Lamb shift and the gravitational binding energy for binary black holes

Rafael A. Porto
ICTP South American Institute for Fundamental Research,
Rua Dr. Bento Teobaldo Ferraz 271, 01140-070 São Paulo, SP Brazil
(Received 25 March 2017; published 31 July 2017)

We show that the correction to the gravitational binding energy for binary black holes due to the *tail effect* resembles the *Lamb shift* in the Hydrogen atom. In both cases a *conservative* effect arises from interactions with *radiation* modes, and moreover an explicit cancelation between near and far zone divergences is at work. In addition, regularization scheme-dependence may introduce "ambiguity parameters." This is remediated—within an effective field theory approach—by the implementation of the *zero-bin* subtraction. We illustrate the procedure explicitly for the Lamb shift, by performing an ambiguity-free derivation within the framework of nonrelativistic electrodynamics. We also derive the renormalization group equations from which we reproduce Bethe logarithm (at order $\alpha_e^5 \log \alpha_e$), and likewise the contribution to the gravitational potential from the tail effect (proportional to $v^8 \log v$).

DOI: 10.1103/PhysRevD.96.024063

I. INTRODUCTION

Binary coalescences are posed to become standard sources for present and future gravitational wave (GW) observatories [1–3]. GW astronomy will map the contents of the universe to an unprecedented level [4,5], addressing fundamental problems in astrophysics and cosmology. The searches demand state-of-the-art numerical and analytical modeling, to enable the most precise parameter estimation [6–8]. Motivated by the construction of an accurate template bank, the effective field theory (EFT) framework was introduced to solve for the gravitational dynamics of inspiraling binary systems to high level of precision [9–16]. The EFT approach was originally coined nonrelativistic general relativity (NRGR) [9], following similarities with the techniques used for the strong interaction (NRQCD), as well as electrodynamics (NRQED). NRGR has enabled the computation of all the ingredients for the GW phase for spinning compact binary systems up to third post-Newtonian (3PN) order [16–25]. In addition, significant progress has been achieved towards 4PN accuracy in the EFT approach, both for nonspinning [26–28] and rotating bodies [29,30]. Some of these results have been obtained using other (more traditional) methods, see e.g. [6,7] for references.

The gravitational binding potential for binary systems has been recently computed in the Arnowitt, Deser and Misner (ADM) and "Fokker-action" approaches up to 4PN order for nonspinning bodies [31–37]. Despite the remarkable feat, the derivation could not be completed at first, because of regularization ambiguities. Hence, the final expression was obtained after comparison with gravitational self-force calculations [33,36], see also [38]. In a companion paper [39] we describe the procedure which yields the gravitational potential, in NRGR, without the need of "ambiguity parameters." The purpose of the present paper is to demonstrate that the issue at hand is actually more common than it might seem, since similar considerations apply in electrodynamics, and in particular in the derivation of the *Lamb shift* [40–44]. As we shall see, by performing the calculation within the EFT approach NRQED, both infrared (IR) and ultraviolet (UV) divergences are present, as in the gravitational case. We perform the *zero-bin* subtraction [45] and arrive at an ambiguity-free result. We also derive the renormalization group equation for the binding potential, and readily obtain the Bethe logarithm. We then show how the manipulations in electrodynamics closely resemble the computations in gravity. In particular, the renormalization group evolution and logarithmic contributions to the binding energy may be obtained in both cases without worrying about the subtleties of the matching conditions [28]. Throughout this paper we work in $c = \hbar = 1$ units, unless otherwise noted.

II. THE (QUANTUM) BINDING ENERGY IN ELECTRODYNAMICS

Quantum effects in QED contribute to the binding energy of the Hydrogen atom. A celebrated example is the Lamb shift [40–44], which involves a one-loop vertex correction, see Fig. 1. Here we perform the computation

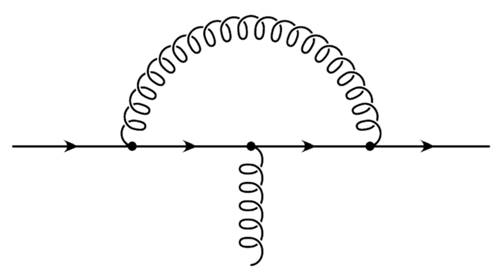

FIG. 1. One loop vertex correction in electrodynamics.

using an EFT approach, highlighting the similarities with the binary inspiral case. We show the existence of IR/UV divergences, discuss the zero-bin subtraction and lack of ambiguities, and the renormalization group structure.

A. Form factors

The full QED vertex (including wave-function renormalization) can be expressed in terms of two form factors,

$$-ie\bar{u}(p_1)\left[F_1(q^2)\gamma^\mu + \frac{i}{2m_e}F_2(q^2)\sigma^{\mu\nu}q_\nu\right]u(p_2), \qquad (2.1)$$

with $q = p_1 - p_2$, γ^μ the Dirac matrices, $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and $u(p)$ a Dirac spinor. The expressions for F_1, F_2 are divergent, and in dimensional regularization (dim. reg.) are given by, e.g. [46],

$$F_1(q^2) = 1 - \frac{\alpha_e(\mu)}{\pi}\frac{q^2}{m_e^2}\left[\frac{1}{3\epsilon_{\rm IR}} + \frac{1}{8} - \frac{1}{6}\log\frac{m_e^2}{\bar{\mu}^2}\right] + \mathcal{O}(q^4), \qquad (2.2)$$

$$F_2(q^2) = \frac{\alpha_e(\mu)}{2\pi}\left[1 + \frac{q^2}{6m_e^2}\right] + \mathcal{O}(q^4), \qquad (2.3)$$

where $\alpha_e \equiv e^2(\mu)/4\pi$ is the fine-structure constant, m_e the mass of the electron, and we have expanded to order q^2/m_e^2 the resulting integrals. The factor of $\bar{\mu}^2 \equiv 4\pi e^{-\gamma_E}\mu^2$, with γ_E the Euler constant, appear in dim. reg. as the "subtraction scale."[1]

We will encounter both IR as well as UV divergences, which in dim. reg. emerge as poles in $\epsilon_{\rm IR/UV} \equiv (d-4)_{\rm IR/UV}$, as we approach $d = 4$ dimensions. While intermedia UV divergences are present, the final expressions for the form factors are UV finite, featuring instead an IR pole (often regularized with a photon mass).[2]

From (2.2) and (2.3) we can derive for instance the one-loop correction to the scattering amplitude in QED, and the Lamb shift. However, in order to draw parallels with computations in gravity, in what follows we will perform the calculation within the framework of non-relativistic QED (NRQED).

[1] In the expressions below we omitted the bar in the $\log\bar{\mu}$'s, for convenience. The distinction is irrelevant for our purposes.

[2] The form factor in (2.2) also enters in the scattering amplitude, and the IR pole is ultimately removed from the cross section by including IR divergences from (ultra-)soft photon emission [47]. However, as we shall see, for the binding energy the low-energy modes contribute a UV divergence instead. This is reminiscent to the gravitational scenario, where the IR divergences in the radiative multipoles turn into UV poles in the computation of the gravitational potential [28] (see below).

