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Canonical realization of (2 + 1)-dimensional Bondi-Metzner-Sachs symmetry

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We construct canonical realizations of the (2+1)-dimensional Bondi-Metzner-Sachs ($\mathfrak{bms}_3$) algebra as symmetry algebras of a free Klein-Gordon (KG) field in $2+1$ dimensions for both massive and massless cases. We consider two types of realizations, one on shell, written in terms of the Fourier modes of the scalar field, and the other off shell, with nonlocal transformations written in terms of the KG field and its momenta. These realizations contain both supertranslations and superrotations, for which we construct the corresponding Noether charges.

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I. INTRODUCTION

Recently, there has been renewed interest in the Bondi-Metzner-Sachs (BMS) group [1]. The BMS invariance of the gravitational scattering matrix was proved in [2] and, as a consequence of this result, Weinberg's soft graviton theorems [3] can be understood as the Ward identities of BMS supertranslations [4–7]. The relation between supertranslations, gravitational memory, and soft graviton theorems has also been studied [8]. There is a proposal that the information paradox [9] could be understood in terms of black hole soft hair associated with supertranslation and superrotation charges [10,11]. On the other hand, the BMS group could play a crucial role in understanding holography in asymptotically flat space-times [12–15]. BMS symmetry is an infinite conformal extension of the Carroll symmetry [16], which was introduced in [17] as a limit of the Poincaré algebra when the velocity of light is scaled down to zero. A pedagogical overview of the role of BMS symmetries in most of these topics is presented in [18].

In this paper we construct a canonical realization of the $\mathfrak{bms}_3$ algebra [19,20] with supertranslations and superrotations [15] associated with a free Klein-Gordon (KG) field in $2+1$ dimensions, for both massive and massless fields. Following the procedure in [21], we consider, in the massive case, the mass-shell hyperboloid representation of the hyperbolic plane H_2, and we compute the associated Laplace-Beltrami operator. It turns out that the three-dimensional momenta is an eigenfunction of the differential operator with eigenvalue $\frac{2}{m^2}$, where m is the mass of the scalar field and the 2 comes from the dimension of the hyperboloid. This property suggests computing all of the eigenfunctions of this operator corresponding to that eigenvalue with the same asymptotic properties as the three momenta. This allows us to generalize the momenta to an infinite set of supermomenta. These momenta yield an infinite-dimensional representation of the $2+1$ Lorentz group and lead to the definition of the generators of the supertranslations in terms of the Fourier modes of the KG field.

The mass-shell condition for a massless scalar field results in a cone, for which a Laplace-Beltrami operator cannot be constructed. To get around this, we consider the massless limit of the Laplace-Beltrami operator on the hyperboloid [22]. Once we have a suitable differential operator, the construction goes in parallel with the massive case.

We also construct a generalization of the Lorentz generators which corresponds to superrotations. In the massless case, the algebra of supertranslations and superrotations is the $\mathfrak{bms}_3$ algebra introduced in [23]. In the massive case, the superrotation generators that we introduce must be separated into two different sets which both contain the Lorentz part, and each set corresponds to a subalgebra of $\mathfrak{bms}_3$. It should be noted that the differential operators appearing in our construction represent one of the two Casimirs of the $2+1$ Lorentz group.

At the quantum level, the Hilbert space of one-particle states supports a unitary irreducible representation of the Poincaré group, and at the same time a unitary reducible representation of the BMS_3 group. In contrast to the gravitational approach, our canonical realization of the supertranslation symmetry is not spontaneously broken. Unitary representations of BMS_3 were also considered in [24,25].

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We study the off-shell (Noether) supertranslation and superrotation symmetries of the massless Klein-Gordon action, and we compute the associated Noether charges. These charges are expressed as nonlocal linear functionals of fields and momenta. The same construction is carried out for the supertranslations in the massive case.

The organization of the paper is as follows. In Sec. II we construct the supertranslations and superrotations in terms of the Fourier modes of the KG field. In Sec. III we construct the transformations in terms of fields and momenta. Section IV is devoted to conclusions and an outlook. Appendix A presents explicit forms for some of the functions that appear in the nonlocal transformations obtained in Sec. III, and Appendix B discusses the geometry of the mass-shell hyperboloid in $2+1$ dimensions. We use the Minkowski metric $(-++)$ throughout the paper.

II. CANONICAL REALIZATION OF BMS_3

A. Canonical realization of Poincaré symmetry for a scalar field

The Lagrangian density for a real massive scalar field is given by

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{1}$$

The solution of the Klein-Gordon equation, in terms of Fourier modes $a(\vec{k})$, is

$$\phi(t,\vec{x}) = \int \tilde{d}k(a(\vec{k})e^{ikx} + \bar{a}(\vec{k})e^{-ikx}), \tag{2}$$

where the phase space Fourier modes have the Poisson bracket

$$\{a(\vec{k}), \bar{a}(\vec{q})\} = -i\Omega(\vec{k})\delta^2(\vec{k}-\vec{q}). \tag{3}$$

The Lorentz invariant integration measure in the hyperbolic plane H_2 is

$$\tilde{d}k = \frac{d^2k}{\Omega(\vec{k})},$$

$$\Omega(\vec{k}) = (2\pi)^2 2k^0(\vec{k}) = (2\pi)^2 2\sqrt{\vec{k}^2+m^2}. \tag{4}$$

Noether's theorem allows us to write the expression for the conserved charge under translations. By use of the solution of the equations of motion (2), the charges on shell can be written as

$$P^\mu = \int \tilde{d}k\,\bar{a}(\vec{k})k^\mu a(\vec{k}), \tag{5}$$

and their action on the Fourier modes is given by

$$\{P^\mu, a(\vec{k})\} = ik^\mu a(\vec{k}). \tag{6}$$

The analogous Lorentz charges on shell are

$$M^{ij} = -i\int \tilde{d}k\,\bar{a}(\vec{k})\left(k^i\frac{\partial}{\partial k^j} - k^j\frac{\partial}{\partial k^i}\right)a(\vec{k}) \tag{7}$$

for rotations, and

$$M^{0j} = tP^j - i\int \tilde{d}k\,\bar{a}(\vec{k})k^0\frac{\partial}{\partial k^j}a(\vec{k}) \tag{8}$$

for boosts. We define the truncated time-independent Lorentz generators

$$M'^{ij} = M^{ij}, \qquad M'^{0j} = M^{0j} - tP^j \tag{9}$$

that satisfy the Poincaré algebra as well and have the following Poisson brackets (we drop the prime and will work with these generators henceforth unless otherwise stated):

$$\{P^\mu, P^\nu\} = 0, \qquad \{M^{\mu\nu}, P^\rho\} = P^\mu\eta^{\nu\rho} - P^\nu\eta^{\mu\rho}, \tag{10}$$

$$\{M^{\mu\nu}, M^{\rho\sigma}\} = M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}. \tag{11}$$

The action of Lorentz generators on the Fourier modes is given by

$$\{M^{\mu\nu}, a(\vec{k})\} = \eta^{\mu\mu'}\eta^{\nu\nu'}D_{\mu'\nu'}a(\vec{k}), \tag{12}$$

where $D_{\mu\nu}$ is a realization of the Lorentz group in terms of the differential operators

$$D_{01} = -\sqrt{\vec{k}^2+m^2}\partial_{k^1} \equiv iK_1, \tag{13}$$

$$D_{02} = -\sqrt{\vec{k}^2+m^2}\partial_{k^2} \equiv iK_2, \tag{14}$$

$$D_{12} = k^1\partial_{k^2} - k^2\partial_{k^1} \equiv iJ. \tag{15}$$

One can check that the generators J, K_1, and K_2 obey the $SO(1,2)$ algebra

$$[K_1, K_2] = -iJ, \quad [K_1, J] = -iK_2, \quad [K_2, J] = iK_1. \tag{16}$$

B. Supertranslations

In order to construct a canonical realization of BMS_3, we follow the procedure of [21] to construct supertranslations. The idea is to generalize the ordinary three-dimensional momenta k^μ to an infinite set of "supermomenta" and to generalize the realization of the charge of the translations on shell (5).

