Canonical Chern-Simons gravity

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We study the canonical description of the axisymmetric vacuum in $2 + 1$-dimensional gravity, treating Einstein's gravity as a Chern-Simons gauge theory on a manifold with the restriction that the dreibein is invertible. Our treatment is in the spirit of Kuchař's description of the Schwarzschild black hole in $3 + 1$ dimensions, where the mass and angular momentum are expressed in terms of the canonical variables and a series of canonical transformations that turn the curvature coordinates and their conjugate momenta into new canonical variables is performed. In their final form, the constraints are seen to require that the momenta conjugate to the Killing time and curvature radius vanish, and what remains is the mass, the angular momentum, and their conjugate momenta, which we derive. The Wheeler-DeWitt equation is trivial and describes time independent systems with wave functions described only by the total mass and total angular momentum.

I. INTRODUCTION

In $2 + 1$ dimensions, many of the problems associated with quantum gravity are expected to be alleviated by the fact that pure gravity in $2 + 1$ dimensions has no local, propagating degrees of freedom. Still, the theory is far from trivial [1]. The vacuum solutions of pure gravity are multiconical spacetimes, obtained by the identification of points in flat space [2], and in the presence of a cosmological constant, one obtains maximally symmetric solutions, viz., the anti-de Sitter (AdS) and de Sitter (dS) spacetimes with a similar identification of points. Such an identification, by a discrete subgroup of $SO(2, 2)$ in AdS spacetime, was shown to give a spinning black hole solution by Bañados, Teitelboim, and Zanelli (BTZ) [3]. The BTZ black hole solution is locally AdS, but globally it is characterized by conserved charges at the boundary of the AdS spacetime [4]. The solution exhibits many of the properties of black holes in $3 + 1$ dimensions and therefore provides a simpler setting for the study of quantum effects. Likewise, gravitational collapse in $2 + 1$ dimensions is rich in structure. The earliest study of gravitational collapse in $2 + 1$ dimensions with and without a cosmological constant was carried out in Ref. [5]. In the context of circularly symmetric, homogeneous dust, the authors showed that collapse to a black hole depends sensitively on the initial data. In the absence of a cosmological constant or in dS spacetime, collapse may or may not occur depending on the initial velocity, but if the dust ball collapses, then it does so to a naked, conical, point source singularity [6]. On the other hand, in AdS spacetime, the BTZ black hole arises naturally as the end state, provided that the initial density is sufficiently large. If not, the end state is again naked conical singularity, but in AdS spacetime. These

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results led to a numerical study of critical phenomena associated with the collapse process in Ref. [7] and were confirmed in studies of inhomogeneous dust collapse in Ref. [8]. Attempts to quantize dust collapse in Refs. [9,10] also taught several lessons. Our ultimate goal is to obtain a description of quantum gravitational collapse in $2 + 1$ dimensions with rotation, for which a classical description was developed in Ref. [11]. This paper is a first step in this program.

Classical $2 + 1$-dimensional gravity and supergravity can also be viewed as Chern-Simons gauge theories of the Poincaré, anti-de Sitter, and de Sitter groups and their supersymmetric generalizations [12,13]. The general procedure is to identify an appropriate (super)group which contains the structure group of the corresponding gravity theory in its even part, construct its Lie algebra with generators $\hat{T}_a$, expand the gauge superfield $A_\mu = A_\mu^a \hat{T}_a$, and construct the Chern-Simons action according to

$$I_{C.S.} = \frac{1}{2}\text{Tr} \int A \wedge \left(dA + \frac{2}{3} A \wedge A \right)$$

$$= \frac{1}{2}\gamma_{ab} \int A^a \wedge \left(dA^b + \frac{1}{3} f_{cd}^b A^c \wedge A^d \right), \quad (1)$$

where f_{bc}^a are the structure constants of the Lie algebra and $\gamma_{ab} = \text{Tr}(T_a T_b)$ plays the role of a metric on the Lie algebra and must be nondegenerate so that the action contains a kinetic term for all components of the gauge field. By construction, the action is invariant under a gauge transformation given by

$$\delta_g A_\mu = -D_\mu \Lambda, \quad (2)$$

where $\Lambda = \Lambda^a T_a$ and $D_\mu = \partial_\mu + [A_\mu,]$ and the classical equations of motion assert that the field strengths vanish

identically. For the action (1) to be an acceptable gauge theory of (super)gravity, gauge transformations must be equivalent to diffeomorphisms. This is indeed true for small diffeomorphisms on shell. It would be incorrect, however, to conclude that Einstein's action in $2 + 1$ dimensions is equivalent to the Chern-Simons action because the latter contains many solutions that have no metric interpretation. Here, we confine our attention to a subspace of solutions that *do* have a metric interpretation. The gauge fields are the dreibein and the spin connection, and the vanishing field strengths simply assert that the torsion vanishes and the curvature is constant. Our aim in this work is to cast the dynamics of Chern-Simons gravity into a canonical form for metric compatible solutions, in the spirit of Kuchař [14]. There is a long history of other approaches in the literature [15]. These approaches focus on solving the constraints and using them to derive a simplified Hamiltonian in a finite number of degrees of freedom or on exploiting the local isometries to begin with a reduced action for the system. The advantage of our approach is that it focuses primarily on simplifying the constraints via a series of canonical transformations. These transformations are then easily modified and continue to be useful in simplifying the constraints in a variety of systems, including the Einstein-Maxwell system [16], Lovelock gravity [17], and when matter is included [18]. This makes it better adapted to the study of dynamical collapse.

In Sec. II, we review the canonical form of the Chern-Simons action for $SO(2,2)$. We employ a general Arnowitt-Deser-Misner (ADM) metric to choose a natural canonical chart consisting of the three functions comprising the spatial metric, $L(r)$, $R(r)$, and $Q(r)$ and their conjugate momenta. We solve three of the six constraints of the Chern-Simons action (corresponding to the vanishing of torsion) and show that the other three are equivalent to the Hamiltonian and momentum constraints that would be obtained from the second order (Einstein) action. In Sec. III, we consider the equations of motion and recover the well-known classical, static solutions describing a spinning particle and the BTZ black hole. We then develop the constraints specific to the spinning particle and the BTZ black hole. We discuss the appropriate fall-off conditions to be imposed on our canonical variables in Sec. IV and determine the boundary action. In Sec. V, by embedding the hypersurfaces from which our ADM metric is constructed in the spacetimes describing the spinning particle and the BTZ black hole (derived in Sec. III), we are able to reconstruct the mass and angular momentum in terms of the canonical variables. This allows us to determine a new canonical chart in which the constraints are greatly simplified in Sec. VI. Taking into account the boundary action, we perform yet another canonical transformation, leading to a description in terms of the area radius, the Killing time, and the mass and angular momentum (and their conjugates) in Sec. VII. The resulting constraints take on a particularly

simple form. When they are imposed as operator constraints on the Wheeler-DeWitt wave functional, the result is as expected: a time independent state, which depends only the ADM mass and angular momentum and, once prepared, remains the same on every spacelike hypersurface. We summarize our results in the concluding Sec. VIII.

