PHYSICAL REVIEW D **96,** 025008 (2017)

Partition function of $\mathcal{N}=2$ supersymmetric gauge theory and two-dimensional Yang-Mills theory

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(Received 13 October 2016; published 13 July 2017)

We study four-dimensional $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with $2N$ fundamental hypermultiplets in the self-dual Ω-background. The partition function simplifies at special points of the parameter space and is related to the partition function of two-dimensional Yang-Mills theory on S^2. We also consider the insertion of a Wilson loop operator in two-dimensional Yang-Mills theory and find the corresponding operator in the four-dimensional $\mathcal{N}=2$ gauge theory.

DOI: 10.1103/PhysRevD.96.025008

I. INTRODUCTION

$\mathcal{N}=2$ supersymmetry in four dimensions imposes powerful constraints on the low energy behavior of supersymmetric theories. All terms with at most two derivatives and four fermions in the Wilsonian effective action are expressed in terms of a single holomorphic quantity, the prepotential $\mathcal{F}$, whose quantum corrections are one-loop exact in the perturbation theory, and generated nonperturbatively only by instantons. The exact form of the prepotential $\mathcal{F}$ was first determined for certain theories by Seiberg and Witten indirectly based on several assumptions on the strong coupling behavior of the theory [1,2]. It was then extended to more general $\mathcal{N}=2$ theories (see [3] for a recent review).

It is useful to deform the supersymmetric theories by putting them on nontrivial supergravity backgrounds [4,5]. The prototypical example is the so-called Ω-background [4], in which the theory is deformed by two parameters ϵ_1, ϵ_2 parametrizing an $SO(4)$ rotation of $\mathbb{R}^4$. The Ω-deformation provides an IR regularization that preserves a part of the deformed supersymmetry. The calculation of the supersymmetric partition function is dramatically simplified and can be performed using equivariant localization techniques. The dependence of the partition function on the parameters ϵ_1, ϵ_2 contains profound physical information. In particular, it gives the prepotential of the low energy effective action of the undeformed theory on $\mathbb{R}^4$, as well as the couplings of the theory to the $\mathcal{N}=2$ supergravity multiplet.

Soon after the exact computation of the partition function in the Ω-background was done, an interesting relation between supersymmetric gauge theory and topological string theory was discovered [6,7]. On the gauge theory side, we have the four-dimensional $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with $2N-2$ fundamental hypermultiplets. Its partition function in the self-dual Ω-background simplifies dramatically at a special point of the parameter space and is identified with the disconnected partition function of A-type topological string theory on S^2. The higher Casimir operators in the four-dimensional gauge theory map to gravitational descendants of the Kähler form in the topological string theory. It was later further generalized in [8] by adding g adjoint hypermultiplets in the four-dimensional gauge theory and replacing S^2 with a genus g Riemann surface.

Inspired by the previous results, we explore the possible simplification of the partition function of the four-dimensional $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with $2N$ fundamental hypermultiplets in this paper. We find that the partition function in the self-dual Ω-background at a special point of the parameter space can be related to the partition function of two-dimensional Yang-Mills theory on S^2 [9,10]. The rank of the gauge group of the two-dimensional theory has nothing to do with the four-dimensional gauge group $U(N)$.

Once the correspondence is established, one may study each side using the information of the other side. In this paper, we consider the Wilson loop operator in the two-dimensional Yang-Mills theory. The exact expectation value of the Wilson loop operator has been known for a long time. We show that inserting a Wilson loop operator in the fundamental representation corresponds to adding a nontrivial operator in the four-dimensional $\mathcal{N}=2$ gauge theory. The generalization to other representations is more involved and will be discussed in the future.

The structure of this paper is as follows. In Sec. II, we review the partition function of four-dimensional $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with $2N$ fundamental hypermultiplets in the Ω-background, and describe the $\mathcal{Y}$-observable that will turn out to be useful in our discussion. We show that the partition function simplifies at special points of the parameter space. In Sec. III, we show that the simplified partition function can be related to the partition function of two-dimensional Yang-Mills

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theory on S^2. We then study the effect of inserting a Wilson loop operator in the two-dimensional Yang-Mills theory. Finally, in Sec. IV, we provide some further discussions.

II. INSTANTON PARTITION FUNCTION OF FOUR-DIMENSIONAL $\mathcal{N}=2$ GAUGE THEORY

In this paper, we are interested in the $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with $2N$ fundamental hypermultiplets. The Lagrangian and the vacua are parametrized by the coupling constant $q=\exp(2\pi i\tau)$; the vacuum expectation value $\mathbf{a}=\mathrm{diag}(a_1,\ldots,a_N)$ of the scalar field in the vector multiplet; and the complex masses $\mathbf{m}=\mathrm{diag}(m_1,\ldots,m_{2N})$ of the matter hypermultiplets. We refer to [11] for a detailed analysis and references for the supersymmetric partition function of very general $\mathcal{N}=2$ supersymmetric gauge theories in the Ω-background.

A. Partition function in the self-dual Ω-background

Let us first recall the partition function of the four-dimensional $\mathcal{N}=2$ gauge theory in the Ω-background [4]. The Ω-background breaks the translational invariance by deforming the theory in a rotationally covariant way, with parameters ϵ_1, ϵ_2. In the following, we always set $\epsilon_1=-\hbar$, $\epsilon_2=\hbar$.

