

$$N_{(B)}^{(8,12)} = -0.366698 + 0.451194\epsilon - 41.675880\epsilon^2 + O(\epsilon^3),$$

$$N_{(C)}^{(8,12)} = -910.687640 + 2668.861873\epsilon$$
$$- 1565.439288\epsilon^2 + O(\epsilon^3) \qquad (5.11)$$

where solution B has real critical couplings whereas the other two cases are purely imaginary. Given the nonunitary nature of solution A and the negative corrections in the ϵ expansion to the low value for the critical value of N defining the conformal window, it would appear that for this theory there is no interesting structure. By contrast, for the theory based on the related group $Sp(N)$, the conformal window is determined from the negative solutions [60]. So that theory would appear to have a conformal window around $N = 910$. A similar feature was observed in the (4,8) case. There were three real solutions for the conformal window boundary for $\beta_i^{(8,14)}(g_1, g_2)$ which are

$$N_{(A)}^{(8,14)} = 602.601144 - 33341.878584\epsilon + O(\epsilon^2),$$

$$N_{(B)}^{(8,14)} = 0.627879 - 1.399181\epsilon + O(\epsilon^2),$$

$$N_{(C)}^{(8,14)} = -186.979023 + 45848.701747\epsilon + O(\epsilon^2). \qquad (5.12)$$

By contrast there is a clear indication of a conformal window here with a relatively high value for N similar to the (4,6) theory [15–17]. The first multicoupling example for the $n = 3$ thread occurs for $\beta_i^{(12,18)}(g_1, g_2)$ and gives

$$N_{(A)}^{(12,18)} = 113.894634 - 1653.078171\epsilon + O(\epsilon^2),$$

$$N_{(B)}^{(12,18)} = 1.116917 - 2.093367\epsilon + O(\epsilon^2),$$

$$N_{(C)}^{(12,18)} = -0.032996 - 0.713266\epsilon + O(\epsilon^2) \qquad (5.13)$$

which has parallels with the previous case. For the remaining theories the increase in the number of couplings and hence β-functions together with a substantial Hessian meant that our computer resources rather than any principle were not powerful enough to solve the system of equations in general.

VI. EXTENSIONS

Our main focus has been on $O(N)$ scalar theories and the new threads of theories which follow from new integer solutions to the canonical relation between the field critical exponents derived from the critical exponent defined by the force-matter vertex. It transpires that such an exercise is not limited to scalar theories. The $O(N)$ symmetric Gross-Neveu (GN) model [36] and the non-Abelian Thirring model (NATM) [37,38] have also been considered in the

context of the Vasil'ev *et al.* large N expansion [61–69]. In the case of the latter theory, it is the large N_f expansion, where N_f is the number of quark flavors, rather than the number of colors of the non-Abelian symmetry which is the expansion parameter. Therefore we have constructed several higher derivative extensions of each of these base theories in the same spirit as the scalar theories. First we recall that the Lagrangian of the two dimensional Gross-Neveu model with an $SU(N)$ symmetry is [36]

$$L_{\text{GN}}^{(2)} = i\bar{\psi}^i \slashed{\partial} \psi^i + \frac{1}{2} g_1^2 (\bar{\psi}^i \psi^i)^2 \qquad (6.1)$$

which like (2.1) can be rewritten in terms of an auxiliary field σ to give

$$L_{\text{GN}}^{(2)} = i\bar{\psi}^i \slashed{\partial} \psi^i + \frac{1}{2} g_1 \sigma \bar{\psi}^i \psi^i - \frac{1}{2}\sigma^2. \qquad (6.2)$$

Equally one can develop a large N expansion for (6.2) using the same approach as [21,22]. In this instance the respective scaling dimensions of ψ and σ are $\tilde{\alpha}$ and $\tilde{\beta}$ and are given by

$$\tilde{\alpha} = \mu + \frac{1}{2}\eta, \qquad \tilde{\beta} = 1 - \eta - \chi \qquad (6.3)$$

giving

$$2\tilde{\alpha} + \tilde{\beta} = d + 1 - \chi. \qquad (6.4)$$

In defining these dimensions we follow [61] and do not include the dimension deriving from $\slashed{\partial}$ in the kinetic term of the Lagrangian. This is so that $\tilde{\alpha}$ corresponds to the exponent in the critical point large N propagators analogous to (2.5). With this convention, for a vertex of the Gross-Neveu form the uniqueness condition for its conformal integration translates to the sum of the vertex exponents is $(d + 1)$ [61].

The Gross-Neveu model is therefore parallel to the $O(N)$ scalar theories in that the vertices are unique in the absence of the vertex anomalous dimension. This has led to the computation of the critical exponents to three orders in the large N expansion [61–66] in the underlying universal theory. In this instance, the four dimensional theory which is in the same universality class as the two dimensional Gross-Neveu model is the Gross-Neveu-Yukawa theory which has been discussed in this context in [39,70]. Specifically the Lagrangian is

$$L_{\text{GN}}^{(4)} = i\bar{\psi}^i \slashed{\partial} \psi^i + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} g_1 \sigma \bar{\psi}^i \psi^i + \frac{1}{24} g_2^2 \sigma^4 \qquad (6.5)$$

where there are two couplings and the quadratic term in σ becomes a kinetic term since σ has a canonical dimension of 1. The process to generalize the Gross-Neveu thread in

the same spirit as the scalar theories is now apparent. One solves the relation for the vertex dimension, (6.4), by allowing for an alternative fermion kinetic term leading to

$$\tilde{\alpha} = \mu - n + \frac{1}{2}\eta, \qquad \tilde{\beta} = 2n + 1 - \eta - \chi \qquad (6.6)$$