II. CHERN-SIMONS GRAVITY

As mentioned in the Introduction, vacuum (super)gravity in $2 + 1$ dimensions can be described as a gauge theory of the gauge groups $ISO(2,1)$ (pure gravity), $SO(2,2)$ (AdS), $SO(3,1)$ (dS), and their supersymmetric extensions, with a Chern-Simons action,

$$I_{C.S.} = \frac{1}{2} \int_M \gamma_{ab} A^a \wedge \left(dA^b + \frac{1}{3} f^b{}_{cd} A^c \wedge A^d \right), \quad (3)$$

where A^a is the gauge connection; $f^a{}_{bc}$ are the structure constants of the corresponding group $\mathcal{G}$; γ_{ab} is the metric of the Lie algebra, i.e., $\gamma_{ab} = 2\mathrm{Tr}(\hat{T}_a \hat{T}_b)$; and the $\hat{T}$'s are generators of the Lie algebra. In what follows, letters from the beginning of the roman alphabet, $\{a, b, c...\}$, will be used for group indices; the greek alphabet, $\{\alpha, \beta...\}$, will be used for spacetime indices; and letters from the middle of the roman alphabet, $\{i, j, k...\}$, will be used for spatial indices. We take the group $\mathcal{G}$ to be the AdS group $SO(2,2)$, with generators $\hat{P}_a$ and $\hat{J}_a$ satisfying the following commutation relations,

$$[\hat{P}_a, \hat{P}_b] = \Lambda \epsilon_{abc} \hat{J}^c, \quad [\hat{P}_a, \hat{J}_b] = \epsilon_{abc} \hat{P}^c, \quad [\hat{J}_a, \hat{J}_b] = \epsilon_{abc} \hat{J}^c, \quad (4)$$

where $\Lambda > 0$ is the cosmological constant and the group indices are raised and lowered using the three-dimensional Minkowski metric. We expand A_μ in the basis of generators

$$A_\mu = e^a{}_\mu \hat{P}_a + \omega^a{}_\mu \hat{J}_a, \quad (5)$$

where $e^a{}_\mu$ and $\omega^a{}_\mu$ are the dreibein and the spin connection, respectively. There are two bilinear invariants (Casimirs), namely $\hat{P} \cdot \hat{J} + \hat{J} \cdot \hat{P}$ and $\hat{P}^2 + \hat{J}^2/\Lambda^2$, which can be used to determine γ_{ab} as

$$\Lambda \mathrm{Tr}(\hat{J}_a \hat{J}_b) = \mathrm{Tr}(\hat{P}_a \hat{P}_b) = \Lambda \eta_{ab} \quad (6)$$

$$\mathrm{Tr}(\hat{J}_a \hat{P}_b) = \mathrm{Tr}(\hat{P}_a \hat{J}_b) = \eta_{ab}, \quad (7)$$

respectively. The first is degenerate in the limit as $\Lambda \to 0$ and would not produce an acceptable Poincaré theory in that limit. With the second, the Chern-Simons action can be cast in the form

$$I_{C.S.} = \frac{1}{2}\eta_{ab}\int_M d^3x \epsilon^{\mu\nu\lambda}\left\{e^a{}_\mu\left[\partial_\nu\omega^b{}_\lambda + \epsilon^b{}_{cd}\left(\omega^c{}_\nu\omega^d{}_\lambda\right.\right.\right.$$
$$\left.\left.\left. + \frac{\Lambda}{3}e^c{}_\nu e^d{}_\lambda\right)\right] + \omega^a{}_\mu\partial_\nu e^b{}_\lambda\right\}. \tag{8}$$

As we are primarily interested in the Hamiltonian formulation, it is convenient to separate the time component in the action and recast it in the form

$$I_{C.S.} = \frac{1}{2}\eta_{ab}\int_M d^3x \epsilon^{ij}\{e^a{}_t[2\partial_i\omega^b{}_j + \epsilon^b{}_{cd}(\omega^c{}_i\omega^d{}_j + \Lambda e^c{}_i e^d{}_j)]$$
$$+ \omega^a{}_t[2\partial_i e^b{}_j + 2\epsilon^b{}_{cd}e^c{}_i\omega^d{}_j] - e^a{}_i\partial_t\omega^b{}_j - \omega^a{}_i\partial_t e^b{}_j\}, \tag{9}$$

making it evident that the dreibein and the spin connection are canonical conjugates of one another. In this first order form, if $\{e^a{}_i, \omega^a{}_i\}$ are treated on an equal footing as configuration space variables, the canonical momenta do not involve time derivatives of the fields and become primary constraints (they are second class). There are then 12 configuration space variables, 12 second class constraints, and 6 first class constraints (the theory has no degrees of freedom). One must proceed by following Dirac's procedure for constrained systems.

Here, we will follow a different approach. We take the spacetime to be of the form $\mathbb{R} \times \Sigma$ and choose $e^a{}_i$ for our configuration space variables. The Chern-Simons action (9) is equivalent to the first order Einstein Hilbert action in the dreibein formulation up to a total derivative, so, discarding the total time derivative, we find that the momentum conjugate to $e^a{}_i$ is

$$\Pi_a{}^i = \eta_{ab}\epsilon^{ij}\omega^b{}_j, \tag{10}$$

where ϵ^{ij} is the two-dimensional Levi-Città tensor. The Hamiltonian density is then

$$\mathcal{H} = -\eta_{ab}\{e^a{}_t F^b[\omega] + \omega^a{}_t F^b[e]\}, \tag{11}$$

where

$$F_a[e] = \epsilon_{acd}\eta^{dm}e^c{}_i\Pi_m{}^i \approx 0$$
$$F_a[\omega] = \partial_i\Pi_a{}^i + \frac{1}{2}\epsilon_{acd}(\epsilon_{kl}\eta^{cm}\eta^{dn}\Pi_m{}^k\Pi_n{}^l + \Lambda\epsilon^{ij}e^c{}_i e^d{}_j) \approx 0 \tag{12}$$

are the six constraints of the theory. The first three enforce the vanishing of torsion, and the second three require the curvature to be constant.