1. Massive case

Consider the $k_0 > 0$ sheet of the mass-shell hyperboloid representation of the hyperbolic plane H_2,

$$-k_0^2 + k_1^2 + k_2^2 = -m^2, \tag{17}$$

in a space with the ambient Minkowski metric

$$ds^2 = -dk_0^2 + dk_1^2 + dk_2^2. \tag{18}$$

The manifold H_2 is invariant under the isometries of the metric, that is, $ISO(1,2)$. We can parametrize (17) for $k_0 > 0$ as

$$k_0 = mz, \tag{19}$$

$$k_1 = m\sqrt{z^2-1}\cos\phi, \tag{20}$$

$$k_2 = m\sqrt{z^2-1}\sin\phi, \tag{21}$$

with $z \in [1, +\infty)$, $\phi \in [0, 2\pi)$. Notice that k_1 and k_2 vanish at $z = 1$. In these coordinates, the Lorentz generators (13), (14), and (15) are given by

$$K_1 = -i\frac{z}{\sqrt{z^2-1}}\sin\phi\frac{\partial}{\partial\phi} + i\sqrt{z^2-1}\cos\phi\frac{\partial}{\partial z}, \tag{22}$$

$$K_2 = i\frac{z}{\sqrt{z^2-1}}\cos\phi\frac{\partial}{\partial\phi} + i\sqrt{z^2-1}\sin\phi\frac{\partial}{\partial z}, \tag{23}$$

$$J = -i\frac{\partial}{\partial\phi}. \tag{24}$$

The metric induced on H_2, which is a Euclidean AdS_2 with an anti–de Sitter (AdS) radius equal to m, is given in this parametrization by

$$ds^2_{\text{induced}} = m^2\frac{1}{z^2-1}dz^2 + m^2(z^2-1)d\phi^2. \tag{25}$$

The boundary is located at $z \to \infty$, and it is a sphere S^1 with radius $m\sqrt{z^2-1}$ and metric

$$ds^2|_{\text{boundary}} = \lim_{z\to\infty}\frac{1}{m^2(z^2-1)}ds^2_{\text{induced}} = d\phi^2.$$

The Laplace-Beltrami operator (Appendix B) is given by

$$\nabla^2 = \frac{1}{m^2}\left((z^2-1)\frac{\partial^2}{\partial z^2} + 2z\frac{\partial}{\partial z} + \frac{1}{z^2-1}\frac{\partial^2}{\partial\phi^2}\right), \tag{26}$$

and it is proportional to a Casimir, the $2+1$ Lorentz group

$$m^2\nabla^2 = -J^2 + K_1^2 + K_2^2. \tag{27}$$

It is immediate to check that the three-dimensional momenta are eigenfunctions of this operator,

$$\nabla^2 k_\mu = \frac{2}{m^2}k_\mu, \qquad \mu = 0, 1, 2. \tag{28}$$

The numerical constant 2 is the dimension of the hyperboloid. To generalize the momenta to infinite supermomenta, we look for general eigenvectors of the Laplacian with the eigenvalue $2/m^2$,

$$\left(-\nabla^2 + \frac{2}{m^2}\right)\Phi(z,\phi) = 0. \tag{29}$$

We look for solutions of the form

$$\Phi(z,\phi) = e^{i\ell\phi}f(z), \tag{30}$$

where $e^{i\ell\phi}$ are the eigenfunctions of S^1. The differential equation for $f(z)$ is

$$(1-z^2)f'' - 2zf' + \left(2 - \frac{\ell^2}{1-z^2}\right)f = 0, \tag{31}$$

with the general solution

$$f(z) = C_1(z-\ell)\left(\frac{z+1}{z-1}\right)^{\frac{\ell}{2}} + C_2(z+\ell)\left(\frac{z-1}{z+1}\right)^{\frac{\ell}{2}}. \tag{32}$$

The first solution does not behave well at $z = 1$ for $\ell > 0$, while the second one is not well behaved for $\ell < 0$, and one of the solutions becomes the other one by changing $\ell \leftrightarrow -\ell$. Since the two-dimensional momenta are regular for $z = 1$, we are interested in the general solution to (29) that is regular at $z = 1$, given by

$$w_\ell(z,\phi) = e^{i\ell\phi}\left(\frac{z-1}{z+1}\right)^{\frac{\ell}{2}}(z+\ell), \qquad \ell \ge 0, \tag{33}$$

$$\hat{w}_\ell(z,\phi) = e^{i\ell\phi}\left(\frac{z+1}{z-1}\right)^{\frac{\ell}{2}}(z-\ell), \qquad \ell < 0, \tag{34}$$

which can also be written in a more compact form as

$$w_\ell(z,\phi) = e^{i\ell\phi}\left(\frac{z-1}{z+1}\right)^{\frac{|\ell|}{2}}(z+|\ell|), \qquad \ell \in \mathbb{Z}. \tag{35}$$

The functions $w_\ell(z,\phi)$ are the infinite set of supermomenta that we are looking for.

Notice that

$$w_\ell(z,\phi) = e^{i\ell\phi}z + O(1/z), \tag{36}$$

and we can define $w_\ell|_{\text{boundary}}(z,\phi) = e^{i\ell\phi}z$. This means that the generalized momenta have, for all ℓ's, the same asymptotic behavior as the ordinary momenta.

Alternatively, one can use the set of real functions

$$u_\ell(z,\phi) = \cos\ell\phi\left(\frac{z-1}{z+1}\right)^{\frac{\ell}{2}}(z+\ell), \qquad \ell \geq 0, \quad (37)$$

$$v_\ell(z,\phi) = \sin\ell\phi\left(\frac{z-1}{z+1}\right)^{\frac{\ell}{2}}(z+\ell), \qquad \ell \geq 0. \quad (38)$$

Notice that the three-dimensional momenta can be written in terms of these functions as

$$u_0(z,\phi) = z = \frac{1}{m}k_0, \quad (39)$$

$$u_1(z,\phi) = (z^2-1)^{\frac{1}{2}}\cos\phi = \frac{1}{m}k_1, \quad (40)$$

$$v_1(z,\phi) = (z^2-1)^{\frac{1}{2}}\sin\phi = \frac{1}{m}k_2. \quad (41)$$

In terms of the two-dimensional momenta, the functions (35) can be written as

$$\omega_\ell(k_1,k_2) = \left(\frac{k_1}{\sqrt{k_0^2-m^2}} + i\frac{k_2}{\sqrt{k_0^2-m^2}}\right)^\ell \cdot \left(\frac{k_0-m}{k_0+m}\right)^{|\ell|/2}\left(\frac{k_0}{m}+|\ell|\right). \quad (42)$$

This can be further simplified to yield

$$\omega_\ell(k_1,k_2) = (k_1+ik_2)^\ell|\vec{k}|^{-\ell}\left(\frac{\sqrt{m^2+|\vec{k}|^2}-m}{\sqrt{m^2+|\vec{k}|^2}+m}\right)^{|\ell|/2} \cdot\left(\sqrt{1+|\vec{k}|^2/m^2}+|\ell|\right), \quad (43)$$

or

$$\omega_\ell(k_1,k_2) = (k_1+ik_2)^\ell\left(\sqrt{m^2+|\vec{k}|^2}+\text{sgn}(\ell)m\right)^\ell \cdot\left(\sqrt{1+|\vec{k}|^2/m^2}+|\ell|\right). \quad (44)$$

