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Topologically nontrivial configurations in the 4d Einstein-nonlinear σ-model system

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We construct exact, regular and topologically nontrivial configurations of the coupled Einstein-nonlinear sigma model in $(3+1)$ dimensions. The ansatz for the nonlinear $SU(2)$ field is regular everywhere and circumvents Derrick's theorem because it depends explicitly on time, but in such a way that its energy-momentum tensor is compatible with a stationary metric. Moreover, the $SU(2)$ configuration cannot be continuously deformed to the trivial Pion vacuum as it possesses a nontrivial winding number. We reduce the full coupled four-dimensional Einstein nonlinear sigma model system to a single second order ordinary differential equation. When the cosmological constant vanishes, such a master equation can be further reduced to an Abel equation. Two interesting regular solutions correspond to a stationary traversable wormhole (whose only "exotic matter" is a negative cosmological constant) and a $(3+1)$-dimensional cylinder whose $(2+1)$-dimensional section is a Lorentzian squashed sphere. The Klein-Gordon equation in these two families of spacetimes can be solved in terms of special functions. The angular equation gives rise to the Jacobi polynomials while the radial equation belongs to the Poschl-Teller family. The solvability of the Poschl-Teller problem implies nontrivial quantization conditions on the parameters of the theory.

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I. INTRODUCTION

The nonlinear sigma model is an important effective field theory with many applications ranging from quantum field theory to statistical mechanics systems like quantum magnetism, the quantum hall effect, meson interactions, superfluid ^{3}He, and string theory [1]. The most relevant application for the $SU(2)$ nonlinear sigma model in particle physics is the description of the low-energy dynamics of pions in $3+1$ dimensions (see for instance [2]; for a detailed review see [3]).

The nonlinear sigma models do not admit static globally regular soliton solutions with nontrivial topological properties in flat, topologically trivial $(3+1)$-dimensional spacetimes. This can be shown using the Derrick's scaling argument [4]. There are two useful strategies to avoid Derrick's argument in the nonlinear sigma model: the first is to search for a time-periodic ansatz such that the energy density of the configuration is still static, as it happens for boson stars [5] in the simpler case of a $U(1)$ charged scalar field (for a detailed review see [6]). The second idea is to couple the nonlinear sigma model with the Einstein theory. In the present paper, we exploit both of them. The possibility of applying the first idea is prompted by the recent generalization of the boson star ansatz to $SU(2)$-valued scalar fields [7–10] and [11–14]. On the other hand, the idea of coupling the nonlinear sigma model to Einstein gravity has been tried, but mostly relying on numerical analyses because of the complexity of the system. Numerical solutions for the Einstein-nonlinear $SU(2)$ σ model with cosmological constant were derived in [15]. Recently exact solutions were presented for the time-dependent gravitating Einstein-Skyrme model [16] by using the ansatz of [7].

In the present paper, we construct two families of analytic topologically nontrivial[1] solutions of the Einstein-nonlinear sigma-model system (both with and without Λ) with quite novel geometrical properties.

In the case in which $\Lambda < 0$, some of these configurations (shortly described in [17]) are smooth and regular everywhere describing wormholes with nonvanishing NUT parameter [18–20] (for a detailed discussion of the NUT geometry, see [21–23] and references therein).

It is worth emphasizing that in [24,25] it has been shown that some typical obstructions to accept solutions with NUT parameter as physically relevant can be removed.

Thus, the present wormholes are supported by a cloud of interacting pions and the only exotic ingredient needed for

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[1] Topologically nontrivial in the sense that the ansatz for the $SU(2)$-valued matter field possesses a nonvanishing third homotopy class. This implies that it cannot be deformed continuously to the trivial pion vacuum.

the construction is a negative cosmological constant (which can hardly be considered as an exotic ingredient).

Moreover, in the $\Lambda = 0$ case, we also construct a second family of regular configurations corresponding to a $(3+1)$-dimensional cylinder whose $(2+1)$-dimensional sections are a Lorentzian squashed sphere. Also in this case the matter field supporting the configuration cannot be deformed continuously to the trivial pion vacuum.

We exploit the gauge invariance of gravitational systems in order to reduce the full coupled system of Einstein nonlinear sigma model equations to a single second order ordinary differential equation (ODE) which, in the vanishing Λ case, can be further reduced to a first order Abel equation. Due to the complexity of the $3+1$-dimensional Einstein $SU(2)$ sigma model, the reduction of the full coupled system to a single second order ODE is a very useful property that can be used by other researchers working in the field. A further nontrivial characteristic of these configurations is that the Klein-Gordon equation on these backgrounds can be fully integrated in terms of special functions.

This paper is organized as follows: in the second section, the action of the system is introduced while in the third, the ansatz for the $SU(2)$-valued matter field is described and the field equations are extracted. In the fourth section, we derive out of the field equations of motion a single second order master equation and interesting particular solutions are analyzed. In Sec. V, the intriguing properties of the Klein-Gordon equation on the two families of regular configurations are disclosed. In the last section, some conclusions are drawn.

II. THE ACTION

We consider the Einstein theory minimally coupled with the $SU(2)$-valued nonlinear sigma model system in four dimensions. The nonlinear sigma model describes the low-energy dynamics of pions, whose degrees of freedom are encoded in an SU(2) group-valued scalar field U [3]. The action of the system is (we follow the notation of [17])

$$S = S_{\mathrm{G}} + S_{\mathrm{pions}}, \tag{1}$$

where the gravitational action S_{G} and the nonlinear sigma model action S_{pions} are given by

$$S_{\mathrm{G}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(\mathcal{R} - 2\Lambda), \tag{2}$$

$$S_{\mathrm{pions}} = \frac{K}{2}\int d^4x\sqrt{-g}\mathrm{Tr}(R^\mu R_\mu), \quad R_\mu = U^{-1}\nabla_\mu U, \tag{3}$$

$$U \in SU(2), \qquad R_\mu = R_\mu^j t_j, \qquad t_j = i\sigma_j, \tag{4}$$

where $\mathcal{R}$ is the Ricci scalar, G is Newton's constant, the parameter K (> 0) is experimentally fixed and σ_j are the Pauli matrices. In our conventions $c = \hbar = 1$, the spacetime signature is $(-,+,+,+)$ (although we shortly discuss Euclidean solutions) and greek indices run over spacetime. Here Λ is the cosmological constant.