B. The EFT framework: NRQED

In addition to the electron's mass, we have two other relevant scales in the bound state problem. There is Bohr's radius,

$$r_B \simeq 1/(m_e v), \qquad (2.4)$$

with v the relative velocity, and the typical frequency scale given by the Rydberg energy

$$E \simeq m_e v^2, \qquad (2.5)$$

which determines the split between levels. In a bound state the virial theorem implies

$$\alpha_e/r_B \sim m_e v^2 \to \alpha_e \sim v. \qquad (2.6)$$

After one eliminates the heavy scale in the theory, m_e, as in the heavy quark effective theory (HQET), we are left with three relevant regions [48–50]: potential modes scaling as

$$(p^0_{\rm pot}, \boldsymbol{p}_{\rm pot}) \sim (m_e v^2, m_e v) \sim (v/r_B, 1/r_B), \qquad (2.7)$$

soft modes,

$$(p^0_S, \boldsymbol{p}_S) \sim (m_e v, m_e v) \sim (1/r_B, 1/r_B), \qquad (2.8)$$

and *ultra-soft* ones,

$$(p^0_{US}, \boldsymbol{p}_{US}) \sim (m_e v^2, m_e v^2) \sim (v/r_B, v/r_B). \qquad (2.9)$$

Notice these power counting rules are similar to the ones in NRGR, for potential and radiation fields.[3] The effective Lagrangian density for NRQED takes the form (ignoring spin interactions for simplicity) [46,50,51]

$$\mathcal{L}_{\rm NRQED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \psi_e^{v\dagger}\left(iD_0 + \frac{\boldsymbol{D}^2}{2m_e} + \frac{\boldsymbol{D}^4}{8m_e^3} + e\frac{c_V}{8m_e^2}\boldsymbol{\nabla}\cdot\boldsymbol{E} + \cdots\right)\psi_e^v + i\psi_p^\dagger D_0\psi_p + \cdots, \qquad (2.10)$$

where D_μ is the covariant derivative, ψ_e^v is given by $\psi_e^v = e^{im_e t}\psi_e$, as in HQET, and we have kept only the terms which are relevant for our purposes. We have also added the contribution from the proton, ψ_p, which we treat as a static source, up to $\mathcal{O}(m_e/m_p)$ corrections. The matching coefficient, c_V, is given by [46]

[3] The (on-shell) soft modes are not present in classical computations, since they *kick* the massive particle (e.g. the electron) off of the mass shell, $E \sim m_e v^2$.

$$c_V = F_1(0) + 2F_2(0) + 8m_e^2 \frac{d}{dq^2} F_1(0), \qquad (2.11)$$

with the form factors in (2.2) and (2.3). In dim. reg. the expression for c_V reads

$$c_V = 1 + \frac{8}{3}\frac{\alpha_e(\mu)}{\pi}\left[-\frac{1}{\epsilon_{\rm IR}} + \log m_e/\mu\right]. \qquad (2.12)$$

Notice we have kept the IR pole explicitly, and will be carried over until the end of the calculation. We will discuss later on in section (2.4) how to properly handle this divergence prior to computing the Lamb shift. As we shall demonstrate, this IR pole will be linked to a UV singularity arising from the ultra-soft sector. (This will be intimately related to cancelation of factors of $\log\mu$.)

The next step is to integrate out the potential and soft modes. This procedure matches NRQED into an effective theory with ultra-soft degrees of freedom only, called "potential" NRQED or pNRQED for short [52]. The binding energy now becomes a matching coefficient. Therefore, we have a Coulomb-type potential of the form [52],[4]

$$\int dt \int d^3\boldsymbol{x}_1 d^3\boldsymbol{x}_2 \psi_p^\dagger(t,\boldsymbol{x}_1)\psi_p(t,\boldsymbol{x}_1)\left(\frac{\alpha_e}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\right) \times \psi_e^\dagger(t,\boldsymbol{x}_2)\psi_e(t,\boldsymbol{x}_2). \qquad (2.13)$$

For the term proportional to c_V we may use Gauss' law, obtaining [52]

$$-c_V\frac{e^2}{8m_e^2}\int dt \int d^3\boldsymbol{x}_1 d\boldsymbol{x}_2 \psi_p^\dagger(t,\boldsymbol{x}_1)\psi_p(t,\boldsymbol{x}_1)\psi_e^\dagger(t,\boldsymbol{x}_2) \times \psi_e(t,\boldsymbol{x}_2)\delta^3(\boldsymbol{x}_1-\boldsymbol{x}_2). \qquad (2.14)$$

Since the typical size of the bound state is given by $r_B \ll 1/E$, the ultra-soft photon field is multipole expanded in powers of $Er_B \sim v \sim \alpha_e$. This is reminiscent of the construction of the radiation theory in NRGR, in terms of a series of multipole moments [24]. At the end of the day, the relevant pieces in the pNRQED Lagrangian are[5] [46,52]

$$L_{\rm pNRQED} = \int d^3\boldsymbol{x}\psi^\dagger(t,\boldsymbol{x})\bigg(i\partial_0 - eA^0_{US}(t,0) + e\boldsymbol{x}^i\boldsymbol{\nabla}_i A^0_{US}(t,0) + \frac{\boldsymbol{\nabla}^2}{2m_e} - V(\boldsymbol{x}) - ie\frac{\boldsymbol{A}_{US}(t,0)\cdot\boldsymbol{\nabla}}{m_e} - c_V\frac{e^2}{8m_e}\delta^3(\boldsymbol{x})\bigg)\psi(t,\boldsymbol{x}) - \frac{1}{4}\int d^3\boldsymbol{x} F^{\mu\nu}_{US}F_{US\mu\nu}, \qquad (2.15)$$

where $V_e = -\alpha_e/|\boldsymbol{x}|$. We dropped the tag on the field, which now represents the wave-function of an electron in the background of a static Coulomb-like source with typical energy/momenta of order $m_e v^2$. Notice the contribution from c_V may be thought of as a local *renormalization* of the potential,

$$\delta V_e(\boldsymbol{x}) = c_V\frac{e^2}{8m_e}\delta^3(\boldsymbol{x}). \qquad (2.16)$$

C. The Lamb shift

The calculation of the Lamb shift can be found in different textbooks, e.g. [54]. Here we derive it following the framework of the EFT approach NRQED. (The use of dim. reg. to regularize the divergences in the computation of the Lamb shift was also advocated in [54–56].)

The ultra-soft contribution to the E_n level of the Hydrogen atom is represented in Fig. 2, and is given by a self-energy type diagram. The computation entails the two-point function

$$G(t,\boldsymbol{x}) \equiv -i\langle 0|T(\psi(0)\psi(t,\boldsymbol{x}))|0\rangle, \qquad (2.17)$$

which it is convenient to transform into Fourier space

$$\tilde{G}(\boldsymbol{x},E) = \int dt e^{iEt}G(t,\boldsymbol{x}). \qquad (2.18)$$

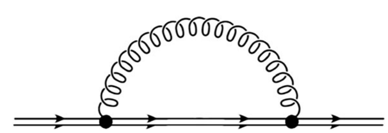

FIG. 2. The one-loop correction in (2.22). The double line represents the bound state, and the dots are the dipole-type coupling from (2.15). A similar diagram—albeit at the classical level—appears in NRGR (see below).

[4]We may construct first an EFT at the scale $m_e v$, integrating out the potential modes. In that case the interaction becomes nonlocal in space, but local in time [53].

[5]The coupling to ultra-soft photons can be rewritten in a manifestly gauge invariant manner in terms of the electric field, $\boldsymbol{E}_{US} = -\partial_0\boldsymbol{A}_{US} - \boldsymbol{\nabla}_i A^0_{US}$, leading to a traditional dipole-type interaction: $e\boldsymbol{x}\cdot\boldsymbol{E}_{US}$. However, the expression in [52] leads to a more transparent derivation of the Lamb shift in Coulomb gauge, since the $\boldsymbol{A}^0_{US}$ is a (nonpropagating) constrained variable in this gauge.

