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Vacuum structure and $\mathcal{PT}$-symmetry breaking of the non-Hermetian ($i\phi^3$) theory

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In this work, we study the $\mathcal{PT}$-symmetric ($i\phi^3$) theory using the effective action formalism. To test the accuracy of the used technique, we apply it first to the $\mathcal{PT}$-symmetric ($-\phi^4$) theory, where we reproduce the same results obtained in the literature using the method of Dyson-Schwinger equations. In $0+1$ space-time dimensions, the one-loop effective potential prediction for the ($i\phi^3$) theory ought to be more accurate than WKB results. The effective potential for the massless $\mathcal{PT}$-symmetric ($i\phi^3$) model is shown to be bounded from below, which is the first analytic result that advocates the vacuum stability of this theory. Our calculations show that the massless theory possesses only one stable vacuum as in the literature, but for the massive theory we find that there exist two stable vacua. For a nonzero magnetic field, we show that the $\mathcal{PT}$-symmetry of the theory is broken for negative imaginary magnetic field, which agrees with the Lee-Yang theorem. We argue that $\mathcal{PT}$-symmetry breaking is a manifestation of the Yang-Lee edge singularity.

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The study of $\mathcal{PT}$-symmetric theories is growing and covers different areas in physics [1–10]. The importance of this trend in research ranges from offering a regime to cure the ghost states in Lee-Wick theories [11–13] to pushing certain Hamiltonians that were rejected in the past to play a role in nature's description [14]. Another manifestation of the importance of such theories is that one can find simple scalar theories possessing the very important asymptotic freedom property for which QCD has been invented [7,15–17]. However, these theories have an extra step of calculations more than those in Hermitian theories. While the Dirac sense metric operator for Hermitian theories is unity, it depends on the Hamiltonian model in non-Hermitian theories, and thus its calculation is necessary for the prediction of physical amplitudes. In other words, the inner product in these theories takes a form that differs from the Dirac sense product used for Hermitian theories. For non-Hermitian theories with real eigenvalues, the metric operator is essential in defining the inner product [18].

The metric operator calculation in a closed form is possible for some $\mathcal{PT}$-symmetric problems [13,19,20], but for most of $\mathcal{PT}$-symmetric theories perturbative methods are employed to obtain it. However, quantum field theoretical techniques have been shown to implement the metric operator without a need for its explicit calculations [21]. Such techniques are useful in studying $\mathcal{PT}$-symmetric quantum field Hamiltonians where explicit calculation of the metric operator is hard to get. Accordingly, in this work we will follow the method of the quantum field effective potential to study the $\mathcal{PT}$-symmetric $i\phi^3$ theory. Although the calculation will be obtained in d space-time dimensions, we will concentrate on $0+1$ dimensions (quantum mechanics) only. The point is that the analytic continuation of the problem in the complex plane for a quantum field theory is not an easy task. Thus, if one thinks that a quantum field approach like the effective potential can do the job, one has to test its power of prediction first. This can be accomplished by comparing its results in $0+1$ space-time dimensions with the available results in the literature. Rigorous work in the literature [22–26] has studied the spectral analysis of the theory at the quantum mechanical level, which can be used as references of comparisons of the effective potential results in $0+1$ space-time dimensions. Then, the more important application of the technique for theories in higher dimensions is direct where techniques used in the quantum mechanical case cannot be applied. However, for higher dimensions one needs to employ a renormalization scheme which is relevant for the study of phase transition of the theory in $5+1$ space-time dimensions, for instance [27].

The renormalization group functions of the theory under consideration in this work describe the Yang-Lee edge singularity of Ising-like models [28]. Since phase transitions are always associated with symmetry breaking, one may wonder about the symmetry to be broken in the model under consideration that is associated with the Yang-Lee edge singularity. The effective potential approach used in this work can play the role of finding the link between $\mathcal{PT}$-symmetry breaking and the zeros of the partition function in statistical models.