where $n = 0$ corresponds to (6.2). However, there is a new feature here in that the fermion kinetic term is formally different depending on whether n is even or odd as a consequence of the necessary γ-algebra. When n is odd one has a product of even numbers of $\slashed{\partial}$ which therefore translate into kinetic terms involving only $\Box$. In other words one ends up with the Klein-Gordon version of a fermionic field and its higher $\Box$ generalizations. We will ignore these types of solutions as one in effect will reproduce results and Lagrangians similar to the scalar theory threads. Therefore the first thread above $n = 0$ to have a kinetic term involving $\slashed{\partial}$ is $n = 2$ with the Lagrangian

$$L_{\text{GN}}^{(6)} = i\bar{\psi}^i \slashed{\partial} \Box \psi^i + \frac{1}{2} g_1 \sigma \bar{\psi}^i \psi^i - \frac{1}{2}\sigma^2 \qquad (6.7)$$

which has critical dimension 6. A similar kinetic term has appeared in a different context in [40]. There a fermionic model was constructed on the assumption of having the fermion kinetic term of (6.7) but requiring the interacting theory of one fermion to be renormalizable in four dimensions. This led to an *eight*-point fermion self-interaction [40] as well as 4- and 6-point interactions. It was noted in [40] that the coupling of the 8-point interaction was asymptotically free. Here our premise is to retain a quartic fermion interaction for each thread as that underlying universal theory is accessible via the large N expansion. It is not clear if the fermionic theory of [40] is amenable to the large N methods of Vasil'ev *et al.* [21–23]. From (6.7) one can build higher dimensional Lagrangians for each thread value n. For instance, the continuation of the $n = 2$ thread gives Lagrangians $L_{\text{GN}}^{(6,d)}$ where we will only consider d to be even. For an odd d one cannot have a σ kinetic term as there would have to be an odd number of derivatives and it is not possible to have a Lorentz singlet. As the σ field of this thread has canonical dimension 3 the theories with $d = 8$ and 10 are the same as $L_{\text{GN}}^{(6)}$ with only a modified σ kinetic term. Equally for $d = 12$ to 16 quartic interactions in σ will be present and involve derivative couplings. Then when $d = 18$ a 6-point σ interaction will be necessary for renormalizability in addition to the independent quartic σ interactions. To summarize we have

$$L_{\text{GN}}^{(6,8)} = i\bar{\psi}^i\partial\Box\psi^i + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}g_1\sigma\bar{\psi}^i\psi^i,$$

$$L_{\text{GN}}^{(6,10)} = i\bar{\psi}^i\partial\Box\psi^i + \frac{1}{2}(\Box\sigma)^2 + \frac{1}{2}g_1\sigma\bar{\psi}^i\psi^i,$$

$$L_{\text{GN}}^{(6,12)} = i\bar{\psi}^i\partial\Box\psi^i + \frac{1}{2}(\Box\partial^\mu\sigma)^2 + \frac{1}{2}g_1\sigma\bar{\psi}^i\psi^i + \frac{1}{24}g_2^2\sigma^4,$$

$$L_{\text{GN}}^{(6,14)} = i\bar{\psi}^i\partial\Box\psi^i + \frac{1}{2}(\Box^2\sigma)^2 + \frac{1}{2}g_1\sigma\bar{\psi}^i\psi^i + \frac{1}{24}g_2^2\sigma^2\partial^\mu\sigma\partial_\mu\sigma,$$

$$L_{\text{GN}}^{(6,16)} = i\bar{\psi}^i\partial\Box\psi^i + \frac{1}{2}(\Box^2\partial^\mu\sigma)^2 + \frac{1}{2}g_1\sigma\bar{\psi}^i\psi^i + \frac{1}{24}g_2^2\sigma^2(\Box\sigma)^2 + \frac{1}{24}g_3^2(\partial^\mu\sigma\partial_\mu\sigma)^2 + \frac{1}{24}g_4^2\sigma^2(\partial^\mu\partial^\nu\sigma)^2,$$

$$L_{\text{GN}}^{(6,18)} = i\bar{\psi}^i\partial\Box\psi^i + \frac{1}{2}(\Box^3\sigma)^2 + \frac{1}{2}g_1\sigma\bar{\psi}^i\psi^i + \frac{1}{24}g_2^2\sigma^3\Box^3\sigma + \frac{1}{24}g_3^2\sigma^2\Box\sigma\Box^2\sigma + \frac{1}{24}g_4^2\sigma^2\partial_\mu\partial_\nu\sigma\Box\partial^\mu\partial^\nu\sigma$$

$$+ \frac{1}{24}g_5^2\sigma^2(\partial_\mu\partial_\nu\partial_\rho\sigma)^2 + \frac{1}{24}g_6^2\sigma^2\Box\partial_\mu\sigma\Box\partial^\mu\sigma + \frac{1}{24}g_7^2\sigma(\Box\sigma)^3 + \frac{1}{24}g_8^2\sigma\partial_\mu\partial_\nu\sigma\partial^\nu\partial^\rho\sigma\partial^\mu\partial_\rho\sigma$$

$$+ \frac{1}{24}g_9^2\sigma\Box\sigma\partial_\mu\partial_\nu\sigma\partial^\mu\partial^\nu\sigma + \frac{1}{40320}g_{10}^4\sigma^6 \tag{6.8}$$

where there are parallels in structure to the $O(N)$ scalar theories but a jump in the complexity when the critical dimension is 18.