As it is well known the nonlinear sigma model can be seen as the $\lambda = 0$ limit (λ being the Skyrme coupling constant) of the Skyrme model (which, on flat spaces, allows the existence of nontrivial topological configurations). We have decided to consider here only the nonlinear sigma model since our analysis provides us with explicit examples of how the coupling with Einstein theory allows the nonlinear sigma model to have smooth configurations with nonvanishing topological charge.

The resulting Einstein equations which follow from the previous action are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{5}$$

where $G_{\mu\nu}$ is the Einstein tensor and κ the gravitational constant. The stress-energy tensor is

$$T_{\mu\nu} = -\frac{K}{2}\mathrm{Tr}\left(R_\mu R_\nu - \frac{1}{2}g_{\mu\nu}R^\alpha R_\alpha\right),$$

which for the nonlinear sigma model can be seen to satisfy the dominant and strong energy conditions [26]. Finally, the matter field equations are

$$\nabla^\mu R_\mu = 0. \tag{6}$$

We adopt the standard parametrization of the $SU(2)$-valued scalar $U(x^\mu)$,

$$U^{\pm 1}(x^\mu) = Y^0(x^\mu)\mathbf{I} \pm Y^i(x^\mu)t_i, \qquad (Y^0)^2 + Y^iY_i = 1, \tag{7}$$

where $\mathbf{I}$ is the 2×2 identity. The last equality implies that $Y^A \coloneqq (Y^0, Y^i)$ is a unit vector in a three sphere, which is naturally accounted for by writing

$$Y^0 = \cos\alpha, \qquad Y^i = n^i\cdot\sin\alpha, \tag{8}$$

$$n^1 = \sin\Theta\cos\Phi, \qquad n^2 = \sin\Theta\sin\Phi, \qquad n^3 = \cos\Theta. \tag{9}$$

III. THE FIELD EQUATIONS

The full coupled Einstein-nonlinear sigma-model system of equations (5) and (6) can be consistently analyzed on the following family of metrics:

$$ds^2 = -F(r)(dt + \cos\theta d\varphi)^2 + N(r)^2dr^2 + \rho^2(r)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{10}$$

As they are already on flat spacetimes the field equations of the nonlinear sigma model are very complicated; it is necessary to take great care in choosing the ansatz for the $SU(2)$-valued scalar field U.

Following the recipe of [7–10] and [11–14], in Ref. [17] the following ansatz time dependent for α, Θ and Φ has been proposed:

$$\Phi = \frac{t+\varphi}{2}, \quad \tan\Theta = \frac{\cot(\frac{\theta}{2})}{\cos(\frac{t-\varphi}{2})}, \qquad \tan\alpha = \frac{\sqrt{1+\tan^2\Theta}}{\tan(\frac{t-\varphi}{2})}. \tag{11}$$

One can verify directly that in any metric of the form in Eq. (10), a pionic configuration of the form in Eqs. (11), (8) and (9) identically satisfies the nonlinear sigma-model field equations [Eq. (6)]. It is also worth emphasizing that the present ansatz is topologically nontrivial as it has a nontrivial winding number along the $\{r = \text{const}\}$ surfaces,

$$W = -\frac{1}{24\pi^2}\int_{\{r=\text{const}\}} tr[(U^{-1}dU)^3] = -\frac{1}{2\pi^2}\int_{\{r=\text{const}\}} \sin^2\alpha \sin\Theta d\alpha d\Theta d\Phi \neq 0.$$

Therefore, the present configuration cannot be deformed continuously to the trivial pion vacuum $U_0 = \mathbf{1}$.

Furthermore, it is worth emphasizing that the nonvanishing winding number of the present configurations is closely related to the nonstandard asymptotic behavior which characterizes both types of solutions constructed in the next sections. Indeed, it is easy to check that if the asymptotic behavior of the metric is anti–de Sitter (AdS), de Sitter (dS) or Minkowski then the present configurations would have vanishing winding number. Such a configuration belongs to a different sector with respect to the usual ones analyzed in the literature on the Einstein-nonlinear sigma-model system.[2]

Despite the fact that such configurations explicitly depend on the timelike coordinate, the energy-momentum tensor is compatible with the above stationary metric,

$$T_t^t = -\frac{K(2F+\rho^2)}{8F\rho^2}, \qquad T_r^r = -\frac{K(2F-\rho^2)}{8F\rho^2},$$
$$T_\theta^\theta = T_\varphi^\varphi = \frac{K}{8F}, \qquad T_t^\varphi = -\frac{K(F+\rho^2)}{4F\rho^2}\cos\theta. \tag{12}$$

Hence, the ansatz in Eqs. (11), (8) and (9) avoids the Derrick theorem since it is explicitly time dependent and, at the same time, compatible with a stationary spacetime. Thus, the present ansatz is the $SU(2)$ generalization of the well-known bosons-starslike ansatz in which a $U(1)$ scalar field depends explicitly on time in such a way that the energy-momentum tensor does not. However, the present case is worth further analyzing for at least two reasons: first of all, the matter field corresponds to pions and secondly, $SU(2)$-valued matter fields may possess nontrivial topological properties. It is also worth emphasizing that the above $T_{\mu\nu}$ has positive energy density. Indeed, as is well known [26], the T_μ^ν of the nonlinear sigma model satisfies both the null and the weak energy conditions [as can be checked directly in Eq. (12)].

A direct computation reveals that the full Einstein-nonlinear sigma-model system reduces, in this sector, to the following three equations (two second order equations and one constraint):

$$0 = 16\rho^3 FN\rho'' + 8\rho^2 FN(\rho')^2 - 16\rho^3 FN'\rho' + N^3[K\kappa\rho^2(2F+\rho^2) + 2F(4\rho^2(\Lambda\rho^2-1)-3F)], \tag{13}$$

$$0 = -8\rho^2 F(\rho')^2 - 8\rho^3\rho' F' + N^2[K\kappa\rho^2(\rho^2-2F) - 2F(4\rho^2(\Lambda\rho^2-1)-F)], \tag{14}$$

$$0 = -8\rho^3 F^2 N\rho'' - 4\rho^4 FNF'' - 4\rho^3 FN\rho' F' + 2\rho^4 N(F')^2 + 4\rho^3 F(2FN'\rho' + \rho N'F') + FN^3(K\kappa\rho^4 - 2F(4\Lambda\rho^4 + F)), \tag{15}$$

where Eqs. (13)–(15) correspond to the $t-t$, $r-r$ and $\theta-\theta$ components of the Einstein equations, respectively, while the prime denotes differentiation with respect to r.