Our next task is to rewrite the constraints above in terms of metric functions. We will eventually be interested in axisymmetric solutions, so we consider a general isotropic line element in Σ, with circular coordinates (r, ϕ),

$$ds^2 = \gamma_{ij}dx^i dx^j = A^2(r)dr^2 + B^2(r)d\phi^2 + C^2(r)drd\phi, \tag{13}$$

and foliate the three-dimensional spacetime with these leaves, which then also become labeled by a time parameter, t. The resulting ADM metric,

$$ds^2 = \bar{N}^2 dt^2 - A^2(dr + \bar{N}^r dt)^2 - B^2(d\phi + \bar{N}^\phi dt)^2$$
$$- C^2(dr + \bar{N}^r dt)(d\phi + \bar{N}^\phi dt), \tag{14}$$

can be written more conveniently as

$$ds^2 = N^2 dt^2 - L^2(dr + N^r dt)^2 - R^2\left(d\phi + N^\phi dt + \frac{Q}{R}dr\right)^2, \tag{15}$$

with the identifications

$$L = \sqrt{A^2 - \frac{C^4}{4B^2}}, \qquad R = B, \qquad Q = \frac{C^2}{2B},$$
$$N^r = \bar{N}^r, \qquad N^\phi = \bar{N}^\phi + \frac{C^2}{2B^2}\bar{N}^r, \qquad N = \bar{N}. \tag{16}$$

A driebein which yields the metric in (15) may be given in lower triangular form,

$$e^a{}_\mu = \begin{pmatrix} N & 0 & 0 \\ N^r L & L & 0 \\ N^\phi R & Q & R \end{pmatrix}, \tag{17}$$

in terms of which the nonvanishing constraints become

$$F_0[e] := L\Pi_2{}^r - R\Pi_1{}^\phi + Q\Pi_1{}^r \approx 0$$
$$F_1[e] := R\Pi_0{}^\phi + Q\Pi_0{}^r \approx 0$$
$$F_2[e] := L\Pi_0{}^r + R' \approx 0$$
$$F_0[\omega] := \partial_r\Pi_0{}^r + \Pi_1{}^r\Pi_2{}^\phi - \Pi_2{}^r\Pi_1{}^\phi + \Lambda LR \approx 0$$
$$F_1[\omega] := \partial_r\Pi_1{}^r + \Pi_0{}^r\Pi_2{}^\phi - \Pi_2{}^r\Pi_0{}^\phi \approx 0$$
$$F_2[\omega] := \partial_r\Pi_2{}^r - \Pi_0{}^r\Pi_1{}^\phi + \Pi_1{}^r\Pi_0{}^\phi \approx 0. \tag{18}$$

We may readily solve the first three constraints, which are purely algebraic. From the third, we have $\Pi_0{}^r = -R'/L$. Inserting this into the second yields $\Pi_0{}^\phi = QR'/LR$ and, from the first equation, $\Pi_1{}^\phi = \frac{L}{R}\Pi_2{}^r - \frac{Q}{R}\Pi_1{}^r$. With the three momenta obtained, the remaining three nontrivial constraints read

$$F_0[\omega] := \Pi_1{}'\Pi_2{}^\phi - \Pi_2{}'\Pi_1{}^\phi + \Lambda LR - \left(\frac{R'}{L}\right)' \approx 0$$

$$F_1[\omega] := \partial_r \Pi_1{}^r - \frac{R'}{L}\Pi_2{}^\phi - \frac{QR'}{LR}\Pi_2{}^r \approx 0$$

$$F_2[\omega] := \partial_r \Pi_2{}^r + \frac{R'}{R}\Pi_2{}^r \approx 0, \tag{19}$$

and, defining $P_L = \Pi_1{}^r$, $P_Q = \Pi_2{}^r$ and $P_R = \Pi_2{}^\phi$, we may write the simplified Hamiltonian density as

$$\mathcal{H} = -N\mathcal{H}^g - N^r \mathcal{H}_r - N^\phi \mathcal{H}_\phi, \tag{20}$$

where

$$\mathcal{H}^g = P_L P_R + \Lambda LR - \frac{L}{R}P_Q{}^2 + \frac{Q}{R}P_Q P_L - \left(\frac{R'}{L}\right)' \approx 0$$

$$\mathcal{H}_r = LP_L{}' - R'P_R - \frac{QR'}{R}P_Q \approx 0$$

$$\mathcal{H}_\phi = (RP_Q)' \approx 0, \tag{21}$$

which are the Hamiltonian and momentum constraints of the theory. The last momentum constraint implies that

$$RP_Q = \alpha(t), \tag{22}$$

and we could also write

$$\mathcal{H}_r = LP_L{}' - R'P_R + QP_Q{}' - \frac{Q}{R}\mathcal{H}_\phi \approx 0. \tag{23}$$

To summarize, the phase space is six dimensional, parametrized by three metric functions, L, R, and Q, and their conjugate momenta. Axisymmetric solutions are obtained by taking $Q = 0$, and circularly symmetric solutions are obtained by taking $Q = N^\phi = 0$. The entire content of the theory is in the constraints; the equations of motion follow by taking Poisson brackets with $\mathcal{H}$, and in the following section, we recover the well-known stationary solutions with which we will work in later sections.

III. HAMILTONIAN EQUATIONS OF MOTION

The Hamiltonian equations of motion are quite generally given by taking Poisson brackets with the smeared constraints,

$$\dot{X} = \{X, -H^g[N] - H_r[N^r] - H_\phi[N^\phi]\}_{P.B.} \tag{24}$$

$$\dot{P}_X = \{P_X, -H^g[N] - H_r[N^r] - H_\phi[N^\phi]\}_{P.B.} \tag{25}$$

For the six phase space variables,

$$\dot{L} = \{L, H\}_{P.B.} = -NP_R - \frac{NQ}{R}P_Q + (N^r L)'$$

$$\dot{R} = \{R, H\}_{P.B.} = -NP_L + N^r R'$$

$$\dot{Q} = \{Q, H\}_{P.B.} = N\left(\frac{2L}{R}P_Q - \frac{Q}{R}P_L\right) + N^r \frac{QR'}{R} + N^{\phi\prime}R$$

$$\dot{P}_L = \{P_L, H\}_{P.B.} = N\Lambda R - \frac{N'R'}{R^2} - \frac{NP_Q{}^2}{R} + N^r P_L{}'$$

$$\dot{P}_R = \{P_R, H\}_{P.B.} = N\Lambda L - \frac{N''}{L} + \frac{N'L'}{L^2} + \frac{NLP_Q{}^2}{R^2}$$

$$\qquad - \frac{Q}{R^2}P_Q P_L + (N^r P_R)' - N^{\phi\prime}P_Q$$

$$\dot{P}_Q = \{P_Q, H\}_{P.B.} = \frac{N}{R}P_Q P_L + N^r P_Q{}' = -\frac{\dot{R}}{R}P_Q. \tag{26}$$

We have made no assumptions about the canonical variables apart from isotropy, so any isotropic, classical solution must satisfy (21) and (26). Combining the third constraint with the last equation of motion, we find that α must be constant.