At leading order, introducing a complete set of states, we have

$$\tilde{G}_0(\boldsymbol{x}, E) = \sum_{n,\ell} \frac{\psi_{n,\ell}(0)\psi^\dagger_{n,\ell}(\boldsymbol{x})}{E - E_n + i\epsilon}, \tag{2.19}$$

where E_n is the unperturbed energy level, with wave functions $\psi_{n,\ell} \equiv \langle 0|\psi|n,\ell\rangle$, obeying

$$\hat{H}_0\psi_{n,\ell} = E_n\psi_{n,\ell}, \tag{2.20}$$

with

$$H_0 = \frac{\boldsymbol{p}^2}{2m_e} + V_e, \tag{2.21}$$

the unperturbed nonrelativistic Hamiltonian. The loop correction in Fig. 2 contributes to the self energy, $\Sigma(E)$, of the electron moving in a Coulomb background [57]. The one-loop diagram can be resumed as a Dyson series, leading to a correction to the Green's function,

$$\left(E - \frac{\boldsymbol{p}^2}{2m_e} - V_e - \Sigma(E)\right) G(\boldsymbol{x}, E) = 1, \tag{2.22}$$

and subsequently to the energy levels. Here $\boldsymbol{p}^i$ is the momentum operator: $\boldsymbol{p}^i = -i\boldsymbol{\nabla}^i$.

The self-energy diagram can be computed in dim. reg. using the Feynman rules from [52], and it reads[6]

$$\Sigma(E) = -i\frac{e^2}{m_e^2}\int \frac{d^dk}{(2\pi)^d}\left(\delta^{ij} - \frac{\boldsymbol{k}^i\boldsymbol{k}^j}{\boldsymbol{k}^2}\right)\frac{1}{k_0^2 - \boldsymbol{k}^2 + i\epsilon}\boldsymbol{p}^i\frac{1}{H_0 - E - k_0 + i\epsilon}\boldsymbol{p}^j. \tag{2.24}$$

Using (see footnote 1)

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k_0^2 - \boldsymbol{k}^2 + i\epsilon}\left(\delta^{ij} - \frac{\boldsymbol{k}^i\boldsymbol{k}^j}{\boldsymbol{k}^2}\right)\frac{1}{\omega - k_0 + i\epsilon} = i\frac{\omega}{6\pi^2}\delta^{ij}\left(\frac{1}{\epsilon_{\rm UV}} + \frac{5}{6} - \log\frac{2\omega}{\mu}\right), \tag{2.25}$$

we obtain,

$$\Sigma(E) = \frac{2\alpha_e}{3\pi}\frac{\boldsymbol{p}^i}{m_e}(H_0 - E)\left(\frac{1}{\epsilon_{\rm UV}} + \frac{5}{6} - \log\frac{2(H_0 - E)}{\mu}\right)\frac{\boldsymbol{p}^i}{m_e}. \tag{2.26}$$

Taking the limit $E \to E_n$, we find for the energy shift:

$$\begin{aligned}(\delta E_{n,\ell})_{US} = \frac{2\alpha_e}{3\pi}\Bigg[& e^2\left(\frac{1}{\epsilon_{\rm UV}} + \frac{5}{6}\right)\frac{|\psi_{n,\ell}(\boldsymbol{x}=0)|^2}{2m_e^2} \\ & - \sum_{m\neq n,\ell}\left\langle n,\ell\left|\frac{\boldsymbol{p}}{m_e}\right|m,\ell\right\rangle^2 (E_m - E_n)\log\frac{2|E_n - E_m|}{\mu}\Bigg]\end{aligned} \tag{2.27}$$

where we used [54]

$$\boldsymbol{p}^i(H_0 - E_n)\boldsymbol{p}^i = \frac{1}{2}\boldsymbol{\nabla}^2 V_e = \frac{e^2}{2}\delta^3(\boldsymbol{x}). \tag{2.28}$$

To complete the relevant part of the calculation we need to add the (local) contribution from the short-distance modes in (2.16), proportional to the Wilson coefficient c_V in (2.12), which yields

$$(\delta E_{n,\ell})_{c_V} = \langle n,\ell|\delta V_e|n,\ell\rangle = \frac{e^2}{8m_e^2}c_V|\psi_{n,\ell}(\boldsymbol{x}=0)|^2 = \frac{4\alpha_e^2}{3m_e^2}\left(-\frac{1}{\epsilon_{\rm IR}} + \log\frac{m_e}{\mu}\right)|\psi_{n,\ell}(\boldsymbol{x}=0)|^2. \tag{2.29}$$

[6]The (ultra-soft) photon propagator in Coulomb gauge is given by

$$D^{ij}_{US}(k_0, \boldsymbol{k}) = \frac{i}{k_0^2 - \boldsymbol{k}^2 + i\epsilon}\left(\delta^{ij} - \frac{\boldsymbol{k}^i\boldsymbol{k}^j}{\boldsymbol{k}^2}\right), \qquad D^{00}_{US}(k_0, \boldsymbol{k}) = \frac{i}{\boldsymbol{k}^2}. \tag{2.23}$$

The nonpropagating component contributes a (tadpole) scaleless integral ($\int \frac{dk_0}{k_0}$) that can be set to zero in dim. reg.

Therefore, combining the two terms together we have

$$\begin{aligned}\delta E_{n,\ell} &= (\delta E_{n,\ell})_{US} + (\delta E_{n,\ell})_{c_V} + \cdots \\ &= \frac{2\alpha_e}{3\pi}\left[\frac{5}{6}e^2\frac{|\psi_{n,\ell}(\boldsymbol{x}=0)|^2}{2m_e^2} - \sum_{m\neq n,\ell}\left\langle n,\ell\left|\frac{\boldsymbol{p}}{m_e}\right|m,\ell\right\rangle^2 (E_m - E_n)\log\frac{2|E_n - E_m|}{m_e}\right] + \cdots \\ &\quad + \frac{4\alpha_e^2}{3m_e^2}\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right)|\psi_{n,\ell}(\boldsymbol{x}=0)|^2. \end{aligned} \tag{2.30}$$

Notice the anticipated link between IR and UV divergences. Provided we identify the IR/UV poles, these two singular terms drop out of the computation, as the factors of $\log\mu$ do. The relevant scale in the logarithm is replaced by m_e. In the next subsection we will describe how to properly implement the cancelation. The remaining terms are the celebrated correction in the Lamb shift at leading order, including Bethe logarithm and the numerical factor of $5/6$ [41–44]. By power counting the (enhanced) logarithmic contribution, we find it scales as (recall $\alpha_e \sim v$)[7]

$$\delta E_{n,\ell} \simeq \alpha_e v^2 m_e v^2 \log(m_e v^2/m_e) \sim m_e \alpha_e^5 \log\alpha_e. \tag{2.31}$$

Notice that, if one treats the local contribution from δV_e in (2.16) independently, we would be misguided to remove the IR pole in (2.2) first, in order to arrive to a finite result. This, in turn, would introduce scheme-dependent ambiguities, since we could subtract from (2.2) either $1/\epsilon_{\rm IR}$ or $1/\epsilon_{\rm IR} + C$, with C some unspecified dimensionless constant. Hence, after removing the UV divergence from the ultra-soft loop with an (independent) counter term, we would need additional information to fix an undetermined contribution [54]

$$\delta V_e^{(C)} = C\frac{4\alpha_e^2}{3m_e^2}\delta^3(\boldsymbol{x}), \tag{2.32}$$

similarly to what occurs in the methodology in [31–37]. We discuss in what follows the steps which enable us to obtain an unambiguous result for the Lamb shift, regardless of the regularization scheme.

