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Physical properties of a source of the Kerr metric: Bound on the surface gravitational potential and conditions for the fragmentation

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We investigate some important physical aspects of a recently presented interior solution for the Kerr metric. It is shown that, as in the spherically symmetric case, there is a specific limit for the maximal value of the surface potential (degree of compactness), beyond which unacceptable physical anomalies appear. Such a bound is related to the appearance of negative (repulsive) gravitational acceleration that is accompanied by the appearance of negative values of the pressure. A detailed discussion on this effect is presented. We also study the possibility of a fragmentation scenario, assuming that the source leaves the equilibrium, and we bring out the differences with the spherically symmetric case.

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I. INTRODUCTION

In a recent paper [1], we have proposed an interior solution for the Kerr metric [2], satisfying the matching conditions on the boundary surface of the matter distribution and endowed with reasonable physical properties, at least for a range of values of the parameters, which includes values considered in the existing literature to describe realistic models of rotating neutron stars and white dwarfs.

For these reasons, after decades of intense theoretical work (see [3–19] and references therein), the solution presented in [1] may be reasonably regarded as a satisfactory solution to the problem of constructing a physically viable source for the Kerr metric. It is the purpose of this work to study some important physical properties of this source.

We shall first establish a bound on the degree of compactness (surface gravitational potential), which of course implies a bound on the gravitational surface redshift of spectral lines from the surface of our source, similar to the limit existing for spherically symmetric sources [20,21]. Two points deserve to be emphasized here:

(i) Our source is generated by an anisotropic fluid.

(ii) In the spherically symmetric case, there is a well-established link between the maximal values of the surface redshift and the local anisotropy of pressure (see [22–32] and references therein). The great interest aroused on this issue is easily justified if we recall that the surface redshift is an observable variable, thereby entitled to provide relevant information about the structure of the source.

In the spherically symmetric case, the bound on the degree of compactness expresses itself through the appearance of different kinds of physical anomalies which render the fluid distribution physically inviable (e.g. singularities of the physical variables, negative energy density, etc.).

In this manuscript, we shall relate the above-mentioned limit to the appearance of repulsive gravitational acceleration, described by means of the acceleration tensor recently introduced by Maluf [33]. This tensor gives the values of the inertial (i.e., nongravitational) accelerations that are necessary to maintain the frame in a given inertial state (stationary in our case). If the frame is maintained stationary in spacetime, then the inertial acceleration is exactly minus the gravitational acceleration imparted to the frame. Obviously, once the acceleration tensor becomes negative, for any piece of the material, the system becomes unstable and leaves the stationary state.

We shall investigate, in detail, the maximal degree of compactness, which excludes the appearance of negative gravitational acceleration for a range of values of the angular momentum of the source. We shall also see that the appearance of negative gravitational accelerations is always accompanied by the appearance of negative pressure of the fluid distribution and a singularity of the pressure at the center of the fluid distribution for exactly the same values of the parameters.

Next, we shall consider our source as the initial state of a fluid distribution which is assumed to leave the equilibrium regime. We shall then evaluate the system on a time scale that is smaller than the hydrostatic time scale. Doing so, we

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shall be able to detect possible scenarios of fragmentation (cracking) of the source, which are absent in the spherically symmetric limit of our source.

II. A SOURCE FOR THE EXTERIOR KERR SOLUTION

In [1], we provided a general method to construct global models of self-gravitating stationary sources. As a particular example, we considered a source for the Kerr metric. The present study concerns this last fluid distribution. In what follows, we shall very briefly summarize the basic equations and the main properties of the solution. We refer the reader to [1] for the details and some intermediate calculations. At this point, we would like to call attention to two misprints in [1] that have been corrected here, namely, Eq. (11) below is the right version of the corresponding Eq. (36) appearing in [1], and a misprint appearing in Eq. (40) in [1] has been corrected in the last of the forthcoming Eqs. (26). It is worth emphasizing, however, that such misprints are irrelevant for the discussion presented in [1] since all the calculations in that reference were carried out using the right expressions, written down here.

A. The exterior metric

The line element for a vacuum stationary and axially symmetric spacetime, in Weyl canonical coordinates, may be written as

$$ds_E^2 = -e^{2\psi}(dt - wd\phi)^2 + e^{-2\psi+2\Gamma}(d\rho^2 + dz^2) + e^{-2\psi}\rho^2 d\phi^2, \tag{1}$$

where $\psi = \psi(\rho, z)$, $\Gamma = \Gamma(\rho, z)$ and $w = w(\rho, z)$ are functions of their arguments.