The study of phase transitions in higher dimensions is interesting, and certainly it is our target in a future work where a link is to be found between existence of more than one stable vacuum for the theory and the ability to make phase transitions when the number of degrees of freedom is infinite (higher dimensions). Our aim in this

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work, however, is to test the effective potential by applying it first to the study of the $\mathcal{PT}$-symmetric $i\phi^3$ theory in $0+1$ dimensions. As we will see in this work, the massive case of the $i\phi^3$ theory has a richer vacuum structure than what is predicted by other quantum field techniques in the literature, where only one vacuum is always assumed. Although the two vacua can be related by an equivalent canonical transformation in $0+1$ space-time dimensions, in higher dimensions the same canonical transformation turns out to be an unitarily inequivalent one, and a roam for phase transition might exist [29].

The existence of more than one vacuum of a theory is due to two different analytic continuations of a theory. In Ref. [30] it has been argued that the number of available spectra depends on the number of noncontiguous Stokes wedges. In fact, the effective potential method can deduce all possible analytic continuations of the theory in the complex plane. The method has the same spirit of the methods used in the literature to study the problem in the quantum mechanical case. For instance, in Refs. [22,23,31] spectral analysis of the theory has been applied. The authors followed the main idea of considering the theory on a contour in the complex plane such that $\psi(z) \to 0$ as $|z| \to \infty$ where $\psi(z)$ is the eigenfunction and $z(x)$ is a complex contour. In fact, making the change of variable $x \to z(x)$ is a canonical transformation (point canonical transformation). Now, the effective potential expands around the classical field v or equivalently makes the transformation $\phi \to \phi + v$, which is also a point canonical transformation. Then, the effective potential is subjected to the stability $\frac{\partial V_{\text{eff}}}{\partial v} = 0$ and mass renormalization $\frac{\partial^2 V_{\text{eff}}}{\partial v^2} = M^2$ conditions. Since renormalized mass squared is always positive, the two conditions select v values (can be complex) that make the effective potential V_{eff} bounded from below. In quantum mechanics, potentials bounded from below are associated with the existence of bound state wave functions. In other words, the condition $\psi(z) \to 0$ as $|z| \to \infty$ applied to quantum mechanical problems shares the same spirit with the conditions $\frac{\partial V_{\text{eff}}}{\partial v} = 0$ and $\frac{\partial^2 V_{\text{eff}}}{\partial v^2} = M^2$ applied to the effective potential in quantum field problems. However, the effective potential method has two advantages over the other methods. The first advantage is that it can be extended easily to higher dimensions (quantum field theories), while the second one is that it implements the employment of the metric operator in the calculations [21].

The conditions applied to the effective potential might lead to more than one stable vacuum. However, the massive $\mathcal{PT}$-symmetric $i\phi^3$ theory has only one pair of Stokes wedges and thus is supposed to have only one real spectra. As we will see in this work, while the massless theory has one stable vacuum, the effective potential predicts two nondegenerate stable vacua for the massive case. These two vacua are in fact related by a unitarily equivalent canonical transformation in $0+1$ dimensions, but in going to higher dimensions the canonical transformation is no longer unitarily equivalent. In the quantum field case (higher dimensions) the number of degrees of freedom is infinite, and thus phase transitions between the two vacua might occur, while, as we will see, it cannot happen in $0+1$ dimensions between unitarily equivalent vacua. Also, in this work we will try to resolve the contradiction between the effective potential results regarding the number of available vacua for the massive case and the number predicted from the analysis in Ref. [30].

Phase transitions always exist in theories with a large number of degrees of freedom (higher dimensions). However, a kind of critical phenomenon can still be investigated even in $0+1$ space-time dimensions. In statistical systems, the zeros of the partition function (Yang-Lee edge singularity) of magnetic systems exist for imaginary magnetic fields [28]. Since the $\mathcal{PT}$-symmetric $i\phi^3$ lies in the same class of universality as the Ising model for that type of critical behavior, then one can test that critical behavior by subjecting the theory to an external magnetic field interaction term of the form $i\gamma\phi$. In this case the Hamiltonian density takes the form

$$H = \frac{1}{2}((\nabla\phi(x))^2 + \pi^2(x)) + i\phi^3 + i\gamma\phi. \qquad (1)$$

The analytic continuation of the problem can lead to $\mathcal{PT}$-symmetry breaking for negative values for the coupling γ even in $0+1$ dimensions. This form allows us to link the Yang-Lee edge singularity in magnetic systems to the $\mathcal{PT}$-symmetry breaking of the theory. The idea is that the $\mathcal{PT}$-symmetry is broken when the effective potential (vacuum energy) turns out to be complex. As we will see at a critical coupling, the vacuum condensate changes from being pure imaginary to a complex quantity that turns the vacuum energy complex, which is a signature of $\mathcal{PT}$-symmetry breaking. We will argue that level crossing at $\mathcal{PT}$-symmetry breaking is equivalent to the existence of the Yang-Lee edge singularity.