D. The zero-bin subtraction

We must implement a procedure in which modes other than the ultra-soft never leave the realm pertinent to the bound state, henceforth avoiding IR divergences. This is known as the zero-bin subtraction [45]. As an example, let us consider any one-loop graph in NRQED with contributions from different regions. Let us concentrate only on the propagating degrees of freedom, namely soft and ultra-soft modes. The soft part of the graph may have UV and IR divergences,

$$I_S = \frac{A_S}{\epsilon_{\rm UV}} + \frac{B_S}{\epsilon_{\rm IR}} + f_S(q,\mu), \tag{2.33}$$

with $q \sim m_e v$. The UV divergence is removed by a counterterm as usual, therefore, without loss of generality, we set $A_S = 0$. On the other hand, for the ultra-soft part,

$$I_{US} = \frac{A_{US}}{\epsilon_{\rm UV}} + \frac{B_{US}}{\epsilon_{\rm IR}} + f_{US}(E,\mu), \tag{2.34}$$

with $E \sim m_e v^2$. The IR divergences in the ultra-soft calculation would match into the IR singularities of the full theory, if any, in the quantity at hand. Let us assume the observable is IR safe in QED, and therefore $B_{US} = 0$. Since the method of regions is designed to reproduce the full theory computation in terms of relevant zones, we must have [48,53]

$$I_{\rm full} = I_S + I_{US} + I_{\rm hard}, \tag{2.35}$$

where the "hard" part corresponds to modes with $k \sim m_e$. This is the contribution which matches into Wilson coefficients, as a series of local terms.[8]

In general, we will find $B_S = -A_{US}$, which will be ultimately related to the cancelation of spurious divergences due to the splitting into regions. Therefore, adding the soft and ultra-soft contributions together,

$$I_S + I_{US} = f_S(q,\mu) + f_{US}(E,\mu) + B_S\left(\frac{1}{\epsilon_{\rm IR}} - \frac{1}{\epsilon_{\rm UV}}\right). \tag{2.36}$$

The role of the zero-bin subtraction is to remove from I_S the IR singularity. In other words, we replace

$$I_S \to I_S - I_{\text{zero-bin}}, \tag{2.37}$$

[7]One can actually think of two contributions, from $\log(Er_B)$ and (minus) $\log(m_e r_B)$, both scaling as $\log v$. In gravity, on the other hand, we only find a logarithm of the ratio between radiation and potential scales, at the desired order. Nevertheless, the basic steps are essentially the same in both cases.

[8]The method of regions and dim. reg. go hand-by-hand, enforcing that contributions from momenta $k \gg m_e$ can be ignored, since they turn into a scaleless integral.

where $I_{\text{zero-bin}}$ corresponds to an asymptotic expansion of the soft integral around the region responsible for the IR poles. This procedure removes the double-counting induced by the overlap between the IR sensitive part of the I_S integral and the contribution from I_{US}.

The zero-bin part may involve a scaleless integral, which in dim. reg. are usually set to zero. That is the case because they entail a *cancelation* between IR and UV poles. However, when IR divergences are present, scaleless integral require some extra care [53]. In dim. reg., the zero bin will often take the form,

$$I_{\text{zero-bin}} = B_S\left(\frac{1}{\epsilon_{\text{IR}}} - \frac{1}{\epsilon_{\text{UV}}}\right) + \text{finite}, \tag{2.38}$$

such that

$$I_S - I_{\text{zero-bin}} + I_{US} = f_S(q,\mu) + f_{US}(E,\mu), \tag{2.39}$$

See [45] for more details.

Returning to the case at hand, there are a few subtleties regarding the IR divergence in (2.2). In principle, the IR pole entered in the matching into NRQED.[9] However, an effective theory is constructed such that all the long-distance physics from the full theory is recovered. Hence, the IR divergence in (2.2), which trickled into c_V in (2.12), should be matched to a similar IR singularity in the effective theory [53]. The IR pole in the EFT side, however, is subtle, since it arises from scaleless integrals which are often ignored [46].[10] At the end of the day, this procedure (keeping scaleless integrals in the long-distance theory) is entirely equivalent to performing a zero-bin subtraction from I_{hard}, removing unwanted soft(er) modes prior to performing the matching. The advantage of implementing the zero-bin prescription is that it enables us to set to zero other scaleless integrals (for example the contribution from A^0_{US} in the calculation of the Lamb shift, see footnote 6), since all quantities are then IR safe. (Moreover, the zero-bin subtraction is independent of the regularization scheme.)

Let us return to the form factor in (2.2). If we denote as $(p, p-q)$ the incoming and outgoing momenta respectively, the vertex correction entails

$$I_{\text{vertex}} = -ie^2 p\cdot(p-q)\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+i\epsilon} \times \frac{1}{(p-k)^2-m_e^2+i\epsilon}\frac{1}{(p-q-k)^2-m_e^2+i\epsilon}. \tag{2.40}$$

The part of the integral with $k \sim m_e v$ is reproduced by the soft modes in NRQED, and likewise for the ultra-soft modes. On the other hand, the contribution from the hard region, which matches into Wilson coefficients, is given by modes with $k \sim m_e$. At leading order in q^2/m_e^2 we have,

$$I_{\text{hard}} = -ie^2 m_e^2 \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+i\epsilon}\left(\frac{1}{k^2 - p\cdot k + i\epsilon}\right)^2 + \mathcal{O}(q^2/m_e^2). \tag{2.41}$$

This integral clearly has an IR divergence, and the result reads

$$I_{\text{hard}} = \frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon_{\text{IR}}} + \log\mu/m_e\right) + \mathcal{O}(q^2/m_e^2). \tag{2.42}$$

The IR pole, however, appears from the region, $k \ll m_e$, which does not belong to I_{hard}. Therefore, we need to perform the (zero-bin) subtraction

$$I_{\text{zero-bin}} = -ie^2\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+i\epsilon}\left(\frac{1}{v\cdot k+i\epsilon}\right)^2, \tag{2.43}$$

where we used $p^\mu = m_e v^\mu$, and $p^2 = m_e^2$. This integral is easy to calculate in the rest frame, with $v^\mu = (1,0,0,0)$, yielding

$$I_{\text{zero-bin}} = \frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon_{\text{IR}}} - \frac{1}{\epsilon_{\text{UV}}}\right), \tag{2.44}$$

such that

$$I_{\text{hard}} - I_{\text{zero-bin}} = \frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon_{\text{UV}}} + \log\mu/m_e\right). \tag{2.45}$$

Iterating this procedure in all the IR sensitive terms transforms the IR pole in (2.12) into a UV singularity,

$$c_V \xrightarrow{\text{zero-bin}} 1 + \frac{8}{3}\frac{\alpha_e(\mu)}{\pi}\left[-\frac{1}{\epsilon_{\text{UV}}} + \log m_e/\mu\right]. \tag{2.46}$$

Following our computation of the Lamb shift, this UV pole now readily cancels against the UV divergence arising in the ultra-soft loop correction, see (2.30), unfolding the ambiguity-free final result. The same would have happened had we used any other regularization scheme.

[9]Technically speaking, QED is first matched into HQET by integrating out m_e. The same happens in the gravitational case, with the finite size scale identified with the hard modes.

[10]Notice that, while adding a scaleless integral from the EFT side may cancel the IR poles on both sides of the matching condition, it also leaves behind a UV divergent term, as in the zero-bin prescription. The latter would likewise cancel out against the UV divergence in the ultra-soft loop.

E. The renormalization group

In the previous calculation within NRQED we ended up without divergences, but also the factors of μ are gone after using (2.46). However, we could have approached the problem differently—from the bottom up—by computing directly in the ultra-soft effective theory. While the matching condition determines the value of the parameters in the effective theory (at a matching scale), the from of the effective Lagrangian can be constructed using the low-energy symmetries and degrees of freedom [51]. There is (at least for our purposes) only one Wilson coefficient, c_V, in the long-distance theory. The computation of the shift in the energy levels follows from the ultra-soft loop, which is UV divergent. From the point of view of the ultra-soft theory we can then use a counter term to renormalize the divergence. Hence, the UV pole may be removed via

$$c_V^{\text{c.t.}} = -\frac{8\alpha_e}{3\pi}\frac{1}{\epsilon_{\text{UV}}}, \tag{2.47}$$

or in terms of the local potential [see (2.16)]

$$\delta V_e^{\text{c.t.}} = -\frac{4\alpha_e^2}{3m_e^2}\frac{1}{\epsilon_{\text{UV}}}\delta^3(\boldsymbol{x}). \tag{2.48}$$