It can be easily seen that these remaining Einstein equations are not independent from one another. Indeed, a direct computation shows that the total derivative of (14) is a combination of (13) and (15). Moreover it is straightforward to see that the field equations can be derived from the variation of a Lagrangian $\mathcal{L} = L(N, F, F', \rho, \rho')$. Due the reparametrization invariance of this Lagrangian, Eqs. (13)–(15) form a singular dynamical system. From the theory of constrained systems we know that Eq. (14)—since it corresponds to a first class constraint in the Hamiltonian formalism of the theory—removes a full degree of freedom from the system (classic references are [27,28]). Thus, between F and ρ only one can be considered as a true physical degree of freedom and its evolution can be described by a single second order equation, which we derive in the following section.

In order to see the classical analogue of that through a gauge fixing process, consider without loss of generality that $N = N(F, \rho)$; then the field equations describe a classical system of two degrees of freedom in which (14) is the Hamiltonian function expressed in velocity phase space variables. It can be seen as a conservation law with a fixed value, which means that it constrains the

[2]As in most of the examples analyzed in the literature the pions have vanishing winding number.

solution. Indeed, if we solve the two second order differential equations, one of the four integration constants is a constraint from (14). Moreover, because (14) is the Hamiltonian function it means that the dynamical system is autonomous and the solution is invariant under translations of the independent variable, in our consideration the radius. Hence, a second integration constant can be eliminated, which means that there are only two integration constants and consequently only one free degree of freedom, which means that there exists a second order differential equation which describes equivalently the solution of the field equations.

IV. DERIVATION OF THE MASTER EQUATION

Let us avoid the typical fixing condition $N = 1$ and exploit the fact that the constraint equation (14) is algebraic with respect to N; its solution is

$$N(r) = \pm 2\rho \left(\frac{\rho' (\rho F)'}{F^2 + \frac{1}{2}\bar{K}\rho^4 - F\rho^2(\bar{K} + 4\Lambda\rho^2 - 4)} \right)^{\frac{1}{2}}, \quad (16)$$

where from now on we consider for simplicity a single constant $\bar{K} = K\kappa$. Upon substitution of (16) in (13) and (15), they both reduce to a single equation[3] for ρ and F, namely

$$\begin{aligned} &F\rho^2 F'(2F^2 + \bar{K}\rho^4 - 2F\rho^2(\bar{K} + 4\Lambda\rho^2 - 4))\rho'' \\ &\quad - F\rho^2\rho'(2F^2 + \bar{K}\rho^4 - 2F\rho^2(\bar{K} + 4\Lambda\rho^2 - 4))F'' \\ &\quad + \rho'[F\rho F'\rho'(4(\bar{K} - 4)F\rho^2 - 14F^2 + 3\bar{K}\rho^4) \\ &\quad + \rho^2(F')^2(\bar{K}\rho^4 - 2F^2) \\ &\quad + 4F^2(\rho')^2(\rho^4(\bar{K} - 4\Lambda F) - 2F^2)] = 0. \end{aligned} \quad (17)$$

We have to note here that solving (14) with respect to $N(r)$ does not constitute a fixing choice, since the constraint equation must be satisfied in any lapse. As a result, we still have the freedom to gauge fix the system by choosing either ρ or F to be some explicit function of r. We make the following choice regarding ρ together with a reparametrization condition for F, by introducing a new function $g(r)$ in its place,[4]

$$\rho(r) = r, \qquad F = \frac{g(r)}{r}. \quad (18)$$

Then, Eq. (17) simplifies to the following second order differential equation:

$$\begin{aligned} &rg(\bar{K}r^6 - 2r^3(\bar{K} - 4 + 4r^2\Lambda)g + 2g^2)g'' \\ &\quad - g'(3\bar{K}r^6 g - 6g^3 + \bar{K}r^7 g' - 2rg^2(8r^4\Lambda + g')) = 0. \end{aligned} \quad (19)$$

As a result, we have succeeded in reducing the initial system of equations into a problem of finding the solution of a single nonautonomous second order ODE. As it is shown in the examples below, all the metric components can be constructed once the solution of the above master equation is known.

In particular, once a solution of the above master equation (19) is known, one can construct a solution of the full system made by Eqs. (13)–(15) just using Eqs. (18) and (16).

What is more, Eq. (19) can be further reduced (when $\Lambda = 0$) to a first order ODE of Abel type.

In the remainder of this section we study sets of particular solutions of (19) as well as its group invariant transformations and specifically the Lie point symmetries.

It is worth emphasizing that the reduction of the full coupled four-dimensional Einstein nonlinear sigma model system in a topologically nontrivial sector to a single second order ODE is quite an achievement in itself and is very useful for researchers in the field. For instance, the above master equation is a very good starting point to analyze whether or not there are black holes in this sector of the theory. We hope to come back on this issue in a future publication.

A. Solution from Lie symmetry

Equation (19) is of the general form $g'' = \Omega(r, g, g')$. It is defined in the jet space $A_T = \{r, g, g', g''\}$; hence in order to be invariant under an infinitesimal point transformation of the form

$$\bar{r} = r + \varepsilon\xi(r, g), \qquad \bar{g} = g + \varepsilon\eta(r, g), \quad (20)$$

whose generator is $X = \xi\frac{\partial}{\partial r} + \eta\frac{\partial}{\partial g}$, the following condition should hold,

$$X^{[2]}(g'' - \Omega) = 0, \quad \mathrm{mod}\ g'' = \Omega, \quad (21)$$

where $X^{[2]}$ is the second prolongation of X, i.e., the extension of the generator into the jet space A_T (for the rigorous mathematical definitions, as well as the general theory of Lie point symmetries, we refer the interested reader to various textbooks on the subject, e.g., [32,33]). If (21) is satisfied, then X generates a Lie point symmetry for the differential equation. Its existence provides a point transformation under which Eq. (19) can be either

[3] This is a further confirmation of the fact that among F and ρ only one can be considered as a true physical degree of freedom.