The $\mathcal{PT}$-symmetric $i\phi^3$ theory has been studied using field theory approach in Refs. [27,32–35]. All of these studies reflect the power of quantum field approach to study $\mathcal{PT}$-symmetric theories. Out of the field approaches are the Schwinger-Dyson equations and effective action treatments of a theory where both stem from the path integral formulation of the problem. In fact, the vacuum to vacuum transition amplitude in path integral formulation can mimic the partition function in statistical systems. Accordingly, when the effective potential is singular or nonanalytic for some coupling values, it is a signature of the Yang-Lee singularity.

To start, let us first introduce the effective action formulation of the problem under consideration. For this, consider the Hamiltonian density operator (for a zero magnetic field) of the $\mathcal{PT}$-symmetric $i\phi^3$ field theory in the form

$$H = \frac{1}{2}((\nabla\phi(x))^2 + \pi^2(x)) + \frac{1}{2}m^2\phi^2(x) + i\frac{g}{3}\phi^3, \quad (2)$$

where m is the mass parameter and g is the coupling constant. Here $\pi = \dot{\phi}$ is the conjugate momentum field. The corresponding Lagrangian density is then

$$\mathcal{L}[\phi] = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2(x) - i\frac{g}{3}\phi^3. \quad (3)$$

The generating functional associated with this Lagrangian is given by the relation [36]

$$Z(J) = \int D\phi \exp\left(i\int d^dx(\mathcal{L}[\phi] + J\phi)\right), \quad (4)$$

where J represents an external source. In introducing the energy functional $E(J)$ such that $Z(J) = \exp(-iE[J])$, the effective action $\Gamma(v)$ can be obtained by the Legendre transform,

$$\Gamma(v) = -E[J] - \int d^4yJ(y)v(y),$$

where v is the vacuum expectation value of the field ϕ. If the vacuum is translational invariant, one can introduce the effective potential $V_{\mathrm{eff}} = -\frac{\Gamma(v)}{VT}$, where VT is the volume of the space-time region over which the functional integral is to be carried out. Since

$$\frac{\delta\Gamma(v)}{\delta(v)} = -J,$$

in the presence of no external source J the effective action satisfies the relation $\frac{\delta\Gamma(v)}{\delta(v)} = 0$, which leads to the constraint $\frac{\partial V_{\mathrm{eff}}}{\partial v} = 0$ on the effective potential. Following the steps in Ref. [36], one expands the effective action around v and makes truncation at the one-loop order of approximation to get

$$\Gamma(v) = \int d^dx\mathcal{L}(v) + \frac{i}{2}\log\det\left(\frac{\partial^2\mathcal{L}(\phi)}{\partial\phi^2}\right).$$

We also can show that

$$\log\det\left(\frac{\partial^2\mathcal{L}(\phi)}{\partial\phi^2}\right) = \mathrm{Tr}\log\left(\frac{\partial^2\mathcal{L}(\phi)}{\partial\phi^2}\right) = -VT\left(i\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\left(\frac{\partial^2V(\phi)}{\partial\phi^2}\right)^{\frac{d}{2}}\right)_{\phi=v}. \quad (5)$$

$V(\phi)$ is the classical potential of the theory, and d is the dimension of the space-time. For a classical potential $V(\phi)$ the effective potential up to one-loop of approximation takes the form

$$V_{\mathrm{eff}}(v) = \left[V(\phi) + \frac{i}{2}\left(i\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\left(\frac{\partial^2V(\phi)}{\partial\phi^2}\right)^{\frac{d}{2}}\right)\right]_{\phi=v}. \quad (6)$$