Putting the pieces together, we find

$$\delta E_{n,\ell} = \left[\left\{\frac{2\alpha_e}{3\pi}\left(\frac{5}{6}+\log\frac{\mu}{m_e}\right)+\frac{c_V^{\text{ren}}(\mu)}{4}\right\}e^2\frac{|\psi_{n,\ell}(\boldsymbol{x}=0)|^2}{2m_e^2} - \frac{2\alpha_e}{3\pi}\sum_{m\neq n,\ell}\left\langle n,\ell\left|\frac{\boldsymbol{p}}{m_e}\right|m,\ell\right\rangle^2(E_m-E_n)\log\frac{2|E_n-E_m|}{m_e}\right]+\cdots. \tag{2.49}$$

Notice two important differences. First of all, the appearance of a renormalized parameter, $c_V^{\text{ren}}(\mu)$, and the $\log\mu$. The binding energy is obviously μ-independent, and therefore one can obtain a renormalization group equation,

$$\mu\frac{d}{d\mu}\delta E_{n,\ell} = 0 \to \mu\frac{d}{d\mu}c_V^{\text{ren}}(\mu) = -\frac{8\alpha_e(\mu)}{3\pi}, \tag{2.50}$$

or, in other words,

$$\mu\frac{d}{d\mu}\delta V_e^{\text{ren}}(\boldsymbol{x},\mu) = -\frac{4\alpha_e^2}{3m_e^2}\delta^3(\boldsymbol{x}). \tag{2.51}$$

By solving this equation we find,[11]

$$c_V^{\text{ren}}(\mu) = c_V^{\text{ren}}(m_e) - \frac{8\alpha_e}{3\pi}\log\frac{\mu}{m_e}, \tag{2.52}$$

and likewise (in momentum space)

$$\delta V_e^{\text{ren}}(\boldsymbol{p},\mu) = \delta V_e^{\text{ren}}(\boldsymbol{p},m_e) - \frac{4\alpha_e^2}{3m_e^2}\log\frac{\mu}{m_e}. \tag{2.53}$$

The utility of this expression is clear. First of all, let us re-write (2.49) as

$$\begin{aligned}\delta E_{n,\ell} &= \langle n,\ell|\delta V_e^{\text{ren}}(\boldsymbol{x},\mu)|n,\ell\rangle \\ &\quad - \frac{2\alpha_e}{3\pi}\sum_{m\neq n,\ell}\left\langle n,\ell\left|\frac{\boldsymbol{p}}{m_e}\right|m,\ell\right\rangle^2(E_m-E_n) \\ &\quad \times\log\frac{2|E_n-E_m|}{\mu}+\cdots.\end{aligned} \tag{2.54}$$

If we now take $\mu\sim m_e v^2$, the second term in (2.54) becomes subdominant (since $\Delta E/(m_e v^2)\sim 1$). Hence, we directly obtain the logarithmic Lamb shift from the renormalization group equation (recall $\alpha_e\sim v$)

$$\begin{aligned}\delta E_{n,\ell} &= \langle n,\ell|\delta V_e^{\text{ren}}(\boldsymbol{x},\mu=m_e v^2)|n,\ell\rangle+\cdots \\ &= -\frac{4\alpha_e^2}{3m_e^2}|\psi_{n,\ell}(\boldsymbol{x}=0)|^2\log v^2+\cdots \\ &= -\frac{8}{3\pi}\frac{\delta_{\ell 0}}{n^3}m_e\alpha_e^5\log\alpha_e+\cdots,\end{aligned} \tag{2.55}$$

where (only the $\ell=0$ states have support at $\boldsymbol{x}=0$)

$$|\psi_{n,\ell}(\boldsymbol{x}=0)|^2 = \frac{\alpha_e^3 m_e^3}{\pi n^3}\delta_{\ell 0}, \tag{2.56}$$

for the Hydrogen atom. In this manner we unambiguously obtain Bethe logarithm directly from the long-distance effective theory. This is similar to what we find in the gravitational case, which we discuss next.

III. THE (CLASSICAL) BINDING ENERGY IN GRAVITY

The two-body problem in gravity, needless to say, is classical in nature, whereas the Lamb shift in QED is rooted

[11] To be consistent we should match pNRQED into NRQED at $\mu_0\sim m_e v$. However, since the zero-bin subtraction removes the double counting, we can *pull up* the matching condition to $\mu_0\sim m_e$. (See Fig. 1 in [45], also [58–60] for the implementation of the "velocity renormalization group" in "vNRQED," which is better suited to handle the $\log v$'s to all orders in α_e [and α_s] in one go, from m_e to $m_e v^2$.)

in quantum effects. Moreover, gravity is in spirit more closely related to the strong interaction, and NRQCD, where the potential and ultra-soft gauge fields can couple not only to fermions but also to each other [53]. Nonetheless, similarities arise between the two EFT approaches. In NRGR, as in NRQED, the IR divergence in the near region is also linked to a UV pole in the far zone. The latter follows from a *conservative* radiative effect, namely the tail contribution to the radiation-reaction force [28]. Moreover, akin to the implementation in electrodynamics, the IR divergences can be removed using the zero-bin subtraction, paving the way to ambiguity-free results [39]. To complete the analogy, in what follows we rederive the logarithmic correction to the binding potential for binary black holes, which bears a close resemblance with our derivation of Bethe logarithm for the Hydrogen atom.

A. The EFT framework: NRGR

The relevant scales for the binary inspiral problem are, the size of the compact object, r_s, the separation, r, and the typical wavelength of the emitted radiation, $\lambda_{\rm rad} \sim r/v$. For a bound state we also have $r_s/r \sim v^2$, and therefore

$$r_s \ll r \ll \lambda_{\rm rad}, \tag{3.1}$$

in the PN regime, $v \ll 1$. Therefore, after the hard scale, r_s, is integrated out we encounter two relevant regions for the binary problem (recall soft modes are not present in classical computations). Namely, the—off-shell—potential,

$$(p^0_{\rm pot}, \boldsymbol{p}_{\rm pot}) \sim (v/r, 1/r), \tag{3.2}$$

and—on-shell—radiation (or ultra-soft) modes,

$$(p^0_{\rm rad}, \boldsymbol{p}_{\rm rad}) \sim (v/r, v/r). \tag{3.3}$$

The NRGR action takes the form $(L = i_i \ldots i_\ell)$[12]

$$S_{\rm NRGR}[x^{(p)}_{\rm cm}(\tau), h_{\mu\nu}] = \sum_p \int d\tau_p \bigg[-M_{(p)}(\tau) - \frac{1}{2}\omega_{\mu ab} S^{ab}_{(p)}(\tau) u^\mu(\tau) + \sum_{\ell=2} \left(\frac{1}{\ell!} I^L_{{\rm src}(p)}(\tau) \nabla_{L-2} E_{i_{\ell-1} i_\ell} - \frac{2\ell}{(2\ell+1)!} J^L_{{\rm src}(p)}(\tau) \nabla_{L-2} B_{i_{\ell-1} i_\ell} \right) \bigg], \tag{3.4}$$

where $x^{(p)}_{\rm cm}(\tau)$ is the center-of-mass worldline of the bodies, $\omega_{\mu ab}$ are the Ricci coefficients, and E_{ij}, B_{ij} are the electric and magnetic components of the Weyl tensor. The metric perturbation, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, has support on modes longer than the hard scale, and it includes both potential and radiation modes. The monopole, M, represents the mass, S_{ab} is the spin tensor, and the $I^L_{\rm src}, J^L_{\rm src}$ are the permanent mass- and current-type source multipole moments, of the compact objects [16].[13]

The EFT for at the radiation scale is constructed similarly to pNRQED (although at the nonlinear level the structure resembles pNRQCD instead), by integrating out the potential modes [16]. Unlike QED, all the calculations remain at the classical level, involving a series of iterations of Green's functions convoluted with external sources. Because of the symmetries of the long-distance theory, i.e. general relativity, the effective action in the radiation sector is exactly the same as in (3.4), but only radiation fields are present. The bodies are replaced by a single worldline at the center-of-mass of the binary, and the Wilson coefficients are now associated with the two-body system. For example, M is the (Bondi) binding energy of the bound state, and $(I^L_{\rm src}, J^L_{\rm src})$ are the corresponding source multipole moments. In principle, the power loss is obtained in terms of their time derivatives, using the equations of motion which follow from the gravitational binding potential [16]. See Fig. 3 for a schematic representation of the relevant scales in NRGR. There is yet one other important contribution to be considered, namely the tail effect, or the scattering of the outgoing radiation off of the Newtonian potential produced by the whole binary. This is responsible for the rich structure of the radiation theory [24,28,63].