This exercise can be repeated for the non-Abelian Thirring model which like the Gross-Neveu model has a quartic fermion interaction and is renormalizable in two dimensions. The Lagrangian is

$$L_{\text{NATM}}^{(2)} = i\bar{\psi}^i\partial\!\!\!/\psi^i + \frac{1}{2}g_1^2(\bar{\psi}^i\gamma^\mu T^a\psi^i)^2 \tag{6.9}$$

where T^a are the generators of the underlying non-Abelian symmetry. It is accessible to large N_f computations where N_f is the number of fermion fields and this parameter is chosen partly not to be confused with the parameter associated with the non-Abelian group but mainly because of the connection of (6.9) with QCD. As noted in [67] the non-Abelian Thirring model is in the same universality class as QCD at the Wilson-Fisher fixed point accessible via the large N_f expansion. So there is a thread of theories parallel to the $O(N)$ scalar and Gross-Neveu ones. The connection is more evident by the introduction of a spin-1 auxiliary field in the adjoint representation of the non-Abelian symmetry group since

$$L_{\text{NATM}}^{(2)} = i\bar{\psi}^i\partial\!\!\!/\psi^i + g_1\bar{\psi}^i T^a\gamma^\mu\psi^i A_\mu^a - \frac{1}{2}A_\mu^a A^{a\mu}. \tag{6.10}$$

In our discussions we will use the shorthand designation of gluon for this spin-1 adjoint field even though this is usually understood to be the quanta of the strong force only in four dimensions. From (6.10) the core quark-gluon vertex of QCD is apparent and it is this interaction which forms the basis of the universality class containing four dimensional QCD and was examined further in [20]. Where

(6.2) and (6.10) differ is that the presence of the γ-matrix in the vertex means that vertex does not have any uniqueness condition in order to be able to develop its conformal integration. From the point of view of carrying out large N_f computations this is not a limitation. Uniqueness can be exploited explicitly within the evaluation of contributing Feynman diagrams once the integrals are rewritten from tensor integrals to scalar ones.

In parallel to (6.7) other threads of theories can be developed with higher derivative fermion kinetic terms. While the quark-gluon vertex of (6.10) is not unique the relation between the exponents of the fields

$$2\tilde{\alpha} + \tilde{\beta} = d + 1 - \chi \tag{6.11}$$

is still valid from the scaling dimensions of the universal interaction where χ here is the vertex anomalous dimension. Like the Gross-Neveu case the solutions (6.6) are still valid leading to a new thread of theories based on the Lagrangian

$$L_{\text{NATM}}^{(6)} = i\bar{\psi}^i\partial\!\!\!/\Box\psi^i + g_1\bar{\psi}^i T^a\gamma^\mu\psi^i A_\mu^a - \frac{1}{2}A_\mu^a A^{a\mu} \tag{6.12}$$

where A_μ^a is to be regarded as an auxiliary field rather than a gauge field. By this we mean that there is no restriction on the number of degrees of freedom of the components A_μ^a. In other words at this stage we are only interested in the structure of the Lagrangians of higher dimensional theories in this $n = 2$ thread. For instance, each interaction will carry its own independent coupling constant which is not related to any others as would be the case in a four dimensional gauge theory. In that case the gauge symmetry would place conditions on the coupling constants which would be preserved in the quantum theory via the

Slavnov-Taylor identities. So, for instance, the first few Lagrangians in the non-Abelian Thirring model thread with $n = 2$ are

$$L_{\text{NATM}}^{(6,8)} = i\bar{\psi}^i\partial\!\!\!/\Box\psi^i + g_1\bar{\psi}^iT^a\gamma^\mu\psi^iA_\mu^a + \frac{\beta}{2}\partial_\mu A_\nu^a\partial^\mu A^{a\nu}$$

$$+ \frac{1}{2}(1-\beta)\partial^\mu A_\mu^a\partial_\nu A^{a\nu},$$

$$L_{\text{NATM}}^{(6,10)} = i\bar{\psi}^i\partial\!\!\!/\Box\psi^i + g_1\bar{\psi}^iT^a\gamma^\mu\psi^iA_\mu^a + \frac{\beta}{2}\partial_\mu\partial_\nu A_\sigma^a\partial^\mu\partial^\nu A^{a\sigma}$$

$$+ \frac{1}{2}(1-\beta)\partial^\mu\partial^\nu A_\nu^a\partial_\mu\partial_\sigma A^{a\sigma} + g_2 f^{abc}A_\mu^a A_\nu^b\partial^\mu A^{c\nu}$$

$$+ g_3 d^{abc}A^{a\mu}A_\mu^b\partial_\nu A^{c\nu} \qquad (6.13)$$

where f^{abc} are the structure constants of the non-Abelian Lie group and d^{abc} is the associated rank 3 symmetric tensor.

The parameter β is not the gauge parameter of the linear covariant gauge fixing in a gauge theory as such. It should be regarded as an interpolating parameter. A parallel situation was noted in [20] as being possible in the quadratic part of the construction of the six dimensional Lagrangian which extends the NATM-QCD Wilson-Fisher fixed point equivalence to six dimensions. However in that situation the use of Bianchi identities for the field strength and the gauge symmetry meant that the Lagrangian could be written in terms of one gauge independent 2-leg gluonic operator. A second quadratic operator in the gluon was, however, necessary to effect the gauge fixing. In $L_{\text{NATM}}^{(6,d)}$ the parameter β would become related to a gauge parameter if a higher symmetry was imposed on the construction. To construct the next Lagrangian in the sequence, $L_{\text{NATM}}^{(6,12)}$, is a relatively straightforward exercise in principal but requires an interplay between the tensors of the color group and the structure of the interactions. For instance, $L_{\text{NATM}}^{(6,12)}$ will have quartic interactions in addition to cubic ones in the gluon. The latter will however involve three derivatives and involve both f^{abc} and d^{abc} tensors. Therefore the number of independent interactions which are present in $L_{\text{NATM}}^{(6,12)}$ will be significantly larger especially since products of the two color tensors will be possible. To gauge the potential structure of such a Lagrangian, similar interactions arise in four dimensional Yang-Mills theory when there is a nonlinear gauge fixing which generalizes the 't Hooft-Veltman gauge of QED, [71,72]. While this ultimately leads to a twelve dimensional Lagrangian with a significantly large number of interactions, it would seem that the connection of the original non-Abelian Thirring model with four dimensional gauge theory is not present in the $n = 2$ thread. This is apparent in the higher dimensional extensions of (6.12) where the square of the field strength does not appear to emerge even allowing for relations between coupling constants. However, we emphasise that our construction is based on the premises of requiring renormalizability with a core interaction which seeds the equivalence of theories at the Wilson-Fisher fixed point. There may be other fixed points where such operators emerge naturally.