The supersymmetric partition function of $\mathcal{N}=2$ theory consists of three parts: the classical, the one-loop, and the instanton parts,

$$Z(\mathbf{a},\mathbf{m},q;\hbar)=Z^{\mathrm{classical}}(\mathbf{a},q;\hbar)Z^{1-\mathrm{loop}}(\mathbf{a},\mathbf{m};\hbar)\times Z^{\mathrm{instanton}}(\mathbf{a},\mathbf{m},q;\hbar). \tag{1}$$

The classical part is simply

$$Z^{\mathrm{classical}}(\mathbf{a},q;\hbar)=q^{\frac{1}{2\hbar^2}\sum_{\alpha=1}^{N}a_\alpha^2}. \tag{2}$$

The one-loop part is given as a product of contributions from the vector multiplet and the matter hypermultiplets using the Barnes double gamma function. The one-loop contribution of a vector multiplet is

$$Z^{1-\mathrm{loop}}_{\mathrm{vector}}(\mathbf{a};\hbar)=\prod_{1\le i<j\le N}[\Gamma_2(a_i-a_j+\hbar|\hbar,-\hbar)\times\Gamma_2(a_i-a_j-\hbar|\hbar,-\hbar)]^{-1}, \tag{3}$$

while the one-loop contribution of fundamental hypermultiplets is

$$Z^{1-\mathrm{loop}}_{\mathrm{fund}}(\mathbf{a},m;\hbar)=\prod_{i=1}^{N}\prod_{f=1}^{2N}\Gamma_2(a_i-m_f|\hbar,-\hbar). \tag{4}$$

The instanton partition function is defined as an equivariant integral over the instanton moduli space. Applying the equivariant localization method, the integral can be reduced to a sum over contributions of the fixed points of the moduli space. There is a one-to-one correspondence between the fixed points and colored partitions $\Lambda=(\lambda^{(\alpha)})_{\alpha=1}^N$, with each partition $\lambda^{(\alpha)}$ being a weakly decreasing sequence of nonnegative integers,

$$\lambda^{(\alpha)}=(\lambda_1^{(\alpha)}\ge\lambda_2^{(\alpha)}\ge\cdots\ge\lambda^{(\alpha)}_{\ell(\lambda^{(\alpha)})}>\lambda^{(\alpha)}_{\ell(\lambda^{(\alpha)})+1}=\cdots=0), \tag{5}$$

whose size is denoted to be $|\lambda^{(\alpha)}|=\sum_i\lambda_i^{(\alpha)}$. Accordingly the instanton partition function becomes a statistical model of random partitions [4],

$$Z^{\mathrm{instanton}}(\mathbf{a},\mathbf{m},q;\hbar)=\sum_{\Lambda}q^{|\Lambda|}\mu_\Lambda(\mathbf{a},\mathbf{m};\hbar), \tag{6}$$

where $|\Lambda|=\sum_{\alpha=1}^N|\lambda^{(\alpha)}|$. The contribution to the measure of a vector multiplet is given by

$$\mu_{\Lambda\mathrm{vector}}(\mathbf{a};\hbar)=\prod_{(\alpha,i)\neq(\beta,j)}\frac{a_\alpha-a_\beta+\hbar(\lambda_i^{(\alpha)}-\lambda_j^{(\beta)}+j-i)}{a_\alpha-a_\beta+\hbar(j-i)}, \tag{7}$$

and the contribution to the measure of fundamental hypermultiplets is

$$\mu_{\Lambda\mathrm{fund}}(\mathbf{a},\mathbf{m};\hbar)=\prod_{\alpha=1}^{N}\prod_{f=1}^{2N}\prod_{\square\in\lambda^{(\alpha)}}(c_\square-m_f)=\hbar^{2N|\Lambda|}\prod_{\alpha=1}^{N}\prod_{f=1}^{2N}\prod_{i}\frac{\Gamma(\frac{a_\alpha-m_f}{\hbar}+1+\lambda_i^{(\alpha)}-i)}{\Gamma(\frac{a_\alpha-m_f}{\hbar}+1-i)}, \tag{8}$$

where for each box $\square=(i,j)\in\lambda^{(\alpha)}$, we define its content as

$$c_\square=a_\alpha+\epsilon_1(i-1)+\epsilon_2(j-1). \tag{9}$$

The contribution to the measure of an antifundamental hypermultiplet with mass m is equal to the contribution to the measure of a fundamental hypermultiplet with mass $-m$ in the self-dual Ω-background.

For the undeformed theory on $\mathbb{R}^4$, we can perturb the theory by adding gauge-invariant chiral operators to the ultraviolet prepotential, while keeping the ultraviolet antiprepotential unchanged,

$$\bar{\mathcal{F}}^{\mathrm{UV}}=\frac{\bar{\tau}}{2}\mathrm{Tr}\bar{\Phi}^2. \tag{10}$$

For example, we can add single-trace operators,

$$\mathcal{F}^{\mathrm{UV}} \to \frac{\tau}{2}\mathrm{Tr}\Phi^2 + \sum_{j=2}^{\infty}\frac{\tau_j}{j}\mathrm{Tr}\Phi^j, \tag{11}$$

which get deformed in the Ω-background. The localization computation still works, and the partition function becomes

$$\begin{aligned} Z(\mathbf{a},\mathbf{m},q;\tau;\hbar) &= Z^{\mathrm{classical}}(\mathbf{a},q;\hbar)Z^{1-\mathrm{loop}}(\mathbf{a},\mathbf{m};\hbar) \\ &\times \sum_{\Lambda} q^{|\Lambda|}\mu_\Lambda(\mathbf{a},\mathbf{m};\hbar) \\ &\times \exp\left(\frac{1}{\hbar^2}\sum_{j=2}^{\infty}\frac{\tau_j}{j}\mathrm{ch}_j(\mathbf{a},\Lambda)\right). \end{aligned} \tag{12}$$

