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Noether's theorems and conserved currents in gauge theories in the presence of fixed fields

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We extend the standard construction of conserved currents for matter fields in general relativity to general gauge theories. In the original construction, the conserved current associated with a spacetime symmetry generated by a Killing field h^μ is given by $\sqrt{-g}T^{\mu\nu}h_\nu$, where $T^{\mu\nu}$ is the energy-momentum tensor of the matter. We show that if in a Lagrangian field theory that has gauge symmetry in the general Noetherian sense some of the elementary fields are fixed and are invariant under a particular infinitesimal gauge transformation, then there is a current $\mathcal{B}^\mu$ that is analogous to $\sqrt{-g}T^{\mu\nu}h_\nu$ and is conserved if the nonfixed fields satisfy their Euler-Lagrange equations. The conservation of $\mathcal{B}^\mu$ can be seen as a consequence of an identity that is a generalization of $\nabla_\mu T^{\mu\nu}=0$ and is a consequence of the gauge symmetry of the Lagrangian. This identity holds in any configuration of the fixed fields if the nonfixed fields satisfy their Euler-Lagrange equations. We also show that $\mathcal{B}^\mu$ differs from the relevant canonical Noether current by the sum of an identically conserved current and a term that vanishes if the nonfixed fields are on shell. For an example, we discuss the case of general, possibly fermionic, matter fields propagating in fixed gravitational and Yang-Mills background. We find that in this case the generalization of $\nabla_\mu T^{\mu\nu}=0$ is the Lorentz law $\nabla_\mu T^{\mu\nu}-F^{a\nu\lambda}\mathcal{J}_{a\lambda}=0$, which holds as a consequence of the diffeomorphism, local Lorentz and Yang-Mills gauge symmetry of the matter Lagrangian. For a second simple example, we consider the case of general fields propagating in a background that consists of a gravitational and a real scalar field.

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I. INTRODUCTION

In general relativity, the most usual way to construct conserved currents associated with spacetime symmetries for matter fields is to contract the Einstein-Hilbert energy-momentum tensor $T^{\mu\nu}$ with the Killing vector field h^μ that generates the relevant symmetry. It is easy to verify that the current $j^\mu=\sqrt{-g}T^{\mu\nu}h_\nu$ (where g denotes the determinant of the metric) obtained in this way is conserved (i.e. $\partial_\mu j^\mu=0$) as a consequence of the Killing equation $\nabla_\mu h_\nu+\nabla_\nu h_\mu=0$ and of the divergencelessness of $T^{\mu\nu}$ ($\nabla_\mu T^{\mu\nu}=0$). The latter property of $T^{\mu\nu}$ is ensured if the matter Lagrangian density transforms as a scalar density under diffeomorphisms and the matter fields satisfy their equations of motion (see e.g. Sec. E. 1 of Ref. [1]). The metric does not need to satisfy its field equations; i.e. it can be a fixed, external field. On the other hand, Noether's first theorem can also be used to construct a conserved current associated with h^μ, if the matter fields admit a Lagrangian description. This current generally differs from $\sqrt{-g}T^{\mu\nu}h_\nu$, and its construction is also apparently completely different. Nevertheless, it can be shown that if the matter fields satisfy their Euler-Lagrange (EL) equations, then the difference between the two currents is an identically conserved current, i.e. a current of the form $\partial_\mu\Sigma^{\mu\nu}$, where $\Sigma^{\mu\nu}$ is antisymmetric [2,3].

To some extent similarly, in electrodynamics, the electric current is usually defined as the Euler-Lagrange derivative of the Lagrangian function with respect to the vector potential, i.e. as the source of the electromagnetic field, but it can also be obtained as the Noether current corresponding to global $U(1)$ gauge transformations.

The primary aim of the present paper is to generalize the construction described above to any gauge theory that can be formulated in the framework applied in Noether's classic paper [4,5] on symmetries and conservation laws in Lagrangian field theory. This is a very general framework that allows one to study various special kinds of gauge theories, for example diffeomorphism covariant theories and Yang-Mills (YM) type gauge theories, on the same footing.

Our main motivation, besides general interest in conservation laws in gauge theories, to look for a generalization of the above construction is that recent developments [6–9] appear to indicate that such a generalization is possible and would be useful for better understanding these developments, for answering certain open questions and for possible further applications.

In Ref. [6], it was proposed that in the case when some matter propagates in fixed electrically charged black hole

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background, its energy and angular momentum can be obtained using the conserved currents

$$j^\mu = \sqrt{-g}(T^\mu{}_\nu h^\nu + h^\nu A_\nu \mathcal{J}^\mu), \tag{1.1}$$

where A_μ is the vector potential of the background electromagnetic field, $\mathcal{J}^\mu$ is the electric current, $T^{\mu\nu}$ is the Einstein-Hilbert energy-momentum tensor of the matter, and h^μ is the Killing field generating time translations [10] or rotations. It is also assumed here that not only the background electromagnetic field but also A_μ is invariant under time translations or rotations. Equation (1.1) was very useful in Ref. [6] because, together with a suitable energy condition on $T^{\mu\nu}$, it allowed the authors to derive general results for charged black holes without more detailed information on the nature of the matter. To justify (1.1), it was shown in Ref. [6] that the conservation of (1.1) can be derived from the generalized Lorentz law

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda}\mathcal{J}_\lambda, \tag{1.2}$$

the Killing equation, the symmetry of $T^{\mu\nu}$, the invariance of A_μ, and the conservation of the electric current. In (1.2), $F^{\mu\nu}$ denotes the electromagnetic field. A justification of the validity of (1.2) was also given in Ref. [6], although it is not analogous to the derivation of $\nabla_\mu T^{\mu\nu} = 0$ mentioned in the first paragraph, and it uses the Maxwell equation $\nabla_\mu F^{\mu\nu} = 0$ for the background electromagnetic field. Nevertheless, a derivation of (1.2) that is analogous to the derivation of $\nabla_\mu T^{\mu\nu} = 0$ is possible; moreover, it can be extended to the case of fixed gravitational + YM background [11].

In Ref. [12], we also found the current (1.1) for the complex Klein-Gordon field by applying Noether's first theorem, and in Ref. [13], we found that in the case of the Dirac field as a matter field the current (1.1) can be obtained by adding a certain identically conserved current to the Noether current, if the Dirac equation is satisfied. The only information on the background metric (or tetrad) and vector potential that was needed for these results was their invariance under time translation or rotation.

An identity and a current similar to (1.2) and (1.1) also appear in Refs. [7,8]. In these works, Eq. (1.2) takes a more general form, since some of the currents associated with the fixed covector fields are not conserved. Further comments concerning the formulas appearing in Refs. [7,8] will be made at the end of Sec. III A. A construction of currents in a general setting that is partly similar to the one in the present paper was given in Sec. II of Ref. [9]. This will be discussed further at the end of Sec. II E.

We note that there is a well-known derivation of the generalized Lorentz law in the case when electromagnetic field and matter propagate, interacting with one another, in a fixed gravitational background (see Sec. 22.4 of Ref. [14]). In this derivation, one obtains the generalized Lorentz law from the Maxwell equations and from the divergencelessness of the matter + electromagnetic total energy-momentum tensor. The derivation can also be extended to non-Abelian gauge theory in a straightforward way. Nevertheless, in this paper, we are interested in the more general situation when the electromagnetic (or YM) field is also part of the background.

Regarding the results mentioned above, several questions can also be raised. Since in Ref. [11] fermionic matter fields were not considered, one would like to derive (1.2) also for the case when some matter fields are fermionic. Equation (1.1) should also be extended to the case of gravitational + YM background. One can ask what form (1.1) will take if the vector potential is invariant under the diffeomorphisms generated by h^μ only up to YM gauge transformation, since the latter is a more natural symmetry requirement on the vector potential. The difference between (1.1) and the Noether current that one obtains by Noether's first theorem should be investigated as well.

The generalization of the construction described in the first paragraph that we present is closely related to Noether's theorems and is relatively easy to find once one has thoroughly understood these theorems; therefore, we review them briefly in a way that is suitable for the purpose of this paper. This review is intended to be quite general but mathematically elementary. For expositions of Noether's theorems in the literature and for related results, see e.g. Refs. [2,15–43], and references therein. Many references can be found in Ref. [5] as well.

As an application and illustration of the general construction, we discuss first the example of matter fields coupled to external gravitational and YM gauge fields (allowing the electromagnetic field as a special case). The type of the matter fields and the precise form of their Lagrangian are left unspecified. For deriving the generalized Lorentz law and the related currents, only the symmetry properties of the gravitational and YM gauge fields and of the matter Lagrangian are needed, but in other parts, for simplicity, we make some assumptions on the nature of the matter fields and on the form of the matter Lagrangian. We allow fermionic matter fields; therefore, we use tetrad fields as elementary gravitational field variables. We discuss the cases of the Dirac field and the scalar field as matter field more explicitly, since these are interesting special cases. The aim of this example is also to answer the questions mentioned above. The second example, which is included on account of its simplicity, is the case of arbitrary fields propagating in the presence of external scalar and gravitational fields.

The paper is organized as follows. Section II contains the review of Noether's theorems and the generalization of the construction described in the first paragraph. The latter can be found in Sec. II E, which is the central part of the paper. In Sec. III, the details of the first example introduced above are given. In particular, the generalized Lorentz law and the currents (1.1) are discussed in Sec. III A. The second

example is described in Sec. IV. A summary is given in Sec. V. Appendix A contains definitions and comments related to the examples.

The signature of metric tensors is $(+,-,-,-)$. The brackets () and [] applied to indices are used to denote symmetrization and antisymmetrization, and these operations are understood to include division by the number of permutations. A coordinate based formalism is employed, and the notions of fiber bundles and differential forms are mostly avoided in order to keep the exposition as elementary as possible. The terms "gauge theory" and "gauge symmetry" are used in the general sense described in Sec. II C, and it is stated explicitly when a special kind of gauge symmetry is meant. The coordinate independent YM gauge transformations are called global.

II. NOETHER'S THEOREMS AND THE CONSTRUCTION OF CONSERVED CURRENTS IN THE PRESENCE OF FIXED FIELDS

In order to describe the situation when some of the fields are fixed, i.e. do not necessarily satisfy their field equations, we divide the complete set of elementary fields into two sets. The fields in these sets are denoted as Φ_i and χ_j, where i and j are general indices labeling the fields and their components. Any of the two sets may be empty. Generally, χ_j will be the fixed fields, but Φ_i, which may be called dynamical fields, are also not assumed to satisfy their field equations unless explicitly stated. The physical role of Φ_i and χ_j, i.e. whether they are matter or another type of fields, is not restricted. In particular, the presence of a metric tensor is not required. Commuting and anticommuting fields are both allowed. For simplicity, the fields are assumed to be real (self-conjugate) in this section. This does not cause any loss of generality, since any complex field is equivalent to two real fields. For derivatives with respect to anticommuting variables, the following sign convention is used: if θ is an anticommuting variable and E is an expression of the form $E_1\theta E_2$, then $\frac{\partial E}{\partial\theta} = (-1)^n E_1 E_2$, where $n = 0$ if E_2 is even and $n = 1$ if E_2 is odd.

The Lagrangian density function $L(x^\mu, \chi_j, \partial_\mu\chi_j, \partial_{\mu\nu}\chi_j, \ldots, \Phi_i, \partial_\mu\Phi_i, \partial_{\mu\nu}\Phi_i, \ldots)$ is assumed to be an even local function of Φ_i and χ_j, but otherwise it is allowed to depend on arbitrarily high derivatives. Further assumptions on the Lagrangian are not made in this section, and it is not specified what kind of physical system it describes. A local function of some fields $\phi_i(x^\mu)$ is defined in this paper to be a function of the form $f(x^\mu, \phi_i(x^\mu), \partial_\nu\phi_i(x^\mu), \partial_{\nu\lambda}\phi_i(x^\mu), \ldots)$, that depends on the fields and on finitely many derivatives of them and may depend explicitly on the coordinates x^μ as well [44]. Although the action integral will not be used, it should be noted that the contribution to it from the domain U on which x^μ are defined is $\int_U d^D x L$, where D denotes the dimension of the base manifold M on which the fields are defined. The integration measure used here is the measure determined by the coordinate chart. It is not necessary to specify the behavior of Φ_i and χ_j under coordinate changes for the purpose of this section.

It is known that higher derivative theories generally have various undesirable features, particularly instabilities and unphysical degrees of freedom, but they have better renormalizability properties than low derivative theories, and the difficulties caused by the mentioned features may be surmountable (see e.g. Ref. [45–53]). As far as symmetries are concerned, there is no major reason to restrict the order of the derivatives that may appear in the Lagrangian.

We distinguish three theorems of Noether, and within the third theorem, we distinguish two parts. The first theorem, which is the most well known, is not specific to gauge theories. In the literature, the third theorem is often included in the second one, but it seems useful to separate them. In Noether's paper [4,5], the statement of the second theorem does not include the third theorem, but the third theorem does not appear as a separate theorem either.

We do not follow rigorously the original formulation of Noether's theorems. In particular, we are interested only in the consequences of symmetries and do not consider reverse statements. For the latter, we refer the reader to the literature, e.g. Ref. [15].

The content of the three theorems can be summarized very briefly as follows. The first theorem states that if the Lagrangian has a symmetry, then there exists a current J^μ (the Noether current) that is conserved (i.e. $\partial_\mu J^\mu = 0$) if the fields satisfy their EL equations. The second theorem states that if the Lagrangian has a gauge symmetry, then the EL derivatives of the Lagrangian with respect to the fields satisfy a differential identity, and thus the EL equations are not independent. The third theorem states that if the Lagrangian has a gauge symmetry, then the Noether currents associated with these symmetries coincide with certain identically conserved currents up to some terms that vanish if all fields satisfy their EL equations.