In the static case, the time derivative of all canonical variables must vanish. Using (22), this implies that the equations of motion, together with the first two constraints (21), will read

$$P_R = \frac{(N^r L)'}{N} - \frac{\alpha Q}{R^2}$$

$$P_L = \frac{N^r R'}{N}$$

$$N\left(\frac{2\alpha L}{R^2} - \frac{Q}{R}P_L\right) + N^r \frac{QR'}{R} + N^{\phi\prime}R = 0$$

$$N\Lambda R - \frac{N'R'}{L^2} - \frac{\alpha^2 N}{R^3} + N^r P_L{}' = 0$$

$$N\Lambda L - \frac{N''}{L} + \frac{N'L'}{L^2} + \frac{\alpha^2 NL}{R^4} - \frac{\alpha Q}{R^3} + (N^r P_R)' - \frac{\alpha}{R}N^{\phi\prime} = 0$$

$$P_L P_R + \Lambda LR - \frac{\alpha^2 L}{R^3} + \frac{\alpha Q}{R^2}P_L - \left(\frac{R'}{L}\right)' = 0$$

$$P_L{}' - \frac{R'}{L}P_R - \frac{\alpha QR'}{LR^2} = 0. \tag{27}$$

We now have to find eight unknown functions from the above seven equations. So there is the freedom to choose one of the unknown functions, and we choose $N^r = 0$. This gives $P_L = 0$, $P_R = -\alpha Q/R^2$, and the last equation is satisfied identically. We are left with four equations in five unknowns, namely

$$\frac{2\alpha NL}{R^2} + N^{\phi'}R = 0$$

$$N\Lambda R - \frac{N'R'}{L^2} - \frac{\alpha^2 N}{R^3} = 0$$

$$N\Lambda L - \frac{N''}{L} + \frac{N'L'}{L^2} + \frac{\alpha^2 NL}{R^4} - \frac{\alpha Q}{R^3} - \frac{\alpha}{R}N^{\phi'} = 0$$

$$\Lambda LR - \frac{\alpha^2 L}{R^3} - \left(\frac{R'}{L}\right)' = 0. \tag{28}$$

Therefore, we can yet choose another function; this we take to be $Q = 0$. Solving the first equation, $N^{\phi'} = -2\alpha NL/R^3$, we find that the remaining equations are

$$N\Lambda R - \frac{N'R'}{L^2} - \frac{\alpha^2 N}{R^3} = 0$$

$$N\Lambda L - \frac{N''}{L} + \frac{N'L'}{L^2} + \frac{3\alpha^2 NL}{R^4} = 0$$

$$\Lambda LR - \frac{\alpha^2 L}{R^3} - \left(\frac{R'}{L}\right)' = 0. \tag{29}$$

The equations are once again not independent; the second can be obtained from the remaining two equations, so there are two independent equations for three unknown functions,

$$N\Lambda R - \frac{N'R'}{L^2} - \frac{\alpha^2 N}{R^3} = 0$$

$$\Lambda LR - \frac{\alpha^2 L}{R^3} - \left(\frac{R'}{L}\right)' = 0, \tag{30}$$

and we are free to choose yet another function. We take $R(r) = r$ below.

A. $\Lambda = 0$: The spinning point particle

With $\Lambda = 0$, the gauge group $SO(2,2)$ turns into the Poincaré group by a Wigner-Inonu contraction. Taking $R(r) = r$, the equations (30) readily yield the following solutions,

$$L(r) = \frac{1/\mu}{\sqrt{1 + \frac{\alpha^2}{\mu^2 r^2}}}$$

$$N(r) = N_+\sqrt{1 + \frac{\alpha^2}{\mu^2 r^2}}$$

$$N^{\phi}(r) = N_+^{\phi} + \frac{N_+\alpha}{\mu r^2}, \tag{31}$$

where μ, N_+, and N_+^{ϕ} are constants of the integration. For example, if we choose $N_+ = 1$ and $N_+^{\phi} = 0$, the line element is given by

$$ds^2 = N^2 dt^2 - \frac{N^{-2}}{\mu^2}dr^2 - r^2\left(d\phi - \frac{j}{\mu r^2}dt\right)^2, \tag{32}$$

where μ can be identified with the mass of the particle and $j = -\alpha$ can be identified with its angular momentum.

B. $\Lambda \neq 0$: The BTZ black hole

With $\Lambda \neq 0$, we find, from the second of (30), that

$$L(r) = \left(\Lambda r^2 - M + \frac{\alpha^2}{r^2}\right)^{-1/2}, \tag{33}$$

where M is a constant of the integration. Using this in the first, we have

$$N(r) = N_+\left(\Lambda r^2 - M + \frac{\alpha^2}{r^2}\right)^{1/2}, \tag{34}$$

and, together, these imply that $N^{\phi} = N_+^{\phi} + N_+\alpha/r^2$. With $N_+ = 1$ and $N_+^{\phi} = 0$, we recover the BTZ solution of mass M and angular momentum $J = -\alpha$ with line element

$$ds^2 = N(r)^2 dt^2 - N(r)^{-2}dr^2 - r^2\left(d\phi - \frac{J}{r^2}dt\right)^2. \tag{35}$$

IV. FALL-OFF CONDITIONS AND BOUNDARY ACTION

The total action in general will combine the bulk action

$$S_\Sigma = \int dt \int dr[P_L\dot{L} + P_R\dot{R} + P_Q\dot{Q} - \mathcal{H}] \tag{36}$$

and a boundary action, $S_{\partial\Sigma}$, the function of which is to cancel unwanted boundary variations and the value of which will depend on the boundary conditions that are imposed. We adopt boundary conditions that enforce every solution to asymptotically approach one of the spacetimes derived in the previous section. For the maximally extended spinning particle, as for the BTZ black hole, the boundaries of spatial hypersurfaces will be taken to lie at $r \to \infty$.

A. Point particle

In the case of a spinning point particle, we assume that the canonical variables have an asymptotic expansion in integer powers of $1/r$ as $r \to \infty$. We adopt the conditions

$$R \longrightarrow r + \mathcal{O}^{\infty}(r^{-2})$$

$$L \longrightarrow \frac{1}{\mu_\pm} - \frac{j_\pm^2}{2\mu_\pm^3}r^{-2} + \mathcal{O}^{\infty}(r^{-3})$$

$$Q \longrightarrow \mathcal{O}^{\infty}(r^{-2})$$

$$P_R \longrightarrow P_{R0} + \mathcal{O}^{\infty}(r^{-1})$$

$$P_L \longrightarrow \mathcal{O}^{\infty}(r^{-1})$$

$$P_Q \longrightarrow j_\pm r^{-1} + \mathcal{O}^{\infty}(r^{-2})$$

$$N \longrightarrow \left[1 + \frac{j_\pm}{2\mu_\pm^2}r^{-2}\right]N_\pm + \mathcal{O}^{\infty}(r^{-3})$$