For vacuum spacetimes, Einstein's field equations imply for the metric functions

$$f(f_{,\rho\rho} + \rho^{-1}f_{,\rho} + f_{,zz}) - f_{,\rho}^2 - f_{,z}^2 + \rho^{-2}f^4(w_{,\rho}^2 + w_{,z}^2) = 0, \tag{2}$$

$$f(w_{,\rho\rho} + \rho^{-1}w_{,\rho} + w_{,zz}) + 2f_{,\rho}w_{,\rho} + 2f_{,z}w_{,z} = 0, \tag{3}$$

with $f \equiv e^{2\psi}$ and

$$\Gamma_{,\rho} = \frac{1}{4}\rho f^{-2}(f_{,\rho}^2 - f_{,z}^2) - \frac{1}{4}\rho^{-1}f^2(w_{,\rho}^2 - w_{,z}^2)$$
$$\Gamma_{,z} = \frac{1}{2}\rho f^{-2}f_{,\rho}f_{,z} - \frac{1}{2}\rho^{-1}f^2 w_{,\rho}w_{,z}. \tag{4}$$

It will be useful to introduce the Erez-Rosen [34] or standard Schwarzschild-type coordinates $\{r, y \equiv \cos\theta\}$ or in spheroidal prolate coordinates $\{x \equiv \frac{r-M}{M}, y\}$ [35],

$$\rho^2 = r(r - 2M)(1 - y^2), \qquad z = (r - M)y, \tag{5}$$

where M is a constant which will be identified later.

In terms of the above coordinates, the line element (1) may be written as

$$ds_E^2 = -e^{2\psi(r,y)}(dt - wd\phi)^2 + e^{-2\psi+2[\Gamma(r,y)-\Gamma^s]}dr^2 + e^{-2[\psi-\psi^s]+2[\Gamma(r,y)-\Gamma^s]}r^2 d\theta^2 + e^{-2[\psi-\psi^s]}r^2\sin^2\theta d\phi^2, \tag{6}$$

where ψ^s and Γ^s are the metric functions corresponding to the Schwarzschild solution, namely,

$$\psi^s = \frac{1}{2}\ln\left(\frac{r-2M}{r}\right) \qquad \Gamma^s = -\frac{1}{2}\ln\left[\frac{(r-M)^2 - y^2M^2}{r(r-2M)}\right], \tag{7}$$

where the parameter M is easily identified as the Schwarzschild mass.

For the specific case of the Kerr metric, in Weyl coordinates, we have the following expressions for the metric functions,

$$f = \frac{(r_1+r_2)^2(1-j^2) - 4M^2(1-j^2) + j^2(r_1-r_2)^2}{(r_1+r_2+2M)^2(1-j^2) + j^2(r_1-r_2)^2}, \tag{8}$$

$$e^{2\Gamma} = \frac{(r_1+r_2)^2(1-j^2) - 4M^2(1-j^2) + j^2(r_1-r_2)^2}{4r_1r_2(1-j^2)}, \tag{9}$$

$$w = \frac{j(2M + r_1 + r_2)(4M^2(1-j^2) - (r_1-r_2)^2)}{(r_1+r_2)^2(1-j^2) - 4M^2(1-j^2) + j^2(r_1-r_2)^2}, \tag{10}$$

where $j \equiv \frac{J}{M^2} = a/M$ denotes the dimensionless parameter representing the angular momentum of the source and is related to the rotation parameter a of the Kerr metric in its well-known Boyer-Lindquist representation.

Also, $r_{1,2} \equiv \sqrt{\rho^2 + (z \pm M\sqrt{1-j^2})^2}$, which in the Erez-Rosen coordinates become

$$r_{1,2}^2 = \left[(r-M) \pm My\sqrt{1-j^2}\right]^2 - M^2j^2(1-y^2). \tag{11}$$

From the study of the relativistic multipole moments (RMM) [36–41] of the Kerr solution, it follows that

$$m_k = M_k = M(ia)^k, \tag{12}$$

where (m_k) are the expansion coefficients of the Ernst potential on the axis of symmetry.

Also, we have that the massive RMM (even orders) and the rotational RMM (odd orders) can be expressed as

$$M_{2l} = (-1)^l M^{2l+1} j^{2l},$$
$$M_{2l+1} = i(-1)^l M^{2l+2} j^{2l+1}. \quad (13)$$

From the above, it follows that the rotation of the object leads to a negative quadrupole massive moment, $q \equiv \frac{M_2}{M^3} = -j^2$, i.e. all the possible sources of the Kerr solution are oblate (see [1] for details).

B. The interior metric

Following the procedure sketched in [1], the following line element is assumed at the interior:

$$ds_I^2 = -e^{2\hat{a}} Z(r)^2 (dt - \Omega d\phi)^2 + \frac{e^{2\hat{g}-2\hat{a}}}{A(r)} dr^2 + e^{2\hat{g}-2\hat{a}} r^2 d\theta^2 + e^{-2\hat{a}} r^2 \sin^2\theta d\phi^2, \quad (14)$$

with

$$\hat{a} \equiv a(r,\theta) - a^s(r), \qquad \hat{g} \equiv g(r,\theta) - g^s(r,\theta), \quad (15)$$

where $\Omega = \Omega(r,\theta)$, and $a^s(r)$ and $g^s(r,\theta)$ are functions that, on the boundary surface, equal the metric functions corresponding to the Schwarzschild solution (7), i.e. $a^s(r_\Sigma) = \psi^s_\Sigma$ and $g^s(r_\Sigma) = \Gamma^s_\Sigma$. Also, $A(r) \equiv 1 - pr^2$ and $Z \equiv \frac{3}{2}\sqrt{A(r_\Sigma)} - \frac{1}{2}\sqrt{A(r)}$, where p is an arbitrary constant and the boundary surface of the source is defined by $r = r_\Sigma = \text{const.}$

The case $w = 0$, $\hat{g} = \hat{a} = 0$ corresponds to a spherically symmetric distribution, more specifically, to the well-known incompressible (homogeneous energy density) perfect fluid sphere, and hence the matching of (14) with the Schwarzschild solution implies $p = \frac{2M}{r_\Sigma^3}$. The simple condition $w = 0$ recovers, of course, the static case.