B. The tail effect

The interaction of the binary's gravitational potential with the outgoing radiation modifies the total emitted power. In practice, the source moments, $I^L_{\rm src}$, which enter in the effective action in (3.4), turn into radiative

[12]As in electrodynamics, the expression in (3.4) applies more generally to the dynamics of an extended objects in a long-wavelength background, prior to considering a two-body bound state [16].

[13]For instance, for a spinning body $I^{ij}_{\rm src} = \frac{1}{2} C_{ES^2} S^{ik} S^j_k$ [10,16–19]. We must also incorporate response terms, e.g. to the background field induced by the companion, $I^{ij}_{\mathcal{R}} = C_E E^{ij} + \cdots$, and likewise for the magnetic components. The $C_{E,B}$ coefficients are known as Love numbers, encoding the information regarding the internal degrees of freedom of the compact bodies. (Surprisingly, all the Love numbers vanish for black holes in $d = 4$, which opens up a unique opportunity to test the *shape* of spacetime in the forthcoming era of precision gravity [13,61,62].)

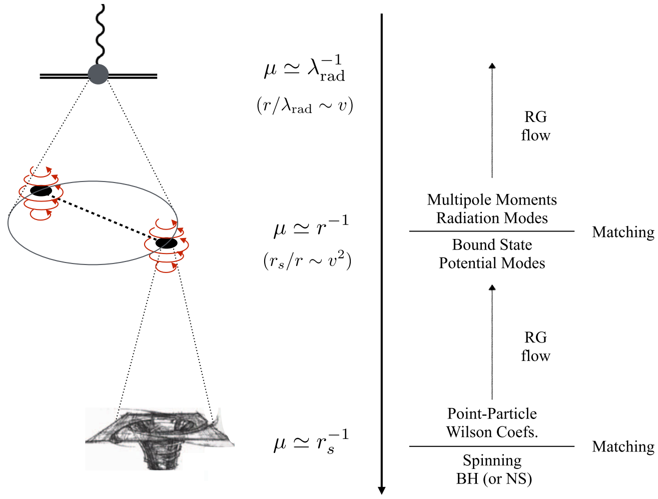

FIG. 3. The EFT approach to the binary inspiral problem. See [16] for a thorough review.

multipoles, $I^L_{\rm rad}$, in the computation of the radiated power [6]. For example, the radiative quadrupole is obtained by computing the Feynman graph in Fig. 4, which follows from the interaction between the quadrupole, $I^{ij}_{\rm src}$, and the monopole, M. The calculation is straightforward, and one obtains a correction of the form [24,64–66],

$$I^{ij}_{\rm rad}(\omega) = I^{ij}_{\rm src}(\omega)\left[1 + GM\omega\left(\text{sign}(\omega)\pi + i\left[\frac{2}{\epsilon_{\rm IR}} + \log\omega^2/\mu^2 + \text{finite}\right]\right)\right], \quad (3.5)$$

which features an IR divergence. It is easy to see all the IR poles cancel out in the radiated power, since they add up to an overall phase [24]. (This type of IR divergence is thus intimately related to the soft factors in QED [47].) However, similarly to what occurred for the Lamb shift, the contribution from the tail effect to radiation-reaction,

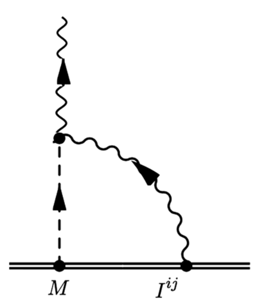

FIG. 4. The tail contribution to the radiative quadrupole moment. Only the lines with an arrow propagate. The double-line represents the two-body system, treated as an external non-propagating source.

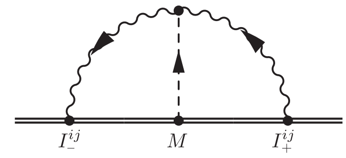

FIG. 5. The tail contribution to radiation reaction. The $(+,-)$ labels are associated to the "in-in" formalism, required to properly compute retardation effects. The wavy line is a radiation mode $p^\mu \sim \lambda_{\rm rad}^{-1}$, whereas the dashed line corresponds to a potential mode with $\boldsymbol{q} \sim \lambda_{\rm rad}^{-1}$. See [28] for more details.

and in particular its *conservative* part, features instead a UV divergence, see Fig. 5,[14]

$$\int dt V_{\rm tail}(\mu) = \frac{G_N^2 M}{5}\int\frac{d\omega}{2\pi}\omega^6 I^{ij}_{\rm src}(-\omega)I^{ij}_{\rm src}(\omega) \times\left[\frac{1}{\epsilon_{\rm UV}} + \log\frac{\omega^2}{\mu^2} + \text{finite}\right]. \quad (3.6)$$

(We drop the "src" label below since all the multipole moments in what follows refer to the source.) The term in (3.6) is the equivalent to (2.27) in the derivation of the Lamb shift. By the same token, the IR divergence in the NRGR potential from the near region (which enters as a local term in the radiation theory) is the analogous to the one in (2.16), through (2.12). All we need is to show that the coefficients of the poles (and the $\log\mu^2$) match, as they do in NRQED.

While the computation of the 4PN gravitational potential within the EFT approach is still undergoing [26,27], we expect to find the following structure in the near region [28,39]

$$\int dt V_{\rm pot}(\mu) = -\frac{G_N^2 M}{5}\int\frac{d\omega}{2\pi}\omega^6 I^{ij}(-\omega)I^{ij}(\omega) \times\left(\frac{1}{\epsilon_{\rm IR}} - 2\log(\mu r)\right) + \text{local/finite}. \quad (3.7)$$

Hence, adding both contributions together, and restricting to a circular orbit (for which $\omega \simeq 2v/r$), we would get [28] (see Fig. 6)

$$V_{\rm full} = V_{\rm pot} + V_{\rm tail} = \frac{2G_N^2 M}{5} I^{ij(3)}(t)I^{ij(3)}(t) \times\left[\log v + \frac{1}{2}\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right)\right] + \text{finite}. \quad (3.8)$$

The upper script (n) represents the n-th time derivative. In [39] we elaborate on the zero-bin prescription to deal

[14] There is also a $i\pi\text{sign}(\omega)$ in the computation which accounts for the radiative part of the tail contribution, see [28].

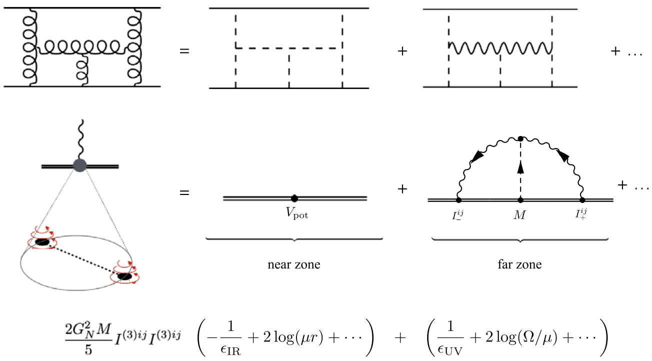

FIG. 6. The full theory computation in general relativity (curly propagators) is split into regions in the EFT formalism: potential (dashed) and radiation (wavy) modes. The subsequent IR/UV divergences appear from the splitting into near and far zones [28]. The calculations are similar to the derivation of the (quantum) Lamb shift described here, except that the gravitational case involves nonlinear couplings (and is fully classical). The logarithmic contribution, scaling as $Mv^8 \log v$, resembles Bethe logarithm in electrodynamics.

with the divergences in (3.8), which are the source of ambiguities in the regularization schemes implemented in [31–37]. The logarithmic correction, on the other hand, is universal [28]. The latter may be obtained unambiguously without the need of any matching condition, as we show next.