$$N^r \longrightarrow \mathcal{O}^{\infty}(r^{-1})$$

$$N^{\phi} \longrightarrow N_\pm^{\phi} + \frac{j_\pm}{\mu_\pm}r^{-2} + \mathcal{O}^{\infty}(r^{-3}), \tag{37}$$

where $\mathcal{O}(r^{-n})$ represents a term of which the asymptotic behavior is as r^{-n} and is multiplied by some function of t and the plus and minus refer to the right and left infinities, respectively. It is easily verified that these fall-off conditions are compatible with the constraints and preserved by the time evolution equations (26). To determine the boundary action, we must consider all terms in the Hamiltonian density, $\mathcal{H}$, the variation of which will lead to boundary terms. As no derivative of P_R appears, a variation of P_R will yield no boundary contribution, and due to the fall-off conditions, contributions from all the variations with respect to R and P_L will fall off much faster than r^{-1}. Only variations with respect to L and P_Q yield boundary contributions, viz.,

$$\int dt \left[NR'\delta\left(\frac{1}{L}\right) - N^\phi R\delta P_Q \right] \Big|_{\partial M}$$
$$= -\int dt [N_+\delta\mu_+ + N_-\delta\mu_- - N^\phi_+\delta j_+ - N^\phi_-\delta j_-]. \quad (38)$$

This must be cancelled by an appropriate boundary action, which we therefore take to be

$$S_{\partial\Sigma} = \int dt [N_+\mu_+ + N_-\mu_- - N^\phi_+ j_+ - N^\phi_- j_-]. \quad (39)$$

The boundary action affirms the role of μ and $-\alpha = j$ as the mass and the angular momentum of the spinning particle.

B. BTZ black hole

In this case, for the asymptotic behavior of our canonical variables, we adopt

$$R \longrightarrow r + \mathcal{O}^\infty(r^{-2})$$
$$L \longrightarrow \frac{r^{-1}}{\sqrt{\Lambda}} + \frac{M_\pm r^{-3}}{2\Lambda^{3/2}} + \mathcal{O}^\infty(r^{-4})$$
$$Q \longrightarrow \mathcal{O}^\infty(r^{-6})$$
$$P_L \longrightarrow \mathcal{O}^\infty(r^{-2})$$
$$P_R \longrightarrow \mathcal{O}^\infty(r^{-4})$$
$$P_Q \longrightarrow -J_\pm r^{-1} + \mathcal{O}^\infty(r^{-2})$$
$$N \longrightarrow \left(\sqrt{\Lambda}r - \frac{M_\pm}{2\sqrt{\Lambda}}r^{-1} \right) N_+ + \mathcal{O}^\infty(r^{-2})$$
$$N^r \longrightarrow \mathcal{O}^\infty(r^{-2})$$
$$N^\phi \longrightarrow N^\phi_\pm + J_\pm r^{-2} + \mathcal{O}^\infty(r^{-4}). \quad (40)$$

Again, it is easy to check that these conditions are compatible with the constraints and preserved by the time evolution equations. As before, only those variables of which the space derivatives appear in the action will contribute to the boundary action. From the action, we see that R, L, P_L, and P_Q are all likely to contribute to the

boundary action. However, by explicitly performing the variation, we find that the variation with respect to R and P_L rapidly approach zero at both boundaries, but variations with respect to L and P_Q contribute at $r \to \infty$. Explicitly, using the asymptotic expressions for corresponding variables as before, we find that the boundary action will be

$$S_{\partial\Sigma} = -\int dt \left[\frac{1}{2}(N_+M_+ + N_-M_-) + N^\phi_+J_+ + N^\phi_-J_- \right]. \quad (41)$$

While the inclusion of a boundary action unfreezes the evolution at infinity, it leads to another problem, which is that the lapse and shift functions may also be varied at the boundaries. This would lead to the conclusion that $\mu_\pm = j_\pm = M_\pm = J_\pm = 0$. Therefore, Kuchař [14] proposed that $N_\pm$ and $N^\phi_\pm$ should be viewed as prescribed functions of t. This "parametrization at infinity" will be exploited in Sec. VII to present a greatly reduced form of the canonical action.

V. EMBEDDING

Our aim now is to develop the action and constraints specific to the two solutions obtained in Sec. III. First, we show how the canonical data determine the mass and the angular momentum of these systems by embedding the hypersurfaces of the ADM metric into the metrics describing the spinning particle and the BTZ black hole, respectively, imagining that they are leaves of a particular foliation of these spacetimes.

A. Spinning particle

We begin with the spinning point particle, expressing the metric in terms of the Killing time and area radius as

$$ds^2 = FdT^2 - \frac{1}{\mu^2F}dR^2 - R^2\left(d\phi - \frac{j}{\mu R^2}dT\right)^2, \quad (42)$$

where $F = (1 + \frac{j^2}{\mu^2R^2})$ and μ and j are the mass and angular momentum, respectively. It is convenient rescale the Killing time according to $\bar{T} = T/\mu$; the metric in (42) can then be written as

$$ds^2 = F_1 d\bar{T}^2 - \frac{1}{F_1}dR^2 - R^2\left(d\phi - \frac{j}{R^2}d\bar{T}\right)^2,$$

where $F_1 = (\mu^2 + \frac{j^2}{R^2})$. Taking $\bar{T}$ and R to be functions of the ADM coordinates, i.e., $\bar{T} = \bar{T}(t,r)$ and $R = R(t,r)$, and comparing with the ADM form of the line element (15), we find

$$N = \frac{R'\dot{T} - \bar{T}'\dot{R}}{L}$$

$$N^r = \frac{F_1^{-1}\dot{R}R' - F_1\dot{\bar{T}}\bar{T}'}{L^2}$$

$$N^\phi = -\frac{j\dot{T}}{R^2}$$

$$L^2 = F_1^{-1}R'^2 - F_1\bar{T}'^2$$

$$Q = -\frac{j\bar{T}'}{R}. \tag{43}$$

Inserting the lapse and shift into the second equation of (26), we then have

$$P_L = \frac{1}{N}(-\dot{R} + N^r R') = -\frac{F_1\bar{T}'}{L} \Rightarrow T' = -\frac{LP_L}{F_1}, \tag{44}$$

which, inserted into the expression for L^2 in (43), gives

$$F_1 = \mu^2 + \frac{j^2}{R^2} = \left(\frac{R'^2}{L^2} - P_L^2\right), \tag{45}$$

and, from the last equation in (43), we also find

$$j = \frac{QR}{LP_L}\left(\frac{R'^2}{L^2} - P_L^2\right). \tag{46}$$

Together, these equations allow us to recover the mass and angular momentum from the canonical data. Furthermore, differentiating (45) with respect to r, we find

$$(F_1)' = \left(\frac{R'^2}{L^2} - P_L^2\right)' = -2P_L P_L' + 2\left(\frac{R'}{L}\right)\left(\frac{R'}{L}\right)'$$

$$= -\frac{2R'}{L}\mathcal{H}_g - \frac{2P_L}{L}\mathcal{H}_r - \frac{2R'}{R}P_Q^2, \tag{47}$$

where $\mathcal{H}^g$ and $\mathcal{H}_r$ are the Hamiltonian and momentum constraints. Therefore,

$$\left(\mu^2 + \frac{j^2}{R^2} - P_Q^2\right)' = -\frac{2R'}{L}\mathcal{H}_g - \frac{2P_L}{L}\mathcal{H}_r - \frac{2P_Q}{R}\mathcal{H}_\phi \tag{48}$$

is a linear combination of the constraints, which, we note, do not require μ' and j' to separately vanish.