Before we apply this formula to the $\mathcal{PT}$-symmetric $i\phi^3$ theory, we need to test its accuracy by first trying to obtain results that exist in the literature. For instance, the $\mathcal{PT}$-symmetric ϕ^4 theory has been studied in Ref. [21] using the Schwinger-Dyson equations. Let us consider the same theory within the effective action formalism. The one-loop effective potential is then

$$V_{\mathrm{eff}}(v) = \frac{1}{2}m^2v^2 - \frac{1}{4}gv^4 + \left(i\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}(M^2)^{\frac{d}{2}}\right),$$

where $M^2 = [\frac{\partial^2V(\phi)}{\partial\phi^2}]_{\phi=v}$ represents the renormalized mass. Besides this relation, the effective potential has to satisfy the relation $\frac{\partial V_{\mathrm{eff}}}{\partial v} = 0$ too, which for $d = 1$ gives

$$-\frac{1}{2}v\frac{-2m^2M + 2gv^2M + 3g}{M} = 0.$$

If $v \neq 0$, it can be written as

$$m^2 - gv^2 - \frac{3}{2M}g = 0,$$

which is exactly Eq. (39) in Ref. [21], taking into account that the two-point function there is given by $G_2(0) = \frac{1}{2M}$. Moreover, the mass renormalization condition of the form $M^2 = [\frac{\partial^2V(\phi)}{\partial\phi^2}]_{\phi=v}$ leads to the equation

$$m^2 - 3gv^2 - \frac{9}{2M^3}g^2v^2 - \frac{3}{2M}g = M^2.$$

Again when keeping terms linear in g only, we get exactly Eq. (41) in Ref. [21]. For more tests of the accuracy of the effective potential method, we consider the massless $\mathcal{PT}$-symmetric $i\phi^3$ theory in $0 + 1$ space-time dimensions. In this case, the one-loop effective potential takes the form

$$V_{\mathrm{eff}}(v) = \frac{1}{3}igv^3 + \left(i\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}(M^2)^{\frac{d}{2}}\right). \quad (7)$$

When applying the conditions $\frac{\partial V_{\mathrm{eff}}}{\partial v} = 0$ and $M^2 = [\frac{\partial^2V(\phi)}{\partial\phi^2}]_{\phi=v}$, we get the following equations:

$$\frac{1}{4\pi}\frac{i\sqrt{\pi}g\sqrt{4\pi} + 4i\pi v^2g\sqrt{M^2}}{\sqrt{M^2}} = 0, \quad (8)$$

$$\frac{1}{4\pi M^4}\left(8i\pi M^4 vg + \sqrt{\pi}g^2\sqrt{4\pi}\sqrt{M^2}\right) = M^2. \quad (9)$$

For $g = 1$, we obtain the result $v = -0.63538i$ compared to its exact value $v = -0.64058i$ from Ref. [33]. When taking $g = \frac{3}{2}$, we get $2V_{\mathrm{eff}} = 1.2555$ compared to the exact value of 1.1562 from Ref. [9] and the WKB result of 1.0942 from the same reference. The massless theory is a critical one and quantities of the same mass dimension have the same behavior. For instance, vacuum energy has the same dimension of mass, while the coupling g has a mass dimension of $\frac{5}{2}$. Accordingly, one expects that the vacuum energy behaves as $V_{\mathrm{eff}} \propto g^{\frac{2}{5}}$. Our calculations proved this interesting result of the power-law behavior of the vacuum energy for the massless case where we obtained the result $V_{\mathrm{eff}} = E_0 = 0.53376g^{\frac{2}{5}}$. This result has been obtained in Ref. [24] as a strong coupling limit which is equivalent to setting $m = 0$. According to these results, the one-loop effective potential appears to be a reliable technique to predict physical amplitudes in $\mathcal{PT}$-symmetric theories. However, the effective potential has a property that makes it preferred over the other techniques. In fact, the shape of the effective potential can reflect the vacuum stability as the conditions $\frac{\partial V_{\mathrm{eff}}}{\partial v} = 0$ and $M^2 = [\frac{\partial^2 V(\phi)}{\partial \phi^2}]_{\phi=v}$ associated with it lead to an effective potential bounded from below. These conditions can be considered as the reflections of the quantization condition $\psi(x) \to 0$ as $|x| \to 0$, used in $\mathcal{PT}$-symmetric quantum mechanics, in the effective action formulation. The point is that a potential bounded from below leads to a localized wave function. Accordingly, the bounded-from-below effective potential and the condition $\psi(x) \to 0$ as $|x| \to 0$ are two sides of the same coin.