In Sec. II B, we present the first theorem in a form that is adapted to the situation when fixed fields are present. The distinction between Φ_i and χ_j is irrelevant in the second and third theorems, but we keep it for later use in the construction in Sec. II E. The definition of gauge symmetry in the general sense can be found in Sec. II C.

After Noether's three theorems, we present the generalization of the construction described in the first paragraph of Sec. I. This generalized construction can also be regarded as an extension of Noether's theorems. The construction described in Sec. I has three parts: the first one is the construction of the energy-momentum tensor and the derivation of its divergencelessness, the second one is the construction of the current from the energy-momentum tensor and from the Killing field and the derivation of its conservation, and the third one is the

result that the current constructed in this way differs from the Noether current associated with the Killing field in an identically conserved current if the matter fields satisfy their EL equations. Each part is generalized in Sec. II E.

In general relativity, the divergencelessness of the energy-momentum tensor can also be seen as a consequence of the Einstein equation if the metric is not fixed. Similarly, in electrodynamics, the conservation of the electric current can be seen as a consequence of the Maxwell equation. These observations are also generalized in Sec. II E.

In the next subsection, an auxiliary formula is described, which is of central importance in the subsequent derivations.

A. Partial integration formula

Let G be a quantity that can be written as

$$G = G_\alpha \epsilon^\alpha + G_\alpha^\nu \partial_\nu \epsilon^\alpha + G_\alpha^{\nu\lambda} \partial_{\nu\lambda} \epsilon^\alpha + \ldots, \tag{2.1}$$

where ϵ^α is a function with several components indexed by α; the functions $G_\alpha^{\nu\lambda}$, $G_\alpha^{\nu\lambda\rho}$, … are completely symmetric in the upper indices; and the sum on the right-hand side contains only finitely many terms. By straightforward application of the basic differentiation rule $(uv)' = u'v + uv'$, one can show that

$$G = \hat{G}_\alpha \epsilon^\alpha + \partial_\nu \mathcal{G}^\nu, \tag{2.2}$$

where

$$\hat{G}_\alpha = G_\alpha - \partial_\nu G_\alpha^\nu + \partial_{\nu\lambda} G_\alpha^{\nu\lambda} - \ldots \tag{2.3}$$

and

$$\begin{aligned}\mathcal{G}^\nu = {} & G_\alpha^\nu \epsilon^\alpha + (G_\alpha^{\nu\lambda} \partial_\lambda \epsilon^\alpha - \partial_\lambda G_\alpha^{\nu\lambda} \epsilon^\alpha) \\ & + (G_\alpha^{\nu\lambda\rho} \partial_{\lambda\rho} \epsilon^\alpha - \partial_\rho G_\alpha^{\nu\lambda\rho} \partial_\lambda \epsilon^\alpha + \partial_{\lambda\rho} G_\alpha^{\nu\lambda\rho} \epsilon^\alpha) + \cdots\end{aligned} \tag{2.4}$$

The $(n+1)$th term of the sum on the right-hand side of (2.3) is $(-1)^n \partial_{\nu_1\nu_2\ldots\nu_n} G_\alpha^{\nu_1\nu_2\ldots\nu_n}$. The nth group of terms on the right-hand side of (2.4) is

$$\begin{aligned}& G_\alpha^{\nu\lambda_1\lambda_2\ldots\lambda_{n-1}} \partial_{\lambda_1\lambda_2\ldots\lambda_{n-1}} \epsilon^\alpha - \partial_{\lambda_{n-1}} G_\alpha^{\nu\lambda_1\lambda_2\ldots\lambda_{n-1}} \partial_{\lambda_1\lambda_2\ldots\lambda_{n-2}} \epsilon^\alpha \\ & \quad + \partial_{\lambda_{n-2}\lambda_{n-1}} G_\alpha^{\nu\lambda_1\lambda_2\ldots\lambda_{n-1}} \partial_{\lambda_1\lambda_2\ldots\lambda_{n-3}} \epsilon^\alpha - \ldots \\ & \quad + (-1)^{n-1} \partial_{\lambda_1\lambda_2\ldots\lambda_{n-1}} G_\alpha^{\nu\lambda_1\lambda_2\ldots\lambda_{n-1}} \epsilon^\alpha.\end{aligned}$$

Since the sum on the right-hand side of (2.1) is finite, the sums in (2.3) and (2.4) are also finite. Moreover, if G_α, G_α^ν, $G_\alpha^{\nu\lambda}$, … do not depend on ϵ^α (and on its derivatives), then if all terms beyond the first n terms are zero on the right-hand side of (2.1), then $\mathcal{G}^\nu$ does not depend on higher than $(n-2)$th derivatives of ϵ^α.

We call (2.2) the partial integration formula, since it can be regarded as a generalization of the differentiation rule $(uv)' = u'v + uv'$ on which partial integration is based. It is clear that adding an arbitrary conserved current to $\mathcal{G}^\nu$ preserves (2.2), but we always use the definition (2.4).

B. First theorem

A one-parameter transformation of the fields can be written after linearization in the parameter, denoted by s, as

$$\Phi_i \to \Phi_i + s\delta\Phi_i, \qquad \chi_j \to \chi_j + s\delta\chi_j. \tag{2.5}$$

s is assumed to be real number valued, and $\delta\Phi_i$ and $\delta\chi_j$ are assumed to have the same commutation character as Φ_i and χ_j, respectively. Generally, both $\delta\Phi_i$ and $\delta\chi_j$ may depend on Φ_i and χ_j and on their derivatives. Such a transformation may be induced by a transformation in the base manifold or in the target space of the fields, but may be more general. Supersymmetry transformations, for example, also fit in this framework. The associated first order variation of L is defined as $\delta L = \frac{dL}{ds}|_{s=0} = \delta L_\Phi + \delta L_\chi$ with

$$\delta L_\Phi = \frac{\partial L}{\partial \Phi_i}\delta\Phi_i + \frac{\partial L}{\partial(\partial_\mu \Phi_i)}\partial_\mu \delta\Phi_i + \frac{\partial L}{\partial(\partial_{\mu\nu}\Phi_i)}\partial_{\mu\nu}\delta\Phi_i + \cdots \tag{2.6}$$

$$\delta L_\chi = \frac{\partial L}{\partial \chi_j}\delta\chi_j + \frac{\partial L}{\partial(\partial_\mu \chi_j)}\partial_\mu \delta\chi_j + \frac{\partial L}{\partial(\partial_{\mu\nu}\chi_j)}\partial_{\mu\nu}\delta\chi_j + \cdots \tag{2.7}$$

Applying the partial integration formula (2.2) to (2.6) with $\alpha \to i$, $\epsilon^\alpha \to \delta\Phi_i$, and $G_\alpha^{\mu\nu\ldots} \to \frac{\partial L}{\partial(\partial_{\mu\nu\ldots}\Phi_i)}$ gives

$$\delta L_\Phi = \frac{\delta L}{\delta \Phi_i}\delta\Phi_i + \partial_\mu j_\Phi^\mu, \tag{2.8}$$

where

$$\begin{aligned}\frac{\delta L}{\delta \Phi_i} = {} & \frac{\partial L}{\partial \Phi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \Phi_i)} + \partial_{\mu\nu} \frac{\partial L}{\partial(\partial_{\mu\nu}\Phi_i)} \\ & - \partial_{\mu\nu\lambda} \frac{\partial L}{\partial(\partial_{\mu\nu\lambda}\Phi_i)} + \cdots\end{aligned} \tag{2.9}$$

and

$$j^\mu_\Phi = \frac{\partial L}{\partial(\partial_\mu \Phi_i)}\delta\Phi_i + \left(\frac{\partial L}{\partial(\partial_{\mu\nu}\Phi_i)}\partial_\nu\delta\Phi_i - \partial_\nu\frac{\partial L}{\partial(\partial_{\mu\nu}\Phi_i)}\delta\Phi_i\right) + \left(\frac{\partial L}{\partial(\partial_{\mu\nu\lambda}\Phi_i)}\partial_{\nu\lambda}\delta\Phi_i - \partial_\nu\frac{\partial L}{\partial(\partial_{\mu\nu\lambda}\Phi_i)}\partial_\lambda\delta\Phi_i + \partial_{\nu\lambda}\frac{\partial L}{\partial(\partial_{\mu\nu\lambda}\Phi_i)}\delta\Phi_i\right) + \cdots \tag{2.10}$$

$\frac{\delta L}{\delta\Phi_i}$ has the role of $\hat{G}_\alpha$, and j^μ_Φ has the role of $\mathcal{G}^\nu$. $\frac{\delta L}{\delta\Phi_i}$ is the Euler-Lagrange derivative of L with respect to Φ_i, and the Euler-Lagrange equations for Φ_i are $\frac{\delta L}{\delta\Phi_i} = 0$. We call j^μ_Φ the Euler-Lagrange current corresponding to Φ_i, because it is the counterpart of the Euler-Lagrange derivative in (2.8). We note that a closely related quantity is called a symplectic potential form in Ref. [2]. Formulas similar to (2.8), (2.9), and (2.10) can also be obtained for χ_j by applying (2.2) to (2.7).

Let us consider now a specific transformation and assume that χ_j are in a configuration that is invariant under this transformation, i.e. $\delta\chi_j = 0$. If in addition

$$\delta L_\Phi = \partial_\mu K^\mu \tag{2.11}$$

holds with some K^μ, then this transformation is called a symmetry transformation, and (2.8) implies that

$$\partial_\mu J^\mu_\Phi + \frac{\delta L}{\delta\Phi_i}\delta\Phi_i = 0, \tag{2.12}$$

where J^μ_Φ is defined as

$$J^\mu_\Phi = j^\mu_\Phi - K^\mu \tag{2.13}$$

and is called Noether current. In particular, if Φ_i satisfy their EL equations, then from (2.12), it follows that the current J^μ_Φ is conserved: $\partial_\mu J^\mu_\Phi = 0$. These statements for J^μ_Φ constitute the first theorem, extended to the situation when some of the elementary fields are fixed but their configuration is invariant under the symmetry that is considered.

Usually, $\delta\Phi_i$ are local functions of Φ_i, K^μ is also required to be a local function of Φ_i, and (2.11) is understood to be an identity for Φ_i. In the rest of the paper, these properties will be assumed.

Although (2.11) is required above to hold only for a specific configuration of χ_j, in practice, one often has an identity $\delta L = \partial_\mu K^\mu$ without any restriction on χ_j and with K^μ, $\delta\chi_j$, $\delta\Phi_i$ that are local functions of both Φ_i and χ_j, and then this identity reduces to (2.11) if χ_j is such that $\delta\chi_j = 0$.

It is clear that K^μ is not uniquely determined in (2.11); therefore, the application of the above theorem involves making a suitable choice. The simplest possibility, which is suitable for many cases, is $K^\mu = 0$ (see Sec. III for examples).

The local conservation law $\partial_\mu J^\mu_\Phi = 0$ can be rewritten in integral form by applying Stokes's theorem. Let $\mathcal{U}$ be a D-dimensional domain within U. By using Stokes's theorem, one obtains

$$\int_{\partial\mathcal{U}} n_\mu J^\mu_\Phi = 0, \tag{2.14}$$

where $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$ and n_μ is the normal vector field of $\partial\mathcal{U}$. n_μ is normalized using the flat Euclidean metric $\delta_{\mu\nu}$ determined by the coordinate system, i.e. $\delta^{\mu\nu}n_\mu n_\nu = 1$. By restricting this metric to $\partial\mathcal{U}$, one gets a Riemannian metric on $\partial\mathcal{U}$, and the corresponding measure is the one that is used for the integration over $\partial\mathcal{U}$. By choosing $\mathcal{U}$ to be a cylindrical domain $\mathcal{U} = [t_1, t_2]\times\Omega$, where Ω is a $D-1$-dimensional domain, one obtains from (2.14) the charge conservation law

$$\int_\Omega J^0_\Phi|_{x^0=t_2} - \int_\Omega J^0_\Phi|_{x^0=t_1} = -\int_{[t_1,t_2]\times\partial\Omega} n_\mu J^\mu_\Phi, \tag{2.15}$$

where $\partial\Omega$ is the boundary of Ω and n_μ is the outward pointing normal vector of $\partial\Omega$. $\int_\Omega J^0_\Phi|_{x^0=t_1}$ and $\int_\Omega J^0_\Phi|_{x^0=t_2}$ are the charges in Ω at $x^0 = t_1$ and $x^0 = t_2$, respectively, and $\int_{[t_1,t_2]\times\partial\Omega} n_\mu J^\mu_\Phi$ is the charge that flows out of Ω during the x^0 interval $[t_1, t_2]$. The index of x^μ runs from zero to $D-1$.

C. Second theorem

Let us consider a transformation of the fields specified by $\delta\Phi_i$ and $\delta\chi_j$ of the form

$$\delta\Phi_i = \delta\Phi_{i\alpha}\epsilon^\alpha + \delta\Phi^\mu_{i\alpha}\partial_\mu\epsilon^\alpha + \delta\Phi^{\mu\nu}_{i\alpha}\partial_{\mu\nu}\epsilon^\alpha + \cdots \tag{2.16}$$

$$\delta\chi_j = \delta\chi_{j\alpha}\epsilon^\alpha + \delta\chi^\mu_{j\alpha}\partial_\mu\epsilon^\alpha + \delta\chi^{\mu\nu}_{j\alpha}\partial_{\mu\nu}\epsilon^\alpha + \cdots, \tag{2.17}$$

where ϵ^α is a function that can have several components indexed by α and may be commuting or anticommuting, and there are finitely many terms in the sums on the right-hand sides. It is assumed that $\delta\Phi_{i\alpha}$, $\delta\Phi^\mu_{i\alpha}$, $\delta\Phi^{\mu\nu}_{i\alpha}$, …, $\delta\chi_{j\alpha}$, $\delta\chi^\mu_{j\alpha}$, $\delta\chi^{\mu\nu}_{j\alpha}$, … are local functions of Φ_i and χ_j and do not depend on ϵ^α nor on its derivatives. Transformations of this form are called (infinitesimal) gauge transformations, parametrized by ϵ^α. We do not assume any group property of these transformations.