It should be noticed that, for simplicity, we consider here only matching surfaces of the form $r = r_\Sigma = \text{const}$. In particular, this choice simplifies considerably the treatment of the matching conditions (see below). Besides, this type of boundary surface allows us to describe quite appropriately the shape that we expect for a relativistic rotating body. Of course, more general surfaces with axial symmetry, of the form $r = r_\Sigma(\theta)$, could be considered as well.

In order to satisfy the matching (Darmois) conditions [42], the following equations have to be satisfied,

$$\begin{aligned}
&a_\Sigma = \psi_\Sigma, \qquad a'_\Sigma = \psi'_\Sigma, \qquad g_\Sigma = \Gamma_\Sigma, \qquad g'_\Sigma = \Gamma'_\Sigma,\\
&a^s_\Sigma = \psi^s_\Sigma, \qquad (a^s)'_\Sigma = (\psi^s)'_\Sigma,\\
&g^s_\Sigma = \Gamma^s_\Sigma, \qquad (g^s)'_\Sigma = (\Gamma^s)'_\Sigma,\\
&\Omega_\Sigma = w_\Sigma, \qquad \Omega'_\Sigma = w'_\Sigma,
\end{aligned} \quad (16)$$

where prime denotes partial derivative with respect to r, and subscript Σ indicates that the quantity is evaluated on the boundary surface. It is important to keep in mind that we are using global coordinates $\{r,\theta\}$ on both sides of the boundary.

Indeed, the Darmois matching conditions require the continuity of the first and the second fundamental form across the boundary surface of the source.

The first fundamental form is just the induced metric on the boundary surface. Therefore, the first set of Darmois conditions requires

$$(ds_E^2)_\Sigma - (ds_I^2)_\Sigma \equiv [ds^2] \overset{\Sigma}{=} 0, \quad (17)$$

where ds_E^2 and ds_I^2 are given by (6) and (14), respectively, and the square bracket denotes the discontinuity of any enclosed quantity, across the boundary surface of the source. Now, since the matching surface considered is $r = r_\Sigma = \text{const}$, the first fundamental form is continuous on that surface whenever $a_\Sigma = \psi_\Sigma$, $g_\Sigma = \Gamma_\Sigma$ and $\Omega_\Sigma = w_\Sigma$.

Next, we have to require the continuity of the second fundamental form, which implies that the extrinsic curvature evaluated on both sides of the matching surface must be equal.

This second fundamental form (II) evaluated on the boundary surface is defined by

$$II = -(n_{\mu;\nu} dx^\nu dx^\mu)_\Sigma \equiv (K_{ab} dx^a dx^b)_\Sigma, \quad (18)$$

where the indexes a,b stand for $\{t,\theta,\phi\}$, and n_μ denotes the unit, normal vector to the boundary surface.

In our case, the boundary surface equation is given by

$$f \equiv r - r_\Sigma = 0; \qquad r_\Sigma = \text{constant}, \quad (19)$$

implying that the unit vector, normal to the boundary surface, is defined by

$$n_\mu = \frac{\partial_\mu f}{\sqrt{\partial_\alpha f \partial_\beta f g^{\alpha\beta}}}. \quad (20)$$

From (19) and (20), it follows that $K_{ab} = -\Upsilon^1_{ab}$, where Υ^i_{jk}, denote the Christoffel symbols of the corresponding (interior or exterior) metric. Then, after some simple calculations, we obtain

$$\begin{aligned}
0 &\overset{\Sigma}{=} [K_{tt}] \Rightarrow e^{4\hat{a}-2\hat{g}} A(\hat{a}' Z^2 + ZZ') \overset{\Sigma}{=} e^{4\psi - 2\hat{\Gamma}}(\psi'),\\
0 &\overset{\Sigma}{=} [K_{\theta\theta}] \Rightarrow A((\hat{g}' - \hat{a}')r^2 + r) \overset{\Sigma}{=} e^{2\psi^s}((\hat{\Gamma}' - \hat{\psi}')r^2 + r),\\
0 &\overset{\Sigma}{=} [K_{\phi\phi}] \Rightarrow (-\hat{\psi}' r^2 + r)\sin^2\theta e^{-2\hat{\psi}} - (\psi' w^2 + ww')e^{-2\psi} \overset{\Sigma}{=}\\
&\quad \times (-\hat{a}' r^2 + r)\sin^2\theta e^{-2\hat{a}}\\
&\quad - (\hat{a}' Z^2 \Omega^2 + ZZ'\Omega^2 + Z^2 \Omega\Omega')e^{2\hat{a}},\\
0 &\overset{\Sigma}{=} [K_{\phi t}] \Rightarrow 2\psi' w + w' \overset{\Sigma}{=} (2\hat{a}' Z^2 \Omega + 2ZZ'\Omega + Z^2\Omega')e^{-2\psi^s},
\end{aligned} \quad (21)$$

where $\hat{\psi} \equiv \psi - \psi^s$, $\hat{\Gamma} \equiv \Gamma - \Gamma^s$ and the square bracket denotes the discontinuity of any enclosed quantity, across the boundary surface of the source. From the above expressions, it follows at once that the continuity of K_{tt} requires that $\hat{a}'_\Sigma = \hat{\psi}'_\Sigma$. The continuity of $K_{\theta\theta}$ implies that $\hat{g}'_\Sigma = \hat{\Gamma}'_\Sigma$, and finally the continuity of $K_{\phi\phi}$ and $K_{\phi t}$ imposes that $\Omega'_\Sigma = w'_\Sigma$. These conditions, together with those obtained from the continuity of the first fundamental form, are exactly conditions (16).