B. BTZ black hole

Similarly, for the BTZ black hole, the metric is expressed as

$$F(R)dT^2 - \frac{1}{F(R)}dR^2 - R^2\left(d\phi - \frac{J}{R^2}dT\right)^2, \tag{49}$$

where $F(R) = \Lambda R^2 - M + \frac{J^2}{R^2}$ and M and J are the mass and angular momentum, respectively. Embedding (15) into the BTZ metric, we obtain (43). Then, inserting the lapse and shift into the second equation of (26), we obtain T' and substitute its value into the expressions for L^2 and Q; this gives

$$F = \Lambda R^2 - M + \frac{J^2}{R^2} = \frac{R'^2}{L^2} - P_L^2$$

$$J = \frac{QR}{LP_L}\left(\frac{R'^2}{L^2} - P_L^2\right), \tag{50}$$

and, again, one recovers the mass and angular momentum in terms of the canonical data. Furthermore,

$$(F - \Lambda R^2)' = \left(\frac{R'^2}{L^2} - P_L^2 - \Lambda R^2\right)'$$

$$= -2P_L P_L' - 2\Lambda RR' + 2\left(\frac{R'}{L}\right)\left(\frac{R'}{L}\right)'$$

$$= -\frac{2R'}{L}\mathcal{H}^g - \frac{2P_L}{L}\mathcal{H}_r - \frac{2R'}{R}P_Q^2, \tag{51}$$

and we may write

$$\left(-M + \frac{J^2}{R^2} - P_Q^2\right)' = -\frac{2R'}{L}\mathcal{H}^g - \frac{2P_L}{L}\mathcal{H}_r - \frac{2P_Q}{R}\mathcal{H}_\phi \approx 0. \tag{52}$$

As before, the constraints do not require M' and J' to separately vanish. Equations (48) and (52) are identical apart from the sign of the mass terms. If these solutions are regarded as end states of collapse, then the signs strongly depend on the initial data, as discovered in Ref. [5] and noted in the Introduction.

VI. NEW CANONICAL VARIABLES

We have determined the mass and angular momentum in terms of the canonical variables. Following Kuchař [14], we now seek a new set of canonical variables in which the constraints are simplified. From the expressions for $\mu(M)$ and $j(J)$, however, it appears that both the mass and the angular momentum cannot be a part of the same canonical chart because their Poisson brackets do not vanish. In the quantum theory, they are not simultaneously observable. A more transparent configuration space variable is provided by the time-time component of the metrics. We will show how this comes about.

We will work with a nonzero cosmological constant as the spinning particle is the $\Lambda \to 0$ limit of the same together with $M \to -\mu^2$. From the expression for F in (50), it is straightforward to show that

$$Z = \frac{R'^2}{L^2} - P_L^2 - \Lambda R^2 - P_Q^2, \qquad P_Z = -\frac{LP_L}{2F} \qquad (53)$$

are conjugate variables, i.e., $\{Z, P_Z\}_{P.B.} = 1$. However, the Poisson brackets of Z with Q and P_R, as well as the Poisson bracket of P_Z with P_R, are nonvanishing,

$$\{Z, R\}_{P.B.} = 0 \qquad \{Z, P_R\}_{P.B.} = -\left(\frac{2R'}{L^2}\right)' - 2\Lambda R$$

$$\{Z, Q\}_{P.B.} = 2P_Q \qquad \{Z, P_Q\}_{P.B.} = 0$$

$$\{P_Z, R\}_{P.B.} = 0 \qquad \{P_Z, P_R\}_{P.B.} = -\left(\frac{R'P_L}{F^2L}\right)'$$

$$\{P_Z, Q\}_{P.B.} = 0 \qquad \{P_Z, P_Q\}_{P.B.} = 0. \qquad (54)$$

We wish to replace L and P_L by Z and P_Z in the canonical chart, and the Poisson brackets above tell us that a canonical transformation to new variables, $\bar{P}_R$ and $\bar{Q}$, is required. However, one explicitly checks that

$$\bar{Q} = Q + \frac{LP_L P_Q}{F} \qquad (55)$$

does, in fact, have vanishing Poisson brackets with Z, P_Z, and R and is conjugate to P_Q. The only remaining problem is to find $\bar{P}_R$, and there is a standard procedure for achieving this. The canonical transformation from the original chart, $\{L, R, Q, P_L, P_R, P_Q\}$, to the new chart, $\{Z, R, \bar{Q}, P_Z, \bar{P}_R, P_Q\}$, is found to be generated by

$$G[L, R, P_L, P_Q]$$
$$= \int dr \left[LP_L \left(1 - \frac{P_Q^2}{F}\right) - R'\tanh^{-1}\left(\frac{R'}{LP_L}\right) \right], \qquad (56)$$

and $\bar{P}_R$ is determined to be

$$\bar{P}_R = P_R - \frac{\Lambda R L P_L}{F} - \frac{(R'/LP_L)'}{1 - (R'/LP_L)^2}. \qquad (57)$$

The fall-off of the new canonical variables is easily determined from the fall-off conditions (40).