[4] The reasoning behind this choice and parametrization can be traced to a minisuperspace analysis of the system and the fact that it simplifies a nonlocal integral of motion constructed out of a conformal vector of the minisuperspace metric. For similar techniques used in a minisuperspace context in order to achieve integrability see for example [29,30] and [30] and in regular systems [31].

transformed to an autonomous equation or reduced to a first order equation.

From the symmetry condition (21) we find that Eq. (19) admits a Lie point symmetry if and only if the cosmological constant is 0. The corresponding symmetry vector is found to be

$$X = \partial_r + 3g\partial_g \tag{22}$$

and can be used to further simplify the equation when $\Lambda = 0$.

By using this symmetry vector we can introduce the normal coordinates,

$$r = \exp\left(\frac{R}{3}\right), \text{ and, } g = e^R S(R), \tag{23}$$

with the help of which Eq. (19) assumes the following form:

$$3S_{,RR} + (2S + 5S_{,R}) - \frac{3(\bar{K} - 2S^2)(S + S_{,R})(2S + S_{,R})}{S(\bar{K} - 2(\bar{K} - 4)S + 2S^2)} = 0. \tag{24}$$

The latter is an autonomous second order differential equation due to the fact that the variables we introduced transformed the symmetry vector into $X = \partial_R$.

1. The Abel master equation with $\Lambda = 0$

In the sector with vanishing cosmological constant it is possible to further simplify the master equation in Eq. (19). Indeed, from the application of the differential invariants, (24) can be reduced to a first order Abel equation of the second kind

$$v_{,z} = \lambda_0(z)\left[\lambda_1(z)v + \frac{\lambda_2(z)}{v} + \lambda_3(z)\right], \tag{25}$$

where $z = S(R)$, $v(z) = S_{,R}$ or $R = \int \frac{1}{v(z)} dz + R_0$. Coefficients $\lambda_{0,1,2,3}$ are as follows,

$$\lambda_0(z) = [3z(2(\bar{K} - 4)z - 2z^2 - \bar{K})]^{-1},$$
$$\lambda_1(z) = (6z^2 - 3\bar{K}), \tag{26}$$

$$\lambda_2(z) = 4(4z^4 + (4 - \bar{K})z^3 - \bar{K}z^2), \tag{27}$$

$$\lambda_3(z) = 28z^3 + 10(4 - \bar{K})z^2 - 4\bar{K}z. \tag{28}$$

Of course with a change of variables Eq. (25) can be written as an Abel equation of the first kind [34].

In the following we study some particular solutions.

B. Particular solutions

It can be easily verified that Eq. (19) admits a power law solution that is given by

$$g(r) = \sigma r^3 \tag{29}$$

whenever $\sigma = \frac{\bar{K}}{4}$ or $\sigma = -1$. We begin our analysis by examining the former case. Solutions of that form indicate that $F = \sigma\rho^2$ as we can see from (18). In what follows we study those solutions in the original coordinates of the spacetime.

1. The $\sigma = \frac{\bar{K}}{4}$ case: the wormhole

This solution corresponds to the Lorentzian wormhole constructed in [17]. Here, as we can observe from (16) and (18), it is expressed in a gauge where the "lapse" $N(r)$ of the metric is

$$N(r) = \pm\frac{4\sqrt{3}}{\sqrt{24 - 3\bar{K} - 16\Lambda r^2}}. \tag{30}$$

At this point, to see how the solution is expressed in the original variables, we can perform a transformation $r \to \tilde{r}$ that returns us to the gauge $N(\tilde{r}) = 1$. For the case when $\Lambda \neq 0$ and $\bar{K} \neq 8$ this is achieved by

$$\int N(r)dr = \tilde{r} \Rightarrow r = \lambda\sqrt{\frac{3(\bar{K} - 8)}{16(-\Lambda)}}\cosh\left(\frac{\sqrt{-\Lambda}\tilde{r}}{\sqrt{3}}\right) + \sqrt{1 - \lambda^2}\sqrt{\frac{3(\bar{K} - 8)}{16\Lambda}}\sinh\left(\frac{\sqrt{-\Lambda}\tilde{r}}{\sqrt{3}}\right), \tag{31}$$

where λ is an integration constant which can be either real or imaginary; it depends on the behavior of the rest of the parameters $\bar{K}$ and Λ. We have to note here that the original configuration space variables F and ρ that appear on the metric are now given as

$$\rho(\tilde{r}) = r(\tilde{r}) = \lambda\sqrt{\frac{3(\bar{K} - 8)}{16(-\Lambda)}}\cosh\left(\frac{\sqrt{-\Lambda}\tilde{r}}{\sqrt{3}}\right) + \sqrt{1 - \lambda^2}\sqrt{\frac{3(\bar{K} - 8)}{16\Lambda}}\sinh\left(\frac{\sqrt{-\Lambda}\tilde{r}}{\sqrt{3}}\right), \tag{32a}$$

$$F(\tilde{r}) = \frac{g(\tilde{r})}{r(\tilde{r})} = \frac{\bar{K}}{4}r(\tilde{r})^2 = \frac{\bar{K}}{4}\rho^2(\tilde{r}). \tag{32b}$$

Since only ρ^2 appears in the metric, all combinations for which (31) is real (or pure imaginary) are acceptable for a Lorentzian (or Euclidean) signature solution. Let us note that for the rest of this subsection—and for reasons of simplicity—we omit the tilde, understanding that we are going to restrict our analysis solely in the gauge $N = 1$ in which we decide to symbolize the dynamical variable from now on as r.