In the spherically symmetric case, we have $\hat{a} = \hat{g} = 0$, and the physical variables are obtained from the field equations for a perfect fluid, the result is well known and reads (in relativistic units)

$$-T_0^0 \equiv \mu = \frac{3p}{8\pi},$$
$$T_1^1 = T_2^2 = T_3^3 \equiv P = \mu\left(\frac{\sqrt{A} - \sqrt{A_\Sigma}}{3\sqrt{A_\Sigma} - \sqrt{A}}\right), \tag{22}$$

with $A = 1 - \frac{2m(r)}{r} = 1 - pr^2 = 1 - \frac{2Mr^2}{r_\Sigma^3}$, where μ and P denote the energy density and the isotropic pressure, respectively, and for the mass function $m(r)$, we have

$$m(r) = -4\pi \int_0^r r^2 T_0^0 dr, \tag{23}$$

implying

$$M \equiv m(r_\Sigma) = -4\pi \int_0^{r_\Sigma} r^2 T_0^0 dr = \frac{pr_\Sigma^3}{2}. \tag{24}$$

This model, which describes the well-known incompressible perfect fluid sphere, is further restricted by the requirement that the pressure be regular and positive everywhere within the fluid distribution, which implies $\tau \equiv \frac{r_\Sigma}{M} > \frac{9}{4}$, where τ measures the inverse of the degree of compactness. As is evident from (22), the pressure vanishes at the boundary surface.

We shall now proceed to consider the general, nonspherical case. In [1], we provided a general procedure to choose the interior metric functions $\hat{a}$, $\hat{g}$ and Ω producing physically meaningful models. With this aim, besides the fulfilment of the junction conditions (16), we required that all physical variables be regular within the fluid distribution and the energy density be positive. Following this procedure, we have, for the interior of the Kerr metric,

$$\hat{a}(r,\theta) = \hat{\psi}_\Sigma s^2(3-2s) + r_\Sigma \hat{\psi}'_\Sigma s^2(s-1),$$
$$\hat{g}(r,\theta) = \hat{\Gamma}_\Sigma s^3(4-3s) + r_\Sigma \hat{\Gamma}'_\Sigma s^3(s-1),$$
$$\Omega(r,\theta) = w_\Sigma s^4(5-4s) + r_\Sigma w'_\Sigma s^4(s-1), \tag{25}$$

with $s \equiv r/r_\Sigma \in [0,1]$, and

$$\hat{\psi}_\Sigma \equiv \psi_\Sigma - \psi_\Sigma^s = \frac{1}{2}\ln \times\left\{\frac{\tau}{\tau-2}\frac{N + r_1^\Sigma r_2^\Sigma(2j^2-1)}{N + r_1^\Sigma r_2^\Sigma(2j^2-1) - 2(1-j^2)(r_1^\Sigma + r_2^\Sigma + 2)}\right\},$$
$$\hat{\Gamma}_\Sigma \equiv \Gamma_\Sigma - \Gamma_\Sigma^s = \frac{1}{2}\ln\left\{\frac{(\tau-1)^2 - y^2}{\tau(\tau-2)}\frac{N + r_1^\Sigma r_2^\Sigma(2j^2-1)}{2r_1^\Sigma r_2^\Sigma(j^2-1)}\right\},$$
$$w_\Sigma = Mj\frac{(N + r_1^\Sigma r_2^\Sigma)(2 + r_1^\Sigma + r_2^\Sigma)}{-N + r_1^\Sigma r_2^\Sigma(1-2j^2)}, \tag{26}$$

with

$$r_{1,2}^\Sigma = \sqrt{\left(\tau - 1 \pm y\sqrt{1-j^2}\right)^2 - j^2(1-y^2)}. \tag{27}$$

$$N \equiv -\tau(\tau-2) + (1-y^2) - j^2. \tag{28}$$

The metric functions so obtained satisfy the junction conditions (16) and produce physical variables (Eqs. (17)–(21) in [1]), which are regular within the fluid distribution. Furthermore, the vanishing of $\hat{g}$ on the axis of symmetry, as required by the regularity conditions, necessary to ensure elementary flatness in the vicinity of the axis of symmetry and, in particular, at the center [43–45], is assured by the fact that $\hat{\Gamma}_\Sigma$ and $\hat{\Gamma}'_\Sigma$ vanish on the axis of symmetry. Also, the good behavior of the function Ω on the symmetry axis is fulfilled since w_Σ and w'_Σ vanish when $y = \pm 1$.

III. BOUND ON THE SURFACE REDSHIFT AND THE ACCELERATION TENSOR

We shall now proceed to establish the limit on the degree of compactness of our source. In doing so, we shall make use of the concept of the acceleration tensor introduced in [33].