Here $\mathrm{ch}_j(\mathbf{a},\Lambda) = \sum_{\alpha=1}^{N}\mathrm{ch}_j(a_\alpha,\lambda^{(\alpha)})$, with

$$\begin{aligned} \mathrm{ch}_j(a,\lambda) = a^j + \sum_{i=1}^{\infty}&((a+\hbar(\lambda_i+1-i))^j \\ &- (a+\hbar(\lambda_i-i))^j - (a+\hbar(1-i))^j \\ &+ (a-\hbar i)^j). \end{aligned} \tag{13}$$

For example,

$$\mathrm{ch}_2(a,\lambda) = a^2 + 2\hbar^2|\lambda|, \tag{14}$$

$$\mathrm{ch}_3(a,\lambda) = a^3 + 6\hbar^2 a|\lambda| + 3\hbar^3\sum_i \lambda_i(\lambda_i+1-2i). \tag{15}$$

Multitrace operators can also be added and can be analyzed using the Hubbard-Stratonovich transformation. The full set of gauge-invariant chiral operators can be expressed as

$$\begin{aligned} &\mathcal{F}^{\mathrm{UV}} \to \frac{\tau}{2}\mathrm{Tr}\Phi^2 + \sum_{\vec{k}}^{\infty} t_{\vec{k}}\prod_{j=1}^{\infty}\frac{1}{k_j!}\left(\frac{1}{j}\mathrm{Tr}\Phi^j\right)^{k_j}, \\ &\vec{k} = (k_1,k_2,\ldots), \end{aligned} \tag{16}$$

and the partition function is deformed to be

$$\begin{aligned} Z(\mathbf{a},\mathbf{m},q;\mathbf{t};\hbar) &= Z^{\mathrm{classical}}(\mathbf{a},q;\hbar)Z^{1-\mathrm{loop}}(\mathbf{a},\mathbf{m};\hbar) \\ &\times \sum_{\Lambda} q^{|\Lambda|}\mu_\Lambda(\mathbf{a},\mathbf{m};\hbar) \\ &\times \exp\left(\frac{1}{\hbar^2}\sum_{\vec{k}}^{\infty} t_{\vec{k}}\prod_{j=1}^{\infty}\frac{1}{k_j!}\left(\frac{1}{j}\mathrm{ch}_j(a,\lambda)\right)^{k_j}\right). \end{aligned} \tag{17}$$

B. $\mathcal{Y}$-observable

With the identification of the instanton partition function with a statistical model (6), we can compute the expectation value of observables in the Ω-background as

$$\langle \mathcal{O} \rangle = \frac{\sum_\Lambda q^{|\Lambda|}\mu_\Lambda \mathcal{O}[\Lambda]}{\sum_\Lambda q^{|\Lambda|}\mu_\Lambda}, \tag{18}$$

where $\mathcal{O}[\Lambda]$ is the value of $\mathcal{O}$ at the fixed point labeled by Λ.

An important observable in the analysis of nonperturbative information of four-dimensional $\mathcal{N}=2$ gauge theory is the $\mathcal{Y}$-observable, which is defined using the gauge-invariant polynomials of the adjoint scalar field ϕ in the vector multiplet, evaluated at the fixed point of the rotational symmetry $SO(4)$,

$$\mathcal{Y}(x) = x^N \exp\left(-\sum_{j=1}^{\infty}\frac{1}{jx^j}\mathrm{Tr}(\phi(0))^j\right). \tag{19}$$

Classically, it is given by

$$\mathcal{Y}(x)^{\mathrm{classical}} = \det(x-\phi(0)) = \prod_{\alpha=1}^{N}(x-a_\alpha). \tag{20}$$

However, there are quantum corrections due to instantons. Denote the outer and the inner boundaries of the partition λ as $\partial_+\lambda$ and $\partial_-\lambda$, respectively. The value of $\mathcal{Y}(x)$ in the self-dual Ω-background at the fixed point labeled by Λ is [12]

$$\begin{aligned} \mathcal{Y}(x)[\Lambda] &= \prod_{\alpha=1}^{N}\frac{\prod_{\boxplus\in\partial_+\lambda^{(\alpha)}}(x-c_\boxplus)}{\prod_{\boxminus\in\partial_-\lambda^{(\alpha)}}(x-c_\boxminus)} \\ &= \prod_{\alpha=1}^{N}\prod_{i}^{\infty}\frac{x-a_\alpha-\hbar(\lambda_i^{(\alpha)}-i+1)}{x-a_\alpha-\hbar(\lambda_i^{(\alpha)}-i)}. \end{aligned} \tag{21}$$

Notice that the expression (21) is highly redundant, and there can be many cancellations between the numerator and the denominator. For example, the contribution from the box $(n+1,\lambda_{n+1}^{(\alpha)}+1)\in\widetilde{\partial_+\lambda^{(\alpha)}}$ cancels the contribution from the box $(n,\lambda_n^{(\alpha)})\in\widetilde{\partial_-\lambda^{(\alpha)}}$ for $n>\ell(\lambda^{(\alpha)})$. Hence, $\mathcal{Y}(x)[\Lambda]$ does not change if we truncate the range of the index i to $1\le i\le n$ for an arbitrary integer $n\ge\ell(\lambda^{(\alpha)})$.

C. Simplification of partition function

Up to this point we assumed that the expectation values $a_1,\ldots,a_N$ and masses $m_1,\ldots,m_{2N}$ are generic. Then the partition function (6) contains an infinite sum over colored partitions. For a special value of the masses, the partitions Λ that we sum over can be constrained. As a result, the partition function (6) gets simplified.