VII. LOWER DIMENSION COMPLETENESS

While we have demonstrated that there is a large class of new renormalizable quantum field theories which are the ultraviolet completeness of a higher derivative ϕ^4 theory, one question which arises and which we can speculate on is whether there is a set of theories *below* the critical dimension of the defining theory $L^{(D)}$ which we will call *lower dimension* completeness. This is partly motivated by the fact that our $n = 2$ and 3 large N critical exponents can in principle be expanded around dimensions less than the critical dimension of $L^{(D)}$. Therefore it is worth exploring the potential Lagrangians which could be present in the lower part of the tower of theories for these new threads. Earlier we chose to begin with (2.1) as the base Lagrangian. However it is well known that theory is in the same universality class as the nonlinear σ model which is

$$L^{(4,2)} = \frac{1}{2}(\partial_\mu\phi^i)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i - \frac{1}{2}\sigma \qquad (7.1)$$

in the notation we use here as it is renormalizable in two dimensions. While one could launch the tower of theories with this Lagrangian we chose not to do so as $L^{(4)}$ does not suffer from infrared issues. By this we mean that in two dimensions the basic $1/k^2$ bosonic propagator within a Feynman diagram is infrared divergent unlike in four dimensions. This is not unrelated to the fact that in two dimensions ϕ^i is dimensionless. By contrast $L^{(4)}$ is infrared finite. While (7.1) is used for the large N expansion, for perturbation theory the constraint is eliminated and the canonical form of the nonlinear σ model is used which is

$$L^{(4,2)} = \frac{1}{2}g_{ab}(\pi)\partial^\mu\pi^a\partial_\mu\pi^b \qquad (7.2)$$

where $1 \leq a \leq (N-1)$. For instance, in [73] the parametrization

$$\phi^i = \left(\pi^a, \sqrt{\frac{1}{g_1} - \pi^b\pi^b}\right) \qquad (7.3)$$

was used to establish the renormalization of (7.2). This version of the Lagrangian, (7.2), has a rich geometric structure as indicated by the presence of the metric of the underlying manifold. In this form there are an infinite number of interactions but one coupling constant with the theory retaining its renormalizability. See, for instance, [74] for a review article.

With this brief overview of the situation with the $n = 1$ thread we can now speculate on the potential for the lower dimensional completeness of the subsequent threads. For instance, the parallel to $L^{(4,2)}$ for $L^{(8)}$, (2.12), would be

$$L^{(8,4)} = \frac{1}{2}(\Box \phi^i)^2 + \frac{g_1}{2}\sigma \phi^i \phi^i - \frac{1}{2}\sigma. \qquad (7.4)$$

Clearly this has similar infrared issues as $L^{(4,2)}$ since the $1/(k^2)^2$ propagator is not infrared safe in four dimensions. Again this is related to the dimensionlessness of ϕ^i in this Lagrangian. The elimination of the constraint does not lead to the same elegant geometric Lagrangian as $L^{(4,2)}$ due to the nature of the 2-point term although there will be an infinite number of interactions again. Despite this the theory should be renormalizable. This would need to be established prior to completing any explicit computations. However in order to extract any renormalization group functions one would have to add masslike terms to regularize the infrared. One such term cannot be $\phi^i \phi^i$ in $L^{(8,4)}$ since that disappears with the elimination of the constraint. Instead terms such as $\frac{1}{2}m^2(\partial_\mu \phi^i)^2$ could be used to facilitate the infrared regularization. Such an analysis is beyond the scope of the present article as our focus here is on the Wilson-Fisher fixed point connections which can be accessed through the renormalization of *massless* quantum field theories. Moreover, the verification of the lower dimension completeness may not be as straightforward as the ultraviolet one. For instance, there is a clue in the large N exponents $\eta^{(8)}$ and $\chi^{(8)}$ which we have computed here. If $L^{(8,4)}$ is the lower dimension completeness of $L^{(8)}$ then the renormalization group functions must produce exponents which agree with the ϵ expansion of the critical exponents of the eight dimensional large N theory. Analyzing these near four dimensions indicates that there is a well-behaved ϵ expansion. This is not a minor point. For instance, in the extension of $L^{(8)}$ we have constructed theories in higher dimensions in steps of two. Equally there are parallels of the $n = 2$ thread with the $n = 1$ one. For the latter the large N exponents of [21,22] are divergent when one formally expands near *one* dimension which is one dimension below where σ can appear linearly in the Lagrangian. The same situation arises for $n = 2$ where the corresponding large N exponents diverge near three dimensions but have well-defined expansions for dimensions above three.