The partial integration formula (2.2) can be applied to $\frac{\delta L}{\delta\Phi_i}\delta\Phi_i$ and to $\frac{\delta L}{\delta\chi_j}\delta\chi_j$, giving

$$\frac{\delta L}{\delta \Phi_i}\delta\Phi_i = B_{\Phi\alpha}\epsilon^\alpha + \partial_\mu \mathcal{B}^\mu_\Phi \tag{2.18}$$

$$\frac{\delta L}{\delta \chi_j}\delta\chi_j = B_{\chi\alpha}\epsilon^\alpha + \partial_\mu \mathcal{B}^\mu_\chi, \tag{2.19}$$

where

$$B_{\Phi\alpha} = \frac{\delta L}{\delta \Phi_i}\delta\Phi_{i\alpha} - \partial_\mu\left(\frac{\delta L}{\delta \Phi_i}\delta\Phi^\mu_{i\alpha}\right) + \partial_{\mu\nu}\left(\frac{\delta L}{\delta \Phi_i}\delta\Phi^{\mu\nu}_{i\alpha}\right) - \cdots \tag{2.20}$$

$$\mathcal{B}^\mu_\Phi = \frac{\delta L}{\delta \Phi_i}\delta\Phi^\mu_{i\alpha}\epsilon^\alpha + \left[\frac{\delta L}{\delta \Phi_i}\delta\Phi^{\mu\nu}_{i\alpha}\partial_\nu\epsilon^\alpha - \partial_\nu\left(\frac{\delta L}{\delta \Phi_i}\delta\Phi^{\mu\nu}_{i\alpha}\right)\epsilon^\alpha\right] + \cdots, \tag{2.21}$$

and similar formulas can also be written for χ_j. Using (2.18) and (2.19), one gets

$$\begin{aligned}\delta L &= \delta L_\Phi + \delta L_\chi = \frac{\delta L}{\delta \Phi_i}\delta\Phi_i + \frac{\delta L}{\delta \chi_j}\delta\chi_j + \partial_\mu j^\mu_\Phi + \partial_\mu j^\mu_\chi \\ &= B_{\Phi\alpha}\epsilon^\alpha + \partial_\mu \mathcal{B}^\mu_\Phi + \partial_\mu j^\mu_\Phi + B_{\chi\alpha}\epsilon^\alpha + \partial_\mu \mathcal{B}^\mu_\chi + \partial_\mu j^\mu_\chi.\end{aligned} \tag{2.22}$$

Let us assume that the transformation specified by (2.16) and (2.17) is a symmetry for arbitrary ϵ^α functions (and without any assumption on χ_j), i.e.

$$\delta L = \partial_\mu K^\mu, \tag{2.23}$$

with some K^μ. K^μ is also assumed to be a homogeneous linear local function of ϵ^α, i.e.

$$K^\mu = K^\mu_\alpha\epsilon^\alpha + K^{\mu\nu}_\alpha\partial_\nu\epsilon^\alpha + K^{\mu\nu\lambda}_\alpha\partial_{\nu\lambda}\epsilon^\alpha + \cdots, \tag{2.24}$$

where K^μ_α, $K^{\mu\nu}_\alpha$, $K^{\mu\nu\lambda}_\alpha$, … do not depend on ϵ^α nor on its derivatives and $K^{\mu\nu\lambda\ldots}_\alpha$ are symmetric in the upper indices $\nu\lambda\ldots$ following μ. K^μ_α, $K^{\mu\nu}_\alpha$, $K^{\mu\nu\lambda}_\alpha$, … are also local functions of Φ_i and χ_j. In this case, L is said to have a gauge symmetry, and it follows from (2.22) that

$$(B_{\Phi\alpha} + B_{\chi\alpha})\epsilon^\alpha = -\partial_\mu(\mathcal{B}^\mu_\Phi + \mathcal{B}^\mu_\chi + J^\mu), \tag{2.25}$$

where

$$J^\mu = j^\mu_\Phi + j^\mu_\chi - K^\mu. \tag{2.26}$$

If ϵ^α and sufficiently many of its derivatives vanish on the boundary of an open domain Ω, then by applying Stokes's theorem, we get

$$\int_\Omega d^D x(B_{\Phi\alpha} + B_{\chi\alpha})\epsilon^\alpha = 0 \tag{2.27}$$

from (2.25). Since (2.23) holds for arbitrary ϵ^α, Eq. (2.27) implies that

$$B_{\Phi\alpha} + B_{\chi\alpha} = 0. \tag{2.28}$$

This result is the second theorem. From (2.20), it can be seen that (2.28) is a differential identity for the EL derivatives of L with respect to Φ_i and χ_j.

Equation (2.28) applied to the Einstein-Hilbert Lagrangian with the diffeomorphism symmetry as the gauge symmetry gives $\nabla_\mu G^{\mu\nu} = 0$, where $G^{\mu\nu}$ is the Einstein tensor. This is the twice contracted Bianchi identity; therefore, Eq. (2.28) can be called the generalized Bianchi identity. $B_{\Phi\alpha}$ and $B_{\chi\alpha}$ will be called Bianchi expressions.

Since $\mathcal{B}^\mu_\Phi$ and $\mathcal{B}^\mu_\chi$ are the counterparts of $B_{\Phi\alpha}$ and $B_{\chi\alpha}$ in (2.18) and (2.19), we call them Bianchi currents. Note that $\mathcal{B}^\mu_\Phi = 0$ if Φ_i satisfy their EL equations, and obviously the same is true for χ_j and $\mathcal{B}^\mu_\chi$. Moreover, $\mathcal{B}^\mu_\Phi = 0$ also if the coefficients of all derivatives of ϵ^α on the right-hand side of (2.16) are zero, and the same can be said of $\mathcal{B}^\mu_\chi$.

If L has a gauge symmetry, then the corresponding variation δL of L is also a homogeneous linear local function of ϵ^α with some coefficients δL_α, δL^μ_α, $\delta L^{\mu\nu}_\alpha$, …, and (2.23) holds for arbitrary ϵ^α; therefore, the coefficients of ϵ^α, $\partial_\mu\epsilon^\alpha$, $\partial_{\mu\nu}\epsilon^\alpha$, … on the two sides of (2.23) have to be equal, i.e.

$$\delta L_\alpha = \partial_\mu K^\mu_\alpha \tag{2.29}$$

$$\delta L^\mu_\alpha = K^\mu_\alpha + \partial_\nu K^{\nu\mu}_\alpha \tag{2.30}$$

$$\delta L^{\mu\nu}_\alpha = K^{\mu\nu}_\alpha + \partial_\lambda K^{\lambda\mu\nu}_\alpha \quad \cdots \tag{2.31}$$

These equations are known as Klein-Noether identities.

D. Third theorem

In this section, we continue to consider Lagrangian systems with gauge symmetry. The current J^μ introduced in (2.26) is the standard Noether current corresponding to the symmetry transformation (2.16), (2.17) in the absence of any fixed field; thus, it is conserved if both Φ_i and χ_j satisfy their EL equations. On the other hand, from (2.25) and (2.28), it follows that the current

$$I^\mu = \mathcal{B}^\mu_\Phi + \mathcal{B}^\mu_\chi + J^\mu \tag{2.32}$$

is also conserved, regardless of the EL equations. Nevertheless, if Φ_i and χ_j satisfy their EL equations, then $I^\mu = J^\mu$. Thus, by adding $\mathcal{B}^\mu_\Phi + \mathcal{B}^\mu_\chi$ to J^μ, we get a current which is conserved regardless of the EL equations but which nevertheless coincides with J^μ if the EL equations of

all fields are satisfied. This is the first part of the third theorem.

The second part is the following: since I^μ is a homogeneous linear local function of ϵ^α and is conserved for arbitrary ϵ^α, it can be written as

$$I^\mu = \partial_\nu \Sigma^{\mu\nu}, \tag{2.33}$$

where $\Sigma^{\mu\nu}$, which is called the superpotential, is antisymmetric, and is given by the formula

$$\Sigma^{\mu\nu} = \frac{1}{2}(I^{\mu\nu}_\alpha - I^{\nu\mu}_\alpha)\epsilon^\alpha + \left[\frac{2}{3}(I^{\mu\nu\rho}_\alpha - I^{\nu\mu\rho}_\alpha)\partial_\rho\epsilon^\alpha - \frac{1}{3}\partial_\rho(I^{\mu\nu\rho}_\alpha - I^{\nu\mu\rho}_\alpha)\epsilon^\alpha\right] + \left[\frac{3}{4}(I^{\mu\nu\rho\lambda}_\alpha - I^{\nu\mu\rho\lambda}_\alpha)\partial_{\rho\lambda}\epsilon^\alpha - \frac{2}{4}\partial_\lambda(I^{\mu\nu\rho\lambda}_\alpha - I^{\nu\mu\rho\lambda}_\alpha)\partial_\rho\epsilon^\alpha + \frac{1}{4}\partial_{\rho\lambda}(I^{\mu\nu\rho\lambda}_\alpha - I^{\nu\mu\rho\lambda}_\alpha)\epsilon^\alpha\right] + \cdots \tag{2.34}$$

with the nth group of terms on the right-hand side being

$$\sum_{i=1}^{n}(-1)^{i+1}\frac{n+1-i}{n+1}\partial_{\lambda_{n-i+1}\ldots\lambda_{n-1}}(I^{\mu\nu\lambda_1\ldots\lambda_{n-1}}_\alpha - I^{\nu\mu\lambda_1\ldots\lambda_{n-1}}_\alpha) \times \partial_{\lambda_1\lambda_2\ldots\lambda_{n-i}}\epsilon^\alpha. \tag{2.35}$$

The $I^{\mu\nu\lambda\ldots}_\alpha$ appearing in these formulas are the coefficients in the expansion

$$I^\mu = I^\mu_\alpha\epsilon^\alpha + I^{\mu\nu}_\alpha\partial_\nu\epsilon^\alpha + I^{\mu\nu\lambda}_\alpha\partial_{\nu\lambda}\epsilon^\alpha + \cdots. \tag{2.36}$$

$I^{\mu\nu\lambda\ldots}_\alpha$ are symmetric in the upper indices $\nu\lambda\ldots$ following μ. The two conditions mentioned above (2.33) are sufficient for (2.33) and (2.34); i.e. no more details on the properties of I^μ are needed. This second part of the third theorem is proved, for example, in Refs. [16,54].

It should be noted that, although from the Poincaré lemma it follows that in any simply connected domain there exists some antisymmetric $\Sigma^{\mu\nu}$ for which (2.33) holds, the statement that the $\Sigma^{\mu\nu}$ given by (2.34) satisfies (2.33) is stronger. Equation (2.34) implies that $\Sigma^{\mu\nu}$ is also a local function of the fields, and therefore by applying Stokes's theorem, the total charge corresponding to I^μ in a $D-1$-dimensional domain Ω can be expressed as an integral of a local expression of the fields over the boundary of Ω.

Currents of the form $\partial_\nu\Sigma^{\mu\nu}$, where $\Sigma^{\mu\nu}$ is antisymmetric, are identically conserved because of the antisymmetry of $\Sigma^{\mu\nu}$. Moreover, from Stokes's theorem, it follows that if $\Sigma^{\mu\nu}$ falls off sufficiently fast at infinity then the total charge associated with the current $\partial_\nu\Sigma^{\mu\nu}$ is zero. Due to these facts, the currents that have the form $\partial_\nu\Sigma^{\mu\nu}$ are often called trivial. Nevertheless, the $\Sigma^{\mu\nu}$ introduced above does not always fall off very fast at infinity, and the associated charge can be different from zero. For further discussion of the definition of total charges in gauge theories, especially in general relativity, the reader is referred to the literature, e.g., Refs. [1–3,14,18,22–24,26,32,36,40–42].

E. Conserved currents in gauge theories in the presence of fixed fields

In this section, the generalization of the construction mentioned at the beginning of Sec. I is described. It is assumed that L has a gauge symmetry in the general sense described in Sec. II C. This means that L is assumed to have a gauge symmetry with respect to the complete set of fields, but with respect to the actual dynamical fields, it does not necessarily have any gauge symmetry.

If Φ_i satisfy their EL equations, then $B_{\Phi\alpha} = 0$, as can be seen from (2.20), and thus from the generalized Bianchi identity (2.28), it follows that

$$B_{\chi\alpha} = 0, \tag{2.37}$$

without any assumption on χ_j. This is the first part of the construction. We call (2.37) a partial Bianchi identity.

Next, let us consider the situation in which $\delta\chi_j = 0$ for some particular ϵ^α, i.e. the configuration of χ_j is invariant with respect to a particular infinitesimal gauge transformation. In this case, it follows from (2.19) that

$$0 = B_{\chi\alpha}\epsilon^\alpha + \partial_\mu\mathcal{B}^\mu_\chi. \tag{2.38}$$

Moreover, if Φ_i satisfy their EL equations, then $B_{\chi\alpha} = 0$, as we have seen; thus, Eq. (2.38) reduces to

$$0 = \partial_\mu\mathcal{B}^\mu_\chi. \tag{2.39}$$

This means that, in addition to the standard Noether current J^μ_Φ, there is another conserved current, $\mathcal{B}^\mu_\chi$. This is the second part of the construction.