It is easy to see that if $a_\alpha = m_f$ for some $\alpha\in\{1,2,\ldots,N\}$ and $f\in\{1,2,\ldots,2N\}$, then $\lambda^{(\alpha)}=\emptyset$; otherwise (8) is zero. Therefore, if we choose a particular point on the parameter space

$$a_\alpha = m_{2\alpha-1} = m_{2\alpha}, \qquad \alpha = 1,\ldots,N, \tag{22}$$

the partitions $\lambda^{(\alpha)} = \emptyset$ for all $\alpha = 1, 2, \ldots, N$, and the instanton partition function is trivially 1. This simplification of the instanton partition function has been known for a long time. Physically, when one of the a_α's is equal to two masses, two of the hypermultiplets become massless, and can be Higgsed so that the $U(N)$ theory with $2N$ flavors is reduced to a $U(N-1)$ theory with $2N-2$ flavors. However, the instanton partition function will not change since it is a Coulomb-branch quantity which is independent of the manipulation on the hypermultiplet side.

Now let us relax the condition (22) a little bit. We still fix

$$a_\alpha = m_{2\alpha-1} = m_{2\alpha}, \qquad \alpha = 2, \ldots, N, \tag{23}$$

so that the partitions $\lambda^{(\alpha)} = \emptyset$ for $\alpha = 2, \ldots, N$. We effectively reduce the $U(N)$ gauge theory with $2N$ fundamental hypermultiplets to the $U(1)$ theory with two fundamental hypermultiplets. At the same time, we choose

$$a_1 = m_1 + n\hbar = m_2 + n\hbar, \tag{24}$$

where n is a positive integer. We see from (8) that if $\lambda^{(1)}_{n+1} \geq 1$, then the contribution of the box $\square = (n+1, 1) \in \lambda^{(1)}$ makes $\mu_{\Lambda\text{fund}}$ vanish. Hence, the length of the partition $\lambda^{(1)}$ is at most n. We can set the length of the partition $\lambda^{(1)}$ to be n by adding zeros to the end of the partition if its precise length is less than n. In this case, the measure in the instanton partition function simplifies.

The case $n = 1$ is special, since now $\lambda^{(1)}$ is no longer a two-dimensional partition. The measure of the vector multiplet completely cancels the measure of the fundamental hypermultiplets, and the instanton partition function is

$$Z^{\text{instanton}} = \sum_{\lambda_1^{(1)}=0}^{\infty} q^{\lambda_1^{(1)}} = \frac{1}{1-q}. \tag{25}$$

In the following, we always assume that $n \geq 2$. In this case, the measure of the vector multiplet (7) becomes

$$\begin{aligned}
\mu_{\Lambda\text{vector}} &= \left(\prod_{i\neq j} \frac{\hbar(\lambda_i^{(1)} - \lambda_j^{(1)} + j - i)}{\hbar(j-i)}\right)\left(\prod_{\beta=2}^{N}\prod_{i,j} \frac{a_1 - a_\beta + \hbar(\lambda_i^{(1)} + j - i)}{a_1 - a_\beta + \hbar(j-i)}\right)^2 \\
&= \left(\prod_{1\leq i<j\leq n} \frac{\lambda_i^{(1)} - \lambda_j^{(1)} + j - i}{j-i}\right)^2 \left(\prod_{i=1}^{n} \frac{\Gamma(n+1-i)}{\hbar^{\lambda_i^{(1)}}\Gamma(n+1+\lambda_i^{(1)}-i)}\right)^2 \\
&\quad \times \left(\prod_{\beta=2}^{N}\prod_{i=1}^{n} \frac{\Gamma(\frac{a_1-a_\beta}{\hbar} - i + 1)}{\hbar^{\lambda_i^{(1)}}\Gamma(\frac{a_1-a_\beta}{\hbar} - i + \lambda_i^{(1)} + 1)}\right)^2,
\end{aligned} \tag{26}$$

while the measure of the fundamental hypermultiplets (8) becomes

$$\begin{aligned}
\mu_{\Lambda\text{fund}} &= \prod_{f=1}^{2N}\prod_{i=1}^{n} \frac{\Gamma(\frac{a_1-m_f}{\hbar} + 1 + \lambda_i^{(1)} - i)}{\Gamma(\frac{a_1-m_f}{\hbar} + 1 - i)} \\
&= \hbar^{2N|\lambda^{(1)}|}\left(\prod_{i=1}^{n} \frac{\Gamma(n+1+\lambda_i^{(1)}-i)}{\Gamma(n+1-i)}\right)^2 \prod_{\alpha=2}^{N}\left(\prod_{i=1}^{n} \frac{\Gamma(\frac{a_1-a_\alpha}{\hbar} + 1 + \lambda_i^{(1)} - i)}{\Gamma(\frac{a_1-a_\alpha}{\hbar} + 1 - i)}\right)^2.
\end{aligned} \tag{27}$$

After many cancellations between $\mu_{\Lambda\text{vector}}$ and $\mu_{\Lambda\text{fund}}$, the remaining measure is

$$\mu_\Lambda = \mu_{\Lambda\text{vector}}\mu_{\Lambda\text{fund}} = \left(\prod_{1\leq i<j\leq n} \frac{\lambda_i^{(1)} - \lambda_j^{(1)} + j - i}{j-i}\right)^2. \tag{28}$$