Let us consider for example the case when $\bar{K} > 8$, $\Lambda < 0$ and at the same time $|\lambda| \geq 1$. By setting $|\lambda| = \cosh m$ inside solution (32a) and applying the appropriate trigonometric identity we get

$$\rho(r) = \pm\frac{1}{4}\sqrt{\frac{3(\bar{K}-8)}{-\Lambda}}\cosh\left(\sqrt{\frac{-\Lambda}{3}}r \pm m\right) \quad (33)$$

with the $\pm$ signs not being of importance, since only ρ^2 appears on the metric and m being able to be either positive or negative. By checking the metric components it is obvious that this latter constant is not even essential for the geometry because it can be easily absorbed by a translation in the r variable. Hence, without any loss of generality, we can assume m to be equal to 0 and write the most general solution, for this specific range of validity of the parameters $\bar{K}$ and Λ,

$$\rho(r) = \frac{1}{4}\sqrt{\frac{3(\bar{K}-8)}{-\Lambda}}\cosh\left(\sqrt{\frac{-\Lambda}{3}}r\right), \quad (34a)$$

$$F(r) = \frac{\bar{K}}{4}\rho(r)^2, \qquad \bar{K} - 8 > 0. \quad (34b)$$

As it has been discussed in [17], the metric corresponding to Eq. (33) is a traversable Lorentzian wormhole constructed using only ingredients from the standard model and a negative cosmological constant. To analyze the global structure of the solution it is useful to compare it with the NUT-AdS line element,

$$ds^2 = -f(r)(dt + 2n\cos\theta d\phi)^2 + \frac{dr^2}{f(r)} + (r^2+n^2)d\Omega_2^2, \quad (35)$$

where

$$f(r) = \frac{r^2 + N^2 + \ell^{-2}(r^4 - 6N^2r^2 - 3N^4)}{r^2 - N^2}, \quad (36)$$

which in the asymptotical limit $r \to +\infty$ can be written as

$$f_\infty(r) = \ell^{-2}r^2 + (1 - 5\ell^{-2}N^2) + \mathcal{O}(r^{-2}). \quad (37)$$

The corresponding asymptotical form of line element (35) can be transformed under $r = \ell(1-\frac{5N^2}{\ell^2})^{1/2}\sinh(\bar{r}/\ell)$ [assuming $\ell^{-2} > 5N^2$ and $n = (\ell^2 - 5N^2)^{1/2}$] into

$$\begin{aligned} ds^2 = &-\frac{(\ell^2-5N^2)\cosh^2(\bar{r}/\ell)}{l^2} \\ &\times (dt + 2(\ell^2-5N^2)^{1/2}\cos\theta d\phi)^2 \\ &+ d\bar{r}^2 + (\ell^2-5N^2)\cosh^2(\bar{r}/\ell)d\Omega_2^2, \end{aligned} \quad (38)$$

which belongs to the general form of our solution. Thus, the asymptotic form of a NUT-AdS solution coincides with our result, the difference being of course that the present solution has two identical asymptotic regions for $r \to \pm\infty$.

Moreover, the ensuing spacetime we get from (10) does not exhibit any curvature singularity. For instance, the Kretschmann scalar is

$$\begin{aligned} K_s = R_{\kappa\lambda\mu\nu}R^{\kappa\lambda\mu\nu} = &\frac{\Lambda^2}{9(\bar{K}-8)^2\cosh^4(\sqrt{\frac{-\Lambda}{3}}r)} \\ &\times\Big[3(\bar{K}-8)^2\cosh\Big(4\sqrt{\frac{-\Lambda}{3}}r\Big) \\ &- 4(\bar{K}-8)(\bar{K}+16)\cosh\Big(2\sqrt{\frac{-\Lambda}{3}}r\Big) \\ &+ 57\bar{K}^2 + 272\bar{K} + 1088\Big] \end{aligned} \quad (39)$$

and it is regular for all $r \in \mathbb{R}$. The special properties of the Klein-Gordon equation on this metric are described in the next sections.

For the sake of brevity we state in Table I all the admissible Lorentzian solutions corresponding to the various choices of parameters. The procedure is similar and in all cases the integration constant can be absorbed by a simple translation in the r variable.

The corresponding Euclidean solutions can be obtained in each case by simply violating the conditions given for $\bar{K}$ in the first column of Table I. For example, if we consider $\bar{K} < 8$ and $\Lambda < 0$ the first solution is written as $\rho(r) = \frac{i}{4}\sqrt{\frac{3(8-\bar{K})}{-\Lambda}}\cosh\left(\sqrt{\frac{-\Lambda}{3}}r\right)$ and leads to a Euclidean space without a singularity. An interesting pattern may be observed in regard to these solutions: The curvature singularities appear for $\bar{K} < 8$ in the Lorentzian manifolds, while the situation is inverted for Euclidean spacetimes, where they appear only for $\bar{K} > 8$.

TABLE I. Admissible values of essential parameters and singularities.

Parameters	$\rho(r)$	Singularities
$\bar{K} > 8$, $\Lambda < 0$	$\frac{1}{4}\sqrt{\frac{3(\bar{K}-8)}{-\Lambda}}\cosh\left(\sqrt{\frac{-\Lambda}{3}}r\right)$	No
$\bar{K} < 8$, $\Lambda < 0$	$\frac{1}{4}\sqrt{\frac{3(8-\bar{K})}{-\Lambda}}\sinh\left(\sqrt{\frac{-\Lambda}{3}}r\right)$	$r = 0$
$\bar{K} < 8$, $\Lambda > 0$	$\frac{1}{4}\sqrt{\frac{3(8-\bar{K})}{\Lambda}}\sin\left(\sqrt{\frac{\Lambda}{3}}r\right)$	$r = \sqrt{\frac{3}{\Lambda}}k\pi, k \in \mathbb{Z}$

Of course we have still to examine three special cases spanned by $\Lambda = 0$ and/or $\bar{K} = 8$, which were excluded in transformation (31). Let us begin by considering $\Lambda = 0$. In this case, one only needs to perform a scaling in the radial variable to express the result in the gauge $N = 1$. The corresponding solution reads

$$\rho(r) = \frac{1}{4}\sqrt{8 - \bar{K}}r, \tag{40a}$$

$$F(r) = \frac{\bar{K}}{4}\rho(r)^2, \tag{40b}$$

where $\bar{K} \neq 8$. The line element is Lorentzian if $\bar{K} < 8$ and Euclidean when $\bar{K} > 8$. In both cases the manifold exhibits a curvature singularity at the origin $r = 0$.