The momentum constraint $\mathcal{H}_r$, written in terms of the new variables, now reads

$$\mathcal{H}_r = Z'P_Z - R'\bar{P}_R + \bar{Q}P_Q' - \left(\frac{\bar{Q} - 2P_Z P_Q}{R}\right)\mathcal{H}_\phi, \qquad (58)$$

and by substituting the new variables into the Hamiltonian constraint, we also find

$$\mathcal{H}^g = \frac{2FP_Z}{RL}[\bar{Q}P_Q + R\bar{P}_R] - \frac{R'}{2FRL}[2P_Q(RP_Q)' + RZ']. \qquad (59)$$

This last expression can be greatly simplified by exploiting (48); after some algebra, we find

$$\mathcal{H}^g = +\frac{F}{RP_L}[\bar{Q}P_Q + R\bar{P}_R] - \frac{R'}{L^2 P_L}\mathcal{H}_r, \qquad (60)$$

so the full Hamiltonian can now be written in terms of new constraints,

$$\tilde{\mathcal{H}}^g = R\bar{P}_R + \bar{Q}P_Q$$
$$\tilde{\mathcal{H}}_r = Z'P_Z - R'\bar{P}_R + \bar{Q}P_Q'.$$
$$\mathcal{H}_\phi = (RP_Q)', \qquad (61)$$

and adjoined to the action by means of new multipliers. Explicitly,

$$\mathcal{H} = -\tilde{N}\tilde{\mathcal{H}}^g - \tilde{N}^r\tilde{\mathcal{H}}_r - \tilde{N}^\phi\tilde{\mathcal{H}}_\phi, \qquad (62)$$

where

$$\tilde{N} = \frac{NF}{RP_L}$$
$$\tilde{N}^r = N^r + \frac{NR'}{L^2 P_L}$$
$$\tilde{N}^\phi = N^\phi - \left(N^r + \frac{NR'}{L^2 P_L}\right)\left(\frac{\bar{Q}}{R}\right) \qquad (63)$$

with $\bar{Q}$ given in (55). We also notice that

$$\frac{R'}{R}\tilde{\mathcal{H}}^g + \tilde{\mathcal{H}}_r - \frac{\bar{Q}}{R}\mathcal{H}_\phi = Z'P_Z, \qquad (64)$$

so we could just as well consider the constraint system

$$\tilde{\mathcal{H}}^g = R\bar{P}_R + \bar{Q}P_Q$$
$$\mathcal{H}_Z = Z'P_Z$$
$$\mathcal{H}_\phi = (RP_Q)' \qquad (65)$$

and adjoin these [instead of (61)] to the bulk action by means of new multipliers. In the next section, we will absorb the boundary action into the bulk action, and by doing so, we will be able to simplify the constraint system even further.

VII. BOUNDARY ACTION

We will make one more canonical transformation, a trivial one interchanging coordinates and momenta,

$$\bar{Q} = -P_Y, \qquad P_Q = Y. \qquad (66)$$

In terms of the new variables, the Chern-Simons action takes the form

$$S[Z, R, \Omega; P_Z, \bar{P}_R, P_\Omega]$$

$$= \int dt \int_{-\infty}^{\infty} dr[P_Z\dot{Z} + \bar{P}_R\dot{R} + P_Y\dot{Y} - \tilde{N}\tilde{\mathcal{H}}^g$$

$$- \tilde{N}^r\tilde{\mathcal{H}}_r - \tilde{N}^\phi\tilde{\mathcal{H}}_\phi] + S_{\partial\Sigma}, \qquad (67)$$

where the boundary action is given by (41) in terms of the old variables. If the lapse and shift functions on the boundary were allowed to be freely varied, it would imply that the mass and angular momentum of the black hole both vanish at infinity. To avoid this conclusion and allow for a nonvanishing mass and angular momentum, they must be treated as prescribed functions of the ADM time parameter, t; i.e., the lapse and shifts must have fixed ends. To determine these functions, we compare the asymptotic ADM metric in (15) at fixed r,

$$ds^2 = (N_\pm^2 - R^2 N_\pm^{\phi\,2})dt^2 - 2R^2 N_\pm^\phi dt d\phi - R^2 d\phi^2, \qquad (68)$$

with the asymptotic metric in comoving coordinates [11] (also at fixed r)

$$ds^2 = d\mathfrak{t}_\pm^2 + 2\Omega_\pm R^2 d\tau d\phi - R^2 d\phi^2, \qquad (69)$$

where $\mathfrak{t}$ is the proper time and Ω is the angular velocity. Evidently, we must take

$$N_\pm = \pm\sqrt{1 + r^2\Omega_\pm^2}\,\dot{\mathfrak{t}}_\pm \overset{\text{def}}{=} \pm\dot{\tau}_\pm,$$

$$N_\pm^\phi = \mp\,\Omega_\pm\dot{\mathfrak{t}}_\pm \overset{\text{def}}{=} \pm\dot{\omega}_\pm, \qquad (70)$$

where $\mathfrak{t}_\pm(t)$ represents the proper time and $\Omega_\pm(t)$ represents the angular velocity function as measured along constant r world lines at the infinities. The surface action now reads

$$S_{\partial\Sigma} = -\int dt\left[\frac{1}{2}(M_+\dot{\tau}_+ - M_-\dot{\tau}_-) + J_+\dot{\omega}_+ - J_-\dot{\omega}_-\right]. \qquad (71)$$

First, consider the Liouville form

$$\Theta_1 := \int_{-\infty}^{\infty} dr P_Z\delta Z - \frac{1}{2}(M_+\delta\tau_+ - M_-\delta\tau_-), \qquad (72)$$

and note that, according to the fall-off conditions, $\lim_{r\to\infty}Z(r) = -M_\pm$. We therefore define the function $\Gamma(r)$ by

$$Z(r) = -M_- - \int_{-\infty}^{r} dr'\Gamma(r'), \qquad Z'(r) = -\Gamma(r) \qquad (73)$$

and rewrite Θ_1 as follows:

$$\Theta_1 := \int_{-\infty}^{\infty} dr P_Z(r)\left[-\delta M_- - \int_{-\infty}^{r} dr'\delta\Gamma(r')\right] - \frac{1}{2}\delta(\tau_+ M_+ - \tau_- M_-) + \frac{1}{2}\tau_+\delta M_+ - \frac{1}{2}\tau_-\delta M_-$$

$$= \delta M_-\left(-\frac{1}{2}\tau_- - \int_{-\infty}^{\infty} dr P_Z(r)\right) - \int_{-\infty}^{\infty} dr P_Z(r)\int_{-\infty}^{r} dr'\delta\Gamma(r')$$

$$+ \frac{1}{2}\tau_+\left(\delta M_- + \int_{-\infty}^{\infty} dr'\delta\Gamma(r')\right) - \frac{1}{2}\delta(\tau_+ M_+ - \tau_- M_-)$$

$$= \delta M_-\left(\frac{1}{2}(\tau_+ - \tau_-) - \int_{-\infty}^{\infty} dr P_Z(r)\right) - \int_{-\infty}^{\infty} dr P_Z(r)\int_{-\infty}^{r} dr'\delta\Gamma(r')$$

$$+ \frac{1}{2}\tau_+\int_{-\infty}^{\infty} dr'\delta\Gamma(r') - \frac{1}{2}\delta(\tau_+ M_+ - \tau_- M_-). \qquad (74)$$

This allows us to identify the conjugate variables,

$$m = M_-, \qquad p_m = \frac{1}{2}(\tau_+ - \tau_-) - \int_{-\infty}^{\infty} dr P_Z(r), \qquad (75)$$

and we have, apart from an exact form,

$$\Theta_1 = p_m\delta m + \int_{-\infty}^{\infty} dr\left[\frac{1}{2}\tau_+\delta\Gamma(r) - P_Z(r)\int_{-\infty}^{r} dr'\delta\Gamma(r')\right]. \qquad (76)$$