The acceleration tensor gives the values of the inertial (i.e., nongravitational) accelerations that are necessary to maintain the frame adapted to a field of observers in spacetime in a given inertial state. If the frame is maintained static (stationary) in spacetime, then the inertial acceleration is exactly minus the gravitational acceleration imparted to the frame.

Thus, we may write

$$\vec{a} = \frac{1}{2(-g^{00})^{3/2}}\left(\frac{\partial_1 g^{00}}{\sqrt{g_{11}}}\vec{r} + \frac{\partial_2 g^{00}}{\sqrt{g_{22}}}\vec{\theta}\right), \tag{29}$$

with $\vec{a} \equiv a_r\vec{r} + a_\theta\vec{\theta}$.

In our case, the following expression verifies

$$g^{00} = -\frac{e^{-2\hat{a}}}{Z^2} + e^{2\hat{a}}\frac{\Omega^2}{r^2\sin^2\theta}, \tag{30}$$

then the radial component of the acceleration (the only one we need for our discussion) reads

$$a_r = \frac{e^{-\hat{g}(s,y)}\sqrt{A}}{r_\Sigma(-g^{00})^{3/2}}\left[\frac{e^{-\hat{a}(s,y)}}{Z(s)^3}\left(Z(s)\partial_s\hat{a}(s,y) + \frac{s}{\sqrt{\tau(\tau-2s^2)}}\right) + e^{3\hat{a}(s,y)} \times \frac{\hat{\Omega}^2}{s^2\sin^2\theta}\left(\partial_s\hat{a} + \frac{\partial_s\hat{\Omega}}{\hat{\Omega}} - \frac{1}{s}\right)\right], \tag{31}$$

where $s \equiv r/r_\Sigma \in [0, 1]$, and $\hat{\Omega} \equiv \Omega(s, y) = \Omega(r, y)/r_\Sigma$.

As we have already mentioned, for the spherically symmetric incompressible fluid, we have a bound for the degree of compactness (surface gravitational potential) given by $\tau \equiv \frac{r_\Sigma}{M} > \frac{9}{4}$. Usually, such a limit is related to the appearance of a singularity of the pressure at the center of the distribution.

We shall start our study by imposing only the positivity of the energy density and the absence of singularities in the physical variables. Doing so, the evaluation of (31) produces the following results:

(i) The change of sign in a_r (from positive to negative), occurs always for a critical value of τ_c, ($\tau_c = 9/4 = 2.25$). A specific example is depicted in Fig. 1, for a given value of the parameter j and a given value of the angular coordinate θ. The continuous line corresponds to a value of $\tau > \tau_c$. As is apparent from this figure, the range of values of the radial coordinate s, for which the acceleration is negative, depends on the values of τ. Figure 2 is a plot of the acceleration as function of s and τ for a given value of y. Again the negative values of the acceleration are clearly exhibited

(ii) The behavior of a_r depends, although very slightly, on the angular coordinate y. This is shown in the Fig. 3, for very small values of τ (very compact objects), for the value of the radial coordinate $s = 0.3$. Observe the smaller (in absolute value) values of a_r, close to the axis of symmetry. This can also be appreciated in the Fig. 4, for a value of $\tau = 2.1$.

(iii) In all the examples analyzed, the positive energy condition (P.E.D.) is always satisfied ($-T^0_0 > 0$), at least for $j < 0.1$. This is clearly indicated in the Table I, where minimal values of τ (maximal degree of compactness), compatible with P.E.D., are given for different values of j. As is apparent in this table, for values $j < 0.1$ the minimal value of τ preserving the P.D.E., is smaller than $\tau_c = 2.25$, thereby indicating that negative accelerations are compatible with positive energy density.

(iv) Although, the value of j affects the value of a_r when it is positive, the former neither affects the value of a_r when it is negative, nor does it affect the value of s for which the acceleration changes of sign. This is clearly shown in the Fig. 5. It is worth noting that this behavior also holds for $j = 0$, as it is apparent from the corresponding curve in Fig. 5. The general behavior is always the same: negative acceleration appears in the inner part of the source, whereas it is

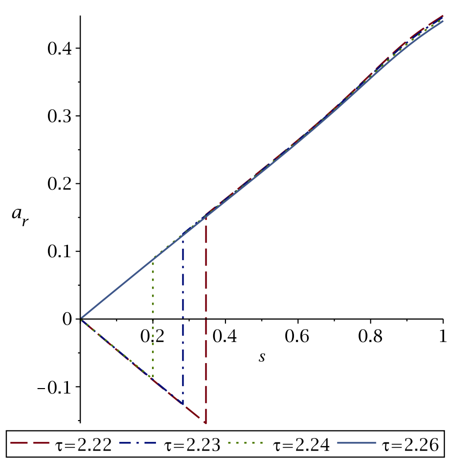

FIG. 1. Graphics of a_r as function of s, for different values of τ, with $j = 0.1$, $y \equiv \cos\theta = 0.5$.