In this case, the $\mathcal{Y}$-observable (21) also simplifies,

$$\begin{aligned}
\mathcal{Y}(x)[\Lambda] &= \frac{\prod_{i=1}^{n+1}(x - a_1 - \hbar(\lambda_i^{(1)} + 1 - i))}{\prod_{i=1}^{n}(x - a_1 - \hbar(\lambda_i^{(1)} - i))} \\
&= (x - a_1 + n\hbar)\prod_{i=1}^{n} \frac{(x - a_1 - \hbar(\lambda_i^{(1)} + 1 - i))}{(x - a_1 - \hbar(\lambda_i^{(1)} - i))}.
\end{aligned} \tag{29}$$

As we see, at the point (23)–(24) of the parameter space, the instanton partition function is independent of the gauge group rank N, and the difference for different N values in the full partition function is an overall constant which is irrelevant to our discussion. Therefore, we shall concentrate on the case $N = 1$ in the following discussion and drop some of the subscripts 1. Notice that the $U(1)$ gauge theory with two fundamental hypermultiplets is nontrivial due to the inexplicit noncommutative deformation.

III. RELATION TO TWO-DIMENSIONAL YANG-MILLS THEORY

In this section, we shall relate the partition function discussed in Sec. II to the partition function of two-dimensional Yang-Mills theory on S^2.

A. Partition function of two-dimensional Yang-Mills theory

Two-dimensional Yang-Mills theory is an exactly solvable model and has been extensively studied from many different points of view (see [10] for a review). Its partition function on a Riemann surface Σ of genus g is defined as

$$Z_\Sigma^{\mathrm{YM2}}(\varepsilon,\mathcal{A}(\Sigma),G) = \frac{1}{\mathrm{Vol}(G)}\int \mathcal{D}A\mathcal{D}\phi \exp\left(\mathrm{i}\int_\Sigma \mathrm{Tr}\phi F_A + \frac{\varepsilon}{2}\int_\Sigma d\mu \mathrm{Tr}\phi^2\right), \tag{30}$$

where ε is the coupling constant; $\mathcal{A}(\Sigma)$ is the area of the Riemann surface Σ; and Tr denotes the invariant, negative-definite quadratic form on the Lie algebra $\mathfrak{g}$ of the gauge group G. The partition function (30) can be expressed as a sum over all finite-dimensional irreducible representations R of the gauge group G [9,13,14],

$$Z_\Sigma^{\mathrm{YM2}}(\varrho,G) = e^{-\beta(2-2g)-\gamma\varepsilon\mathcal{A}(\Sigma)}\sum_R (\dim R)^{2-2g} \times \exp\left(-\frac{\varrho}{2}C_2(R)\right), \tag{31}$$

where the prefactor is the regularization-dependent ambiguity, $\dim R$ is the dimension of the representation R, $C_2(R)$ is the quadratic Casimir of the representation R, and $\varrho=\varepsilon\mathcal{A}(\Sigma)$ is the dimensionless coupling constant.

B. Matching the parameters

We would like to find the precise relation between the partition function (17) and the partition function of two-dimensional Yang-Mills theory (31), both for the group $SU(n)$ and for the group $U(n)$.

1. $SU(n)$ theory

For the group $G=SU(n)$, the irreducible representations R are parametrized by the partition $(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{n-1}\geq\lambda_n=0)$. The dimension and the quadratic Casimir of the representation R are

$$\dim R = \prod_{1\leq i<j\leq n}\frac{\lambda_i-\lambda_j+j-i}{j-i}, \tag{32}$$

$$C_2(R) = \sum_{i=1}^n \lambda_i(\lambda_i-2i+1) + n|\lambda| - \frac{|\lambda|^2}{n}. \tag{33}$$

We see that both the dimension and the quadratic Casimir are independent of the overall shift of λ's. Therefore, the difference between the summation over $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{n-1}\geq\lambda_n\geq0$ and $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{n-1}\geq\lambda_n=0$ in the partition function is merely an irrelevant overall constant.

To identify the partition function of two-dimensional $SU(n)$ Yang-Mills theory on S^2 with the partition function of the four-dimensional $\mathcal{N}=2$ $U(1)$ gauge theory with two fundamental hypermultiplets at the degenerate point of the parameter space, we need to set $a=0$ and turn on operators with couplings $t_{0,1}$, $t_{0,2}$, and $t_{0,0,1}$ in (17). The partition function becomes

$$\begin{aligned} Z(a=0,m_1=m_2=-n\hbar,q;\tau;\hbar) &= \Gamma_2(n\hbar|\hbar,-\hbar)^2\sum_\lambda q^{|\lambda|}\left(\prod_{1\leq i<j\leq n}\frac{\lambda_i-\lambda_j+j-i}{j-i}\right)^2 \\ &\quad\times\exp\left\{\frac{1}{\hbar^2}\left[\frac{t_{0,1}}{2}\mathrm{ch}_2(0,\lambda)+\frac{t_{0,2}}{8}(\mathrm{ch}_2(0,\lambda))^2+\frac{t_{0,0,1}}{3}\mathrm{ch}_3(0,\lambda)\right]\right\} \end{aligned} \tag{34}$$

$$\begin{aligned} &= \Gamma_2(n\hbar|\hbar,-\hbar)^2\sum_\lambda q^{|\lambda|}\left(\prod_{1\leq i<j\leq n}\frac{\lambda_i-\lambda_j+j-i}{j-i}\right)^2 \\ &\quad\times\exp\left\{\left[t_{0,1}|\lambda|+\frac{t_{0,2}\hbar^2}{2}|\lambda|^2+t_{0,0,1}\hbar\sum_i\lambda_i(\lambda_i+1-2i)\right]\right\}. \end{aligned} \tag{35}$$