If $\delta\chi_j = 0$, then $j^\mu_\chi = 0$, and thus $J^\mu = J^\mu_\Phi$. From (2.32) and (2.33), it follows then that

$$-\mathcal{B}^\mu_\chi = J^\mu_\Phi + \mathcal{B}^\mu_\Phi - I^\mu = J^\mu_\Phi + \mathcal{B}^\mu_\Phi - \partial_\nu\Sigma^{\mu\nu}; \tag{2.40}$$

therefore, the difference of $-\mathcal{B}^\mu_\chi$ and J^μ_Φ is the sum of the terms $\mathcal{B}^\mu_\Phi$ and $-\partial_\nu\Sigma^{\mu\nu}$, the first of which vanishes when Φ_i satisfy their EL equations and the second of which is the

divergence of an antisymmetric matrix. This is the third part of the construction.

Equation (2.40) shows that, although J^μ_Φ and $\mathcal{B}^\mu_\chi$ are defined completely differently, they are equivalent in the usual sense of the equivalence of conserved currents. It is also worth mentioning that J^μ_Φ depends on the choice of K^μ, whereas $\mathcal{B}^\mu_\chi$ does not.

Until now, it has been assumed that L is the total Lagrangian of the system under consideration, but it is clear that if L takes the form $L = L_1 + L_2$, where L_1 does not depend on Φ_i and their derivatives and L_2 has gauge symmetry in itself, then the above construction can also be applied to L_2. For example, in the standard case in general relativity, one constructs $\sqrt{-g}T^{\mu\nu}h_\nu$ from the matter Lagrangian rather than from the total Lagrangian, and the same will be done in the examples in Secs. III and IV as well.

If L_1 has a gauge symmetry in itself and χ_j satisfy their EL equations $\frac{\delta L_1}{\delta \chi_j} + \frac{\delta L_2}{\delta \chi_j} = 0$, then the generalized Bianchi identity for L_1 together with the EL equations of χ_j imply an equation that formally coincides with the partial Bianchi identity (2.37) obtained from L_2. In order to see this, one should consider that the expression that appears on the left-hand side of the generalized Bianchi identity is given in the present case by the formula (2.20) for Bianchi expressions, with $L \to L_1$ and $\Phi \to \chi$. In this expression, $\frac{\delta L_1}{\delta \chi_j}$ can be replaced by $-\frac{\delta L_2}{\delta \chi_j}$ as a consequence of the EL equations of χ_j, and the resulting expression is just -1 times the one that appears in (2.37); thus, indeed (2.37) is obtained. In this argument, it is not necessary to use any symmetry property of L_2, and Φ_i do not have to satisfy their EL equations. The example in Sec. III shows that the observations that the Einstein equation implies the divergencelessness of the energy-momentum tensor and the Maxwell equation implies the conservation of the electric current are special cases of the result described in this paragraph.

We note that the second part of the construction is not used in the derivation of (2.40), and (2.40) also implies the conservation of $\mathcal{B}^\mu_\chi$. On the other hand, it is the derivation given in the second part that is the generalization of the usual derivation of the conservation of $\sqrt{-g}T^{\mu\nu}h_\nu$ in general relativity.

The results in Sec. II of Ref. [9] are similar to (2.40), but they are less explicit and are restricted to the special case when the particular infinitesimal gauge transformation under which χ_j are invariant is rigid, i.e. is such that ϵ^α is a constant function. In principle, it is not a severe restriction to consider only rigid transformations, since a nonrigid gauge transformation can be transformed into a rigid one by suitable reparametrization. For example, let us choose n different functions ϵ^α_k, $k = 1, \ldots, n$, so that ϵ^α_1 corresponds to the particular transformation under which $\delta\chi_j = 0$, and introduce the new parameter functions ξ^k via $\epsilon^\alpha = \xi^k \epsilon^\alpha_k$. In terms of ξ^k, the transformation specified by $\epsilon^\alpha = \epsilon^\alpha_1$ is rigid, since it corresponds to $\xi^1 = 1$, $\xi^k = 0$, $k = 2, \ldots, n$. Nevertheless, one can deal with nonrigid transformations directly, as we have seen in this section.

III. MATTER FIELDS IN THE PRESENCE OF FIXED GRAVITATIONAL AND YANG-MILLS GAUGE FIELDS

In this section, the case of matter fields in the presence of gravitational and YM gauge fields is discussed. Since fermionic fields are also considered, the basic gravitational field variable is taken to be an orthonormal tetrad field $V^{\bar\mu}_\mu$. The metric can be expressed in terms of $V^{\bar\mu}_\mu$ as $g_{\mu\nu} = V^{\bar\mu}_\mu V^{\bar\nu}_\nu g_{\bar\mu\bar\nu}$, where $g_{\bar\mu\bar\nu} = \mathrm{diag}(1, -1, -1, -1)$. Here and in the following, an overbar is used to distinguish internal Lorentz vector indices. Such indices can be raised and lowered by $g_{\bar\mu\bar\nu}$ and its inverse, and $V^{\bar\mu}_\mu$ and its inverse can be used to turn spacetime vector indices into internal Lorentz vector indices and vice versa. For a detailed introduction to the tetrad formalism and its use for including half-integer spin fields in general relativity, see, e.g., Ref. [55]. A different formalism is described, e.g., in Ref. [56].

The Lie algebra of the global YM gauge group is taken to be a direct sum of compact simple and $u(1)$ algebras. It would be straightforward to consider several YM gauge groups with distinct coupling constants, but for simplicity, only one is taken. The structure constants of the Lie algebra of the global YM gauge group are denoted by $f_{ab}{}^c$. The basis of the Lie algebra is chosen so that δ^{ab} is invariant under the adjoint action; therefore, there is no significant difference between upper and lower Lie algebra indices. Nevertheless, upper and lower indices will be distinguished, and δ^{ab} will be used to raise and lower them. f^{abc} is completely antisymmetric and real. The YM field strength is defined as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \kappa f^a{}_{bc} A^b_\mu A^c_\nu$, where κ is the YM coupling constant and A^a_μ is the YM vector potential.

The Lagrangian density function L of the matter fields is assumed to be an even real local function of the tetrad field, the YM vector potential, and the matter fields, and it is assumed to have diffeomorphism, local Lorentz, and YM gauge symmetry in the sense described in Sec. II C. The precise form of L and of the Lagrangian of the gravitational and YM gauge fields does not need to be specified. For fermionic fields, Lorentz transformations mean $SL(2, \mathbb{C})$ transformations.

The application of the first two parts of the construction described in Sec. II E is discussed in Sec. III A, and the third part is discussed in Sec. III B. The matter fields will take the role of Φ_i, whereas the tetrad and YM gauge fields together will take the role of χ_j. As was mentioned in Sec. II E, the construction will be applied to the matter Lagrangian described above, rather than to the total

Lagrangian. The total gauge symmetry group that will be considered is the group generated by the symmetries mentioned above.

A. Bianchi currents and partial Bianchi identities

For the application of the first two parts of the construction described in Sec. II E, it is necessary to know the transformation properties of the tetrad and of the YM gauge fields. The K^μ quantities in the symmetry condition (2.23) and the transformation properties of the matter fields are not needed.

About the Bianchi currents, the following general preliminary remarks can be made: since the first order variations of each field that we consider depend only on ϵ^α and $\partial_\mu\epsilon^\alpha$ and not on higher derivatives, the corresponding Bianchi currents do not depend on the derivatives of ϵ^α at all. Specifically, the first term on the right-hand side of (2.21) gives the Bianchi currents. Furthermore, if the first order variation of a field does not depend on $\partial_\mu\epsilon^\alpha$, then the corresponding Bianchi current is zero.

The first order variation of the tetrad and the YM vector potential under a YM gauge transformation parametrized by ϵ^a, which takes the role of ϵ^α and has values in the coadjoint representation of the global gauge group, is

$$\delta V_\mu^{\bar\mu} = 0 \tag{3.1}$$

$$\delta A_\mu^a = D_\mu\epsilon^a. \tag{3.2}$$

D_μ denotes the covariant differential operator for the whole gauge group—for more detail on its definition, see Appendix A. In (3.2), $D_\mu\epsilon^a = \partial_\mu\epsilon^a + \kappa f^a{}_{bc}A_\mu^b\epsilon^c$.

In the case of local Lorentz transformations, the role of ϵ^α is taken by $\omega^{\bar\mu\bar\nu}$, which is antisymmetric. The first order variation of the tetrad and the YM vector potential is

$$\delta V_\mu^{\bar\mu} = -\omega^{\bar\lambda\bar\rho}(L_{\bar\lambda\bar\rho})_{\bar\nu}{}^{\bar\mu}V_\mu^{\bar\nu} \tag{3.3}$$

$$\delta A_\mu^a = 0. \tag{3.4}$$

The $(L_{\bar\lambda\bar\rho})_{\bar\nu}{}^{\bar\mu} = g_{\bar\lambda\bar\nu}\delta_{\bar\rho}^{\bar\mu} - g_{\bar\rho\bar\nu}\delta_{\bar\lambda}^{\bar\mu}$ appearing in (3.3) are the generators of the Lorentz group in the Minkowski representation.

In the case of diffeomorphisms, the role of ϵ^α is taken by the vector fields h^μ, which generate diffeomorphisms. The first order variation of the tetrad and the YM vector potential is

$$\delta V_\mu^{\bar\mu} = -h^\nu\partial_\nu V_\mu^{\bar\mu} - \partial_\mu h^\nu V_\nu^{\bar\mu} \tag{3.5}$$

$$\delta A_\mu^a = -h^\nu\partial_\nu A_\mu^a - \partial_\mu h^\nu A_\nu^a. \tag{3.6}$$

Equation (3.5) implies that $\delta g_{\mu\nu} = -h^\lambda\partial_\lambda g_{\mu\nu} - g_{\nu\lambda}\partial_\mu h^\lambda - g_{\mu\lambda}\partial_\nu h^\lambda = -\nabla_\mu h_\nu - \nabla_\nu h_\mu$. Here and in subsequent formulas, ∇_μ denotes the usual Levi-Cività covariant differential operator associated with the metric $g_{\mu\nu}$. ∇_μ acts only on the spacetime vector and covector indices. Equations (3.5) and (3.6) can also be written as $\delta V_\mu^{\bar\mu} = -h^\nu\nabla_\nu V_\mu^{\bar\mu} - \nabla_\mu h^\nu V_\nu^{\bar\mu}$, $\delta A_\mu^a = -h^\nu\nabla_\nu A_\mu^a - \nabla_\mu h^\nu A_\nu^a$.

Using these transformation properties, one finds that in the case of the YM gauge symmetry the partial Bianchi identity (2.37) takes the form

$$B_{Aa} = \sqrt{-g}D_\mu\mathcal{J}_a^\mu = 0, \tag{3.7}$$

where $\mathcal{J}_a^\mu$ is defined as

$$\sqrt{-g}\mathcal{J}_a^\mu = -\frac{\delta L}{\delta A_\mu^a} \tag{3.8}$$

and g denotes the determinant of the metric. B_{Va} is obviously zero in this case. We note that (3.7) was also obtained, e.g., in Ref. [11]. As is well known, Eq. (3.7) also follows from the YM equation, if the YM field is not fixed. The Bianchi currents $\mathcal{B}_A^\mu$ and $\mathcal{B}_V^\mu$, corresponding to the YM gauge field and to the tetrad, are found to be

$$\mathcal{B}_A^\mu = -\sqrt{-g}\epsilon^a\mathcal{J}_a^\mu, \qquad \mathcal{B}_V^\mu = 0; \tag{3.9}$$

thus, $\mathcal{B}_\chi^\mu = \mathcal{B}_A^\mu + \mathcal{B}_V^\mu$ is

$$\mathcal{B}_\chi^\mu = -\sqrt{-g}\epsilon^a\mathcal{J}_a^\mu. \tag{3.10}$$

$\mathcal{B}_\chi^\mu$ is conserved if the first order variation of A_μ^a corresponding to ϵ^a is zero, i.e. if $D_\mu\epsilon^a = 0$, and the matter fields satisfy their EL equations. It is easy to see that this is a consequence of the partial Bianchi identity (3.7), $\partial_\mu\mathcal{B}_\chi^\mu = -\sqrt{-g}\nabla_\mu(\epsilon^a\mathcal{J}_a^\mu) = -\sqrt{-g}(D_\mu\epsilon^a\mathcal{J}_a^\mu + \epsilon^a D_\mu\mathcal{J}_a^\mu)$; thus, in virtue of (3.7), $\partial_\mu\mathcal{B}_\chi^\mu = -\sqrt{-g}D_\mu\epsilon^a\mathcal{J}_a^\mu$, and this is zero if $D_\mu\epsilon^a = 0$. This derivation is a specialization of the general derivation of (2.39) in Sec. II E. For flat spacetime and in a slightly different context, it can also be found in Ref. [57].

In the case of the local Lorentz symmetry, the partial Bianchi identity (2.37) takes the form

$$B_{\bar\lambda\bar\rho} = B_{V,\bar\lambda\bar\rho} = -2\sqrt{-g}\mathbb{T}_{\bar\lambda\bar\rho} = 0, \tag{3.11}$$

where $\mathbb{T}_{\bar\lambda\bar\rho}$ is the antisymmetric part in the decomposition

$$V_{\bar\lambda\rho}\frac{\delta L}{\delta V_\rho^{\bar\rho}} = \sqrt{-g}(-T_{\bar\lambda\bar\rho} + \mathbb{T}_{\bar\lambda\bar\rho}) \tag{3.12}$$

of $\frac{1}{\sqrt{-g}}V_{\bar\lambda\rho}\frac{\delta L}{\delta V_\rho^{\bar\rho}}$ into symmetric and antisymmetric parts. The Bianchi currents $\mathcal{B}_V^\mu$ and $\mathcal{B}_A^\mu$ are obviously zero for local Lorentz symmetry; thus, also $\mathcal{B}_\chi^\mu = 0$.