However, the dimensionalities of the fields and the parallel to $L^{(4)}$ directed us to $L^{(8,4)}$ which has skipped two dimensions. Examining the exponents $\eta^{(8)}$ and $\chi^{(8)}$ with respect to six dimensions produces exponents which do not all begin with $O(\epsilon)$. This is similar to evaluating the exponents of [21–23] in three dimensions. To understand this it is worth considering the potential form of a lower

dimension completeness to six dimensions. Based purely on the dimensionalities of the fields leads to

$$L^{(8,6)} = \frac{1}{2}(\Box \phi^i)^2 + \frac{1}{2}\sigma\Box^{-1}\sigma + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2^2}{4}(\phi^i\partial_\mu\phi^i)^2$$
$$+ \frac{g_3^2}{2}\phi^i\phi^i\phi^j\Box\phi^j + \frac{g_4^4}{12}(\phi^i\phi^i)^3 \qquad (7.5)$$

where the inverse box operator acts to the right and hence this Lagrangian contains a nonlocal quadratic term for σ. While such a construction is based on the same premises as those for the ultraviolet completion, it is not clear whether this potential Lagrangian is the one which produces the critical exponents of $\eta^{(8)}$ and $\chi^{(8)}$ near six dimensions or not. If it does not or any modifications do not then there may be a doubt as to whether $L^{(8,4)}$ is part of the $n = 2$ thread with a gap at six dimensions or whether there is a break below eight dimensions. In other words the lower dimension completeness for $L^{(4,2)}$ is a special case. Similar remarks are applicable to the $n = 3$ thread where

$$L^{(12,6)} = \frac{1}{2}(\Box \partial_\mu \phi^i)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i - \frac{1}{2}\sigma \qquad (7.6)$$

would be the lower dimension completeness with the large N exponents also being divergent in ϵ near the odd dimensions five and below. Again in the critical dimension ϕ^i is dimensionless. So $L^{(12,6)}$ would also require infrared regularization with masslike terms in order to extract the renormalization group functions to compare with the large N exponents. For the twelve dimensional universal theory these exponents have ϵ expansions which are $O(\epsilon)$ near six dimensions. However near ten dimensions $\chi_1^{(12)}$ has a pole in ϵ which does not have a parallel in the lower thread towers. This suggests some sort of pathology in the lower dimension completeness hypothesis in this and other intermediate theories which may be related to nonlocalities in the corresponding Lagrangians. Given this observation it may be the case that one has to consider only local Lagrangians in the lower dimension completeness construction. As a side remark we note that like $n = 1$ the next threads have finite exponents in odd dimensions. For instance,

$$\eta^{(8)}|_{d=7} = -\frac{256}{315\pi^2 N} - \frac{1960214528}{31255875\pi^4 N^2} + O\left(\frac{1}{N^3}\right),$$

$$\chi^{(8)}|_{d=7} = \frac{15872}{315\pi^2 N} + O\left(\frac{1}{N^2}\right) \qquad (7.7)$$

and

$$\eta^{(12)}\big|_{d=11} = \frac{131072}{405405\pi^2 N} - \frac{569571965136797696}{7403290525756125\pi^4 N^2}$$
$$+ O\left(\frac{1}{N^3}\right),$$

$$\chi^{(12)}\big|_{d=11} = \frac{65536}{567\pi^2 N} + O\left(\frac{1}{N^2}\right). \qquad (7.8)$$

Nonlocalities, however, can be accommodated within a local quantum field theory context if the nonlocality can be localized. This is the case for the Gribov operator in QCD [75] where the operator was localized by introducing localizing ghost fields [76–81]. The resulting Gribov-Zwanziger Lagrangian was shown to be renormalizable and amenable to multiloop renormalization. Again there are parallels here since $L^{(8,6)}$ can be localized in the same vein to give

$$L^{(8,6)} = \frac{1}{2}(\Box\phi^i)^2 + \frac{1}{2}\rho\Box\rho + \rho\sigma + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2^2}{4}(\phi^i\partial_\mu\phi^i)^2$$
$$+ \frac{g_3^2}{2}\phi^i\phi^i\phi^j\Box\phi^j + \frac{g_4^4}{12}(\phi^i\phi^i)^3 + \frac{g_5}{6}\rho^3 + \frac{g_6^2}{4}\rho^2\phi^i\phi^i$$
$$+ \frac{g_7}{2}\rho\partial^\mu\phi^i\partial_\mu\phi^i + \frac{g_8}{2}\rho\phi^i\Box\phi^i. \qquad (7.9)$$

We have followed the localization prescription used in the construction of the Gribov-Zwanziger Lagrangian [76–81]. In this an auxiliary field, which in our case is ρ, is initially introduced in such a way that when its equation of motion is derived in (7.9) and included in the kinetic part of the Lagrangian then the nonlocal σ 2-point term of (7.5) is produced. More specifically the ρ equation of motion produces

$$\rho = \Box^{-1}\sigma. \qquad (7.10)$$

In (7.9) the original interactions are retained. However, with the introduction of a new field there is the potential to have new interactions. These are represented by additional terms in (7.9) with couplings g_5 to g_8 and there are no such corresponding terms present in the original Lagrangian of (7.5) as it stands. These interactions arise because ρ has dimension 1 and therefore on power counting grounds they would be required to ensure renormalizability. In other words if we ignore for the moment the notion of lower dimension completeness of $L^{(8)}$ and one were given scalar fields ϕ^i, σ, and ρ of respective canonical dimensions 1, 4, and 2 then (7.9) would be the most general renormalizable quantum field theory one could construct in the critical dimension six. The σ-ρ 2-point term is present on dimensional grounds. While such a 2-point term may appear to be unnatural in a Lagrangian, a term with a similar structure is present in the localized renormalizable Gribov-Zwanziger Lagrangian [76–81]. Specifically there is a 2-point term involving the spin-1 adjoint and the bosonic localizing

ghost fields. While it leads to a matrix of propagators this is not a hindrance to performing perturbative computations. In other words one can regard (7.9) as a local renormalizable Lagrangian with which to perform computations. Whether its renormalization group functions and resulting critical exponents then have any connection to the underlying universal theory of the $L^{(8)}$ thread is not clear and beyond the scope of the present article.