Ignoring the unimportant prefactor coming from the one-loop contribution, the partition function is equal to the partition function of two-dimensional Yang-Mills theory on S^2 (31) with gauge group $SU(n)$ when

$$\log(q)|\lambda| + t_{0,1}|\lambda| + \frac{t_{0,2}\hbar^2}{2}|\lambda|^2 + t_{0,0,1}\hbar\sum_i \lambda_i(\lambda_i + 1 - 2i)$$
$$= -\frac{\varrho}{2}\left(\sum_{i=1}^{n}\lambda_i(\lambda_i - 2i + 1) + n|\lambda| - \frac{|\lambda|^2}{n}\right), \tag{36}$$

which gives

$$t_{0,1} = -\frac{\varrho n}{2} - \log(q), \qquad t_{0,2} = \frac{\varrho}{n\hbar^2}, \qquad t_{0,0,1} = -\frac{\varrho}{2\hbar}. \tag{37}$$

2. $U(n)$ theory

For the group $U(n)$, the irreducible representations $\mathcal{R}$ are parametrized by n integers $(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n)$ without positivity restriction. It is convenient to use the decomposition of the representation $\mathcal{R}$ of $U(n)$ in terms of representation R of $SU(n)$ and the $U(1)$ charge p,

$$\mu_i = \lambda_i + r, \qquad i = 1, 2, \ldots, \qquad n-1$$
$$\mu_n = r,$$
$$p = |\lambda| + nr, \qquad r \in \mathbb{Z}. \tag{38}$$

The dimension of representation $\mathcal{R}$ of group $U(n)$ has the same form (32) as the group $SU(n)$, while the quadratic Casimir is given by

$$C_2(\mathcal{R}) = C_2(R) + \frac{p^2}{n} = \sum_{i=1}^{n}\lambda_i(\lambda_i - 2i + 1) + (n + 2r)|\lambda| + nr^2. \tag{39}$$

To relate the four-dimensional theory to two-dimensional Yang-Mills theory with gauge group $U(n)$, we no longer need to turn on the double-trace operators. Instead, we turn on operators with parameter τ_2 and τ_3 in (12),

$$Z(a, \mathbf{m}, q; \tau; \hbar)$$
$$= \Gamma_2(n\hbar|\hbar, -\hbar)^2 \sum_\lambda \left(\prod_{1\leq i<j\leq n}\frac{\lambda_i - \lambda_j + j - i}{j - i}\right)^2$$
$$\times \exp\left[(\tau_2 + \log(q))\left(\frac{a^2}{2\hbar^2} + |\lambda|\right) + \tau_3\left(\frac{a^3}{3\hbar^2} + 2a|\lambda| + \hbar\sum_i \lambda_i(\lambda_i + 1 - 2i)\right)\right]. \tag{40}$$

We now set

$$a = m_1 + n\hbar = m_2 + n\hbar = r\hbar, \tag{41}$$

where $r \in \mathbb{Z}$. Ignoring the irrelevant prefactor coming from the one-loop contribution, the partition function becomes

$$Z(r\hbar, (r - n)\hbar, q; \tau; \hbar)$$
$$= \sum_\lambda \left(\prod_{1\leq i<j\leq n}\frac{\lambda_i - \lambda_j + j - i}{j - i}\right)^2$$
$$\times \exp\left[(\tau_2 + \log(q))\left(\frac{r^2}{2} + |\lambda|\right) + \tau_3\hbar\left(\frac{r^3}{3} + 2r|\lambda| + \sum_i \lambda_i(\lambda_i + 1 - 2i)\right)\right]. \tag{42}$$

Now we consider the sum over $r \in \mathbb{Z}$ with a possible weight depending on r,

$$\sum_{r\in\mathbb{Z}} \exp(-f_2 r^2 - f_3 r^3) Z(r\hbar, (r - n)\hbar, q; \tau; \hbar)$$
$$= \sum_{r\in\mathbb{Z}}\sum_\lambda \left(\prod_{1\leq i<j\leq n}\frac{\lambda_i - \lambda_j + j - i}{j - i}\right)^2$$
$$\times \exp\left[(\tau_2 + \log(q))\left(\frac{r^2}{2} + |\lambda|\right) + \tau_3\hbar\left(\frac{r^3}{3} + 2r|\lambda| + \sum_i \lambda_i(\lambda_i + 1 - 2i)\right) - f_2 r^2 - f_3 r^3\right], \tag{43}$$

which is equal to the partition function of two-dimensional Yang-Mills theory on S^2 (31) with gauge group $U(n)$ when

$$\tau_3\hbar\sum_i \lambda_i(\lambda_i + 1 - 2i) + (\tau_2 + \log(q))|\lambda| + 2\tau_3\hbar|\lambda|r + \left(\frac{\tau_2 + \log(q)}{2} - f_2\right)r^2 + \left(\frac{\tau_3\hbar}{3} - f_3\right)r^3$$
$$= -\frac{\varrho}{2}\left[\sum_{i=1}^{n}\lambda_i(\lambda_i - 2i + 1) + n|\lambda| + 2r|\lambda| + nr^2\right], \tag{44}$$

which gives that

$$\tau_2 = -\frac{\varrho n}{2} - \log(q), \qquad \tau_3 = -\frac{\varrho}{2\hbar}, \qquad f_2 = \frac{\varrho n}{4}, \qquad f_3 = -\frac{\varrho}{6\hbar}. \tag{45}$$

Therefore, we have the relation

$$\sum_{r\in\mathbb{Z}} \exp\left(-\frac{\varrho n}{4} r^2 + \frac{\varrho}{6\hbar} r^3\right) Z\left(r\hbar, (r-n)\hbar, q; \tau_2 = -\frac{\varrho n}{2} - \log(q), \tau_3 = -\frac{\varrho}{2\hbar}; \hbar\right) = Z_{S^2}^{\mathrm{YM2}}(\varrho, U(n)). \tag{46}$$

C. Wilson loop operator in two-dimensional Yang-Mills theory

The correspondence was hitherto at the level of the partition functions. We would like to deepen it by studying the Wilson loop operator in the two-dimensional Yang-Mills theory.