One can check that the subspace of functions spanned by u_0, u_1, v_1, or, alternatively, by w_0, w_1, and w_{-1}, is invariant under the action of the $2+1$ Lorentz group. In general, the action of the Lorentz generators on the w_ℓ is given by

$$K_1 w_\ell = -\frac{i}{2}(\ell-1)w_{\ell+1} + \frac{i}{2}(\ell+1)w_{\ell-1}, \quad (45)$$

$$K_2 w_\ell = -\frac{1}{2}(\ell-1)w_{\ell+1} - \frac{1}{2}(\ell+1)w_{\ell-1}, \quad (46)$$

$$J w_\ell = \ell w_\ell. \quad (47)$$

Defining $K_\pm = K_1 \pm iK_2$, one has

$$K_\pm w_\ell = i(1\mp\ell)w_{\ell\pm1}, \quad (48)$$

and therefore $K_\pm w_\ell$ are raising and lowering operators.

Since each function w_ℓ defines a (super)translation in the phase space of a massless scalar particle, we define the supertranslation generators as

$$P_\ell = \int \tilde{\mathrm{d}}k\, w_\ell(\vec{k})\bar{a}(\vec{k})a(\vec{k}). \quad (49)$$

It is easy to check that these supertranslations commute,

$$\{P_\ell, P_{\ell'}\} = 0. \quad (50)$$

Their action on the Fourier modes is given by

$$\{P_l, a(\vec{k})\} = iw_\ell a(\vec{k}). \quad (51)$$

Let us see how Lorentz generators act on them. A Lorentz generator $\mathcal{O}$ is represented by

$$\mathcal{O} = \int \tilde{\mathrm{d}}k\, \bar{a}(\vec{k})O_{\vec{k}}a(\vec{k}), \quad (52)$$

with $O_{\vec{k}}$ being a first order differential operator in $\vec{k}$. For instance, for the rotation J, one has

$$O_{\vec{k}} = -i\left(k_1\frac{\partial}{\partial k_2} - k_2\frac{\partial}{\partial k_1}\right),$$

while for the boost generators K_i,

$$O_{\vec{k}} = i\sqrt{k_1^2+k_2^2+m^2}\frac{\partial}{\partial k_i}.$$

One can show that

$$\{\mathcal{O}, P_\ell\} = -i\int \tilde{\mathrm{d}}k\, a(\vec{k})\bar{a}(\vec{k})O_{\vec{k}}w_\ell(\vec{k}), \quad (53)$$

and hence it suffices to know the action of the generators on the functions w_ℓ. Specifically, one gets

$$\{J, P_\ell\} = -i\ell P_\ell, \quad (54)$$

$$\{K_1, P_\ell\} = \frac{1}{2}(1-\ell)P_{\ell+1} + \frac{1}{2}(1+\ell)P_{\ell-1}, \quad (55)$$

$$\{K_2, P_\ell\} = -\frac{i}{2}(1-\ell)P_{\ell+1} + \frac{i}{2}(1+\ell)P_{\ell-1}, \tag{56}$$

$$\{K_\pm, P_\ell\} = (1 \mp \ell)P_{\ell\pm1}. \tag{57}$$

This is the analog in $2+1$ dimensions of the four-dimensional BMS algebra [1,21]. The generalization of the algebra to include superrotations will be discussed after we consider the massless case.

Relations (50) and (54)–(57) imply, at the quantum level, that the Hilbert space of one-particle states supports a unitary irreducible representation of the Poincaré group, and at the same time a unitary reducible representation of the BMS_3 group. In contrast to the gravitational approach, our canonical realization of the supertranslations is unbroken.

2. Massless case

Now we want to construct the canonical realization of BMS associated with a three-dimensional massless free scalar field. In this case what previously was a hyperboloid is now a cone,

$$-k_0^2 + k_1^2 + k_2^2 = 0, \tag{58}$$

that can be parametrized as

$$k_0 = r, \tag{59}$$

$$k_1 = r\cos\phi, \tag{60}$$

$$k_2 = r\sin\phi, \tag{61}$$

with $r > 0$ for $k_0 > 0$ and $\phi \in [0, 2\pi)$. This can be obtained from (19)–(21) by putting $z = r/m$ and letting $m \to 0$ [22]. However, a nondegenerate induced metric does not exist and the standard construction of the Laplace operator fails. Instead, we scale the operator in (29) by m^2 and replace $z = r/m$,

$$D \equiv -m^2\Delta + 2 = -\left(\left\{\left(\frac{r}{m}\right)^2 - 1\right\}m^2\partial_r^2 + 2r\partial_r + \frac{1}{(\frac{r}{m})^2 - 1}\partial_\phi^2\right) + 2. \tag{62}$$

In the limit $m \to 0$, one gets

$$D_{\text{massless}} = -r^2\partial_r^2 - 2r\partial_r + 2, \tag{63}$$

which turns out to be independent of ϕ. As in the massive case, we look for solutions of

$$D_{\text{massless}}\Phi(r,\phi) = 0 \tag{64}$$

of the form

$$\Phi(r,\phi) = e^{i\ell\phi}f(r).$$

The function $f(r)$ must obey

$$-r^2 f'' - 2rf' + 2f = 0. \tag{65}$$

This equation has independent solutions $f_1(r) = r$ and $f_2(r) = 1/r^2$, and hence the regular solution at $r = 0$ is $f(r) = r$. The supermomenta that we are looking for are

$$w_\ell(r,\phi) = re^{i\ell\phi}, \qquad \ell \in \mathbb{Z}, \tag{66}$$

up to a normalization constant. The expression in terms of momenta is given by

$$\omega_\ell(\vec{q}) = \frac{(q^1 + iq^2)^\ell}{((q^1)^2 + (q^2)^2)^{\frac{\ell-1}{2}}}. \tag{67}$$

Notice that, in the massless case, the dependence of r in the boundary and in the bulk is the same; that is, there are no corrections in $1/r$ like in the massive case.

The $SO(1,2)$ generators on the cone are given by

$$J = -i\frac{\partial}{\partial\phi}, \tag{68}$$

$$K_1 = ir\cos\phi\frac{\partial}{\partial r} - i\sin\phi\frac{\partial}{\partial\phi}, \tag{69}$$

$$K_2 = ir\sin\phi\frac{\partial}{\partial r} + i\cos\phi\frac{\partial}{\partial\phi}. \tag{70}$$

As in the massive case, it turns out that the purely differential part of D_{massless} is proportional (actually equal in this case) to one of the two Casimirs of $SO(2,1)$,

$$-r^2\frac{\partial^2}{\partial r^2} - 2r\frac{\partial}{\partial r} = -J^2 + K_1^2 + K_2^2. \tag{71}$$

Notice that these generators—or, more precisely, J, $K_\pm = K_1 \pm iK_2$—can be written as particular cases ($n = 0, \pm1$) of the more general expression

$$L_n = e^{in\phi}\left(-\frac{\partial}{\partial\phi} + inr\frac{\partial}{\partial r}\right). \tag{72}$$

The action of $SO(1,2)$ generators on the eigenfunctions w_ℓ is exactly the same found for the massive case

$$Jw_\ell = \ell w_\ell, \tag{73}$$

$$K_1 w_\ell = -\frac{i}{2}(\ell-1)w_{\ell+1} + \frac{i}{2}(\ell+1)w_{\ell-1}, \tag{74}$$

$$K_2 w_\ell = -\frac{1}{2}(\ell-1)w_{\ell+1} - \frac{1}{2}(\ell+1)w_{\ell-1}, \tag{75}$$

which, in particular, shows that the subspace spanned by w_0, w_1, and w_{-1} is invariant under $SO(1,2)$.