Given this excursion to discuss the local Lagrangian (7.9) it is worth reviewing (7.5) in light of what we have constructed. Our Lagrangian (7.5) was written down purely on the grounds of the dimensionality of the fields and the ethos of one common force-matter interaction corresponding to the interaction with coupling g_1. The other underlying assumption was one of locality but that has been loosened a little by the presence of the nonlocal kinetic term. The absence of couplings corresponding to g_5 to g_8 in (7.9) may appear to be inconsistent. However, in light of (7.9) and completely dropping the locality assumption would lead to

$$L_{\text{nl}}^{(8,6)} = \frac{1}{2}(\Box\phi^i)^2 + \frac{1}{2}\sigma\Box^{-1}\sigma + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2^2}{4}(\phi^i\partial_\mu\phi^i)^2$$
$$+ \frac{g_3^2}{2}\phi^i\phi^i\phi^j\Box\phi^j + \frac{g_4^4}{12}(\phi^i\phi^i)^3 + \frac{\tilde{g}_5}{6}(\Box^{-1}\sigma)^3$$
$$+ \frac{\tilde{g}_6^2}{4}(\Box^{-1}\sigma)^2\phi^i\phi^i + \frac{\tilde{g}_7}{2}(\Box^{-1}\sigma)\partial^\mu\phi^i\partial_\mu\phi^i$$
$$+ \frac{\tilde{g}_8}{2}(\Box^{-1}\sigma)\phi^i\Box\phi^i \qquad (7.11)$$

on dimensional grounds. We have introduced parallel couplings $\tilde{g}_5$ to $\tilde{g}_8$ as they are not necessarily equivalent to those of (7.9). Again it is not clear if this is the lower dimension complete Lagrangian for the $L^{(8)}$ thread in six dimensions. With the presence of nonlocal interactions now it is not clear how (7.11) can be localized. Like the nonlocal 2-point term of (7.5) this more nonlocal Lagrangian has parallels in four dimensional gauge theories. For instance, the gauge invariant nonlocal mass operator

$$\mathcal{O} \equiv \frac{1}{2} \{ \overset{\text{min}}{U} \} \int d^4 x (A_\mu^{aU})^2 \qquad (7.12)$$

has nonlocal cubic terms and higher in its coupling constant expansion. See, for example, the review article [82] for more background. Here A_μ^U is a gauge invariant gauge field by construction and

$$A_\mu^U = U A_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger. \qquad (7.13)$$

Moreover, these can be ordered into a series of gauge invariant operators [83,84]. For instance,

$$
\begin{aligned}
\mathcal{O} = {} & -\frac{1}{2} G^{a\mu\nu} \frac{1}{D^2} G^{a\mu\nu} \\
& + g f^{abc} \left(\frac{1}{D^2} G^{a\mu\nu} \right) \left(\frac{1}{D^2} D^\sigma G^b_{\sigma\mu} \right) \left(\frac{1}{D^2} D^\rho G^c_{\rho\nu} \right) \\
& - g f^{abc} \left(\frac{1}{D^2} G^{a\mu\nu} \right) \left(\frac{1}{D^2} D^\sigma G^b_{\sigma\rho} \right) \left(\frac{1}{D^2} D^\rho G^c_{\mu\nu} \right) \\
& + O(g^2)
\end{aligned}
\tag{7.14}
$$

where $G^a_{\mu\nu}$ is the non-Abelian field strength and D_μ is the covariant derivative. The expansion has a structure which is similar to the nonlocal interactions in (7.11). The sticking point of treating the gauge invariant mass operator is that the localization of a nonlocal 3-point interaction has not been established yet for (7.12) as well as the infinite number of nonlocal operators in the series. So to ascertain whether (7.11) is in fact the lower dimension completeness of $L^{(8)}$ is not immediately possible. In discussing the construction of (7.5) and its potential completeness another assumption has been implicitly dropped in (7.11). That is that there is only one force-matter interaction which is the cubic one. In (7.11) as well as in (7.9) by extension there are now additional cubic as well as quartic force-matter interactions. The upshot of naturally examining what we have termed the lower dimension completeness of the $L^{(8)}$ thread is to open up a more complex set of potential Lagrangians which would need to be analyzed and is beyond the scope of the present article.

VIII. DISCUSSION

One of the main features of our investigation of the $O(N)$ scalar field theories is the observation that the universal theory based on a ϕ^4 interaction has an infinite number of universality classes. The core force-matter interaction, $\sigma\phi^i\phi^i$, defines the linear relation between the dimensions of the separate fields. In [21–23] a specific solution was examined at length which we have denoted by the $n = 1$ thread here. In some sense one ordinarily regards the kinetic terms as the canonical starting point for constructing a Lagrangian rather than the interaction. In the way we have considered the Lagrangian construction from a critical point perspective, the interaction by contrast informs the kinetic term. The n referred to relates to or classifies the powers of derivatives in the kinetic term. For integer values of $n > 1$ higher derivative kinetic terms emerge. While this increases the critical dimension which a Lagrangian is renormalizable in it opens up a host of new Lagrangians which can be studied within the developing d-dimensional conformal field theory formalism. While free field higher derivative kinetic terms have been investigated in [29–31], for instance, there is now an opportunity to study interacting cases. This can be used as a new laboratory to study connections with the AdS/CFT ideas as well as a starting point to classify and more importantly connect scalar