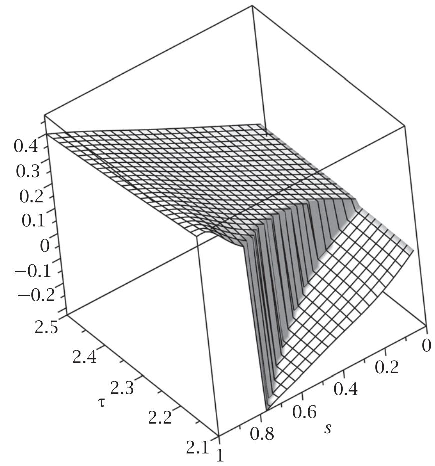

FIG. 2. Plot of a_r as function of s and τ, for $j = 0.1$, $y = 0.2$.

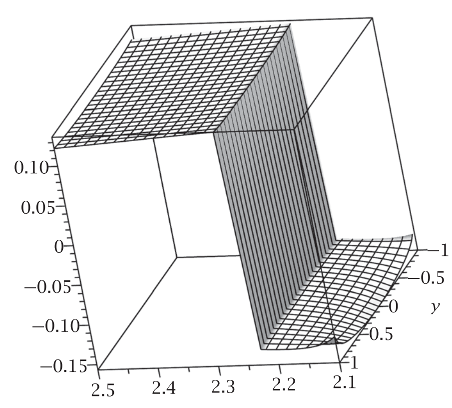

FIG. 3. Plot of a_r as function of τ and y, with $j = 0.1$, $s = 0.3$.

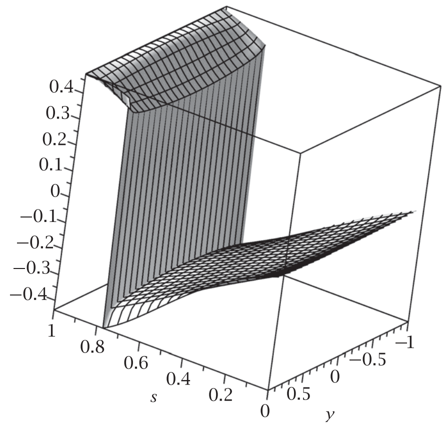

FIG. 4. Plot of a_r, as function of s and y, with $j = 0.1$, $\tau = 2.1$.

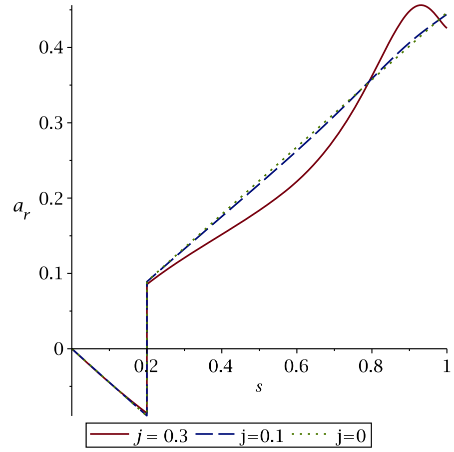

FIG. 5. Curves depicting a_r as function of s, for different values of the rotation parameter j, with $\tau = 2.24$, $y = 0.5$.

positive at the outer one, and there is no range of values of τ and/or j, for which this behavior inverts.

(v) The issue mentioned in the point above, suggests that the change of sign in a_r and the existence of a critical value τ_c, should be related with restrictions already appearing in the spherically symmetric case. This is indeed the case. In fact, below τ_c, the radial pressure becomes negative in some regions of the source and becomes singular at the origin. Furthermore, the range of the radial coordinate for which the acceleration becomes negative ($s \in [0, s_c]$), where the value of s_c depends on τ but not on j, is exactly the same range for which the radial pressure becomes negative.

(vi) The exact value of s_c, can be determined by evaluating the radial pressure (expression (19) in [1]). There is however a much simpler way of doing that. Indeed, as it can be seen from (22), the pressure P (in the spherically symmetric case) is always positive, unless the denominator in that expression becomes negative. On the other hand, the small corrections to the radial pressure introduced by the nonsphericity in the source under consideration, do not affect the conclusion above, since they are smaller than P in their absolute value (see fig 3 in ([1]). In other words the radial pressure is negative only if $P < 0$.

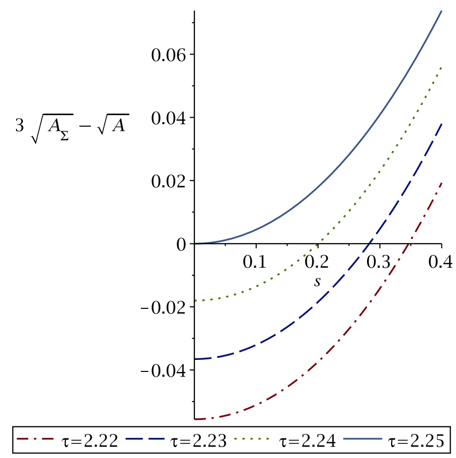

FIG. 6. Curves delimiting the range of values of s, for which the radial pressure is negative, (the curves cut the axis at s_c), for different values of τ.

(vii) In Fig. 6 we plot the denominator of (22) as a function of s, for different values of τ. The roots of these curves define the range $s \in [0, s_c]$ for which the acceleration and the radial pressure become negative. Finally, in Table II, we show some values of s_c for different values of τ.

TABLE I. $\tau_{\rm min}$ denotes the minimal value of the inverse compactness factor compatible with the positive energy density condition Positive Energy Density (P.E.D.) ($-T^0_0 > 0$).