When $\bar{K} = 8$ (but $\Lambda \neq 0$), we are led to the solution (again expressed in the $N = 1$ gauge)

$$\rho(r) = \omega e^{\frac{\sqrt{-\Lambda}r}{\sqrt{3}}}, \tag{41a}$$

$$F(r) = 2\rho(r)^2. \tag{41b}$$

The cosmological constant has to be negative, while the constant ω can be seen not to be essential for the geometry and can be absorbed by a translation in r. Thus, without loss of generality we can consider $\omega = 1$ (or $\omega = \mathbb{i}$) for a Lorentzian (or Euclidean) manifold. The emerging spacetime is regular for $r > 0$, since the Kretschmann scalar is

$$K_s = 27e^{-\frac{4\sqrt{-\Lambda}r}{\sqrt{3}}} + 4\Lambda e^{-\frac{2\sqrt{-\Lambda}r}{\sqrt{3}}} + \frac{8\Lambda^2}{3}. \tag{42}$$

Note here that apart from (41), the solution $\rho(r) = \omega e^{-\frac{\sqrt{-\Lambda}r}{\sqrt{3}}}$ is also valid. The situation however is inverted and the space is now regular for $r < 0$.

2. Cylindrical solution

Another interesting solution is retrieved when both $\Lambda = 0$ and $\bar{K} = 8$, which has to be examined separately. The functions $F(r)$ and $\rho(r)$ in this case are constants, $F(r) = \frac{\bar{K}}{4}\rho^2(r) = \frac{\bar{K}}{4}\rho_0^2 = \text{const}$. This leads to a regular manifold described by the metric

$$ds^2 = \rho_0^2[-2(dt + \cos\theta d\varphi)^2 + (d\theta^2 + \sin^2\theta d\varphi^2)] + dr^2, \tag{43}$$

which admits five Killing fields: the four of the general line element (10)

$$\xi_1 = \partial_t, \qquad \xi_2 = \partial_\phi,$$
$$\xi_3 = \frac{\sin\phi}{\sin\theta}\partial_t + \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi,$$
$$\xi_4 = -\frac{\cos\phi}{\sin\theta}\partial_t + \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \tag{44}$$

plus the obvious from the form of the metric (43) $\xi_5 = \partial_r$. The special properties of the Klein-Gordon equation on this metric are described in the next sections.

3. The $\sigma = -1$ case

The treatment is exactly the same as in the previous case. For brevity we only demonstrate the final resulting solutions in the gauge $N = 1$. In the line element we set $F(r) = -\rho(r)^2$, with $\rho(r)$ being given for various values of the parameters in Table II.

All the above solutions lead to Euclidean line elements. Lorentzian solutions can be obtained by letting $\bar{K}$ possess values that violate the conditions of the first column. However, the signature of the metric becomes $(-,+,-,-)$, indicating that r has effectively assumed the role of the time parameter; hence these solutions should rather be considered as cosmological and we refrain from their analysis here.

From the geometrical perspective the solutions expressed in the tabular are not different from the corresponding Euclidean solutions of the previous sections, since only a combination of constants differs in the line element. Nevertheless, it is to be noted that the critical value for the constant $\bar{K}$ in the line element has changed from 8 to 2.

Finally, we conclude this analysis by presenting the particular solutions for three special cases that arise again: $\Lambda = 0$ and/or $\bar{K} = 2$. Let us start with the cosmological constant being 0 while $\bar{K} \neq 2$. Then, the function $\rho(r)$ reads

$$\rho(r) = \frac{1}{2}\sqrt{\frac{2 - \bar{K}}{2}}r \tag{45}$$

and the resulting manifold is Euclidean when $\bar{K} < 2$ (the Lorentzian case $\bar{K} > 2$ being again cosmological) with a curvature singularity at the origin $r = 0$.

The situation when $\Lambda \neq 0$ but $\bar{K} = 2$ is quite different. The solution is given by

TABLE II. Admissible values of essential parameters and singularities.

Parameters	$\rho(r)$	Singularities
$\bar{K} > 2,\ \Lambda < 0$	$\frac{1}{2}\sqrt{\frac{3(\bar{K}-2)}{-2\Lambda}}\cosh\left(\sqrt{\frac{-\Lambda}{3}}r\right)$	No
$\bar{K} < 2,\ \Lambda < 0$	$\frac{1}{2}\sqrt{\frac{3(2-\bar{K})}{-2\Lambda}}\sinh\left(\sqrt{\frac{-\Lambda}{3}}r\right)$	$r = 0$
$\bar{K} < 2,\ \Lambda > 0$	$\frac{1}{2}\sqrt{\frac{3(2-\bar{K})}{2\Lambda}}\sin\left(\sqrt{\frac{\Lambda}{3}}r\right)$	$r = \sqrt{\frac{3}{\Lambda}}k\pi,\ k \in \mathbb{Z}$

$$\rho(r) = \sigma e^{\frac{\sqrt{-\Lambda}r}{\sqrt{3}}} \tag{46}$$

with σ being an integration constant which if it is real the resulting line element is Euclidean and is given by

$$ds^2 = e^{\frac{2\sqrt{-\Lambda}r}{\sqrt{3}}}dt^2 + e^{\frac{2\sqrt{-\Lambda}r}{\sqrt{3}}}\cos\theta dt d\phi + dr^2 + e^{\frac{2\sqrt{-\Lambda}r}{\sqrt{3}}}(d\theta^2 + d\phi^2). \tag{47}$$

The constant σ is not essential for the geometry and has been set equal to 1. The spacetime is regular for $r > 0$ and, as with a previous special case, the solution is also valid under a parity transformation $r \to -r$, $\rho(r) = \sigma e^{-\frac{\sqrt{-\Lambda}r}{\sqrt{3}}}$ with the manifold being now regular for $r < 0$.

At the end we complete this analysis of particular solutions by checking the case where $\Lambda = 0$ and $\bar{K} = 2$. This results in the solution where both F and ρ are constants with $F(r) = -\rho(r)^2 = -\rho_0^2$, which lead to a regular Euclidean manifold when ρ_0 is real and Lorentzian but a cosmological manifold (with r as time) whenever ρ_0 is purely imaginary.