To show vacuum stability, we consider the one-loop effective potential for $\mathcal{PT}$-symmetric massless $i\phi^3$ that takes the form

$$V_{\mathrm{eff}} = \frac{1}{3}igv^3 + \frac{1}{2}M. \quad (10)$$

In applying the condition $\frac{\partial V_{\mathrm{eff}}}{\partial v} = 0$, we get

$$M = \sqrt{\frac{1}{4v^4}}, \quad (11)$$

which leads to the result

$$V_{\mathrm{eff}} = \frac{1}{3}igv^3 + \frac{1}{4}\sqrt{\frac{1}{v^4}}. \quad (12)$$

This form represents an effective potential bounded from below for a negative imaginary condensate for positive g.

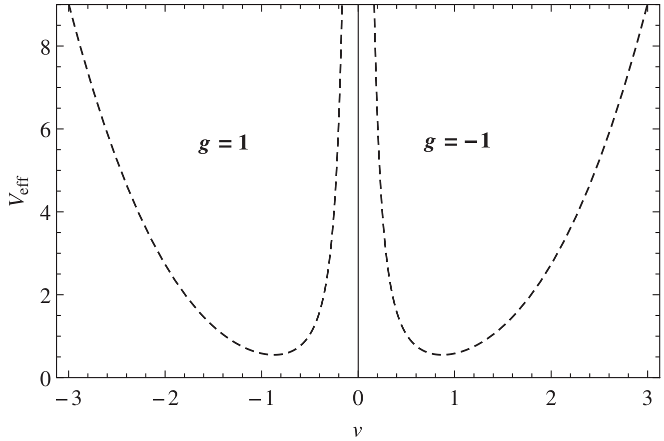

FIG. 1. The effective potential versus vacuum condensate (v/i) for the massless $\mathcal{PT}$-symmetric $i\phi^3$ theory. The left part for $g = 1$ gives a stable vacuum for a negative imaginary condensate, while the right part represents the effective potential for $g = -1$ and is stable for a positive imaginary condensate.

For negative g, however, the vacuum is stable but for a positive imaginary condensate as shown in Fig. 1. In this figure the massless $\mathcal{PT}$-symmetric $i\phi^3$ is shown to have only one stable vacuum using the effective potential method above. Moreover, the vacuum condensate has been shown to be negative imaginary (for positive g) as listed in the literature using WKB and other methods. In fact, the number of available real spectra (equivalently, the number of stable vacua with real energy) has been conjectured in Ref. [30] to be equal to the number of noncontiguous $\mathcal{PT}$-symmetric pairs of Stokes wedges in the complex ϕ-plane. The Stokes wedges in the quantum field approach are generated from considering the controlling factor $\int_\Gamma d\phi \exp(-V(\phi))$ [30]. Here Γ is a contour in a complex plane and $V(\phi)$ is the classical potential in the theory under consideration. For $\mathcal{PT}$-symmetric $i\phi^3$ (see Fig. 2), we do

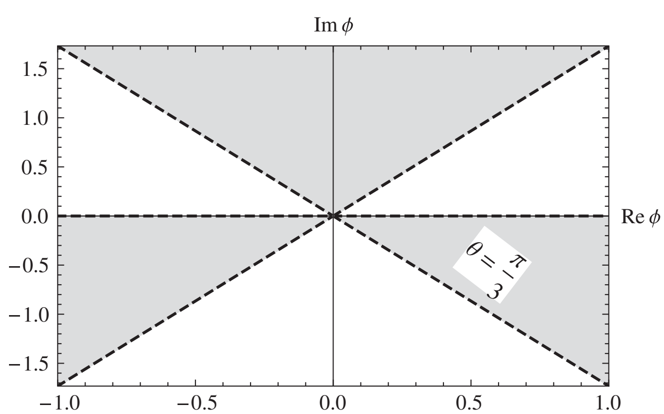

FIG. 2. The Stokes wedges for the massless $\mathcal{PT}$-symmetric $i\phi^3$ theory. There exists only one pair of noncontiguous $\mathcal{PT}$-symmetric wedges, and thus according to the conjecture in Ref. [30] the theory possesses only one stable vacuum.

have only one pair, and thus the effective potential plotted in Fig. 1, which has one stable vacuum, agrees with that conjecture.