Suppose that a loop Γ decomposes S^2 into two disjoint connected components Σ_1 and Σ_2. Associated to the curve Γ we have a representation R_Γ of the gauge group and we define a Wilson loop operator

$$W(\Gamma, R_\Gamma) = \mathrm{Tr}_{R_\Gamma} P \exp \oint_\Gamma A. \tag{47}$$

The expectation value of the Wilson loop operator $W(\Gamma, R_\Gamma)$ is given by

$$\begin{aligned}\langle W(\Gamma, R_\Gamma)\rangle^{\mathrm{YM2}} = {}& Z_{S^2}^{\mathrm{YM2}}(\varepsilon\mathcal{A}(\Sigma_1 \cup \Sigma_2))^{-1} \sum_{R_1,R_2} (\dim R_1)(\dim R_2) \\ & \times \exp\left(-\frac{\varepsilon\mathcal{A}(\Sigma_1)}{2} C_2(R_1) - \frac{\varepsilon\mathcal{A}(\Sigma_2)}{2} C_2(R_2)\right) \mathfrak{N}(R_1 \otimes R_\Gamma, R_2),\end{aligned} \tag{48}$$

where $\mathfrak{N}(R_1 \otimes R_\Gamma, R_2)$ is the fusion number defined by the decomposition of a tensor product into irreducible representations:

$$R_1 \otimes R_\Gamma = \bigoplus_{R_2} \mathfrak{N}(R_1 \otimes R_\Gamma, R_2) R_2. \tag{49}$$

In this paper, we are interested in the simple case that R_Γ is the fundamental representation. The fusion number is 1 if the Young diagram associated to R_2 is obtained by adding a box in the Young diagram associated to R_1, and 0 otherwise. We can make an analogy with (18) and write

$$\langle W(\Gamma, \Box)\rangle^{\mathrm{YM2}} = Z_{S^2}^{\mathrm{YM2}}(\varepsilon\mathcal{A}(\Sigma_1 \cup \Sigma_2))^{-1} \sum_R (\dim R)^2 \exp\left(-\frac{\varepsilon\mathcal{A}(\Sigma_1\cup\Sigma_2)}{2} C_2(R)\right) W(\Gamma, \Box)[R]. \tag{50}$$

Here $W(\Gamma, \Box)[R]$ is the value of $W(\Gamma, \Box)$ evaluated at the representation R,

$$\begin{aligned}W(\Gamma, \Box)[R] = {}& \sum_{R_+ = R\otimes\Box} \frac{\dim R_+}{\dim R} \\ & \times \exp\left(-\frac{\varepsilon\Delta\mathcal{A}}{2}(C_2(R_+) - C_2(R))\right),\end{aligned} \tag{51}$$

where $\Delta\mathcal{A} = \mathcal{A}(\Sigma_2) - \mathcal{A}(\Sigma_1)$.

First we consider the case when the gauge group is $SU(n)$. Suppose that the Young diagram associated to the representation R is $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ and becomes the Young diagram associated to the representation R_+ by adding a box in the lth row. From (32) and (33), we obtain that

$$\frac{\dim R_+}{\dim R} = \prod_{i\neq r} \frac{\lambda_i - (\lambda_l + 1) + l - i}{\lambda_i - \lambda_l + l - i}, \tag{52}$$

$$C_2(R_+) - C_2(R) = 2(\lambda_l - l + 1) + \frac{n^2 - 1 - 2|\lambda|}{n}. \tag{53}$$

It is interesting to notice that

$$\mathrm{Res}_{x = a_1 + \hbar(\lambda_l^{(1)} + 1 - l)} \left(\frac{x + n\hbar}{\mathcal{Y}(x)[\Lambda]}\right) = \frac{\dim R_+}{\dim R}. \tag{54}$$

The appearance of the $\mathcal{Y}$-observable should not be surprising. Recall that the physical meaning of the Y-observable is to add or remove a pointlike instanton. Hence, the four-dimensional operator corresponding to $W(\Gamma,\square)[R]$ is

$$\frac{1}{2\pi i}\oint dx \frac{x+n\hbar}{\mathcal{Y}(x)[\Lambda]} e^{-\varepsilon\Delta\mathcal{A}x} \exp\left(-\varepsilon\Delta\mathcal{A}\left(\frac{n^2-1}{2n}-\frac{1}{n}q\frac{\partial}{\partial q}\right)\right). \tag{55}$$

For the case of $U(n)$, the equations (52) and (54) still hold. The difference between the Casimirs now is simpler:

$$C_2(\mathcal{R}_+)-C_2(\mathcal{R}) = 2(\lambda_l - l + 1) + n + 2r. \tag{56}$$

Hence, the four-dimensional operator corresponding to $W(\Gamma,\square)[R]$ is now

$$\frac{1}{2\pi i}\oint dx \frac{x+n\hbar}{\mathcal{Y}(x)[\Lambda]} \exp\left(-\varepsilon\Delta\mathcal{A}\left(x+\frac{n}{2}\right)\right). \tag{57}$$

IV. DISCUSSIONS

In this paper, we study a generalization of the correspondence between four-dimensional $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with $2N-2$ fundamental hypermultiplets and A-type topological string theory on S^2. In our correspondence, the partition function of the four-dimensional $U(N)$ gauge theory with $2N$ fundamental hypermultiplets at a suitable nongeneric point of the parameter space is related to the partition function of two-dimensional Yang-Mills theory on S^2. We also study the expectation value of a Wilson loop operator in the fundamental representation in the two-dimensional Yang-Mills theory. The corresponding operator in the four-dimensional theory can be found for the fundamental representation. It appears that the correspondence is more complicated than the old correspondence in [6–8].