The supertranslations are still given by

$$P_\ell = \int \tilde{d}k w_\ell(\vec{k})\bar{a}(\vec{k})a(\vec{k}). \tag{76}$$

Together with the Lorentz generators, they constitute a realization of the $\mathfrak{bms}_3$ algebra. The action of supertranslations on the Fourier modes is analogous to the massive case. Induced representations of BMS_3 were constructed in [24,25].

C. Superrotations

We will construct here infinite families of operators generalizing the Lorentz algebra for both massless and massive cases.

1. Massless case

In the massless case, one can generalize (72) for an arbitrary $n \in \mathbb{Z}$ and write

$$L_n = e^{in\phi}\left(-\frac{\partial}{\partial\phi} + inr\frac{\partial}{\partial r}\right), \qquad n \in \mathbb{Z}. \tag{77}$$

One can check to see that these L_n's also obey the Witt algebra

$$[L_n, L_m] = i(n-m)L_{n+m}, \tag{78}$$

and that

$$L_n w_\ell = i(n-\ell)w_{n+\ell}. \tag{79}$$

In terms of $\vec{k}$, the differential operators (77) can be written as

$$L_n = (k_1^2 + k_2^2)^{-n/2}(k_1 + ik_2)^n \times \left(\{ink_1 + k_2\}\frac{\partial}{\partial k_1} + \{ink_2 - k_1\}\frac{\partial}{\partial k_2}\right), \tag{80}$$

and, analogously to the case of supertranslations, we define the on-shell generators of superrotations as

$$\mathcal{R}_n = \int \tilde{d}k\, \bar{a}(\vec{k})L_n a(\vec{k}), \tag{81}$$

which, due to (78) and (79), realize the algebra

$$\{\mathcal{R}_m, \mathcal{R}_n\} = -i\int \tilde{d}k\, \bar{a}(\vec{k})[L_m, L_n]a(\vec{k}) = (m-n)\mathcal{R}_{m+n}, \tag{82}$$

$$\{\mathcal{R}_m, P_n\} = -i\int \tilde{d}k a(\vec{k})\bar{a}(\vec{k})L_m w_n(\vec{k}) = (m-n)P_{m+n}, \tag{83}$$

$$\{P_n, P_m\} = 0. \tag{84}$$

This is the $\mathfrak{bms}_3$ algebra introduced in [23].

2. Massive case

In the massive case, we do not have an initial guess for the form of the superrotations. One may notice, however, that Lorentz generators, once written in the form $\xi^\alpha\partial_\alpha$, $\alpha = z, \phi$, satisfy the following equations:

$$D\xi^z = 0, \qquad \nabla_\alpha \xi^\alpha = 0, \tag{85}$$

where $D = -m^2\Delta + 2$. Thus, one can try, in the massive case, to generalize these operators by first solving $D\xi^z = 0$ and then computing ξ^ϕ from $\nabla_z\xi^z + \nabla_\phi\xi^\phi = 0$.

Clearly,

$$\xi^z = e^{in\phi}\left(\frac{z-1}{z+1}\right)^{|n|/2}(|n| + z), \qquad n \in \mathbb{Z} \tag{86}$$

since this is the solution of the partial differential equation for massive supertranslations. From the divergence equation, and using $\nabla_z\xi^z + \nabla_\phi\xi^\phi = \partial_z\xi^z + \partial_\phi\xi^\phi$ (see Appendix B), one can integrate the angular term to obtain

$$\xi^\phi = -e^{in\phi}\left(\frac{z-1}{z+1}\right)^{|n|/2}\frac{(|n|(|n|+z) + z^2 - 1)}{in(z^2-1)} + f(z), \tag{87}$$

with $f(z)$ being an arbitrary function that we set to zero.

Thus, one may try to define superrotation generators as

$$T_n = e^{in\phi}\left(\frac{z-1}{z+1}\right)^{|n|/2}\left(-\frac{|n|(|n|+z) + z^2 - 1}{z^2-1}\frac{\partial}{\partial\phi} + in(|n|+z)\frac{\partial}{\partial z}\right), \qquad n \in \mathbb{Z}, \tag{88}$$

where we have multiplied all terms by a factor in. However, these operators do not form an algebra (this can be seen when computing the commutator of T_n with opposed sign indices, except in the case $n = \pm 1$). Instead, we can define two infinite-dimensional set of generators, each containing the Lorentz part, according to

$$\mathcal{L}_n = T_n, \qquad n \geq -1, \tag{89}$$

$$\mathcal{Q}_n = T_n, \qquad n \leq 1. \tag{90}$$

Both sets of differential operators satisfy the algebra

$$[\mathcal{L}_n, \mathcal{L}_m] = i(n-m)\mathcal{L}_{n+m}, \quad n, m \geq -1, \tag{91}$$

$$[\mathcal{Q}_n, \mathcal{Q}_m] = i(n-m)\mathcal{Q}_{n+m}, \quad n, m \leq 1. \tag{92}$$

One has, for the lowest values of n,

$$\mathcal{Q}_0 = \mathcal{L}_0 = -iJ, \qquad \mathcal{L}_1 = \mathcal{Q}_1 = K_+, \qquad \mathcal{L}_{-1} = \mathcal{Q}_1 = -K_-. \tag{93}$$

Furthermore, the functions w_n associated with each set provide a realization of the corresponding algebras,

$$\mathcal{L}_n w_m = i(n-m)w_{n+m}, \quad n, m \geq -1, \tag{94}$$

$$\mathcal{Q}_n w_m = i(n-m)w_{n+m}, \quad n, m \leq 1. \tag{95}$$

Defining now the generators of superrotations as in (81) for each set, one can construct realizations of two subalgebras of the $\mathfrak{bms}_3$ algebra. To sum up, it is possible to extend the set of Lorentz generators to the right with the $\mathcal{L}_n$ and to the left with the $\mathcal{Q}_n$, but, in contrast to what happens in the massless case, it is not possible to merge both extensions into a single algebra.

The first equation in (85), which we obtained by generalizing the one satisfied by the Lorentz generators, is clearly noncovariant, but we will show next that, due to the geometry of the mass-shell manifold, it is, in fact, one of the components of a geometrical equation.

The Lorentz generators are the only solutions of the Killing equation

$$g^{\mu\alpha}\nabla_\alpha \xi^\nu + g^{\nu\alpha}\nabla_\alpha \xi^\mu = 0. \tag{96}$$

In order to generalize the generators of Lorentz transformations, one could consider an equation of the form

$$g^{\mu\alpha}\nabla_\alpha \xi^\nu + g^{\nu\alpha}\nabla_\alpha \xi^\mu = G^{\mu\nu}, \tag{97}$$

with G being symmetric and covariantly divergenceless,

$$\nabla_\mu G^{\mu\nu} = 0. \tag{98}$$

Condition (98) is instrumental for what we want to do. Notice, however, that we are not assuming that G is proportional to the metric, and hence (97) is different from the conformal Killing equation.