The effective potential of the massive $\mathcal{PT}$-symmetric $i\phi^3$ in $0+1$ space-time dimensions can be obtained from the one-loop formula given by

$$V_{\rm eff}(v) = \left[V(\phi) + \frac{i}{2}\left(i\frac{\Gamma(-\frac{1}{2})}{(4\pi)^{\frac{1}{2}}}\left(\frac{\partial^2 V(\phi)}{\partial\phi^2}\right)^{\frac{d}{2}}\right)\right]_{\phi=v}. \quad (13)$$

The condition $\frac{\partial V_{\rm eff}}{\partial v} = 0$ leads to the gap equation

$$m^2 v + i v^2 g + \frac{1}{2}\frac{i}{M} g = 0. \quad (14)$$

This equation is exactly a Schwinger-Dyson equation for the theory up to cutting the series at the two-point function contribution. Note that in $6-\epsilon$, our work leads to a gap equation that can be compared with the one obtained in Ref. [27] using mean field approach where our result include an extra term coming from the two-point function contribution and thus more accurate than the mean field result. The gap equation in $0+1$ space-time dimensions above predicts two vacua since it has the solutions;

$$v = \frac{i}{g}\left(\pm\frac{1}{2}\sqrt{\frac{1}{M}(Mm^4 + 2g^2)} + \frac{1}{2}m^2\right). \quad (15)$$

The existence of two vacua has been obtained in Ref. [27] for $6-\epsilon$ space-time dimensions but is always overlooked in $0+1$ dimensions in the literature. One can realize from the v solutions above that the two vacua have nonzero vacuum expectation values. Thus, it would be interesting to show them in the shape of the effective potential. In fact, the one-loop effective potential predicts two nondegenerate stable vacua as shown in Fig. 3. Note that, for positive g, the effective potential is stable for a negative imaginary condensate, while the opposite is correct for negative g. This result is new for the field theoretic predictions in $0+1$ dimensions, but does this agree with the analysis of the associated Stokes wedges shown above? The answer is yes but with careful analysis of the Stokes wedges structure. For the massive $\mathcal{PT}$-symmetric $i\phi^3$, the classical potential $V(\phi)$ has two couplings, and thus the Stokes wedges have what we can call fine structure. The integral $\int_\Gamma d\phi \exp(-V(\phi))$ does exist if $\mathrm{Re}(V(\phi)) \to \infty$ as $|\phi| \to \infty$. This can happen in two ways (or modes). The first possibility is that the real part of the mass term and that of the interacting part are both positive (the dark gray wedges in Fig. 4). The other possibility is that the dominant interacting term has a positive real part too, while the mass term has a negative real part (the light gray wedges in Fig. 4). This structure suggests the existence of two stable vacua (equivalent to the existence of two spectra) for the massive case. Thus, again the effective potential prediction

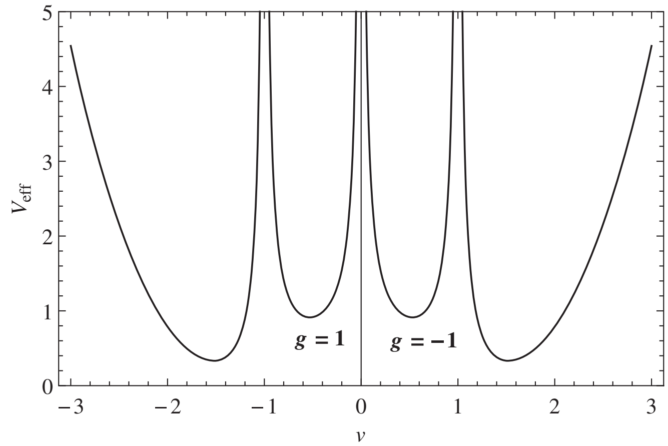

FIG. 3. The effective potential versus vacuum condensate for the massive $\mathcal{PT}$-symmetric $i\phi^3$ theory. The left part for $g=1$ gives two stable vacua for a negative imaginary condensate, while the right part represents the effective potential for $g=-1$ with stable vacua for a positive imaginary condensate.

in the massive case agrees with the conjecture in Ref. [30]. Note that in $0+1$ space-time dimensions there are no phase transitions since there exists one degree of freedom, but it does exist in higher dimensions where the thermodynamic limit is satisfied.