Let us note that all the above solutions of the $F = -\rho^2$ case admit six Killing vector fields instead of just the four of the initial general line element (44), the extra two being

$$\xi_5 = \cot\theta\cos t\partial_t + \sin t\partial_\theta - \frac{\cos t}{\sin\theta}\partial_\phi, \tag{48a}$$

$$\xi_6 = \cot\theta\sin t\partial_t - \cos t\partial_\theta - \frac{\sin t}{\sin\theta}\partial_\phi. \tag{48b}$$

V. THE KLEIN-GORDON EQUATION IN THE TWO REGULAR LORENTZIAN FAMILIES

In this section a quite nontrivial property of the two regular Lorentzian configurations of the Einstein nonlinear sigma model system is discussed. Namely, the Klein-Gordon equation on the metrics corresponding to Eqs. (34a), (34b) and (43), respectively, is not only separable but it is also integrable in terms of very well-known solvable potentials.

A. The wormhole solution

The Klein-Gordon equation

$$(\Box - m^2)\Psi = 0 \tag{49}$$

on the metrics corresponding to Eqs. (34a) and (34b) (which are allowed when $\bar{K} - 8 > 0$) can be solved by separation of variables. That is possible since Eq. (49) is a linear equation, i.e., the Lie symmetry vector $\Psi\partial_\Psi$ exists, and secondly the underlying manifold which defines the box operator is a locally rotational spacetime and admits a four-dimensional Killing algebra, the $A_1 \oplus \mathfrak{so}(3)$ [35], where A_1 indicates the autonomous symmetry vector ∂_t. Recall that the isometry vectors of the spacetime (10) generate Lie point symmetries for Eq. (49); for details see [36].

It is known that any Lie point symmetry $X = \partial_I + \alpha\Psi\partial_\Psi$ is equivalent with the linear Lie-Bäcklund symmetry $\bar{X} = (\Psi_{,I} - \alpha\Psi)\partial_\Psi$, and by the definition of the symmetry vector that transforms solutions into solutions we have $\bar{X}\Psi = (\gamma - \alpha)\Psi$, or $\Psi_{,I} = \gamma\Psi$, that is, $\Psi = \Psi(x^i)e^{\gamma x^I}$; solutions of that form are called invariant solutions.

Hence, for Eq. (49) with the use of the autonomous symmetry, the rotation ∂_ϕ, and the Casimir invariant of the $\mathfrak{so}(3)$ algebra, we find the following general form for an invariant solution,

$$\Psi(t,r,\theta,\phi) = U(r)Y(\theta)e^{i(\mu\phi+\omega t)}, \tag{50}$$

where $U(r), Y(\theta)$, satisfies the following set of linear second order differential equations,

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}Y(\theta)\right) + \left(\frac{\omega\cos\theta - \mu}{\sin\theta}\right)^2 Y(\theta) = \lambda Y(\theta), \tag{51}$$

$$U''(r) + \sqrt{-3\Lambda}\tanh\left(\sqrt{\frac{-\Lambda}{3}}r\right)U'(r) + \frac{16\Lambda(\lambda\bar{K} - 4\omega^2)}{3(\bar{K} - 8)\bar{K}\cosh^2(\sqrt{\frac{-\Lambda}{3}}r)}U(r) = m^2 U(r), \tag{52}$$

where λ is the separation constant. Equations (51) and (52) are linear and maximally symmetric, which implies that there exists a transformation $\{r, U, \theta, Y\} \to \{\bar{r}, \bar{U}, \bar{\theta}, \bar{Y}\}$, where they can be written in the form of those of the free particle [37]; that is,

$$\frac{d^2\bar{U}}{d\bar{r}^2} = 0, \qquad \frac{d^2\bar{Y}}{d\bar{\theta}^2} = 0. \tag{53}$$

Even though this property holds for any general form of the spacetime (10), i.e., for arbitrary functions $F(r)$, $\rho(r)$, and such a transformation always exists, it is not always possible to express the latter in a closed form. However, as we see below for the case at hand, the radial term of the Klein-Gordon equation reduces to a well-known linear equation in which the solution can be written in closed form.

In Eq. (52) we do the following change of variable,

$$U(r) = \frac{\psi(r)}{\left[\cosh\left(\sqrt{\frac{-\Lambda}{3}}r\right)\right]^{3/2}}; \tag{54}$$

then we find

$$\psi'' + \frac{\Sigma}{\left[\cosh\left(\sqrt{\frac{-\Lambda}{3}}r\right)\right]^2}\psi = \left(-\frac{3}{4}\Lambda + m^2\right)\psi, \quad (55)$$

$$\frac{\Lambda[(24 + 64\lambda - 3\bar{K})\bar{K} - 256\omega^2]}{12\bar{K}(\bar{K} - 8)} = \Sigma. \quad (56)$$

The angular Eq. (51) reduces to the equation for the rotation matrices which appear in the scattering problem of a monopole. In particular, Eq. (51) reduces to Eqs. (3.10) and (3.11) of [38] with the identifications

$$\lambda = j(j + 1) - (m')^2, \quad (57)$$

$$\omega = m', \qquad \mu = \bar{m}, \quad (58)$$

$$j \geq |m'|, \qquad |\bar{m}| \Rightarrow \lambda > 0. \quad (59)$$

In the notation of [38], m' is the strength of the Dirac monopole and can take only integer and half-integer values; $\bar{m}$ is the azimutal quantum number (restricted to be integer) while j is the eigenvalue of the total angular momentum. Therefore, just from the generic properties of the rotation matrices, one gets that the frequency ω appearing in the invariant solution (50) is restricted to be integer or half integer. Moreover, the separation constant λ is quantized as in Eq. (57) and strictly positive.