We now take the covariant derivative ∇_μ of (97). Using that $\nabla_\alpha g^{\mu\nu} = 0$ and (98), imposing $\nabla_\mu \xi^\mu = 0$, and using $[\nabla_\mu, \nabla_\alpha]\xi^\mu = R^\mu{}_{\beta\mu\alpha}\xi^\beta$, one arrives at

$$g^{\mu\alpha}\nabla_\mu\nabla_\alpha \xi^\nu + g^{\nu\alpha}R^\mu{}_{\beta\mu\alpha}\xi^\beta = 0. \tag{99}$$

Using the explicit form of the components of the Riemann curvature tensor given in Appendix B, it turns out that

$$g^{\nu\alpha}R^\mu{}_{\beta\mu\alpha}\xi^\beta = -\frac{1}{m^2}\xi^\nu, \tag{100}$$

and (99) boils down to [26]

$$g^{\mu\alpha}\nabla_\mu\nabla_\alpha \xi^\nu = \frac{1}{m^2}\xi^\nu. \tag{101}$$

Evaluating (101) for $\nu = z, \phi$ yields a pair of coupled equations,

$$\Delta_S \xi^z - \frac{2z}{m^2}(\partial_z \xi^z + \partial_\phi \xi^\phi) - \frac{2}{m^2}\xi^z = 0, \tag{102}$$

$$\Delta_S \xi^\phi + \frac{2z}{m^2}\left(\partial_z \xi^\phi + \frac{1}{(z^2-1)^2}\partial_\phi \xi^z\right) = 0, \tag{103}$$

where Δ_S is the scalar Beltrami-Laplace operator (26). However, since

$$0 = \nabla_z \xi^z + \nabla_\phi \xi^\phi = \partial_z \xi^z + \partial_\phi \xi^\phi, \tag{104}$$

equation (102) can be simplified to

$$\Delta_S \xi^z - \frac{2}{m^2}\xi^z = 0. \tag{105}$$

Equations (105) and (104) were our starting point for constructing the superrotation generators and now have received a sound geometrical foundation, i.e., (99). Furthermore, one can check that (103) is satisfied by the ∂_ϕ parts of $\mathcal{L}_n$ and $\mathcal{Q}_n$.

Let us finally note that the 1-forms l_n associated with $\mathcal{L}_n$ (and likewise for $\mathcal{Q}_n$),

$$l_n = m^2 e^{in\phi}\left(\frac{z-1}{z+1}\right)^{n/2}\left(i\frac{n(n+z)}{z^2-1}\mathrm{d}z - (n(n+z) + z^2 - 1)\mathrm{d}\phi\right), \quad n \in \mathbb{Z}. \tag{106}$$

turn out to be eigenvectors of the Hodge–Laplace–de Rham operator $\tilde{\Delta}$ [28],

$$\tilde{\Delta} l_n = -\frac{2}{m^2} l_n. \tag{107}$$

This adds to the geometrical meaning of our construction, and it could be useful for further generalizations.

III. NONLOCAL BMS SYMMETRIES OF THE KLEIN-GORDON LAGRANGIAN

In this section we will prove that the KG action is invariant under supertranslations and superrotations, and we will construct the corresponding Noether charges [29]. We will present explicit expressions only for the massless case.

A. Noether charges of supertranslations

For the classical Klein-Gordon field, the Fourier modes can be written in terms of the fields ϕ and π as

$$a(\vec{k}) = \int d^2x e^{-ikx}(k^0\phi(t,\vec{x}) + i\pi(t,\vec{x})), \quad (108)$$

$$\bar{a}(\vec{k}) = \int d^2x e^{ikx}(k^0\phi(t,\vec{x}) - i\pi(t,\vec{x})), \quad (109)$$

where $k^0 = |\vec{k}|$.

When doing a supertranslation transformation on the fields using $P_\ell = \int \tilde{dk}\,\bar{a}(\vec{k})a(\vec{k})\omega_\ell$ as the generator, one obtains

$$\delta_{ST}\phi = \{\phi, \epsilon^\ell P_\ell\} = \int \tilde{dk}(-i)\epsilon^\ell \omega_\ell (a(\vec{k})e^{ikx} - \bar{a}(\vec{k})e^{-ikx}), \quad (110)$$

$$\delta_{ST}\pi = \{\pi, \epsilon^\ell P_\ell\} = \int \tilde{dk}(-1)k^0\epsilon^\ell \omega_\ell (a(\vec{k})e^{ikx} + \bar{a}(\vec{k})e^{-ikx}), \quad (111)$$

which can be written in terms of the fields using (108) and (109) as [32]

$$\delta_{ST}\phi = \epsilon^\ell \int d^2y[f_\ell(\vec{x}-\vec{y})\phi(t,\vec{y}) + g_\ell(\vec{x}-\vec{y})\pi(t,\vec{y})], \quad (112)$$

$$\delta_{ST}\pi = \epsilon^\ell \int d^2y[h_\ell(\vec{x}-\vec{y})\phi(t,\vec{y}) + f_\ell(\vec{x}-\vec{y})\pi(t,\vec{y})], \quad (113)$$

where

$$f_\ell(\vec{x}) = 2\int \tilde{dk}\omega_\ell(\vec{k})k^0 \sin(\vec{k}\cdot\vec{x}), \quad (114)$$

$$g_\ell(\vec{x}) = 2\int \tilde{dk}\omega_\ell(\vec{k}) \cos(\vec{k}\cdot\vec{x}), \quad (115)$$

$$h_\ell(\vec{x}) = -2\int \tilde{dk}\omega_\ell(\vec{k})k^{02} \cos(\vec{k}\cdot\vec{x}). \quad (116)$$

Notice the symmetry properties $f_\ell(-\vec{x}) = -f_\ell(\vec{x})$, $g_\ell(-\vec{x}) = g_\ell(\vec{x})$, and $h_\ell(-\vec{x}) = h_\ell(\vec{x})$, and also that $\nabla^2 g_\ell(\vec{x}) = h_\ell(\vec{x})$.

Another important aspect to notice here concerns the values of f_ℓ, g_ℓ, and h_ℓ depending on the parity of ℓ. One can check that for ℓ odd, $g_\ell = h_\ell = 0$ since their integrands are odd functions, and, for ℓ even, $f_\ell = 0$ for the same reason. This observation implies that a particular supertranslation will not simultaneously use information from a field and its momentum, but instead from only one of them. Thus, if ℓ is even, $\delta_{ST}\phi$ will depend only on the field momentum, whereas if ℓ is odd, $\delta_{ST}\phi$ will need only the value of the field itself.

Now we would like to see if we can extend the on-shell symmetry to an off-shell Noether symmetry of the massless KG Lagrangian. We consider the off-shell realization of (112) and (113). The variation of the Lagrangian (1) with $m = 0$ under this transformation is

$$\begin{aligned}\delta L = \int \mathrm{d}^2x\mathrm{d}^2y[&h(\vec{x}-\vec{y})\phi(t,\vec{y})\dot{\phi}(t,\vec{x}) \\ &+ f(\vec{x}-\vec{y})\pi(t,\vec{y})\dot{\phi}(t,\vec{x}) + f(\vec{x}-\vec{y})\dot{\phi}(t,\vec{y})\pi(t,\vec{x}) \\ &+ g(\vec{x}-\vec{y})\dot{\pi}(t,\vec{y})\pi(t,\vec{x}) - h(\vec{x}-\vec{y})\phi(t,\vec{y})\pi(t,\vec{x}) \\ &- f(\vec{x}-\vec{y})\pi(t,\vec{y})\pi(t,\vec{x}) \\ &- \vec{\nabla}_x f(\vec{x}-\vec{y})\phi(t,\vec{y})\cdot\vec{\nabla}\phi(t,\vec{x}) \\ &- \vec{\nabla}_x g(\vec{x}-\vec{y})\pi(t,\vec{y})\cdot\vec{\nabla}\phi(t,\vec{x})]. \end{aligned} \quad (117)$$