In $0+1$ space-time dimensions, one can still investigate a critical phenomenon of the theory. To see this, we recall that the zeros of the partition function (Yang-Lee edge singularity) for Ising models are known to exist on the imaginary magnetic fled axis [28]. The approach we use can mimic the partition function by considering the vacuum-to-vacuum transition amplitude $Z(J) = \exp(-iE[J])$. The zeros of the partition function, or equivalently, a singularity

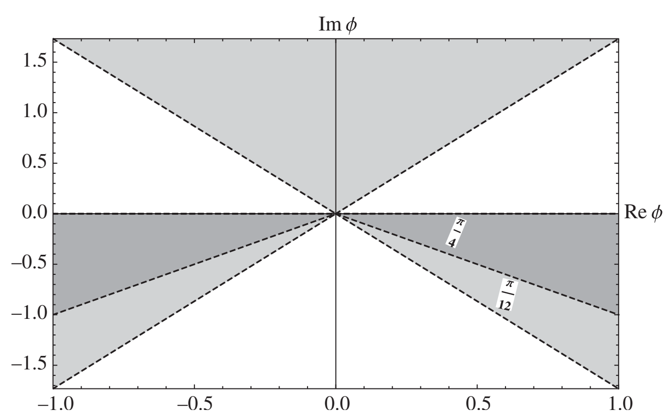

FIG. 4. The Stokes wedges for the massive $\mathcal{PT}$-symmetric $i\phi^3$ theory. The one pair of noncontiguous $\mathcal{PT}$-symmetric wedges for the massless case is now divided into two pairs. The dark gray wedges having an opening angle of $\frac{\pi}{4}$ represent the case where real parts of both terms in the potential are positive. The light gray wedges in the lower half-plane having an opening angle of $\frac{\pi}{12}$ represent the case of a positive real part of the dominant term in the potential, while the massive term has a negative real part.

or nonanalyticity of the vacuum energy, are a manifestation of critical points at which models in the same class of universality behave in a similar way. Accordingly, the $\mathcal{PT}$-symmetric $i\phi^3$ with an external magnetic field has to have a critical point at the imaginary magnetic field axis. To relate the $\mathcal{PT}$-symmetry breaking to the Yang-Lee edge singularity, consider the Hamiltonian form

$$H = \frac{1}{2}((\nabla\phi(x))^2 + \pi^2(x)) + i\frac{\phi^3}{3} + i\gamma\phi, \tag{16}$$

where $i\gamma$ mimics the magnetic field in spin systems. For that form, the one-loop effective potential takes the form

$$V_{\text{eff}}(v) = \frac{1}{2}m^2v^2 - \frac{g}{\alpha}(iv)^\alpha + i\gamma v + \frac{i}{2}\left(i\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}(M^2)^{\frac{d}{2}}\right), \tag{17}$$

where $V_{\text{eff}}(v)$ is subjected to the constraints $\frac{\partial V_{\text{eff}}}{\partial v} = 0$ and $\frac{\partial^2 V_{\text{eff}}}{\partial v^2} = M^2$. The vacuum energy as a function of the coupling γ is shown in Fig. 5. As expected from the critical behavior of Ising models where the zeros of the partition function exist for imaginary magnetic fields, the model under consideration behaves very similarly. In fact the zeros of the partition function are a manifestation of level crossing since we have the relation

$$\langle 0|0\rangle_J = Z(J) = \int D\phi \exp\left(i\int d^dx(\mathcal{L}[\phi] + J\phi)\right), \tag{18}$$

where $\langle 0|0\rangle_J$ is the vacuum-vacuum transition amplitude. Since $Z(J) = \exp(-iE[J])$ and the energy functional is related to the effective action, a zero of the partition function will be reflected in a singularity or nonanalyticity of the effective potential, which is equivalent to self orthogonality of the vacuum state. At the critical point, a singularity in the vacuum energy represents a zero of the partition function. In Ref. [26], the same result has been obtained where self-orthogonality has been verified at the critical coupling. In other words, the Yang-Lee edge singularity is equivalent to $\mathcal{PT}$-symmetry breaking.