It is well known that (3.11) holds if all fields except the tetrad satisfy their EL equations [55], but as we have seen,

due to $\delta A^a_\mu = 0$, Eq. (3.11) holds even when the YM vector potential is also fixed. It is clear that (3.11) holds generally if the fixed fields, with the exception of the tetrad, are scalar under local Lorentz transformations.

For the diffeomorphism symmetry, the Bianchi expressions $B_{V\mu}$ and $B_{A\mu}$ are

$$B_{V\mu} = \sqrt{-g}(-\nabla_\nu T^\nu{}_\mu + \nabla_\nu \mathbb{T}^\nu{}_\mu + \mathbb{T}^{\lambda\nu} V_{\lambda\bar{\mu}} \nabla_\mu V^{\bar{\mu}}_\nu) \quad (3.13)$$

$$B_{A\mu} = \sqrt{-g}(-\mathcal{J}^\nu_a F^a_{\nu\mu} - D_\nu \mathcal{J}^\nu_a A^a_\mu). \quad (3.14)$$

The partial Bianchi identity (2.37) is $B_{V\mu} + B_{A\mu} = 0$, which can nevertheless be simplified to the form

$$B_\mu = \sqrt{-g}(-\nabla_\nu T^\nu{}_\mu - \mathcal{J}^\nu_a F^a_{\nu\mu}) = 0 \quad (3.15)$$

by using (3.7) and (3.11). $T^{\nu\mu}$, defined in (3.12), is the Einstein-Hilbert energy-momentum tensor.

The Bianchi currents $\mathcal{B}^\mu_V$ and $\mathcal{B}^\mu_A$ are

$$\mathcal{B}^\mu_V = \sqrt{-g}(T^\mu{}_\nu - T^\mu{}_\nu)h^\nu, \qquad \mathcal{B}^\mu_A = \sqrt{-g} h^\nu A^a_\nu \mathcal{J}^\mu_a. \quad (3.16)$$

If the matter fields satisfy their EL equations, then $\mathcal{B}^\mu_\chi = \mathcal{B}^\mu_V + \mathcal{B}^\mu_A$ can be simplified to the form

$$\mathcal{B}^\mu = \sqrt{-g}(T^\mu{}_\nu h^\nu + h^\nu A^a_\nu \mathcal{J}^\mu_a) \quad (3.17)$$

by using (3.11). $\mathcal{B}^\mu$ is conserved if the matter fields satisfy their EL equations, h^μ is a Killing vector field, and A^a_μ is also invariant under the diffeomorphisms generated by h^μ. It should be noted that the invariance of the tetrad field is not necessary for the conservation of $\mathcal{B}^\mu$; the invariance of the metric (and of the vector potential) is sufficient. The reason for this is explained in the paragraph below (3.19).

The conservation of $\mathcal{B}^\mu$ can again be seen as a consequence of (3.15) and of the symmetry properties of the metric and the vector potential. Equation (3.7) is also needed, because it was also used to bring B_μ to the form (3.15). The first step of the derivation is to take the divergence of $\mathcal{B}^\mu$: $\partial_\mu \mathcal{B}^\mu = \sqrt{-g}(\nabla_\mu T^\mu{}_\nu h^\nu + T^\mu{}_\nu \nabla_\mu h^\nu + D_\mu \mathcal{J}^\mu_a A^a_\nu h^\nu + \mathcal{J}^\mu_a D_\mu A^a_\nu h^\nu + \mathcal{J}^\mu_a A^a_\nu \nabla_\mu h^\nu)$. If h^μ is a Killing vector field, then $T^\mu{}_\nu \nabla_\mu h^\nu = 0$. In virtue of the partial Bianchi identity (3.7), also $D_\mu \mathcal{J}^\mu_a A^a_\nu h^\nu = 0$. According to (3.15), $\nabla_\mu T^\mu{}_\nu h^\nu$ can be replaced by $-\mathcal{J}^\mu_a F^a_{\mu\nu} h^\nu$. Thus, the expression for $\partial_\mu \mathcal{B}^\mu$ can be written as $\sqrt{-g}(-\mathcal{J}^\mu_a F^a_{\mu\nu} h^\nu + \mathcal{J}^\mu_a D_\mu A^a_\nu h^\nu + \mathcal{J}^\mu_a A^a_\nu \nabla_\mu h^\nu)$. Using the equation $h^\mu \nabla_\mu A^a_\nu + \nabla_\nu h^\mu A^a_\mu = 0$, which expresses the invariance of the vector potential, $\mathcal{J}^\mu_a A^a_\nu \nabla_\mu h^\nu$ can be replaced by $-\mathcal{J}^\mu_a \nabla_\nu A^a_\mu h^\nu$; thus, $\partial_\mu \mathcal{B}^\mu = \sqrt{-g}(-\mathcal{J}^\mu_a F^a_{\mu\nu} h^\nu + \mathcal{J}^\mu_a D_\mu A^a_\nu h^\nu - \mathcal{J}^\mu_a \nabla_\nu A^a_\mu h^\nu)$. This is clearly zero, since $F^a_{\mu\nu} = D_\mu A^a_\nu - \nabla_\nu A^a_\mu$.

More generally, it is interesting to consider the situation when h^μ is a Killing vector field but the vector potential is invariant under the generated diffeomorphisms only up to a YM gauge transformation (see, e.g., Ref. [58] regarding such invariance conditions). This means that there exists a gauge transformation parameter ϵ^a so that the vector potential is invariant under the joint infinitesimal diffeomorphism and YM gauge transformation corresponding to (h^μ, ϵ^a):

$$\delta A^a_\nu = -h^\mu \partial_\mu A^a_\nu - \partial_\nu h^\mu A^a_\mu + D_\nu \epsilon^a = 0. \quad (3.18)$$

The corresponding Bianchi current

$$\mathcal{B}^\mu = \sqrt{-g}[T^\mu{}_\nu h^\nu + (h^\nu A^a_\nu - \epsilon^a)\mathcal{J}^\mu_a] \quad (3.19)$$

is the combination of (3.10) and (3.17). The conservation of (3.19) can be derived in a similar way, using (3.18), as the conservation of (3.10) and (3.17). In this derivation, the equivalent form $\delta A^a_\nu = -h^\mu \nabla_\mu A^a_\nu - \nabla_\nu h^\mu A^a_\mu + D_\nu \epsilon^a = 0$ of (3.18) is useful.

In principle, the cases when the tetrad field is invariant under the diffeomorphisms generated by h^μ and when it is invariant only up to local Lorentz transformations are also different, but since the Bianchi current corresponding to local Lorentz transformations is zero, the Bianchi currents (3.17) or (3.19) are the same in both cases. This is not true for the Noether currents; see Secs. III B 3 and III B 5.

In the case when the YM gauge field is the electromagnetic field, the Bianchi current (3.17) is precisely the current (1.1) proposed in Ref. [6], and the partial Bianchi identity (3.15) is the Lorentz law (1.2). It should be noted that in the (charged black hole) configurations considered in Ref. [6] the background electromagnetic field satisfies the Maxwell equation $\nabla_\mu F^{\mu\nu} = 0$, and this property was used in Ref. [6] in the derivation of the Lorentz law, but in the present derivation, this is not needed.

Making a small digression, we mention that if the YM symmetry of the Lagrangian is not required and A^a_μ are merely some fixed covector fields, then $D_\nu \mathcal{J}^\nu_a = 0$ does not necessarily hold, and from (3.13), (3.14), and (3.11), one obtains the partial Bianchi identity

$$B_\mu = \sqrt{-g}(-\nabla_\nu T^\nu{}_\mu - \mathcal{J}^\nu_a F^a_{\nu\mu} - D_\nu \mathcal{J}^\nu_a A^a_\mu) = 0, \quad (3.20)$$

where D_μ and $F^a_{\nu\mu}$ are defined in the same way as in Abelian gauge theory. This is the identity that appears, e.g., in Refs. [7,8]. The formula (3.17) for the Bianchi current for a diffeomorphism symmetry of the metric and A^a_μ remains unchanged.

B. Noether currents and superpotentials

In this section, we discuss the Noether currents and the superpotentials for the diffeomorphism, local Lorentz, and

YM gauge symmetries, concentrating on the third part of the construction in Sec. II E.

For simplicity, we assume that the Lagrangian density of the matter fields takes the form

$$L = \sqrt{-g}\hat{L}(V^{\bar{\mu}}_{\mu}, \psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}, \bar{\psi}_{\check{\nu}}{}^{\check{\rho}\alpha k}, D_{\mu}\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}, D_{\mu}\bar{\psi}_{\check{\nu}}{}^{\check{\rho}\alpha k}), \quad (3.21)$$

where $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ is a complex matter field and $\bar{\psi}_{\check{\nu}}{}^{\check{\rho}\alpha k}$ is its (generalized) Dirac conjugate. We take only one matter field, but it is completely straightforward to include more of them. The above form of L implies that the matter field is coupled minimally to the gravitational and YM fields and that L does not depend on higher than first derivatives of the matter field. Matter fields can be real as well; in this case, the Dirac conjugate field is not needed. $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ has n covector indices and m vector indices, denoted collectively as $\check{\nu}$ and $\check{\rho}$, i.e. $\check{\nu} \equiv \nu_1\nu_2\ldots\nu_n$ and $\check{\rho} \equiv \rho_1\rho_2\ldots\rho_m$.

Under the global YM gauge group, $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ transforms according to a not necessarily irreducible finite-dimensional unitary representation $\mathcal{R}$; the corresponding index is denoted by k. The basis in this representation is chosen to be orthogonal and normalized to 1. The generators of the global YM Lie algebra in the representation $\mathcal{R}$ are denoted, after they are multiplied by i, by $(t_a)_k{}^l$. These matrices satisfy the commutation relations $[t_a, t_b] = \mathrm{i}f_{ab}{}^c t_c$ of the Lie algebra of the global gauge group and are self-adjoint (i.e. $t_a^{\dagger} = t_a$).

Under Lorentz transformations, $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ transforms according to a not necessarily irreducible finite-dimensional real representation; the corresponding index is denoted by α. A real representation is a direct sum of the irreducible real representations $(\frac{l_1}{2}, \frac{l_2}{2}) \oplus (\frac{l_2}{2}, \frac{l_1}{2})$, $l_1 \neq l_2$, l_1, $l_2 \in \mathbb{N}$, and $(\frac{l}{2}, \frac{l}{2})$, $l \in \mathbb{N}$, where $(\frac{l_1}{2}, \frac{l_2}{2})$, l_1, $l_2 \in \mathbb{N}$, denote the finite-dimensional irreducible representations of the Lorentz group. The simplest examples of these irreducible real representations are the Dirac spinor representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ and the Minkowski representation $(\frac{1}{2}, \frac{1}{2})$. A representation $(\frac{l_1}{2}, \frac{l_2}{2})$ is fermionic if $l_1 + l_2$ is odd and bosonic if $l_1 + l_2$ is even. $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are the Weyl spinor representations. $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ can be taken to be number valued, but if it transforms according to a fermionic representation, then it is also reasonable to take it to be anticommuting. The formulas written in this section are valid for both options.

The generalized Dirac conjugation is defined as $\bar{\psi}_{\check{\nu}}{}^{\check{\rho}\beta k}$

$$\bar{\psi}_{\check{\nu}}{}^{\check{\rho}\beta k} = \psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}{}^{*}\epsilon^{\alpha\beta}, \quad (3.22)$$

where * denotes complex conjugation [59] and $\epsilon^{\alpha\beta}$ is a nondegenerate matrix with the properties $\epsilon^{\alpha\beta *} = \epsilon^{\beta\alpha}$ and $(L_{\bar{\mu}\bar{\nu}})_{\delta}{}^{\alpha *}\epsilon^{\delta\beta} + (L_{\bar{\mu}\bar{\nu}})_{\delta}{}^{\beta}\epsilon^{\alpha\delta} = 0$. The latter property is needed for the Lorentz covariance of the Dirac conjugation, and the $L_{\bar{\mu}\bar{\nu}}$ appearing in it denote the generators of the Lorentz group in the representation according to which $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ transforms. An $\epsilon^{\alpha\beta}$ matrix with these properties can be found for any finite-dimensional real representation of the Lorentz group. If $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ transforms as a Dirac spinor and one uses, e.g., the Weyl basis for Dirac spinors, then $\epsilon^{\alpha\beta}$ can be taken to be the Dirac gamma matrix $\gamma^{\bar{0}}$. For the purpose of the present paper, it is not necessary to specify $\epsilon^{\alpha\beta}$ explicitly for the other representations. In general, it would be necessary to insert into (3.22) a matrix similar to $\epsilon^{\alpha\beta}$ for the YM indices as well, but with the special choice of basis in $\mathcal{R}$ mentioned above, this matrix is the unit matrix.

The restrictions on the possible indices of the matter field above are done in order to avoid writing large formulas. Nevertheless, it is straightforward to extend the various formulas to the cases when ψ transforms under Lorentz transformations according to a nonreal representation (e.g., according to the Weyl spinor representations), or when ψ has several indices corresponding to Lorentz transformations.

We also assume for simplicity that $\hat{L}$ is invariant under YM gauge transformations and local Lorentz transformations and transforms as a scalar function under diffeomorphisms. In this case, the simplest and most natural choice for K^{μ} in the symmetry condition (2.23) is $K^{\mu} = 0$ for YM gauge transformations and local Lorentz transformations and $K^{\mu} = -h^{\mu}L$ for diffeomorphisms.

It should be noted that in the literature the matter fields are often regarded as sections of suitable vector bundles over M. In a coordinate based formalism, it is usually assumed that a trivialization is fixed for these vector bundles over the coordinate patch U that is under consideration.