The second and third terms cancel each other out due to $f(-\vec{x}) = -f(\vec{x})$, while the sixth and seventh terms [the latter upon using $\vec{\nabla}_x f(\vec{x}-\vec{y}) = -\vec{\nabla}_y f(\vec{x}-\vec{y})$ and integration by parts with respect to y] cancel each one out by themselves. Finally, the eighth term can be made to cancel the fifth one by integrating by parts $\vec{\nabla}\phi(t,\vec{x})$ and imposing

$$\nabla^2 g = h, \quad \text{and} \quad g(\vec{x}) = g(-\vec{x}) \quad (118)$$

[which implies also that $h(\vec{x}) = h(-\vec{x})$]. One is left then with

$$\begin{aligned}\delta L &= \int \mathrm{d}^2x\mathrm{d}^2y[h(\vec{x}-\vec{y})\phi(t,\vec{y})\dot{\phi}(t,\vec{x}) \\ &\quad + g(\vec{x}-\vec{y})\dot{\pi}(t,\vec{y})\pi(t,\vec{x})] \\ &= \dot{F}, \end{aligned} \quad (119)$$

with

$$\begin{aligned}F = \frac{1}{2}\int \mathrm{d}^2x\mathrm{d}^2y[&h(\vec{x}-\vec{y})\phi(t,\vec{y})\phi(t,\vec{x}) \\ &+ g(\vec{x}-\vec{y})\pi(t,\vec{y})\pi(t,\vec{x})], \end{aligned} \quad (120)$$

and where $g(-\vec{x}) = g(\vec{x})$ and $h(-\vec{x}) = h(\vec{x})$ have also been used. The conserved charge is given by

$$Q = \int d^2x\pi(t,\vec{x})\delta\phi(t,\vec{x}) - F, \tag{121}$$

and one immediately gets

$$Q = \int d^2x d^2y\bigg(f(\vec{x}-\vec{y})\pi(t,\vec{x})\phi(t,\vec{y}) + \frac{1}{2}g(\vec{x}-\vec{y})\pi(t,\vec{x})\pi(t,\vec{y}) - \frac{1}{2}h(\vec{x}-\vec{y})\phi(t,\vec{y})\phi(t,\vec{x})\bigg), \tag{122}$$

which has the form of the canonical generators P_ℓ (49) (in terms of ϕ and π) for the supertranslations.

B. Noether charges of superrotations

When acting over the Fourier modes, assuming they vanish sufficiently quickly for high momentum (that is, boundary terms can be ignored), one obtains the simple transformation

$$\{\mathcal{R}_n, a(\vec{q})\} = iL_n a(\vec{q}), \tag{123}$$

$$\{\mathcal{R}_n, \bar{a}(\vec{q})\} = iL_n \bar{a}(\vec{q}). \tag{124}$$

Again, one can lift from on shell to off shell the variations of the fields ϕ and π in the Hamiltonian formalism under a superrotation

$$\delta_{SR}\phi = \int \tilde{dk}(\delta a(\vec{k})e^{ikx} + \delta\bar{a}(\vec{k})e^{-ikx}) = \int d^2y\{\phi(t,\vec{y})F_n(\vec{x},\vec{y}) + \pi(t,\vec{y})G_n(\vec{x},\vec{y})\}, \tag{125}$$

$$\delta_{SR}\pi = \int \tilde{dk}(-ik^0)(\delta a(\vec{k})e^{ikx} - \delta\bar{a}(\vec{k})e^{-ikx}) = \int d^2y\{\phi(t,\vec{y})\tilde{H}_n(\vec{x},\vec{y}) + \pi(t,\vec{y})\tilde{I}_n(\vec{x},\vec{y})\}, \tag{126}$$

where we have used the expressions of Fourier modes in terms of the field and momentum (108) and (109) off shell, where

$$F_n(\vec{x},\vec{y}) = -i\int \tilde{dk}[e^{ikx}(L_n e^{-iky}k^0) + e^{-ikx}(L_n e^{iky}k^0)], \tag{127}$$

$$G_n(\vec{x},\vec{y}) = \int \tilde{dk}[e^{ikx}L_n e^{-iky} - e^{-ikx}L_n e^{iky}], \tag{128}$$

$$\tilde{H}_n(\vec{x},\vec{y}) = -\int \tilde{dk}k^0[e^{ikx}(L_n e^{-iky}k^0) - e^{-ikx}(L_n e^{iky}k^0)] \tag{129}$$

$$\tilde{I}_n(\vec{x},\vec{y}) = -i\int \tilde{dk}k^0[e^{ikx}L_n e^{-iky} + e^{-ikx}L_n e^{iky}]. \tag{130}$$

In contrast to the supertranslation case, the functions involved in the nonlocal transformation do not depend solely on the difference $\vec{y}-\vec{x}$ but on different combinations of these variables. More explicitly,

$$F_n(\vec{x},\vec{y}) = -i\int \tilde{dk}2\omega_n(\vec{k})[in\cos(\vec{k}(\vec{y}-\vec{x})) - (in\vec{y}\cdot\vec{k} + \vec{y}\times\vec{k})\sin(\vec{k}(\vec{y}-\vec{x}))], \tag{131}$$

$$G_n(\vec{x},\vec{y}) = i\int \tilde{dk}2\frac{\omega_n(\vec{k})}{k^0}[(in\vec{y}\cdot\vec{k} + \vec{y}\times\vec{k})\cos(\vec{k}(\vec{y}-\vec{x}))], \tag{132}$$

$$\tilde{H}_n(\vec{x},\vec{y}) = i\int \tilde{dk}2k^0\omega_n(\vec{k})[in\sin(\vec{k}(\vec{y}-\vec{x})) + (in\vec{y}\cdot\vec{k} + \vec{y}\times\vec{k})\cos(\vec{k}(\vec{y}-\vec{x}))], \tag{133}$$

$$\tilde{I}_n(\vec{x},\vec{y}) = i\int \tilde{dk}2\omega_n(\vec{k})[(in\vec{y}\cdot\vec{k} + \vec{y}\times\vec{k})\sin(\vec{k}(\vec{y}-\vec{x}))], \tag{134}$$

where $\vec{y}\times\vec{k} \equiv y_1k_2 - y_2k_1$. Here, there is an important remark to make concerning the parity of $|n|$: if $|n|$ is odd, $F_n = \tilde{I}_n = 0$, and if $|n|$ is even, $G_n = \tilde{H}_n = 0$. For the case of rotations, $L_0 = -iJ$:

$$F_0(\vec{x},\vec{y}) = 2i\int \tilde{dk}k^0[\vec{y}\times\vec{k}]\sin(\vec{k}(\vec{y}-\vec{x})), \tag{135}$$

$$G_0(\vec{x},\vec{y}) = 2i\int \tilde{dk}[\vec{y}\times\vec{k}]\cos(\vec{k}(\vec{y}-\vec{x})). \tag{136}$$

By symmetry properties, $G_0(\vec{x},\vec{y}) = 0$, and one can substitute F_0 into (125) and show that the usual rotation is recovered:

$$\delta_{SR_0}\phi = i(x_1\partial_{x_2}\phi(t,\vec{x}) - x_2\partial_{x_1}\phi(t,\vec{x})). \tag{137}$$

Recall that when we defined superrotations in Sec. II C, we used the truncated (time-independent) form of Lorentz generators constructed in (9). Thus, when trying to recover ordinary boosts, which involve time, we will need to redefine superrotations to take this under consideration. The final form for superrotations will be

$$\delta_{\text{ordinary SR}_n}\phi = -nt\delta_{ST_n}\phi + \delta_{SR_n}\phi, \tag{138}$$

which now accounts for time translations. With this definition, one can see that, for $n = 1$, a combination of ordinary boosts is recovered,

$$\delta_{\text{ordinary SR}_1}\phi = t\partial_{x_1}\phi(t,\vec{x}) + it\partial_{x_2}\phi(t,\vec{x}) - x_1\pi(t,\vec{x}) - ix_2\pi(t,\vec{x}). \tag{139}$$