C. The renormalization group

As we did for the Lamb shift, let us proceed from the bottom up, where the gravitational potential from the near zone becomes a matching coefficient in the far zone. Therefore, as before [see e.g. (2.48)], we split the local contribution from the near region into a renormalized part and a counter term. The latter is chosen to renormalize the—conservative—contribution from the tail effect [28]

$$V_{\rm c.t.} = -\frac{G_N^2 M}{5} I^{ij(3)}(t) I^{ij(3)}(t) \frac{1}{\epsilon_{\rm UV}}, \tag{3.9}$$

so that we end up with a full gravitational potential of the form

$$V_{\rm full} = V_{\rm ren}(\mu) + \frac{G_N^2 M}{5} \int \frac{d\omega}{2\pi} \omega^6 I^{ij}(-\omega) I^{ij}(\omega) \times \left[\log\frac{\omega^2}{\mu^2} + {\rm finite}\right]. \tag{3.10}$$

This expression is similar to (2.54). Hence, by demanding the μ independence of the (physical) gravitational potential [28] we find,

$$\mu \frac{d}{d\mu} V_{\rm full} = 0 \to \mu \frac{d}{d\mu} V_{\rm ren}(\mu) = \frac{2G_N^2 M}{5} I^{ij(3)}(t) I^{ij(3)}(t), \tag{3.11}$$

which is the equivalent of (2.51). Once again, considering a circular orbit and choosing $\mu \sim v/r$, the renormalization group equation carries the information about the logarithmic contribution,

$$V_{\rm full}^{\rm log} = \frac{2G_N^2 M}{5} I^{ij(3)}(t) I^{ij(3)}(t) \log v, \tag{3.12}$$

reproducing (3.8). From here, following the step described in [28], we derived the logarithm entering in the (conserved) binding energy at 4PN order,

$$E_{\rm log} = -2G_N^2 M \langle I^{ij(3)}(t) I^{ij(3)}(t)\rangle \log v, \tag{3.13}$$

which agrees with the result in [67].

IV. CONCLUDING REMARKS

In this paper we studied the Lamb shift using NRQED, illustrating an ambiguity-free derivation of the binding energy within an EFT framework. The parallel with the gravitational case was already emphasized in [31,32], quote: "It is worth pointing out that also the Lamb shift calculation of Ref. [54] shows up an undefined constant in the IR sector, which gets fixed by some dimensional matching."[15] Indeed, an IR singularity appears in the near zone calculations in NRQED, resembling the situation in gravity. Likewise, a UV pole arises from an ultra-soft loop in the far region, echoing the calculation of the (conservative part of) the tail effect in NRGR [28]. Yet, as we showed, the IR/UV divergences in the Lamb shift can be removed without the need to introduce ambiguities. The procedure is implemented for NRGR in [39]. We also rederived the renormalization group equations from which we reproduce both logarithmic contributions, to the—quantum—shift in the energy levels of the Hydrogen atom and the—classical—gravitational binding potential for binary black holes.

ACKNOWLEDGMENTS

I am grateful to Ira Rothstein for enlightening conversations and Aneesh Manohar for comments on a draft. I thank the participants of the workshop "Analytic Methods in General Relativity" held at ICTP-SAIFR, [68] (supported

[15]The prescription in [54] is akin to a cancelation between IR and UV poles in dim. reg. (also advocated in [28]). This is correct, yet conceptually distinct to the zero-bin subtraction. The latter may be applied to any regularization scheme (e.g. momentum cut-off [45]), including those used in [31–37], whereas the procedure in [54] only applies in dim. reg.

by the São Paulo Research Foundation (FAPESP) Grant No. 2016/01343-7) for very fruitful discussions, in particular to Luc Blanchet, Guillaume Faye, and Gerhard Schäfer. I thank Gerhard for bringing to my attention Ref. [54], which prompted the ambiguity-free derivation of the Lamb shift presented here. I also thank the theory group at DESY (Hamburg) for hospitality while this paper was being completed. This work was supported by the Simons Foundation and FAPESP Young Investigator Awards, Grants No. 2014/25212-3 and No. 2014/10748-5.

[1] J. Abadie *et al.* (Virgo Collaboration, LIGO Scientific Collaboration), Predictions for the rates of compact binary coalescences observable by ground-based gravitational-wave detectors, Classical Quantum Gravity **27**, 173001 (2010).

[2] B. P. Abbott *et al.* (Virgo Collaboration, LIGO Scientific Collaboration), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[3] A. Sesana, Prospects for Multiband Gravitational-Wave Astronomy after GW150914, Phys. Rev. Lett. **116**, 231102 (2016).

[4] B. P. Abbott *et al.* (Virgo Collaboration, LIGO Scientific Collaboration), Astrophysical Implications of the Binary Black-Hole Merger GW150914, Astrophys. J. **818**, L22 (2016).

[5] B. P. Abbott *et al.* (Virgo Collaboration, LIGO Scientific Collaboration), Binary Black Hole Mergers in the first Advanced LIGO Observing Run, Phys. Rev. X **6**, 041015 (2016).

[6] L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries, Living Rev. Relativ. **17**, 2 (2014).

[7] A. Buonanno and B. S. Sathyaprakash, Sources of gravitational waves: Theory and observations, arXiv:1410.7832.

[8] A. Le Tiec, The overlap of numerical relativity, perturbation theory and post-Newtonian theory in the binary black hole problem, Int. J. Mod. Phys. **D23**, 1430022 (2014).

[9] W. D. Goldberger and I. Z. Rothstein, An effective field theory of gravity for extended objects, Phys. Rev. D **73**, 104029 (2006).

[10] R. A. Porto, Post-Newtonian corrections to the motion of spinning bodies in NRGR, Phys. Rev. D **73**, 104031 (2006).

[11] W. D. Goldberger, Les Houches lectures on effective field theories and gravitational radiation, arXiv:hep-ph/0701129.

[12] R. A. Porto and R. Sturani, Scalar gravity: Post-Newtonian corrections via an effective field theory approach, arXiv:gr-qc/0701105.

[13] I. Z. Rothstein, Progress in effective field theory approach to the binary inspiral problem, Gen. Relativ. Gravit. **46**, 1726 (2014).

[14] V. Cardoso and R. A. Porto, Analytic approximations, perturbation theory, effective field theory methods and their applications, Gen. Relativ. Gravit. **46**, 1682 (2014).

[15] S. Foffa and R. Sturani, Effective field theory methods to model compact binaries, Classical Quantum Gravity **31**, 043001 (2014).

[16] R. A. Porto, The effective field theorist's approach to gravitational dynamics, Phys. Rep. **633**, 1 (2016).

[17] R. A. Porto and I. Z. Rothstein, The Hyperfine Einstein-Infeld-Hoffmann Potential, Phys. Rev. Lett. **97**, 021101 (2006).

[18] R. A. Porto and I. Z. Rothstein, Spin(1)spin(2) effects in the motion of inspiralling compact binaries at third order in the post-Newtonian expansion, Phys. Rev. D **78**, 044012 (2008).

[19] R. A. Porto and I. Z. Rothstein, Next to leading order spin(1)spin(1) effects in the motion of inspiralling compact binaries, Phys. Rev. D **78**, 044013 (2008).

[20] J. B. Gilmore and A. Ross, Effective field theory calculation of second post-Newtonian binary dynamics, Phys. Rev. D **78**, 124021 (2008).

[21] S. Foffa and R. Sturani, Effective field theory calculation of conservative binary dynamics at third post-Newtonian order, Phys. Rev. D **84**, 044031 (2011).