On the other hand, the radial Eq. (55) [with the coefficient in Eq. (56)] corresponds to the Poschl-Teller potential (which is one of the most famous exactly solvable potentials in quantum mechanics: see, for a review, [39]). Thus, one can extract a lot of information and nontrivial constraints from well-known results on the Poschl-Teller potential. First of all, the coefficient Σ is required to be positive, which implies the following requirement on the coupling constant $\bar{K}$ to be large enough:

$$(24 + 64\lambda - 3\bar{K})\bar{K} - 256\omega^2 < 0. \quad (60)$$

Secondly, one observes that the would-be eigenvalue E^2 of the Schrodinger equation with Poschl-Teller potential (55) is

$$E^2 = -\left(-\frac{3}{4}\Lambda + m^2\right)$$

and therefore negative. Hence, the problem is solvable if the Poschl-Teller potential in Eq. (55) with the coefficient in Eq. (56) admits bound states. Well-known results on the Schrodinger equation with the Poschl-Teller potential tell us the number of bound states and the corresponding eigenvalues. To proceed, let us first write Eq. (55) with standard normalization changing the variable from r to $x = \sqrt{\frac{-\Lambda}{3}}r$,

$$-\frac{d^2\psi}{dx^2} - \frac{3\Sigma}{(-\Lambda)[\cosh(x)]^2}\psi = -\frac{(-\frac{3}{4}\Lambda + m^2)}{(\frac{-\Lambda}{3})}\psi. \quad (61)$$

Then, let us write the coefficient of the potential as

$$\frac{3\Sigma}{(-\Lambda)} = N(N + 1), \qquad N > 0. \quad (62)$$

The number n_B of bound states in the Poschl-Teller potential is

$$n_B = \lfloor N \rfloor, \quad (63)$$

where $\lfloor X \rfloor$ denotes the integer part of X. The corresponding eigenvalues are given by the condition

$$\frac{\Gamma(\zeta + 1)\Gamma(\zeta)}{\Gamma(\zeta + N + 1)\Gamma(\zeta - N)} = 0, \quad (64)$$

$$\frac{(-\frac{3}{4}\Lambda + m^2)}{(\frac{-\Lambda}{3})} = -\zeta^2, \quad (65)$$

where Γ is the Euler gamma function. Therefore, assuming ζ to be positive, the eigenvalues are determined by the condition that $\zeta - N$ is a negative integer,

$$-(\zeta - N) \in \mathbb{N}, \qquad \zeta - N < 0. \quad (66)$$

In particular, the allowed values for m^2 in the Klein-Gordon equation are fixed by the quantization condition in Eq. (66). Finally, in order to have at least one bound state in the radial Schrodinger problem one must require that

$$\frac{3\Sigma}{(-\Lambda)} = N(N + 1) \geq 2. \quad (67)$$

B. The cylindrical solution

The Klein-Gordon equation

$$(\Box - m^2)\Psi = 0,$$

on the metrics corresponding to Eq. (43) (which are allowed when $\bar{K} = 8$ and $\Lambda = 0$) and as before is invariant under the same group of transformations; hence it can be solved with the method of separation of variables. Therefore, the group invariant ansatz (50), substituted in the Klein-Gordon equation, results in the following system,

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}Y(\theta)\right) + \left(\frac{\omega\cos\theta - \mu}{\sin\theta}\right)^2 Y(\theta) = \lambda Y(\theta), \quad (68)$$

$$U''(r) - \left(m^2 + \frac{\lambda - 2\omega^2}{4\rho_0^2}\right)U(r) = 0, \quad (69)$$

where λ is the separation constant.

In the present case, the angular equation is the same as in the previous subsection and can be analyzed following [38]. In particular, the solvability of the problem once again requires the quantization conditions in Eqs. (57) and (58) (which implies that $\lambda > 0$) for C and ω. In this case, the radial equation is simpler and one gets the typical behavior of a free massive field when

$$m^2 + \frac{\lambda - 2\omega^2}{4\rho_0^2} < 0, \tag{70}$$

in which case one gets periodic solutions along the axis of the cylinder. On the other hand, when

$$m^2 + \frac{\lambda - 2\omega^2}{4\rho_0^2} > 0,$$

one gets solutions which are unbounded on, at least, one side of the cylinder (either $r \to -\infty$ or $r \to +\infty$). Therefore, the inequality in Eq. (70) must be satisfied.

In conclusion, in both cases the solvability of the Klein-Gordon equation implies strong constraints on the parameters of the theory.

VI. CONCLUSIONS AND PERSPECTIVES

Exact, regular and topologically nontrivial configurations of the Einstein-nonlinear sigma model system in $(3+1)$ dimensions have been constructed. The ansatz is the $SU(2)$ generalization of the usual boson star ansatz for charged $U(1)$ fields. Moreover, the $SU(2)$ configuration cannot be deformed continuously to the trivial pion vacuum as it possesses a nontrivial winding number. Due to the fact that, with the chosen ansatz for the metric, the nonlinear sigma model satisfies identically the corresponding field equations, the full coupled Einstein nonlinear sigma model system can be successfully reduced to a problem that involves a single second order ODE (our master equation). When the cosmological constant vanishes, such a master equation can be further reduced to an Abel equation. This is quite a technical achievement in itself and it can be used as a very convenient starting point by researchers working in this field. In particular, such a master equation can reveal whether or not there are black holes in this sector of the theory. We hope to come back on this issue in a future publication.

Such a master equation can describe both Lorentzian and Euclidean spacetimes. Here we have described in more detail the properties of regular and smooth Lorentzian solutions. However, the regular Euclidean solutions, due to their nontrivial topological properties, could be interpreted as instantons of the theory. We hope to come back on this interesting issue in a future publication.

The most interesting regular Lorentzian configurations correspond to a stationary traversable wormhole with NUT parameter and a $(3+1)$-dimensional cylinder whose $(2+1)$-dimensional sections are Lorentzian squashed spheres.

The use of the theory of group invariant transformations reveals that the Klein-Gordon equations in these two families of spacetimes are not only separable but the reduced equation leads to well-known quantum systems where the solution can be written in terms of special functions. The angular equation reduces to the equation for the Jacobi polynomial (typical of the Schrodinger problem in the field of a Dirac monopole) while the effective potential appearing in the radial equation belongs to the Poschl-Teller family. The solvability requirement for the Poschl-Teller equation determines a quantization condition for the parameters of the theory.

In the case in which the cosmological constant vanishes, the solutions of the master Eq. (24) that we have found are asymptotic solutions; that is, asymptotically, the general solution is described by one of these solutions. The reason for that is that the families of regular configurations that we have found are fixed points for Eq. (24). We hope to extend the present analysis in the case of the Euclidean configurations in a future work.

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