Hence, the true superrotation generators will be a combination of the already constructed ones plus a proportional term depending on supertranslations. This can be written as follows:

$$\mathcal{G}_n = -ntP_n + \mathcal{R}_n. \tag{140}$$

The generators $\mathcal{R}_n$ can be written off shell as

$$\begin{aligned}\mathcal{R}_n = \frac{1}{2}\int d^2y d^2x[&\phi(t,\vec{x})\phi(t,\vec{y})(\tilde{H}_n + i\tilde{F}_n)\\ &- i\phi(t,\vec{x})\pi(t,\vec{y})(\tilde{G}_n - i\tilde{I}_n)\\ &- i\pi(t,\vec{x})\phi(t,\vec{y})(H_n + iF_n)\\ &+ \pi(t,\vec{x})\pi(t,\vec{y})(G_n - iI_n)],\end{aligned} \tag{141}$$

where $\tilde{F}_n$ and $\tilde{G}_n$ are as the functions F_n and G_n in (131) and (132), respectively, but with an extra k^0 factor under the integral sign, while H_n and I_n are defined as the corresponding functions, but with an additional $1/k^0$ factor.

The $\mathcal{G}_n$ given in (140) are constants of motion

$$\begin{aligned}\frac{d\mathcal{G}_n}{dt} &= \partial_t\mathcal{G}_n + \{\mathcal{G}_n, H\}\\ &= -nP_n - nt\{P_n, P_0\} + \{\mathcal{R}_n, P_0\}\\ &= -nP_n + nP_n = 0,\end{aligned} \tag{142}$$

where we have used $H = P_0$. This was expected, since the Lagrangian is invariant under Lorentz transformations.

The new field variations, $\delta_{\mathcal{G}_n}\phi = -nt\delta_{ST}\phi + \delta_{SR}\phi$, are solutions on shell of the massless Klein-Gordon equation:

$$\begin{aligned}\Box\delta_{\mathcal{G}_n}\phi &= -nt\Box\delta_{ST}\phi + n\partial_t\delta_{ST}\phi + \Box\delta_{SR}\phi\\ &= n\partial_t\delta_{ST}\phi + \Box\delta_{SR}\phi\\ &= n\epsilon^n\int d^2y[f_n(\vec{x}-\vec{y})\dot{\phi}(t,\vec{y}) + g_n(\vec{x}-\vec{y})\dot{\pi}(t,\vec{y})]\\ &\quad + \epsilon^n\int d^2y[\{\nabla^2_{\vec{x}}F_n(\vec{x},\vec{y}) - \nabla^2_{\vec{y}}F_n(\vec{x},\vec{y})\}\phi(t,\vec{y})\\ &\quad + \{\nabla^2_{\vec{x}}G_n(\vec{x},\vec{y}) - \nabla^2_{\vec{y}}G_n(\vec{x},\vec{y})\}\pi(t,\vec{y})].\end{aligned} \tag{143}$$

Using now the on-shell condition $\dot{\pi} = \ddot{\phi} = \nabla^2\phi$, integrating by parts and using symmetry properties of g in the first integral, and expanding the second one, and then using $h_n = \nabla^2 g_n$, we immediately see that $\Box\delta_{\mathcal{G}_n}\phi = 0$. The algebra of the charges is

$$\{\mathcal{G}_n, \mathcal{G}_m\} = (n-m)\mathcal{G}_{n+m}. \tag{144}$$

Thus, we have found another realization of the superrotations, which now reduce to the true Lorentz generators as defined in (8). Indeed,

$$\mathcal{G}_0 = \mathcal{R}_0 = -i\int \tilde{dk}\,\bar{a}(\vec{k})Ja(\vec{k}) = M^{12}, \tag{145}$$

$$\begin{aligned}\mathcal{G}_1 &= -tP_1 + \mathcal{R}_1\\ &= -t\int \tilde{dk}\,\bar{a}(\vec{k})(k_1 + ik_2)a(\vec{k}) + \int \tilde{dk}\,\bar{a}(\vec{k})K_+a(\vec{k})\\ &= -M^{01} - iM^{02},\end{aligned} \tag{146}$$

$$\begin{aligned}\mathcal{G}_{-1} &= tP_{-1} + \mathcal{R}_{-1}\\ &= -t\int \tilde{dk}\,\bar{a}(\vec{k})(k_1 - ik_2)a(\vec{k}) - \int \tilde{dk}\,\bar{a}(\vec{k})K_-a(\vec{k})\\ &= -M^{01} + iM^{02}.\end{aligned} \tag{147}$$

IV. CONCLUSIONS AND OUTLOOK

Using the canonical formalism for a real scalar field, a realization of the BMS group in three dimensions has been constructed in the space of Fourier modes for both massive and massless cases. In the massless case, the superrotation extension of this group can also be constructed by generalizing the Lorentz group in a manner similar to that used for supertranslations.

In the massive case, we have constructed in a heuristic way a set of generators which generalize those of the Lorentz group and reproduce the corresponding part of the $\mathfrak{bms}_3$ algebra. We have shown how our starting equations arise in a geometrical setup, and we have also obtained an equation for the 1-forms associated with the generators.

However, unlike what happens in the massless case, the superrotation generators must be split into two different extensions of the Lorentz algebra, each spanning a subalgebra of $\mathfrak{bms}_3$.

At the quantum level, the Hilbert space of one-particle states supports a unitary irreducible representation of the Poincaré group, and at the same time a unitary reducible representation of the BMS_3 group. Both are realized in an unbroken way.

The BMS_3 transformations are realized as symmetries of the KG action in terms of linear nonlocal functionals of the field and the canonical momentum. The corresponding conserved Noether charges have been computed.

In addition to obtaining a better understanding of the extension of superrotations in the massive case, some further questions are still open for future work.

There was the belief, in the gravitational approach, that BMS was not present in higher dimensions. From the viewpoint considered in this paper, there is no reason to think so, and the method presented here could help to investigate it. In fact, it can be proved that the canonical realization of BMS in higher dimensions does exist [33].

One could also try to add a fermionic field to the present model in order to get a field supersymmetric theory and to see whether there are still conserved charges generated by the extended BMS transformations. In the gravitational approach, this was studied in [34–37].

Finally, there is the question relating to the physical interpretation of the BMS symmetries and charges in the framework that we have used. A possible way to throw light on this issue is to try to construct particle models exhibiting these symmetries using the method of nonlinear realizations [38]. We also conjecture that the nonlocality of the transformations is due to the fact that they are computed for fields depending only on the standard space-time coordinates, and that they would become local for fields depending also on the *supercoordinates* associated with the supermomenta, i.e., the generators of supertranslations.

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APPENDIX A: BEHAVIOR OF f_ℓ, g_ℓ, AND h_ℓ

In this appendix we study in some detail the functions which appear in the nonlocal transformations constructed in Sect. III. We consider only the massless case, although similar—but more complicated—expressions can be obtained in the massive case using (43).

The functions $f_\ell(\vec{x})$ can be written as

$$f_\ell(\vec{x}) = \frac{1}{(2\pi)^2}\int d^2k\,\omega_\ell(\vec{k})\sin(\vec{k}\cdot\vec{x}),$$

where (in the massless case)

$$\omega_\ell(\vec{k}) = |\vec{k}|e^{i\ell\phi_k},$$

with

$$\cos\phi_k = \frac{k_1}{|\vec{k}|}, \qquad \sin\phi_k = \frac{k_2}{|\vec{k}|}.$$

Notice that $\omega_{-\ell}(\vec{k}) = \omega^*_\ell(\vec{k})$.