quantum field theories. Moreover, given that our initial motivation was in $O(N)$ scalar theories we have shown that the higher threads of n are accessible via the large N expansion technique developed in [21–23]. In addition we have constructed the ultraviolet completions of several of the theories in each thread and shown by perturbative analysis that they do indeed lie in the same universality class. These are nontrivial checks and required the use of various connecting techniques such as the Tarasov method for relating d-dimensional Feynman integrals with similar integrals in $(d + 2)$ dimensions. We also had to compute new large N d-dimensional critical exponents for the $n = 2$ and 3 threads as these have to be in total agreement with the perturbative renormalization group functions of the fixed dimension Lagrangians lying in the tower of each thread. The next stage in this will be the computation of other large N critical exponents such as ν as well as η at $O(1/N^3)$. The latter in the $n = 1$ thread used the early conformal bootstrap method of [23]. Equally we have concentrated on the $n = 2$ and 3 threads but there is no reason why the analysis we have given here cannot be extended aside from the practical computation limitation which we encountered.

With the core ϕ^4 Lagrangians in the $n > 1$ threads having a critical dimension greater than 4 there is a new potential feature which is what we termed lower dimension completeness. While this is more speculative as to whether there are connecting Lagrangians in the same universality class, the complicating feature appears to be the presence of nonlocalities. At a critical point this would not be as major an issue as trying to construct a viable nonlocal Lagrangian away from criticality. There are examples, such as that of Gribov, [75], which can be renormalized after the localization process introduced by Zwanziger [76–81]. In principle this provides a potential route to study lower dimension complete Lagrangians and we have suggested a toy model in our $L^{(8,6)}$ hypothesis as a place to begin. Understanding nonlocalities in a Lagrangian context may inform models of color confinement in Yang-Mills theories for which the Gribov construction has already been widely studied. However, this will require going beyond the scalar theories considered in the main part of the article. As the key connecting tool across the dimensions appears to be the large N expansion, we briefly discussed the development of the scalar theory ideas in fermionic models such as the $O(N)$ Gross-Neveu model and the non-Abelian Thirring model. In each there is a parallel defining relation between the dimensions of the matter and force fields which admit higher derivative solutions. The extension of the ideas to these fermionic theories has yet to be analyzed in the same depth perturbatively or in the large N construction which we hope to examine in future work.

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[1] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).

[2] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971).

[3] K. G. Wilson, Phys. Rev. B **4**, 3184 (1971).

[4] K. G. Wilson, Phys. Rev. Lett. **28**, 548 (1972).

[5] E. Brézin, J. C. Le Guillou, J. Zinn-Justin, and B. G. Nickel, Phys. Lett. **44A**, 227 (1973).

[6] A. A. Vladimirov, D. I. Kazakov, and O. V. Tarasov, Sov. Phys. JETP **50**, 521 (1979).

[7] F. M. Dittes, Yu. A. Kubyshin, and O. V. Tarasov, Theor. Math. Phys. **37**, 879 (1978).

[8] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Phys. Lett. **99B**, 147 (1981); **101B**, 457(E) (1981).

[9] K. G. Chetyrkin, S. G. Gorishniy, S. A. Larin, and F. V. Tkachov, Phys. Lett. **132B**, 351 (1983).

[10] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K. G. Chetyrkin, and S. A. Larin, Phys. Lett. B **272**, 39 (1991); **319**, 545(E) (1993).

[11] D. V. Batkovich, K. G. Chetyrkin, and M. V. Kompaniets, Nucl. Phys. **B906**, 147 (2016).

[12] O. Schnetz, arXiv:1606.08598.

[13] M. V. Kompaniets and E. Panzer, *Proc. Sci.*, LL2016 (2016) 038.

[14] M. V. Kompaniets and E. Panzer, arXiv:1705.06483.

[15] L. Fei, S. Giombi, and I. R. Klebanov, Phys. Rev. D **90**, 025018 (2014).

[16] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, Phys. Rev. D **91**, 045011 (2015).

[17] J. A. Gracey, Phys. Rev. D **92**, 025012 (2015).

[18] O. F. de Alcantara Bonfim, J. E. Kirkham, and A. J. McKane, J. Phys. A **13**, L247 (1980); **13**, 3785(E) (1980).

[19] O. F. de Alcantara Bonfim, J. E. Kirkham, and A. J. McKane, J. Phys. A **14**, 2391 (1981).

[20] J. A. Gracey, Phys. Rev. D **93**, 025025 (2016).

[21] A. N. Vasil'ev, Y. M. Pismak, and J. R. Honkonen, Theor. Math. Phys. **46**, 104 (1981).

[22] A. N. Vasil'ev, Y. M. Pismak, and J. R. Honkonen, Theor. Math. Phys. **47**, 465 (1981).

[23] A. N. Vasil'ev, Y. M. Pismak, and J. R. Honkonen, Theor. Math. Phys. **50**, 127 (1982).

[24] D. J. Broadhurst, J. A. Gracey, and D. Kreimer, Z. Phys. C **75**, 559 (1997).

[25] K. Lang and W. Rühl, Z. Phys. C **50**, 285 (1991).

[26] K. Lang and W. Rühl, Nucl. Phys. **B377**, 371 (1992).

[27] K. Lang and W. Rühl, Nucl. Phys. **B400**, 597 (1993).

[28] K. Lang and W. Rühl, Nucl. Phys. **B402**, 573 (1993).