It is useful to introduce the following notation:

$$P^{\mu\check{\nu}}{}_{\check{\rho}}{}^{\alpha k} = \frac{\partial\hat{L}}{\partial D_{\mu}\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}} \qquad \tilde{P}^{\mu\check{\nu}}{}_{\check{\rho}\alpha k} = \frac{\partial\hat{L}}{\partial D_{\mu}\bar{\psi}_{\check{\nu}}{}^{\check{\rho}\alpha k}} \quad (3.23)$$

$$E^{\check{\nu}}{}_{\check{\rho}}{}^{\alpha k} = \frac{1}{\sqrt{-g}}\frac{\delta L}{\delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}} \qquad \tilde{E}^{\check{\nu}}{}_{\check{\rho}\alpha k} = \frac{1}{\sqrt{-g}}\frac{\delta L}{\delta\bar{\psi}_{\check{\nu}}{}^{\check{\rho}\alpha k}} \quad (3.24)$$

$$\begin{aligned}\mathbb{A}_i^{\mu\lambda\nu} = {} & (P^{\mu\ldots\lambda\ldots}{}_{\check{\rho}}{}^{\alpha k} + P^{\lambda\ldots\mu\ldots}{}_{\check{\rho}}{}^{\alpha k})\psi_{\ldots}{}^{\nu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k} - P^{\nu\ldots\lambda\ldots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\mu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k} \\ & + (\tilde{P}^{\mu\ldots\lambda\ldots}{}_{\check{\rho}\alpha k} + \tilde{P}^{\lambda\ldots\mu\ldots}{}_{\check{\rho}\alpha k})\bar{\psi}_{\ldots}{}^{\nu}{}_{\ldots}{}^{\check{\rho}\alpha k} - \tilde{P}^{\nu\ldots\lambda\ldots}{}_{\check{\rho}\alpha k}\bar{\psi}_{\ldots}{}^{\mu}{}_{\ldots}{}^{\check{\rho}\alpha k}\end{aligned} \quad (3.25)$$

$$\mathbb{B}_i^{\mu\lambda\nu} = (P^{\mu\check{\nu}}{}_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\alpha k} - P^{\lambda\check{\nu}}{}_{\ldots}{}^{\mu}{}_{\ldots}{}^{\alpha k})\psi_{\check{\nu}}{}^{\ldots\nu\ldots}{}_{\alpha k} + P^{\nu\check{\nu}}{}_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\mu\ldots}{}_{\alpha k} + (\tilde{P}^{\mu\check{\nu}}{}_{\ldots}{}^{\lambda}{}_{\ldots\alpha k} - \tilde{P}^{\lambda\check{\nu}}{}_{\ldots}{}^{\mu}{}_{\ldots\alpha k})\bar{\psi}_{\check{\nu}}{}^{\ldots\nu\ldots\alpha k} + \tilde{P}^{\nu\check{\nu}}{}_{\ldots}{}^{\lambda}{}_{\ldots\alpha k}\bar{\psi}_{\check{\nu}}{}^{\ldots\mu\ldots\alpha k} \tag{3.26}$$

$$\mathbb{Q}^{\mu\bar{\delta}\bar{\lambda}} = \frac{1}{2}P^{\mu\check{\nu}}{}_{\check{\rho}}{}^{\alpha k}\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\beta k}(L^{\bar{\delta}\bar{\lambda}})_{\alpha}{}^{\beta} - \frac{1}{2}\tilde{P}^{\mu\check{\nu}}{}_{\check{\rho}\alpha k}\bar{\psi}_{\check{\nu}}{}^{\check{\rho}\beta k}(L^{\bar{\delta}\bar{\lambda}})_{\beta}{}^{\alpha} \tag{3.27}$$

In (3.25), for example, $P^{\mu\ldots\lambda\ldots}{}_{\check{\rho}}{}^{\alpha k}$ at the beginning of the right-hand side means $P^{\mu\nu_1\nu_2\ldots\nu_{i-1}\lambda\nu_{i+1}\ldots\nu_n}{}_{\check{\rho}}{}^{\alpha k}$, and similar notation is used subsequently.

$\mathbb{Q}^{\mu\bar{\delta}\bar{\lambda}}$ is antisymmetric in the last two indices. The reality of L implies that $P^{\mu\check{\nu}}{}_{\check{\rho}}{}^{\alpha k}$ is the Dirac conjugate of $\tilde{P}^{\mu\check{\nu}}{}_{\check{\rho}\alpha k}$ and $E^{\check{\nu}}{}_{\check{\rho}}{}^{\alpha k}$ is the Dirac conjugate of $\tilde{E}^{\check{\nu}}{}_{\check{\rho}\alpha k}$ if $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ is an even field and $P^{\mu\check{\nu}}{}_{\check{\rho}}{}^{\alpha k}$ is -1 times the Dirac conjugate of $\tilde{P}^{\mu\check{\nu}}{}_{\check{\rho}\alpha k}$ and $E^{\check{\nu}}{}_{\check{\rho}}{}^{\alpha k}$ is -1 times the Dirac conjugate of $\tilde{E}^{\check{\nu}}{}_{\check{\rho}\alpha k}$ if $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ is anticommuting.

1. Euler-Lagrange currents

For a preliminary step, one determines the currents j_V^μ, j_A^μ, and j_ψ^μ for arbitrary variations of the tetrad, the Yang-Mills vector potential, and the matter field. One finds

$$j_A^\mu = 0, \tag{3.28}$$

$$j_\psi^\mu = \sqrt{-g}(P^{\mu\check{\nu}}{}_{\check{\rho}}{}^{\alpha k}\delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} + \tilde{P}^{\mu\check{\nu}}{}_{\check{\rho}\alpha k}\delta\bar{\psi}_{\check{\nu}}{}^{\check{\rho}\alpha k}), \tag{3.29}$$

and

$$j_V^\mu = j_g^\mu + j_w^\mu \tag{3.30}$$

$$j_g^\mu = \sqrt{-g}\left[-\frac{1}{2}\sum_{i=1}^{n}\mathbb{A}_i^{\mu(\nu\lambda)} + \frac{1}{2}\sum_{i=1}^{m}\mathbb{B}_i^{\mu(\nu\lambda)} + \mathbb{Q}^{(\nu\lambda)\mu}\right]\gamma_{\nu\lambda} \tag{3.31}$$

$$j_w^\mu = \sqrt{-g}\mathbb{Q}^{\mu\nu\lambda}w_{[\nu\lambda]}. \tag{3.32}$$

$\gamma_{\mu\nu} = \delta g_{\mu\nu} = 2w_{(\mu\nu)}$, $w_{\mu\nu} = V_{\nu\bar{\mu}}\delta V_\mu^{\bar{\mu}}$. The quantities $\mathbb{A}_i^{\mu\nu\lambda}$, $\mathbb{B}_i^{\mu\nu\lambda}$, and $\mathbb{Q}_i^{\mu\nu\lambda}$ in (3.31) and (3.32) come from the variation of the connection.

With the above formulas at hand, one can proceed to determine the Noether currents and the superpotentials for the various gauge symmetries. For this, it is also necessary to know the variation of the matter field under the gauge transformations. The Noether currents J^μ are obtained as $J^\mu = j_V^\mu + j_\psi^\mu - K^\mu$, specializing δA_μ^a, $\delta V_\mu^{\bar{\mu}}$, and $\delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ in the above formulas for j_V^μ and j_ψ^μ according to the particular transformation rules (3.1)–(3.6), (3.34), (3.36), and (3.38). $J_\Phi^\mu \equiv J_\psi^\mu$ is $j_\psi^\mu - K^\mu$. Using (3.34), (3.36), or (3.38) one can calculate $\mathcal{B}_\psi^\mu$; then, I^μ is obtained as $I^\mu = J^\mu + \mathcal{B}_A^\mu + \mathcal{B}_V^\mu + \mathcal{B}_\psi^\mu$, and finally $\Sigma^{\mu\nu}$ is given by (2.34). This general procedure for determining I^μ and $\Sigma^{\mu\nu}$ can be simplified further in the present example in the following way: I^μ does not depend on higher than first derivatives of ϵ^α, i.e. $I^\mu = \epsilon^\alpha I_\alpha^\mu + \partial_\nu\epsilon^\alpha I_\alpha^{\mu\nu}$; thus, according to (2.34), $\Sigma^{\mu\nu} = \epsilon^\alpha I_\alpha^{[\mu\nu]}$. This shows that $I_\alpha^{\mu\nu}$ completely determines $\Sigma^{\mu\nu}$ and I^μ. In particular, from $I^\mu = \partial_\nu\Sigma^{\mu\nu}$, we get $I_\alpha^\mu = \partial_\nu I_\alpha^{[\mu\nu]}$ and $I_\alpha^{\mu\nu} = I_\alpha^{[\mu\nu]}$. Furthermore, since the Bianchi currents and K^μ do not depend on the derivatives of ϵ^α, $I_\alpha^{\mu\nu} = J_\alpha^{\mu\nu} = j_{V,\alpha}^{\mu\nu} + j_{\psi,\alpha}^{\mu\nu}$, i.e.

$$\Sigma^{\mu\nu} = \epsilon^\alpha(j_{V,\alpha}^{\mu\nu} + j_{\psi,\alpha}^{\mu\nu}). \tag{3.33}$$

The notation $J_\alpha^{\mu\nu}$, $j_{V,\alpha}^{\mu\nu}$, $j_{\psi,\alpha}^{\mu\nu}$ is understood in the same way as $I_\alpha^{\mu\nu}$ and the index α is used here in the same sense as in Sec. II.

2. Yang-Mills gauge symmetry

The first order variation of the matter field under a YM gauge transformation parametrized by ϵ^a is

$$\delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} = \mathrm{i}\kappa\epsilon^a(t_a)_k{}^l\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha l}. \tag{3.34}$$

One finds that for YM gauge transformations the Noether current is

$$J^\mu = \sqrt{-g}\epsilon^a\mathcal{J}_a^\mu, \tag{3.35}$$

which is obtained by specializing $\delta\psi_k$ in (3.29) according to (3.34). Since $K^\mu = j_A^\mu = j_V^\mu = 0$, $J^\mu = J_\psi^\mu = j_\psi^\mu$. J^μ does not depend on the derivatives of ϵ^a, I^μ and $\Sigma^{\mu\nu}$ are zero, and the Bianchi current $\mathcal{B}_\chi^\mu$ is identical to $-J^\mu$ even if the matter field does not satisfy its EL equation. The Bianchi currents $\mathcal{B}_A^\mu$ and $\mathcal{B}_V^\mu$ are given in (3.9), and $\mathcal{B}_\psi^\mu = 0$.

3. Local Lorentz symmetry

The first order variation of the matter field under a local Lorentz transformation is

$$\delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} = \omega^{\bar{\mu}\bar{\nu}}(L_{\bar{\mu}\bar{\nu}})_\alpha{}^\beta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\beta k}. \tag{3.36}$$

$\mathcal{B}_\psi^\mu$ is obviously zero, and in Sec. III A, we saw that also $\mathcal{B}_V^\mu = \mathcal{B}_A^\mu = 0$. After some calculation, one finds that $J^\mu = 0$ as well; thus, $I^\mu = 0$. On the other hand, J_ψ^μ is not zero,

$$J_\psi^\mu = 2\sqrt{-g}\mathbb{Q}^{\mu\bar{\nu}\bar{\eta}}\omega_{\bar{\nu}\bar{\eta}}. \tag{3.37}$$

A nonzero tetrad is not invariant under any Lorentz transformation; therefore, J^μ_ψ is generally not conserved. Nevertheless, J^μ_ψ has an important role when the tetrad and the vector potential are invariant under a combined transformation (see Sec. III B 5).

4. Diffeomorphism symmetry

The first order variation of the matter field under a diffeomorphism generated by h^μ is the Lie derivative

$$\begin{aligned}\delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} = &-h^\mu\partial_\mu\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}\\ &-\partial_{\nu_1}h^\mu\psi_{\mu\nu_2\ldots\nu_n}{}^{\check{\rho}}{}_{\alpha k}-\partial_{\nu_2}h^\mu\psi_{\nu_1\mu\nu_3\ldots\nu_n}{}^{\check{\rho}}{}_{\alpha k}-\cdots-\partial_{\nu_n}h^\mu\psi_{\nu_1\ldots\nu_{n-1}\mu}{}^{\check{\rho}}{}_{\alpha k}\\ &+\partial_\mu h^{\rho_1}\psi_{\check{\nu}}{}^{\mu\rho_2\cdots\rho_m}{}_{\alpha k}+\partial_\mu h^{\rho_2}\psi_{\check{\nu}}{}^{\rho_1\mu\rho_3\cdots\rho_m}{}_{\alpha k}+\cdots+\partial_\mu h^{\rho_m}\psi_{\check{\nu}}{}^{\rho_1\cdots\rho_{m-1}\mu}{}_{\alpha k}.\end{aligned}\tag{3.38}$$