The Fourier transform of f_ℓ is

$$\hat{f}_\ell(\vec{q}) = \int d^2x f_\ell(\vec{x})e^{-i\vec{q}\cdot\vec{x}} = \frac{1}{2i}(\omega_\ell(\vec{q}) - \omega_\ell(-\vec{q})).$$

Since $\phi_{-q} = \phi_q + \pi$, one has

$$\omega_\ell(-\vec{q}) = \omega_\ell(\vec{q})e^{i\ell\pi} = (-1)^\ell\omega_\ell(\vec{q}) \tag{A1}$$

and

$$\hat{f}_\ell(\vec{q}) = \begin{cases} 0 & \text{if } \ell \text{ is even,} \\ -i\omega_\ell(\vec{q}) & \text{if } \ell \text{ is odd.} \end{cases} \tag{A2}$$

For a general value of ℓ, one can write ω_ℓ as a function of q^1 and q^2 as

$$\omega_\ell(\vec{q}) = \frac{(q^1 + iq^2)^\ell}{((q^1)^2 + (q^2)^2)^{\frac{\ell-1}{2}}}. \tag{A3}$$

Hence, for an odd ℓ, $\hat{f}_\ell(\vec{q})$ grows as $|\vec{q}|$, while $\lim_{|\vec{q}|\to 0}\hat{f}_\ell(\vec{q}) = 0$.

Specifically, for $\ell = 1$ one has $\hat{f}_1(\vec{q}) = -i(q_1 + iq_2)$. The physical components are

$$\frac{\hat{f}_1(\vec{q}) + \hat{f}_{-1}(\vec{q})}{2} = -iq_1,$$
$$\frac{\hat{f}_1(\vec{q}) - \hat{f}_{-1}(\vec{q})}{2i} = -iq_2,$$

which correspond to ordinary translations along the coordinate axes. For $\ell = 2m + 1$, $m \geq 1$, one gets in the denominator of ω_ℓ positive integer powers of $q_1^2 + q_2^2 = |\vec{q}|^2$, which implies that the transformation is nonlocal. For instance, for $\ell = 3$,

$$\hat{f}_3(\vec{q}) = -i\frac{q_1^3 + 3iq_1^2q_2 - 3q_2^2q_1 - iq_2^3}{q_1^2 + q_2^2}.$$

Similarly, one can see that

$$\hat{g}_\ell(\vec{q}) = \begin{cases} \frac{1}{|\vec{q}|}\omega_\ell(\vec{q}), & \text{if } \ell \text{ is even,} \\ 0, & \text{if } \ell \text{ is odd.} \end{cases} \tag{A4}$$

For an even ℓ, one has, explicitly,

$$\hat{g}_\ell(\vec{q}) = \frac{(q_1 + iq_2)^\ell}{(q_1^2 + q_2^2)^{\ell/2}}. \tag{A5}$$

Specifically, for $\ell = 0$, one gets $\hat{g}_0(\vec{q}) = 1$, that is, $g_0(\vec{x}) = \delta(\vec{x})$, which yields a local transformation, corresponding to translations in time, while for $\ell \geq 2$ the transformations are nonlocal.

Finally, since $h_\ell(\vec{x}) = \nabla^2 g_\ell(\vec{x})$, one has $\hat{h}_\ell(\vec{q}) = -|\vec{q}|^2 \hat{g}_\ell(\vec{q})$ and

$$\hat{h}_\ell(\vec{q}) = \begin{cases} -|\vec{q}|\omega_\ell(\vec{q}), & \text{if } \ell \text{ is even,} \\ 0, & \text{if } \ell \text{ is odd.} \end{cases} \tag{A6}$$

For an even ℓ, this is

$$\hat{h}_\ell(\vec{q}) = -\frac{(q_1 + iq_2)^\ell}{(q_1^2 + q_2^2)^{\ell/2-1}}. \tag{A7}$$

This yields local transformations for $\ell = 0$, 2 and is nonlocal for $\ell \geq 4$.

APPENDIX B: GEOMETRY OF THE MASS-SHELL HYPERBOLOID IN (2+1) DIMENSIONS

We list here some results for the geometry of the mass-shell hyperboloid of a massive particle in 2 + 1 dimensions that are useful for the construction of supertranslations and superrotations.

The metrics in (z, ϕ) coordinates (25) and its inverse are, in matrix form,

$$g = \begin{pmatrix} \frac{m^2}{z^2-1} & 0 \\ 0 & m^2(z^2-1) \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} \frac{z^2-1}{m^2} & 0 \\ 0 & \frac{1}{m^2(z^2-1)} \end{pmatrix}. \tag{B1}$$

The nonzero Christoffel symbols are

$$\Gamma^z_{zz} = -\frac{z}{z^2-1}, \quad \Gamma^z_{\phi\phi} = -z(z^2-1), \quad \Gamma^\phi_{z\phi} = \Gamma^\phi_{\phi z} = \frac{z}{z^2-1}. \tag{B2}$$

Given a vector field ξ on the manifold,

$$\xi = \xi^z \partial_z + \xi^\phi \partial_\phi, \tag{B3}$$

we can construct an associated 1-form using g,

$$\omega_\xi = \frac{m^2}{z^2-1}\xi^z \mathrm{d}z + m^2(z^2-1)\xi^\phi \mathrm{d}\phi. \tag{B4}$$

The divergence of a vector field is

$$\begin{aligned} \nabla_z \xi^z + \nabla_\phi \xi^\phi &= \partial_z \xi^z + \Gamma^z_{z\alpha}\xi^\alpha + \partial_\phi \xi^\phi + \Gamma^\phi_{\phi\alpha}\xi^\alpha \\ &= \partial_z \xi^z - \frac{z}{z^2-1}\xi^z + \partial_\phi \xi^\phi + \frac{z}{z^2-1}\xi^z \\ &= \partial_z \xi^z + \partial_\phi \xi^\phi, \end{aligned} \tag{B5}$$

so it coincides with the flat divergence. The Beltrami-Laplace operator acting on a function $f(z, \phi)$ is

$$\begin{aligned} \Delta f &= \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}g^{\alpha\beta}\partial_\beta f) \\ &= \partial_\alpha g^{\alpha\beta}\partial_\beta f + g^{\alpha\beta}\partial_\alpha\partial_\beta f \\ &= \frac{2z}{m^2}\partial_z f + \frac{z^2-1}{m^2}\partial_z^2 f + \frac{1}{m^2(z^2-1)}\partial_\phi^2 f. \end{aligned} \tag{B6}$$

If we denote this scalar Laplacian by Δ_S, on vector fields one has

$$g^{\mu\alpha}\nabla_\mu \nabla_\alpha \xi^\nu = \Delta_S \xi^\nu + g^{\mu\alpha}\nabla_\mu(\Gamma^\nu_{\alpha\beta}\xi^\beta). \tag{B7}$$

The nonzero components of the Riemann curvature tensor are

$$\begin{aligned} &R^z{}_{\phi z\phi} = -(z^2-1), \qquad R^z{}_{\phi\phi z} = z^2-1, \\ &R^\phi{}_{zz\phi} = \frac{1}{z^2-1}, \qquad R^\phi{}_{z\phi z} = -\frac{1}{z^2-1}, \end{aligned} \tag{B8}$$

and the Ricci scalar curvature is

$$R = -\frac{2}{m^2}. \tag{B9}$$

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