P.E.D. ($-T^0_0 > 0$)	
j	$\tau_{\rm min}$
0.1	2.27
0.01	2.07
0.005	2.05
0.001	2.03

TABLE II. s_c denotes the value of the dimensionless radial coordinate s, enclosing the region where negative radial pressure appears; this coincides with the region within which the radial acceleration a_r becomes negative.

$P < 0$, $a_r < 0$, iff $s \in [0, s_c]$	
τ	s_c
2.22	0.3464
2.23	0.2828
2.24	0.1999
2.25	0

IV. THE FRAGMENTATION (CRACKING) OF THE SOURCE

In this section, we shall tackle a different physical issue, related to our source.

Let us consider our source as an initial configuration, submitted to perturbations, under which it is *a priori* unstable. Then we shall evaluate the source immediately after leaving the equilibrium, where "immediately" means at a time scale smaller than the hydrostatic time scale.

For this purpose, let us first calculate the kinematical variables of the source, and the evolution equation for the expansion scalar (Raychaudhuri equation).

Since we choose the fluid to be comoving in our coordinates, we may write the four-velocity as

$$V^\alpha = \left(\frac{1}{\sqrt{-g_{00}}}, 0, 0, 0\right);$$
$$V_\alpha = \left(-\sqrt{-g_{00}}, 0, 0, \frac{g_{30}}{\sqrt{-g_{00}}}\right), \tag{32}$$

whereas, for the vorticity vector, we have

$$\omega_\alpha = \frac{1}{2}\eta_{\alpha\beta\mu\nu}V^{\beta;\mu}V^\nu = \frac{1}{2}\eta_{\alpha\beta\mu\nu}\Omega^{\beta\mu}V^\nu, \tag{33}$$

where $\Omega_{\alpha\beta} = V_{[\alpha;\beta]} + a_{[\alpha}V_{\beta]}$ and $\eta_{\alpha\beta\mu\nu}$ denote the vorticity tensor and the Levi-Civita tensor, respectively.

For the covariant derivative of the four-velocity, we have the well-known expression

$$V_{\alpha;\beta} = \sigma_{\alpha\beta} + \Omega_{\alpha\beta} - a_\alpha V_\beta + \frac{1}{3}h_{\alpha\beta}\Theta, \tag{34}$$

where as usual, Θ, $\sigma_{\alpha\beta}$ and a_α denote the expansion scalar, the shear tensor and the four-acceleration, respectively, and are defined as

$$a_\alpha = V^\beta V_{\alpha;\beta}, \qquad \Theta = V^\alpha_{;\alpha}. \tag{35}$$

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}, \tag{36}$$

where $h_{\alpha\beta}$ is the projector onto the hypersurface orthogonal to the four-velocity.

Now, the Ricci identities for the vector V_α read

$$R^\mu_{\alpha\beta\nu}V_\mu = V_{\alpha;\beta;\nu} - V_{\alpha;\nu;\beta}, \tag{37}$$

then using (36) we obtain

$$\frac{1}{2}R^\rho_{\alpha\beta\mu}V_\rho = a_{\alpha;[\beta}V_{\mu]} + a_\alpha V_{[\mu;\beta]} + \sigma_{\alpha[\beta;\mu]} + \Omega_{\alpha[\beta;\mu]} + \frac{1}{3}h_{\alpha[\beta}\Theta_{,\mu]} + \frac{1}{3}\Theta h_{\alpha[\beta;\mu]}. \tag{38}$$

Contracting Eq. (38) with $V^\beta g^{\alpha\mu}$, we find the Raychaudhuri equation for the evolution of the expansion

$$\Theta_{;\alpha}V^\alpha + \frac{1}{3}\Theta^2 + \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \Omega^{\alpha\beta}\Omega_{\alpha\beta} - a^\alpha_{;\alpha} = -V_\rho V^\beta R^\rho_\beta. \tag{39}$$

Let us now assume that our source is initially stationary (at some $t = 0$), implying that both the shear and the expansion vanish. However, immediately after leaving the equilibrium, these quantities are still negligible, but not so their time derivatives, since as we have already mentioned, "immediately" means on a time scale which is smaller than the hydrostatic time scale.

Then, at the time scale under consideration, (39) becomes

$$\Theta_{;\alpha}V^\alpha - \Omega^{\alpha\beta}\Omega_{\alpha\beta} - a^\alpha_{;\alpha} = -V_\rho V^\beta R^\rho_\beta. \tag{40}$$

In order to evaluate the time derivative of the expansion scalar from (40), we need first to extract some information from the conditions of the vanishing of the expansion scalar and the shear tensor.

Thus, the condition $\Theta = 0$ produces

$$4A\dot{\hat{g}} - 6A\dot{\hat{a}} - \dot{A} = 0, \tag{41}$$

where the dot over the functions denotes time derivative, whereas $\sigma_{\alpha\beta} = 0$ implies

$$A\dot{\hat{g}} - \dot{A} = 0$$
$$2A\dot{\hat{g}} + \dot{A} = 0$$
$$4A\dot{\hat{g}} - \dot{A} = 0. \tag{42}$$

The above equations impose the constraints $\dot{\hat{a}} = \dot{\hat{g}} = \dot{A} = 0$ on the metric functions; these conditions will be used when we evaluate (40) as well as the Einstein equations.