It is straightforward to obtain explicit expressions for J^μ and J^μ_ψ, but we do not write them out here, since they are not enlightening. The Bianchi currents are (3.16) and

$$\begin{aligned}\mathcal{B}^\mu_\psi = \sqrt{-g}[&-E^{\mu\nu_2\nu_3\cdots}{}_{\check{\rho}}{}^{\alpha k}h^\lambda\psi_{\lambda\nu_2\nu_3\ldots}{}^{\check{\rho}}{}_{\alpha k}-E^{\nu_1\mu\nu_3\cdots}{}_{\check{\rho}}{}^{\alpha k}h^\lambda\psi_{\nu_1\lambda\nu_3\ldots}{}^{\check{\rho}}{}_{\alpha k}-\cdots\\ &+E^{\check{\nu}}{}_{\lambda\rho_2\rho_3\ldots}{}^{\alpha k}h^\lambda\psi_{\check{\nu}}{}^{\mu\rho_2\rho_3\cdots}{}_{\alpha k}+E^{\check{\nu}}{}_{\rho_1\lambda\rho_3\ldots}{}^{\alpha k}h^\lambda\psi_{\check{\nu}}{}^{\rho_1\mu\rho_3\cdots}{}_{\alpha k}+\cdots\\ &+[\tilde{E}\,\bar{\psi}]],\end{aligned}\tag{3.39}$$

where $[\tilde{E}\,\bar{\psi}]$ represents the complex conjugate of the previous terms. I^μ is found to be

$$\begin{aligned}I^\mu = \sqrt{-g}[&\nabla_\nu[2(\mathbb{Q}^{(\mu\lambda)\nu}h_\lambda-\mathbb{Q}^{(\nu\lambda)\mu}h_\lambda)-\mathbb{Q}^{\lambda\mu\nu}h_\lambda]\\ &-\frac{1}{2}\sum_{i=1}^{n}\nabla_\nu[P^{\mu\ldots\nu\cdots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_\lambda-P^{\nu\ldots\mu\cdots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_\lambda]\\ &+\frac{1}{2}\sum_{i=1}^{n}\nabla_\nu[P^{\mu\ldots\lambda\cdots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\nu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_\lambda-P^{\nu\ldots\lambda\cdots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\mu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_\lambda]\\ &+\frac{1}{2}\sum_{i=1}^{n}\nabla_\nu[P^{\lambda\ldots\mu\cdots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\nu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_\lambda-P^{\lambda\ldots\nu\cdots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\mu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_\lambda]\\ &-\frac{1}{2}\sum_{i=1}^{m}\nabla_\rho[P^{\mu\check{\nu}}{}_{\ldots}{}^{\rho}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\lambda\ldots}{}_{\alpha k}h_\lambda-P^{\rho\check{\nu}}{}_{\ldots}{}^{\mu}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\lambda\ldots}{}_{\alpha k}h_\lambda]\\ &+\frac{1}{2}\sum_{i=1}^{m}\nabla_\rho[P^{\mu\check{\nu}}{}_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\rho\ldots}{}_{\alpha k}h_\lambda-P^{\rho\check{\nu}}{}_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\mu\ldots}{}_{\alpha k}h_\lambda]\\ &+\frac{1}{2}\sum_{i=1}^{m}\nabla_\rho[P^{\lambda\check{\nu}}{}_{\ldots}{}^{\mu}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\rho\ldots}{}_{\alpha k}h_\lambda-P^{\lambda\check{\nu}}{}_{\ldots}{}^{\rho}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\mu\ldots}{}_{\alpha k}h_\lambda]\\ &+[\tilde{P}\,\bar{\psi}]],\end{aligned}\tag{3.40}$$

where $[\tilde{P}\,\bar{\psi}]$ represents the complex conjugate of the $P\psi$ type terms in the previous six lines. $\Sigma^{\mu\nu}$ is obtained by omitting the ∇ operators in (3.40) and changing the ρ index to ν where necessary.

It is interesting to note that if the matter field is scalar with respect to diffeomorphisms and local Lorentz transformations, then $\mathcal{B}^\mu_\psi = 0$ and $I^\mu = 0$; therefore, the Bianchi current $\mathcal{B}^\mu_\chi$ coincides with $-J^\mu$ even if the matter field does not satisfy its EL equations. $\mathcal{B}^\mu_\psi = 0$ holds even if the matter field is scalar only with respect to diffeomorphisms.

5. Mixed symmetries

A general infinitesimal gauge transformation is a combination of the special transformations discussed in the previous subsections and is characterized by a triple $(h^\mu, \omega^{\bar{\mu}\bar{\nu}}, \epsilon^a)$. For such transformations, $\delta V^{\bar{\mu}}_\mu$, δA^a_μ, and $\delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$, the Bianchi currents, the Noether currents, I^μ, and $\Sigma^{\mu\nu}$ are just the sums of the relevant expressions for the special transformations.

A common case when combined transformations are important is when a spacetime has complete rotation

symmetry [$SO(3)$]. Since it is not possible to choose a tetrad that is invariant under the entire rotation group (understood as a subgroup of the group of diffeomorphisms), some of the rotation symmetry transformations of the tetrad necessarily involve local Lorentz transformations, and this has to be taken into account when one calculates Noether currents.

6. Dirac field

An important example of a matter Lagrangian is the Dirac Lagrangian

$$L = \sqrt{-g}\left[\frac{1}{2} g^{\mu\nu}(\bar{\psi}\mathrm{i}\gamma_\mu D_\nu\psi - D_\nu\bar{\psi}\mathrm{i}\gamma_\mu\psi) - m\bar{\psi}\psi\right], \quad (3.41)$$

where ψ denotes the Dirac field. As usual, the Dirac spinor indices are omitted, and the indices related to YM gauge group representations are also suppressed. γ^μ are defined as $\gamma^\mu = V^\mu_{\bar{\mu}}\gamma^{\bar{\mu}}$, where $\gamma^{\bar{\mu}}$ are the standard Dirac gamma matrices in Minkowski space (see, e.g., Ref. [60]). The generators of the Lorentz group $L^{\bar{\nu}\bar{\lambda}}$ in the Dirac representation are $L^{\bar{\mu}\bar{\nu}} = \frac{1}{2}\sigma^{\bar{\mu}\bar{\nu}}$, where $\sigma^{\bar{\mu}\bar{\nu}} = \frac{1}{2}[\gamma^{\bar{\mu}}, \gamma^{\bar{\nu}}]$.

I^μ [see (3.40)] takes a relatively simple form for the Dirac field. Since the Dirac field does not have vector and covector indices, only the $\mathbb{Q}$ terms are present in (3.40). The quantities $\mathbb{A}^{\mu\lambda\nu}$ and $\mathbb{B}^{\mu\lambda\nu}$ introduced in (3.25) and (3.26) are also absent. The quantity $\mathbb{Q}^{\mu\bar{\delta}\bar{\lambda}}$ introduced in (3.27) is $\mathbb{Q}^{\mu\bar{\delta}\bar{\lambda}} = \frac{1}{8}(\bar{\psi}\mathrm{i}\gamma^\mu\sigma^{\bar{\delta}\bar{\lambda}}\psi + \bar{\psi}\sigma^{\bar{\delta}\bar{\lambda}}\mathrm{i}\gamma^\mu\psi)$. $\gamma^\mu\sigma^{\delta\lambda} + \sigma^{\delta\lambda}\gamma^\mu = \gamma^\lambda\gamma^\mu\gamma^\delta - \gamma^\delta\gamma^\mu\gamma^\lambda$; therefore, $\mathbb{Q}^{\mu\delta\lambda}$ can also be written as

$$\mathbb{Q}^{\mu\delta\lambda} = \frac{1}{8}\mathrm{i}\bar{\psi}(\gamma^\lambda\gamma^\mu\gamma^\delta - \gamma^\delta\gamma^\mu\gamma^\lambda)\psi. \quad (3.42)$$

It is not difficult to see that $\mathbb{Q}^{(\nu\lambda)\mu} = 0$; thus,

$$I^\mu = -\sqrt{-g}\nabla_\nu(\mathbb{Q}^{\lambda\mu\nu}h_\lambda) \quad (3.43)$$

in the case of the Dirac field. This result for I^μ was also found in Ref. [16] in the special case of the electromagnetic field as YM gauge field. In Ref. [16], the approach to treating spinor fields was different from the one applied in this paper.

For a diffeomorphism symmetry under which the tetrad and the vector potential are invariant, the canonical Noether current J^μ_ψ is found to be $-\sqrt{-g}\frac{1}{2}(\bar{\psi}\mathrm{i}\gamma^\mu\nabla_\nu\psi - \nabla_\nu\bar{\psi}\mathrm{i}\gamma^\mu\psi)h^\nu - h^\mu L$. For the Einstein-Hilbert energy-momentum tensor, one finds the expression $T^{\mu\nu} = \frac{1}{4}(\bar{\psi}\mathrm{i}\gamma^\mu D^\nu\psi + \bar{\psi}\mathrm{i}\gamma^\nu D^\mu\psi - D^\mu\bar{\psi}\mathrm{i}\gamma^\nu\psi - D^\nu\bar{\psi}\mathrm{i}\gamma^\mu\psi)$, and $\mathcal{J}^\mu_a = -\kappa\bar{\psi}\gamma^\mu t_a\psi$.

In Ref. [13], it was found in an *ad hoc* manner that subtracting $\sqrt{-g}\nabla_\nu(\mathbb{Q}^{\lambda\mu\nu}h_\lambda)$ from the canonical Noether current $-J^\mu_\psi$ gives (1.1), if the Dirac equation is satisfied. Here, we have been able to derive this within a general formalism. (Note that in Ref. [13] the YM gauge field was the electromagnetic field and a factor -1 was included in the definition of the Noether currents.)

IV. MATTER FIELDS IN THE PRESENCE OF FIXED GRAVITATIONAL AND SCALAR FIELDS

In this example, we discuss only the partial Bianchi identity and the Bianchi current. The matter Lagrangian is again assumed to have the standard form $L = \sqrt{-g}\hat{L}$, and $\hat{L}$ is assumed to be a local function of the tetrad, the real scalar field ϕ, and the other fields. $\hat{L}$ is also assumed to be invariant under local Lorentz transformations and to transform as a scalar function under diffeomorphisms. The total gauge symmetry group in this example is thus the group generated by diffeomorphisms and local Lorentz transformations. The nature of the matter fields does not need to be specified.

The first order variation of ϕ under a diffeomorphism is $\delta\phi = -h^\mu\partial_\mu\phi$. Using this property, one obtains $B_{\phi\mu} = -\sqrt{-g}\mathcal{J}_\phi\partial_\mu\phi$ for the Bianchi expression $B_{\phi\mu}$, with $\mathcal{J}_\phi = \frac{1}{\sqrt{-g}}\frac{\delta L}{\delta\phi}$. $B_{V\mu}$ is given by (3.13). Taking into consideration $\mathbb{T}_{\bar{\mu}\bar{\nu}} = 0$ [see (3.11)], the partial Bianchi identity $B_{V\mu} + B_{\phi\mu} = 0$ is thus

$$B_\mu = \sqrt{-g}(-\nabla_\nu T^\nu{}_\mu - \mathcal{J}_\phi\partial_\mu\phi) = 0. \quad (4.1)$$

The Bianchi current $\mathcal{B}^\mu_\phi$ is obviously zero; therefore, the Bianchi current $\mathcal{B}^\mu_\chi$ is $\sqrt{-g}T^\mu{}_\nu h^\nu$; i.e. it takes the same form as in the absence of the fixed scalar field. $\mathcal{B}^\mu_\chi$ is conserved if h^μ is a Killing field, the first order variation of ϕ with respect to the diffeomorphisms generated by h^μ is also zero, and the matter fields satisfy their EL equations. The conservation of $\mathcal{B}^\mu_\chi$ can be derived from (4.1) and from the symmetry properties of the metric and the scalar field: $\nabla_\mu(T^\mu{}_\nu h^\nu) = \nabla_\mu T^\mu{}_\nu h^\nu + T^\mu{}_\nu\nabla_\mu h^\nu$, and here on the right-hand side, the second term is zero in virtue of the Killing equation, whereas the first term can be rewritten as $\nabla_\mu T^\mu{}_\nu h^\nu = -\mathcal{J}_\phi h^\nu\partial_\nu\phi$ using (4.1). $\mathcal{J}_\phi h^\nu\partial_\nu\phi$ is indeed zero if $\delta\phi = 0$.

It is interesting to note that if only the scalar field is fixed, then the partial Bianchi identity becomes $B_\mu = \sqrt{-g}(-\mathcal{J}_\phi\partial_\mu\phi) = 0$ (and now $\mathcal{J}_\phi$ is obtained from the total Lagrangian), as is also found in Ref. [31]. This identity has the remarkable consequence that ϕ either is constant or it also satisfies its EL equation.

V. CONCLUSION

In this paper, we extended the standard construction of conserved currents associated with spacetime symmetries for matter fields propagating in fixed curved spacetime to general gauge theories, without any restriction on the order of the derivatives of the fields that may appear in the Lagrangian. In particular, we showed that if in a Lagrangian field theory that has gauge symmetry in the general

Noetherian sense some of the elementary fields are fixed and are invariant under an infinitesimal gauge transformation then there exists a current, which we called the Bianchi current, that is analogous to the current $\sqrt{-g}T^{\mu}{}_{\nu}h^{\nu}$ used in general relativity and is conserved if the nonfixed fields satisfy their Euler-Lagrange equations. The conservation of this current can be seen as a consequence of the symmetry of the fixed fields and of an identity, which we called partial Bianchi identity, that is analogous to $\nabla_{\mu}T^{\mu\nu} = 0$ and follows from the gauge symmetry of the Lagrangian. We also showed that the Noether current associated with the symmetry of the fixed fields obtained by applying Noether's first theorem differs from the Bianchi current by the sum of an identically conserved current and a term that vanishes if the nonfixed fields satisfy their Euler-Lagrange equations. We gave explicit formulas for the Bianchi current and for the other quantities appearing in these results, so they can be calculated in any particular model. If the total Lagrangian can be split in the same way as in general relativity to a "matter" and a "gravitational" part, so that the latter depends only on the fixed fields, then the construction can be applied to the matter part separately, as is done in general relativity in the standard case.