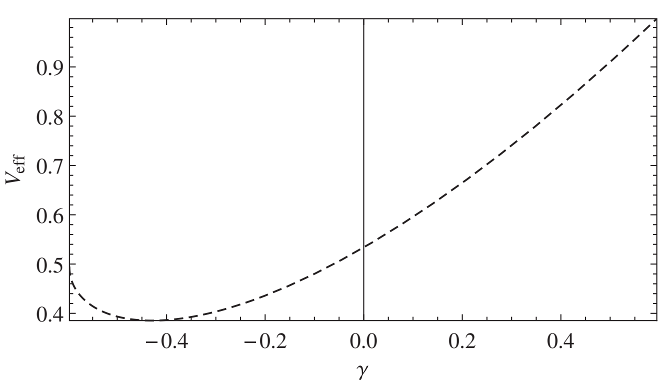

FIG. 5. The effective potential versus the coupling γ for the $\mathcal{PT}$-symmetric $i\phi^3$ theory. For coupling values smaller than $\gamma = \gamma_c = -0.57435$, the vacuum energy as well as the vacuum condensate are complex and the $\mathcal{PT}$-symmetry has been spontaneously broken, which resembles a Yang-Lee edge singularity.

To conclude, we used the effective action formalism to study the $\mathcal{PT}$-symmetric $i\phi^3$ for both massive and massless cases as well as for zero and nonzero external magnetic fields. As a path integral technique, the effective action implements the metric operator, and also the spectrum stability can easily be deduced from the shape of the associated effective potential. To test its accuracy, we employed the effective action for the study of the $\mathcal{PT}$-symmetric $(-\phi^4)$, of which one can find the results in the literature. We were able to exactly reproduce the Dyson-Schwinger prediction for this theory. Moreover, our result for the vacuum condensate of the massless $\mathcal{PT}$-symmetric $i\phi^3$ is very close to its exact result, while the vacuum energy has been shown to be more accurate than the WKB result from the literature.

The effective potential of the massless $\mathcal{PT}$-symmetric $i\phi^3$ has been obtained and plotted. We showed that it is bounded from below, which is an interesting result as it represents the first analytic result that shows the vacuum stability for this theory. Moreover, we found that the effective potential is stable only for a negative imaginary condensate, which agrees well with results found in the literature. We have found only one minimum for the effective potential. Accordingly, our calculations agree well with the conjecture from Ref. [30], where the generation of Stokes wedges of the theory shows only one pair of noncontiguous $\mathcal{PT}$-symmetric wedges. According to that conjecture, the existence of one pair means the existence of one real spectrum (equivalently, one stable vacuum).

For the massive $\mathcal{PT}$-symmetric $i\phi^3$, the effective potential showed two different shapes where each shape has a minimum of energy that differs from the other. This means that the massive case has two stable vacua. To link this result to the conjecture from Ref. [30], we realized that the Stokes wedges now divided into two regions that represent two modes (two vacua). The first mode is where $\exp(-V(\phi))$ goes to zero as $|\phi| \to \infty$ while the real parts of the two terms in the potential are positive (dark gray wedges in Fig. 4). There is another mode (light gray wedges in the lower half of the complex ϕ-plane in Fig. 4) where the mass term real part is negative, while the interacting term has a positive real part. With this analysis of the Stokes wedges of the massive case, the effective potential prediction agrees again with the conjecture in Ref. [30].

For nonzero external magnetic fields, we found that the $\mathcal{PT}$-symmetry is broken for a negative imaginary magnetic field. Since the Ising model and the theory under consideration are in the same class of universality, this result agrees well with the Yang-Lee theory for magnetic systems regarding the existence of zeros of the partition function at imaginary magnetic field values. In $0+1$ space-time dimensions the critical exponent δ is known [28], and one can extract it from the effective field calculation. However, for this case, the effective coupling $\frac{g}{M^{\frac{5}{2}}}$ blows up at the critical point, and the theory is highly nonperturbative. Accordingly, the one-loop calculations cannot lead to an accurate calculation of the critical exponent. This type of calculation needs higher-order calculations followed by a resummation technique (Borel for instance), which is our aim in another work.

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