[29] C. Brust and K. Hinterbichler, J. High Energy Phys. 02 (2017) 066.

[30] C. Brust and K. Hinterbichler, J. High Energy Phys. 02 (2017) 086.

[31] C. Brust and K. Hinterbichler, J. High Energy Phys. 01 (2017) 126.

[32] R. Percacci and O. Zanusso, Phys. Rev. D **81**, 065012 (2010).

[33] R. Flore, A. Wipf, and O. Zanusso, Phys. Rev. D **87**, 065019 (2013).

[34] F. Gliozzi, A. L. Guerrieri, A. C. Petkou, and C. Wen, Phys. Rev. Lett. **118**, 061601 (2017).

[35] F. Gliozzi, A. L. Guerrieri, A. C. Petkou, and C. Wen, J. High Energy Phys. 04 (2017) 056.

[36] D. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).

[37] R. Dashen and Y. Frishman, Phys. Lett. **46B**, 439 (1973).

[38] R. Dashen and Y. Frishman, Phys. Rev. D **11**, 2781 (1975).

[39] J. Zinn-Justin, Nucl. Phys. **B367**, 105 (1991).

[40] R. D. Pisarski, Phys. Rev. D **28**, 1554 (1983).

[41] J. A. Gracey and R. M. Simms, Phys. Rev. D **91**, 085037 (2015).

[42] W. Celmaster and R. J. Gonsalves, Phys. Rev. Lett. **42**, 1435 (1979).

[43] W. Celmaster and R. J. Gonsalves, Phys. Rev. D **20**, 1420 (1979).

[44] M. d'Eramo, L. Peliti, and G. Parisi, Lett. Nuovo Cimento **2**, 878 (1971).

[45] D. I. Kazakov, Joint Institute for Nuclear Research, Report No. JINR-E2-84-410.

[46] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.96.025022 for the renormalization group functions and critical exponents.

[47] S. Laporta, Int. J. Mod. Phys. A **15**, 5087 (2000).

[48] O. V. Tarasov, Phys. Rev. D **54**, 6479 (1996).

[49] O. V. Tarasov, Nucl. Phys. **B502**, 455 (1997).

[50] P. A. Baikov and K. G. Chetyrkin, Nucl. Phys. **B837**, 186 (2010).

[51] M. Misiak and M. Münz, Phys. Lett. B **344**, 308 (1995).

[52] K. G. Chetyrkin, M. Misiak, and M. Münz, Nucl. Phys. **B518**, 473 (1998).

[53] C. Ford, I. Jack, and D. R. T. Jones, Nucl. Phys. **B387**, 373 (1992).

[54] P. Nogueira, J. Comput. Phys. **105**, 279 (1993).

[55] J. A. M. Vermaseren, arXiv:math-ph/0010025.

[56] M. Tentyukov and J. A. M. Vermaseren, Comput. Phys. Commun. **181**, 1419 (2010).

[57] C. Studerus, Comput. Phys. Commun. **181**, 1293 (2010).

[58] A. von Manteuffel and C. Studerus, arXiv:1201.4330.

[59] T. Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982).

[60] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, J. High Energy Phys. 09 (2015) 076.

[61] J. A. Gracey, Int. J. Mod. Phys. A **06**, 395 (1991); **6**, 2755(E) (1991).

[62] S. É. Derkachov, N. A. Kivel, A. S. Stepanenko, and A. N. Vasil'ev, arXiv:hep-th/9302034.

[63] A. N. Vasil'ev, S. É. Derkachov, N. A. Kivel, and A. S. Stepanenko, Theor. Math. Phys. **94,** 127 (1993).

[64] A. N. Vasil'ev and A. S. Stepanenko, Theor. Math. Phys. **97,** 1349 (1993).

[65] J. A. Gracey, Int. J. Mod. Phys. A **09,** 567 (1994).

[66] J. A. Gracey, Int. J. Mod. Phys. A **09,** 727 (1994).

[67] A. Hasenfratz and P. Hasenfratz, Phys. Lett. B **297,** 166 (1992).

[68] J. A. Gracey, Phys. Lett. B **373,** 178 (1996).

[69] M. Ciuchini, S. É. Derkachov, J. A. Gracey, and A. N. Manashov, Nucl. Phys. **B579,** 56 (2000).

[70] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, Prog. Theor. Exp. Phys. (**2016**) 12C105.

[71] G. 't Hooft and M. Veltman, NATO ASI Ser., Ser. B **4,** 177 (1974).

[72] R. Parthasarathy, J. Phys. A **21,** 4593 (1988).

[73] E. Brézin, J. Zinn-Justin, and J. C. Le Guillou, Phys. Rev. D **14,** 2615 (1976).

[74] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Phys. Rep. **116,** 103 (1984).

[75] V. N. Gribov, Nucl. Phys. **B139,** 1 (1978).

[76] D. Zwanziger, Nucl. Phys. **B209,** 336 (1982).

[77] D. Zwanziger, Nucl. Phys. **B321,** 591 (1989).

[78] D. Zwanziger, Nucl. Phys. **B323,** 513 (1989).

[79] D. Zwanziger, Nucl. Phys. **B364,** 127 (1991).

[80] D. Zwanziger, Nucl. Phys. **B378,** 525 (1992).

[81] D. Zwanziger, Nucl. Phys. **B399,** 477 (1993).

[82] M. Lavelle and D. McMullan, Phys. Rep. **279,** 1 (1997).

[83] D. Zwanziger, Nucl. Phys. **B345,** 461 (1990).

[84] M. Lavelle, D. McMullan, and P. Sharma, Phys. Rev. D **85,** 045013 (2012).