For an example, we discussed first the case of general matter fields propagating in backgrounds consisting of a gravitational and a Yang-Mills field. We found (3.17) as the generalization of the current $\sqrt{-g}T^{\mu}{}_{\nu}h^{\nu}$. The extension of (3.17) to the case when the Yang-Mills part of the background, i.e. the vector potential, is invariant under the diffeomorphisms generated by h^{μ} only up to (Yang-Mills) gauge transformations is (3.19). For the generalization of the property $\nabla_{\mu}T^{\mu\nu} = 0$, we found the Lorentz law (3.15). This means that the Lorentz law is found to hold in arbitrary gravitational and Yang-Mills background, if the matter fields satisfy their Euler-Lagrange equations and the matter Lagrangian has diffeomorphism, local Lorentz, and Yang-Mills gauge symmetry. For local Lorentz transformations and Yang-Mills gauge transformations, we found the partial Bianchi identities (3.11) and (3.7). The Bianchi current for a Yang-Mills gauge symmetry of the fixed fields is (3.10). Under not very restrictive assumptions on the type of the matter fields and on the form of the matter Lagrangian, we investigated the difference between the Bianchi currents and the Noether currents. In the case of Yang-Mills gauge symmetries, these currents coincide even if the dynamical fields do not satisfy their Euler-Lagrange equations. In the case of diffeomorphism symmetries, the Bianchi currents are generally not identical with the Noether currents, except if the matter fields are scalar fields, as the known results for zero fixed Yang-Mills field also indicate. We obtained the formulas (3.39) and (3.40) for the characterization of the difference between the Bianchi and Noether currents. In the case of the Dirac field, Eq. (3.40) reduces to (3.43).

If the requirement of the Yang-Mills symmetry of the Lagrangian is omitted but fixed covector fields are nevertheless present, then the generalization of $\nabla_{\mu}T^{\mu\nu} = 0$ becomes (3.20) instead of (3.15), whereas the Bianchi current for a diffeomorphism symmetry has the unchanged form (3.17).

The second example was the case of fields propagating in backgrounds consisting of a gravitational and a real scalar field. For the generalization of $\nabla_{\mu}T^{\mu\nu} = 0$, we found (4.1), whereas the Bianchi current turned out to have the same form, $\sqrt{-g}T^{\mu}{}_{\nu}h^{\nu}$, as in the absence of the fixed scalar field.

The construction presented in this paper can be applied in a very wide variety of models, for example, in metric-affine gravitation theory or in other extended models of gravitation coupled with Yang-Mills type gauge fields and matter fields. It would be interesting to see if the Bianchi currents that can be constructed in these models can be used to obtain results similar to those in Ref. [6]. p-form field theory and other higher spin gauge theories are further examples that could be investigated.

Although in the examples that we discussed one of the fixed fields was the gravitational field, one can also apply the construction in cases when the gravitational field is not among the fixed fields. Examples of partial Bianchi identities for such cases have already been given in Ref. [31].

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APPENDIX: AUXILIARY FORMULAS AND REMARKS

In Sec. III, the complete gauge group is the group generated by the diffeomorphisms, the local Lorentz transformations, and the YM gauge transformations. The covariant derivative of the matter field $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$—introduced at the beginning of Sec. III B—for this gauge group is given by the formula

$$D_{\mu}\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} = \nabla_{\mu}\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} + S_{\alpha}{}^{\beta}{}_{\mu}\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\beta k} - \mathrm{i}\kappa A^{a}_{\mu}(t_a)_{k}{}^{l}\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha l}, \tag{A1}$$

where ∇_{μ} denotes the Levi-Cività covariant differentiation corresponding to $g_{\mu\nu}$ and $S_{\alpha}{}^{\beta}{}_{\mu} = \frac{1}{2}(L^{\bar{\nu}\bar{\lambda}})_{\alpha}{}^{\beta}\mathcal{S}_{\bar{\nu}\bar{\lambda}\mu}$, $\mathcal{S}^{\bar{\lambda}}_{\bar{\eta}\mu} = -V^{\nu}_{\bar{\eta}}\nabla_{\mu}V^{\bar{\lambda}}_{\nu}$. $(L^{\bar{\nu}\bar{\lambda}})_{\alpha}{}^{\beta}$ and $(t_a)_{k}{}^{l}$ are the generators of the Lorentz group and of the global YM gauge group, respectively, in the representations according to which the matter field transforms. The second term on the right-hand side of (A1) describes the action of D_{μ} on the Lorentz group related indices, and the third term gives the action of D_{μ} on the YM indices. More details concerning the second term can be found, e.g., in

Ref. [55]. One also applies D_μ as an operator to other fields that have the same types of indices as the matter field, even if they transform somewhat differently, as the vector potential A^a_μ, for example.

The local Lorentz transformations and the YM gauge transformations form two normal subgroups in the complete gauge group, but the group of diffeomorphisms is not a normal subgroup either with respect to YM gauge transformations or with respect to local Lorentz transformations. In accordance with this situation, the Lie derivatives of those fields that are not scalar with respect to local Lorentz transformations or YM gauge transformations generally transform in a noncovariant manner under these transformations.

Since the subgroup of diffeomorphisms is not a normal subgroup in the complete gauge group, from a purely group theoretical point of view, the complete gauge group does not have a unique diffeomorphism subgroup; rather, it has many diffeomorphism subgroups conjugate to one another. Nevertheless, in a coordinate based formalism, a definite diffeomorphism subgroup becomes distinguished implicitly, which can be called the subgroup of diffeomorphisms.

[1] R. M. Wald, *General Relativity* (University of Chicago, Chicago, 1984).
[2] V. Iyer and R. M. Wald, Some properties of the Noether charge and a proposal for dynamical black hole entropy, Phys. Rev. D **50**, 846 (1994).
[3] L. B. Szabados, Quasi-local energy-momentum and angular momentum in general relativity, Living Rev. Relativ. **12**, 4 (2009).
[4] E. Noether, Invariante Variationsprobleme, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse **1918**, 235 (1918).
[5] Y. Kosmann-Schwarzbach, *The Noether Theorems: Invariance and Conservations Laws in the Twentieth Century* (Springer, Berlin, 2010).
[6] J. Natario, L. Queimada, and R. Vicente, Test fields cannot destroy extremal black holes, Classical Quantum Gravity **33**, 175002 (2016).
[7] J. Lorenzen and D. Martelli, Comments on the Casimir energy in supersymmetric field theories, J. High Energy Phys. 07 (2015) 001.
[8] B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen, and D. Martelli, The Casimir energy in curved space and its supersymmetric counterpart, J. High Energy Phys. 07 (2015) 043.
[9] V. Borokhov, Belinfante tensors induced by matter gravity couplings, Phys. Rev. D **65**, 125022 (2002).
[10] See Ref. [6] regarding AdS black holes.
[11] H. Balasin, D. N. Blaschke, F. Gieres, and M. Schweda, Wong's equations and charged relativistic particles in non-commutative space, SIGMA **10**, 099 (2014), Appendix A.
[12] G. Z. Toth, Test of the weak cosmic censorship conjecture with a charged scalar field and dyonic Kerr–Newman black holes, Gen. Relativ. Gravit. **44**, 2019 (2012).
[13] G. Z. Toth, Weak cosmic censorship, dyonic Kerr-Newman black holes and Dirac fields, Classical Quantum Gravity **33**, 115012 (2016).
[14] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[15] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, Berlin, 2000).
[16] J. G. Fletcher, Local conservation laws in generally covariant theories, Rev. Mod. Phys. **32**, 65 (1960).
[17] S. G. Avery and B. U. W. Schwab, Noether's second theorem and Ward identities for gauge symmetries, J. High Energy Phys. 02 (2016) 031.
[18] G. Barnich and F. Brandt, Covariant theory of asymptotic symmetries, conservation laws and central charges, Nucl. Phys. **B633**, 3 (2002).
[19] G. Barnich, F. Brandt, and M. Henneaux, Local BRST cohomology in gauge theories, Phys. Rep. **338**, 439 (2000).
[20] S. C. Anco, Generalization of Noether's theorem in modern form to non-variational partial differential equations, arXiv:1605.08734.
[21] B. Julia and S. Silva, Currents and superpotentials in classical gauge-invariant theories: I. Local results with applications to perfect fluids and general relativity, Classical Quantum Gravity **15**, 2173 (1998).
[22] A. N. Petrov and R. R. Lompay, Covariantized Noether identities and conservation laws for perturbations in metric theories of gravity, Gen. Relativ. Gravit. **45**, 545 (2013).
[23] A. N. Petrov and R. R. Lompay, Covariant differential identities and conservation laws in metric-torsion theories of gravitation. I. General consideration, J. Math. Phys. (N.Y.) **54**, 062504 (2013).
[24] A. N. Petrov, *Classical and Quantum Gravity Research*, edited by M. N. Christiansen and T. K. Rasmussen (Nova Science, New York, 2008), p. 79.
[25] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Neeman, Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance, Phys. Rep. **258**, 1 (1995).
[26] R. M. Wald and A. Zoupas, A general definition of "conserved quantities" in general relativity and other theories of gravity, Phys. Rev. D **61**, 084027 (2000).
[27] J. Lee and R. M. Wald, Local symmetries and constraints, J. Math. Phys. (N.Y.) **31**, 725 (1990).
[28] S. Gao and R. M. Wald, The "physical process" version of the first law and the generalized second law for charged and rotating black holes, Phys. Rev. D **64**, 084020 (2001).

[29] R. M. Wald, Black hole entropy is the Noether charge, Phys. Rev. D **48**, R3427 (1993).
[30] D. N. Blaschke, F. Gieres, M. Reboud, and M. Schweda, The energy-momentum tensor(s) in classical gauge theories, Nucl. Phys. **B912**, 192 (2016).
[31] R. Bluhm and A. Sehic, Noether identities in gravity theories with nondynamical backgrounds and explicit spacetime symmetry breaking, Phys. Rev. D **94**, 104034 (2016).
[32] T. Jacobson and A. Mohd, Black hole entropy and Lorentz-diffeomorphism Noether charge, Phys. Rev. D **92**, 124010 (2015).
[33] R. Jackiw and N. S. Manton, Symmetries and conservation laws in gauge theories, Ann. Phys. (N.Y.) **127**, 257 (1980).
[34] Y. N. Obukhov, F. Portales-Oliva, D. Puetzfeld, and G. F. Rubilar, Invariant conserved currents in generalized gravity, Phys. Rev. D **92**, 104010 (2015).
[35] Y. N. Obukhov and G. F. Rubilar, Invariant conserved currents in gravity theories: diffeomorphisms and local gauge symmetries, Phys. Rev. D **76**, 124030 (2007).
[36] Y. N. Obukhov and G. F. Rubilar, Invariant conserved currents in gravity theories with local Lorentz and diffeomorphism symmetry, Phys. Rev. D **74**, 064002 (2006).
[37] M. Montesinos and E. Flores, Symmetric energy-momentum tensor in Maxwell, Yang–Mills, and Proca theories obtained using only Noether's theorem, Rev. Mex. Fis. **52**, 29 (2006).
[38] M. Forger and H. Romer, Currents and the energy momentum tensor in classical field theory: a fresh look at an old problem, Ann. Phys. (Amsterdam) **309**, 306 (2004).
[39] M. Banados and I. A. Reyes, A short review on Noether's theorems, gauge symmetries and boundary terms, Int. J. Mod. Phys. D **25**, 1630021 (2016).
[40] K. Prabhu, The first law of black hole mechanics for fields with internal gauge freedom, Classical Quantum Gravity **34**, 035011 (2017).
[41] K. Hajian and M. M. Sheikh-Jabbari, Solution phase space and conserved charges: a general formulation for charges associated with exact symmetries, Phys. Rev. D **93**, 044074 (2016).
[42] K. Hajian, Conserved charges and first law of thermodynamics for Kerr-de Sitter black holes, Gen. Relativ. Gravit. **48**, 114 (2016).
[43] T. Koivisto, Covariant conservation of energy momentum in modified gravities, Classical Quantum Gravity **23**, 4289 (2006).
[44] In Ref. [15], the term "differential function" is used instead of "local function".
[45] J. Z. Simon, Higher-derivative Lagrangians, nonlocality, problems, and solutions, Phys. Rev. D **41**, 3720 (1990).
[46] R. P. Woodard, Avoiding dark energy with 1/R modifications of gravity, Lect. Notes Phys. **720**, 403 (2007).
[47] H. Motohashi and T. Suyama, Third order equations of motion and the Ostrogradsky instability, Phys. Rev. D **91**, 085009 (2015).
[48] S. W. Hawking and T. Hertog, Living with ghosts, Phys. Rev. D **65**, 103515 (2002).
[49] K. S. Stelle, Renormalization of higher derivative quantum gravity, Phys. Rev. D **16**, 953 (1977).
[50] L. Modesto and I. L. Shapiro, Superrenormalizable quantum gravity with complex ghosts, Phys. Lett. B **755**, 279 (2016).
[51] L. Modesto, Super-renormalizable or finite Lee-Wick quantum gravity, Nucl. Phys. **B909**, 584 (2016).
[52] B. L. Giacchini, On the cancellation of Newtonian singularities in higher-derivative gravity, Phys. Lett. B **766**, 306 (2017).
[53] S. A. Woolliams, Master's thesis, Imperial College, London, 2013.
[54] R. M. Wald, On identically closed forms locally constructed from a field, J. Math. Phys. (N.Y.) **31**, 2378 (1990).
[55] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), Chap. 12, Sec. V.
[56] H. Gies and S. Lippoldt, Fermions in gravity with local spin-base invariance, Phys. Rev. D **89**, 064040 (2014).
[57] L. F. Abbott and S. Deser, Charge definition in nonabelian gauge theories, Phys. Lett. **116B**, 259 (1982).
[58] P. Forgács and N. S. Manton, Space-time symmetries in gauge theories, Commun. Math. Phys. **72**, 15 (1980).
[59] For products of anticommuting quantities, complex conjugation includes a reversal of the order of the factors.
[60] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).