The relation between four-dimensional supersymmetric gauge theory and two-dimensional Yang-Mills theory on S^2 was discovered in many other places. For example, the supersymmetric Wilson loops restricted to an S^2 submanifold of four-dimensional space in $\mathcal{N}=4$ supersymmetric Yang-Mills theory [15,16] can be consistently truncated to a two-dimensional Yang-Mills theory on S^2. However, the number of supersymmetry in four-dimensional gauge theory and the way to identify the Wilson loop operator in their work is quite different from our story. One other similar relation is the identification of the superconformal index of a class of four-dimensional $\mathcal{N}=2$ theories with a deformation of two-dimensional Yang-Mills theory on punctured Riemann surfaces [17]. However, in their correspondence, the four-dimensional gauge theory is a complicated quiver theory, and there are necessarily a number of punctures in the Riemann surface. Hence, all these old relations are indeed different from ours.

So far, the correspondence discussed in this paper is only a mathematical coincidence of two different partition functions. It will be nice if one can embed our correspondence into a string theory setup and provide a physical interpretation of the results we have got. The procedure (23) and (24) is similar to the approach to introduce surface operators or vortices in the previous discussions of AGT correspondence, and one may effectively describe the surface operator as some two-dimensional gauge theory. One may wonder whether the two-dimensional Yang-Mills theory we discuss is somehow related to the gauge theory in this construction. However, we would like to point out that this is not the case. Notice that if we want to have a surface operator in a $U(N)$ gauge theory, we can consider a two-dimensional gauge theory coupled to the $U(N)$ gauge theory, or we can start with a $U(N)\times U(N')$ theory and tune the Coulomb moduli in the $U(N')$ part of the theory. Furthermore, in this case, the two-dimensional gauge theory lives inside the spacetime of the four-dimensional gauge theory. Instead, we suggest that the proper physical origin of our result should come from the compactification of little string theory. The four-dimensional gauge theory and the two-dimensional Yang-Mills theory live in the perpendicular spaces. This is also the case for the old correspondence between supersymmetric gauge theory and topological string theory [6,7].

There are many open problems which remain to be answered.

First, we only studied the Wilson loop operator which is inserted in the two-dimensional Yang-Mills theory in the fundamental representation. We can insert Wilson loop operators in arbitrary representations of the gauge group and define a quantity similar to (51),

$$W(\Gamma,R_\Gamma)[R] = \sum_{R_+}\frac{\dim R_+}{\dim R}\exp\left(-\frac{\varepsilon\Delta\mathcal{A}}{2}(C_2(R_+)-C_2(R))\right)\mathfrak{N}(R\otimes R_\Gamma, R_+). \tag{58}$$

Now $\mathfrak{N}(R\otimes R_\Gamma, R_+)$ is more complicated. What are the corresponding four-dimensional operators?

Second, we only consider the first nontrivial simplification of the instanton partition function at a nongeneric point of the parameter space in this paper. It is natural to extend our analysis to the cases

$$a_1 = m_1 + n_1\hbar = m_2 + n_1\hbar, \qquad a_2 = m_3 + n_2\hbar = m_4 + n_2\hbar, \qquad a_3 = m_5 = m_6, \ldots, a_N = m_{2N-1} = m_{2N}. \tag{59}$$

Then the length of the partition $\lambda^{(1)}$ is at most n_1, the length of the partition $\lambda^{(2)}$ is at most n_2, while all the other partitions are empty. Similar to the case discussed in the paper, there are many cancellations in the measure. The resulting measure is

$$\begin{aligned}\mu=&\left(\prod_{1\le i<j\le n_1}\frac{\lambda_i^{(1)}-\lambda_j^{(1)}+j-i}{j-i}\right)^2\left(\prod_{1\le i<j\le n_2}\frac{\lambda_i^{(2)}-\lambda_j^{(2)}+j-i}{j-i}\right)^2\\&\times\left(\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\frac{a_1-a_2+\hbar(\lambda_i^{(1)}-\lambda_j^{(2)}+j-i)}{a_1-a_2+\hbar(j-i)}\right)^2\\&\times\left(\prod_{i=1}^{n_1}\frac{\Gamma(\frac{a_1-a_2}{\hbar}+n_2+1+\lambda_i^{(1)}-i)}{\Gamma(\frac{a_1-a_2}{\hbar}+n_2+1-i)}\right)^2\left(\prod_{i=1}^{n_2}\frac{\Gamma(\frac{a_2-a_1}{\hbar}+n_1+1+\lambda_i^{(2)}-i)}{\Gamma(\frac{a_2-a_1}{\hbar}+n_1+1-i)}\right)^2.\end{aligned}\tag{60}$$

What is the physical interpretation of this a partition function?

ACKNOWLEDGMENTS

Research was supported in part by the National Science Foundation Grant No. PHY 1404446. X. Z. has greatly benefited from discussions with Nikita Nekrasov. X. Z. would also like to thank Alex DiRe, Saebyeok Jeong, Naveen Prabhakar, Dan Xie, Wenbin Yan, and Peng Zhao for discussions.

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