Then the time derivative of the expansion scalar immediately after the source leaves the equilibrium leads to

$$\dot{\Theta} = \frac{e^{-2\hat{a}}}{Z^2}\left(-3\ddot{\hat{a}} - \frac{\ddot{A}}{2A} + 2\ddot{\hat{g}}\right). \tag{43}$$

The calculation of the energy-momentum tensor from the Einstein equations for the system out of equilibrium may be written as the sum of the terms corresponding to the equilibrium state $T^\beta_{\alpha(\mathrm{equi})}$ and terms which correspond to the system out of equilibrium (oeq); thus, we write

$$T^\beta_\alpha = T^\beta_{\alpha(eq)} + T^\beta_{\alpha(oeq)}. \tag{44}$$

The components of these last terms which contain second time derivatives of the metric functions (that are not null) read

$$T^3_0(oeq) = \frac{\Omega e^{2\hat{a}}}{8\pi r^2 \sin^2\theta}\left(-2\ddot{\hat{a}} - \frac{\ddot{A}}{2A} + 2\ddot{\hat{g}}\right)$$

$$T^3_3(oeq) = \frac{1}{8\pi}\left(-\frac{\Omega^2 e^{2\hat{a}}}{r^2\sin^2\theta} + \frac{e^{-2\hat{a}}}{Z^2}\right)\left(-2\ddot{\hat{a}} - \frac{\ddot{A}}{2A} + 2\ddot{\hat{g}}\right)$$

$$T^1_1(oeq) = \frac{e^{2\hat{a}}}{8\pi r^2\sin^2\theta}\left(-\Omega^2\ddot{\hat{g}} - \frac{\Omega^2\ddot{A}}{Z4\sqrt{A}} - \Omega\ddot{\Omega} - \dot{\Omega}^2\right) + \frac{e^{-2\hat{a}}}{8\pi Z^2}(\ddot{\hat{g}} - 2\ddot{\hat{a}}). \tag{45}$$

It is easy to see from the equations above that

$$\Omega T^3_0(oeq) + T^3_3(oeq) = \frac{1}{8\pi}\dot{\Theta} + \frac{e^{-2\hat{a}}}{8\pi Z^2}\ddot{\hat{a}}, \tag{46}$$

and the terms from the stationary case lead to the identity

$$\Omega T^3_{0(eq)} + T^3_{3(eq)} = \frac{e^{2\hat{a}-2\hat{g}}}{8\pi}(8\pi P - \hat{p}_{zz} + \delta J_+) \tag{47}$$

with the notation used in [1].

Let us now first consider the spherically symmetric case ($j = 0$). In such a case, the evolution equation for the expansion scalar, immediately after leaving the equilibrium, reads

$$\dot{\Theta} = \frac{1}{Z^2}\left(-\frac{\ddot{A}}{2A}\right), \tag{48}$$

which, after using the Einstein equations, becomes

$$T^3_3(oeq) = \frac{1}{8\pi}\dot{\Theta}. \tag{49}$$

We see from (49) that the sign of $\dot{\Theta}$ for any piece of material is the sign of $T^3_3(oeq)$. Now, if we recall that in the spherically symmetric case, $T^3_3(eq) = P$, the physical consequence of (49) is quite obvious: if the exit from the equilibrium state of any piece of material is produced by an increase (decrease) in the pressure, then that region will tend to expand (contract). In other words, all fluid elements in that region will experience an overall expansion (contraction) once the system leaves the equilibrium, while no fragmentation (cracking) [46] will be observed.

This absence of cracking in the spherically symmetric limit, with homogeneous energy density distribution, is expected since it can be rigorously shown that changes in the sign of $\dot{\Theta}$ are a necessary condition for the occurrence of cracking (see [47]). Furthermore, as has been confirmed in several numerical studies, cracking in spherically symmetric configurations with homogeneous energy density requires the perturbation of the pressure isotropy [48], whereas the spherically symmetric limit of our source is strictly isotropic in the pressure.

Let us now turn back to the situation under consideration (nonspherical, rotating source). The corresponding equation for the time derivative of the expansion is (46). As is apparent from this equation, now the sign of $\dot{\Theta}$ not only depends on the variation of one of the diagonal pressure terms, but also on $T^3_0(oeq)$ and the value (and sign) of the vorticity. This opens the way for a large number of scenarios, for which the sign of $\dot{\Theta}$ changes within the fluid distribution, giving rise to the possibility of the fragmentation (cracking) [46] of the source.

V. DISCUSSION

We may summarize the results obtained in this work, through the following points:

(i) There exists a bound for the maximal value of the surface gravitational potential (minimal value of τ), producing a bound in the maximal value of the surface redshift, which can be observed from the source discussed here.

(ii) The above-mentioned bound was established by detecting the critical value of τ for which negative acceleration appears, within the fluid distribution.

(iii) This critical value defines exactly the same range of values as the radial coordinate, for which the radial pressure becomes negative. This fact reinforces further the physical relevance of the acceleration tensor introduced in [33].

(iv) The source under consideration allows the possibility of fragmentation (cracking) once the equilibrium state has been abandoned. This scenario depends strictly on the nonspherical aspects of the source. In particular, it emphasizes the physical relevance of the T^3_0 component of the energy-momentum tensor which, as has been discussed in [1], represents a distinct physical property of rotating fluids.

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