Again, using the identity [14],

$$\int_{-\infty}^{\infty} dr P_Z(r)\int_{-\infty}^{r} dr'\delta\Gamma(r') = -\int_{-\infty}^{\infty} dr\delta\Gamma(r)\int_{\infty}^{r} dr' P_Z(r'), \qquad (77)$$

we find

$$\Theta_1 = p_m\delta m + \int_{-\infty}^{\infty} dr\left[\frac{1}{2}\tau_+ + \int_{\infty}^{r} dr' P_Z(r')\right]\delta\Gamma(r), \qquad (78)$$

which now allows us to identify the conjugate variables

$$\Gamma(r) = -Z'(r), \qquad P_\Gamma(r) = \frac{1}{2}\tau_+ + \int_\infty^r dr' P_Z(r').$$

$$(79)$$

We note that $P_\Gamma' = P_Z = \frac{1}{2}T'$, so the Killing time can be identified with the momentum $2P_\Gamma$ up to a constant. We can choose the constant so that T matches τ_+ at infinity; then,

$$T = 2P_\Gamma = \tau_+ + 2\int_\infty^r dr' P_Z(r'),$$

$$(80)$$

and the momentum conjugate to the Killing time is just $\mathcal{P}_T = -\frac{1}{2}\Gamma$.

Next, consider the Liouville form

$$\Theta_2 := \int_{-\infty}^\infty dr[\bar{P}_R \delta R + P_Q \delta \bar{Q}] - (J_+ \delta\omega_+ - J_- \delta\omega_-), \quad (81)$$

and recall that, under the fall-off conditions, $\lim_{r\to\infty} P_Q = -J_\pm r^{-1}$. If we define

$$R(r)P_Q(r) = -J_- + \int_{-\infty}^r dr' \Sigma(r'),$$

$$(82)$$

then, by the third constraint, $\Sigma(r) = 0$. Therefore $J_+ = J_- = J$, and

$$\begin{aligned}\Theta_2 :&= \int_{-\infty}^\infty dr\left[\bar{P}_R \delta R - \frac{J}{R}\delta\bar{Q}\right] - J\delta(\omega_+ - \omega_-)\\ &= \int_{-\infty}^\infty dr\left(\bar{P}_R + \frac{J\bar{Q}}{R^2}\right)\delta R - J\delta\left[(\omega_+ - \omega_-) + \int_{-\infty}^\infty dr\left(\frac{\bar{Q}}{R}\right)\right]\\ &= \int_{-\infty}^\infty dr\left(\bar{P}_R + \frac{J\bar{Q}}{R^2}\right)\delta R + \left[(\omega_+ - \omega_-) + \int_{-\infty}^\infty dr\left(\frac{\bar{Q}}{R}\right)\right]\delta J\end{aligned}$$

$$(83)$$

up to an exact form. Thus, we identify

$$p_J = (\omega_+ - \omega_-) + \int_{-\infty}^\infty dr\left(\frac{\bar{Q}}{R}\right)$$

$$(84)$$

as the momentum conjugate to J and a new momentum,

$$\mathcal{P}_R = \bar{P}_R + \frac{J\bar{Q}}{R^2},$$

$$(85)$$

conjugate to R. In terms of the new variables, the constraints read

$$\begin{aligned}\mathcal{H}_R &= R\mathcal{P}_R\\ \mathcal{H}_T &= T'\mathcal{P}_T\end{aligned}$$

$$(86)$$

and may be adjoined to the canonical action by means of new Lagrange multipliers. The reduced canonical action,

$$\begin{aligned}S = p_m \dot{m} + p_J \dot{J} + \int_{-\infty}^\infty dr[\mathcal{P}_T \dot{T} + \mathcal{P}_R \dot{R}\\ - (N^T\mathcal{P}_T + N^R\mathcal{P}_R)],\end{aligned}$$

$$(87)$$

shows that the configuration space of vacuum $2+1$-dimensional gravity is covered by the coordinates T, R and two degrees of freedom, m and J. The constraints are straightforward: $\mathcal{P}_T = \mathcal{P}_R = 0$.

Quantization proceeds directly. According to Dirac's quantization program, the momenta are raised to operator status, and the constraints act as operator constraints on the state functional, $\Psi = \Psi(m, J, t; T, R)$. The two constraints tell us that the wave functionals are independent of T and R and the spacetimes are described by wave functions, $\Psi(m, J, t)$, which, moreover, are time independent because the Hamiltonian vanishes,

$$i\dot{\Psi}(m, J, t) = 0 \Rightarrow \Psi(m, J, t) = \Psi(m, J). \quad (88)$$

No further information is available. The wave function, once prepared, stays the same on every spacelike hypersurface.

VIII. CONCLUSION

As mentioned in the Introduction, the principal goal of this paper was to construct the canonical description of axisymmetric, vacuum solutions to Einstein's gravity in $2+1$ dimensions using techniques that are easily extended to the description of dynamical collapse. Just as the original canonical reduction of static spherical geometries in $3+1$ dimensions by Kuchař [14] has proved extremely useful in understanding spherically symmetric dynamical collapse, we expect the reduction presented in this paper to play a pivotal role in describing gravitational collapse with rotation, at least in $2+1$ dimensions [11].

Here, we showed that the mass and angular momentum can be recovered from the canonical data and that the constraints describing the axisymmetric vacuum turn out to be extremely simple after a series of canonical transformations and after absorbing the boundary terms into the hypersurface action. Indeed, one finally arrives at a trivial system (with zero Hamiltonian) described by the only two classical features of this vacuum: the ADM mass and the angular momentum. The quantum mechanics of the system describes a time independent wave function that depends only on these variables. This may seem a bit surprising considering that particle production is expected to occur near the horizon, and in fact, this state of affairs will no longer hold once matter is injected into the spacetime. Studies of spherical quantum dust collapse, employing Kuchař's variables, have shown how Hawking radiation arises in this approach [19], but they have also shown that (modulo a selection rule) the collapse process need not lead to the formation of black holes [20]. Infalling

shells of matter are accompanied by expanding shells emanating from the center, and neither the infalling shells nor the expanding shells may ever cross the horizon.

Our ultimate goal is to couple gravity to matter and describe the quantum evolution of gravitational collapse with rotation. One advantage of the canonical approach is that all information is preserved in the canonical data. Thus, if the leaves of the foliation are chosen carefully, so that they cover all of the spacetime, then one can in principal study the collapse everywhere, without and within the horizon and even in the approach to the singularity. Such a foliation is provided by slices of constant proper time, and we will report on the classical and quantum results from embedding the ADM metric here into the spacetime described in Ref. [11] in a future publication.

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