[22] R. A. Porto, Next to leading order spin-orbit effects in the motion of inspiralling compact binaries, Classical Quantum Gravity **27**, 205001 (2010).

[23] R. A. Porto, A. Ross, and I. Z. Rothstein, Spin induced multipole moments for the gravitational wave flux from binary inspirals to third Post-Newtonian order, J. Cosmol. Astropart. Phys. 03 (2011) 009.

[24] W. D. Goldberger and A. Ross, Gravitational radiative corrections from effective field theory, Phys. Rev. D **81**, 124015 (2010).

[25] A. Ross, Multipole expansion at the level of the action, Phys. Rev. D **85**, 125033 (2012).

[26] S. Foffa and R. Sturani, Dynamics of the gravitational two-body problem at fourth post-Newtonian order and at quadratic order in the Newton constant, Phys. Rev. D **87**, 064011 (2013).

[27] S. Foffa, P. Mastrolia, R. Sturani, and C. Sturm, Effective field theory approach to the gravitational two-body dynamics, at fourth post-Newtonian order and quintic in the Newton constant, Phys. Rev. D **95**, 104009 (2017).

[28] C. R. Galley, A. K. Leibovich, R. A. Porto, and A. Ross, Tail effect in gravitational radiation reaction: Time nonlocality and renormalization group evolution, Phys. Rev. D **93**, 124010 (2016).

[29] M. Levi and J. Steinhoff, Next-to-next-to-leading order gravitational spin-orbit coupling via the effective field theory for spinning objects in the post-Newtonian scheme, J. Cosmol. Astropart. Phys. 01 (2016) 11.

[30] M. Levi and J. Steinhoff, Next-to-next-to-leading order gravitational spin-squared potential via the effective field

theory for spinning objects in the post-Newtonian scheme, J. Cosmol. Astropart. Phys. 01 (**2016**) 008.

[31] P. Jaranowski and G. Schafer, Dimensional regularization of local singularities in the 4th post-Newtonian two-point-mass Hamiltonian, Phys. Rev. D **87**, 081503 (2013).

[32] P. Jaranowski and G. Schafer, Derivation of the local-in-time fourth post-Newtonian ADM Hamiltonian for spinless compact binaries, Phys. Rev. D **92**, 124043 (2015).

[33] T. Damour, P. Jaranowski, and G. Schafer, Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems, Phys. Rev. D **89**, 064058 (2014).

[34] T. Damour, P. Jaranowski, and G. Schafer, On the conservative dynamics of two-body systems at the fourth post-Newtonian approximation of general relativity, Phys. Rev. D **93**, 084014 (2016).

[35] L. Bernard, L. Blanchet, A. Bohe, G. Faye, and S. Marsat, Fokker action of non-spinning compact binaries at the fourth post-Newtonian approximation, Phys. Rev. D **93**, 084037 (2016).

[36] L. Bernard, L. Blanchet, A. Bohe, G. Faye, and S. Marsat, Energy and periastron advance of compact binaries on circular orbits at the fourth post-Newtonian order, Phys. Rev. D **95**, 044026 (2017).

[37] T. Damour and P. Jaranowski, On the four-loop static contribution to the gravitational interaction potential of two point masses, Phys. Rev. D **95**, 084005 (2017).

[38] L. Blanchet and A. Le Tiec, First law of compact binary mechanics with gravitational-wave tails, arXiv:1702.06839.

[39] R. A. Porto and I. Z. Rothstein, On the apparent ambiguities in the post-Newtonian expansion for binary systems, preceding article, Phys. Rev. D **96**, 024062 (2017)..

[40] W. E. Lamb and R. C. Retherford, Fine structure of the hydrogen atom by a microwave method, Phys. Rev. **72**, 241 (1947).

[41] H. A. Bethe, The electromagnetic shift of energy levels, Phys. Rev. **72**, 339 (1947).

[42] F. J. Dyson, The electromagnetic shift of energy levels, Phys. Rev. **73**, 617 (1948).

[43] J. B. French and V. F. Weisskopf, The electromagnetic shift of energy levels, Phys. Rev. **75**, 1240 (1949).

[44] N. M. Kroll and W. E. Lamb, On the self-energy of a bound electron, Phys. Rev. **75**, 388 (1949).

[45] A. V. Manohar and I. W. Stewart, The zero-bin and mode factorization in quantum field theory, Phys. Rev. D **76**, 074002 (2007).

[46] A. V. Manohar, The HQET / NRQCD Lagrangian to order alpha /m-3, Phys. Rev. D **56**, 230 (1997).

[47] S. Weinberg, Infrared photons and gravitons, Phys. Rev. **140**, B516 (1965).

[48] M. Beneke and V. A. Smirnov, Asymptotic expansion of Feynman integrals near threshold, Nucl. Phys. **B522**, 321 (1998).

[49] H. W. Griesshammer, Threshold expansion and dimensionally regularized NRQCD, Phys. Rev. D **58**, 094027 (1998).

[50] B. Grinstein and I. Z. Rothstein, Effective field theory and matching in nonrelativistic gauge theories, Phys. Rev. D **57**, 78 (1998).

[51] W. E. Caswell and G. P. Lepage, Effective Lagrangians for bound state problems in QED, QCD, and other field theories, Phys. Lett. **167B**, 437 (1986).

[52] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Potential NRQCD: An Effective theory for heavy quarkonium, Nucl. Phys. **B566**, 275 (2000).

[53] I. Z. Rothstein, TASI lectures on effective field theories, arXiv:hep-ph/0308266.

[54] L. S. Brown, *Quantum Field Theory* (Cambridge University Press, Cambridge, UK, 1994).

[55] L. S. Brown, New use of dimensional continuation illustrated by dE/dx in a plasma, Phys. Rev. D **62**, 045026 (2000).

[56] A. Pineda and J. Soto, The Lamb shift in dimensional regularization, Phys. Lett. B **420**, 391 (1998).

[57] J. Schwinger, Coulomb Green's Function, J. Math. Phys. (N.Y.) **5**, 1606 (1964).

[58] M. E. Luke, A. V. Manohar, and I. Z. Rothstein, Renormalization group scaling in nonrelativistic QCD, Phys. Rev. D **61**, 074025 (2000).

[59] A. V. Manohar and I. W. Stewart, Logarithms of Alpha in QED Bound State from the Renormalization Group, Phys. Rev. Lett. **85**, 2248 (2000).

[60] A. H. Hoang, A. V. Manohar, and I. W. Stewart, The running coulomb potential and lamb shift in QCD, Phys. Rev. D **64**, 014033 (2001).

[61] R. A. Porto, The tune of love and the nature(ness) of spacetime, Fortschr. Phys. **64**, 723 (2016).

[62] V. Cardoso, E. Franzin, A. Maselli, P. Pani, and G. Raposo, Testing strong-field gravity with tidal Love numbers, Phys. Rev. D **95**, 084014 (2017).

[63] W. D. Goldberger, A. Ross, and I. Z. Rothstein, Black hole mass dynamics and renormalization group evolution, Phys. Rev. D **89**, 124033 (2014).

[64] R. A. Porto, A. Ross, and I. Z. Rothstein, Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 Post-Newtonian order, J. Cosmol. Astropart. Phys. 09 (2012) 028.

[65] L. Blanchet and T. Damour, Hereditary effects in gravitational radiation, Phys. Rev. D **46**, 4304 (1992).

[66] L. Blanchet, Time asymmetric structure of gravitational radiation, Phys. Rev. D **47**, 4392 (1993).

[67] L. Blanchet, S. L. Detweiler, A. Le Tiec, and B. F. Whiting, High-order post-Newtonian fit of the gravitational self-force for circular Orbits in the Schwarzschild geometry, Phys. Rev. D **81**, 084033 (2010).

[68] http://www.ictp-saifr.org/gr2016.