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Einstein equation from covariant loop quantum gravity in semiclassical continuum limit

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In this paper we explain how four-dimensional general relativity and, in particular, the Einstein equation, emerge from the spin-foam amplitude in loop quantum gravity. We propose a new limit that couples both the semiclassical limit and continuum limit of spin-foam amplitudes. The continuum Einstein equation emerges in this limit. Solutions of the Einstein equation can be approached by dominant configurations in spin-foam amplitudes. A running scale is naturally associated to the sequence of refined triangulations. The continuum limit corresponds to the infrared limit of the running scale. An important ingredient in the derivation is a regularization for the sum over spins, which is necessary for the semiclassical continuum limit. We also explain in this paper the role played by the so-called flatness in spin-foam formulation, and how to take advantage of it.

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I. INTRODUCTION

Loop quantum gravity (LQG) is an attempt to move toward the nonperturbative and background independent quantum theory of gravity [1–3]. The covariant approach of LQG is known as the spin-foam formulation [4,5], in which the quantum spacetime is understood by the spin-foam amplitude describing the transition between quantum spatial geometries.

This paper focuses on the semiclassical behavior of the covariant LQG. A consistent quantum theory of gravity must reproduce general relativity (GR) as its semiclassical limit. In this paper, we explain how GR and the Einstein equation emerge from the covariant LQG.

The analysis and results in this paper evolve from the recent extensive studies of spin-foam asymptotics (briefly reviewed in Sec. II; see also, e.g., [6–10]). It has been shown that if one does not consider the spin sum, but considers the spin-foam (partial) amplitude with fixed spins, the large spin asymptotics of the amplitude give the Regge action of gravity, being a discretization of the Einstein-Hilbert action on the triangulation.

However, the discussion on carrying out the sum over spins and its semiclassical limit has been insufficient in the literature, the reason for which is explained in a moment. There has been a proposal of carrying out the spin sum semiclassically in asymptotically large spins while sending the Barbero-Immirzi parameter γ to 0 at the same time [11]. This proposal produces the Regge equation (equation of motion from Regge action) from the spin-foam amplitude. The idea of this type of limit has also been used in the graviton propagator computation from spin foams [12–16].

The present work considers the semiclassical behavior of the spin-foam amplitude with an arbitrarily fixed Barbero-Immirzi parameter, and takes into account the sum over spins. The semiclassical limit in this situation turns out to have more interesting consequences. The reason why this situation was not sufficiently studied is because of the question about the *flatness* in spin-foam amplitudes. It was observed in [17–19] that when one takes into account the sum over spins and studies the semiclassical limit, the spin-foam amplitude is dominant by the flat Regge geometry with all deficit angles vanishing.[1] There has been worry in the LQG community that the flatness might be the obstruction of the spin-foam amplitude having a consistent semiclassical limit. However it has been suggested in [20] that the flatness, if treated properly, is a good property of the spin-foam amplitude, which makes spin foams well behaved near the classical curvature singularity. Moreover, it has also been suggested in [19,21] that the flatness should relate to the continuum limit of spin foams, since deficit angles of discrete geometries indeed approach 0 in the continuum limit. Namely, the flatness means that for spin-foam amplitude, the semiclassical limit should be taken together with the continuum limit.[2] The last point of view is one of the motivations of the present work.

The situation is similar to the subtlety of interchanging limits in mathematical physics. We have two limits involved here: (1) deficit angles $\varepsilon_f \to 0$ and (2) the refinement limit of triangulations. $\varepsilon_f \to 0$ relates to the lattice spacing $\ell \to 0$ in Regge geometries since $\varepsilon_f \sim \ell^2/\rho^2$ where ρ is the curvature scale of the geometry approximated by Regge geometries [22]. If one takes first the limit

[1] More precisely, the dominant geometries there have deficit angles vanishing modulo $4\pi\mathbb{Z}$.

[2] [21] mentioned this limit as an analog of the hydrodynamical limit.

(1) then the limit (2), one only obtains the flat geometry on the continuum. However if both limits are coupled and taken at the same time, instead of one after the other, we can recover arbitrary curved geometry by the limit [23]. In the derivation of the flatness [17–19], the treatment of spin sum effectively leads to $\varepsilon_f \to 0$ on a fixed triangulation (before the refinement limit). In order to implement the proper limit, taking (1) and (2) at the same time, the spin sum has to be treated differently, which should open a window of small but nonvanishing ε_f, to let $\varepsilon_f \to 0$ couple nontrivially to the refinement limit.

The desired window can be given by the treatment in [20], where a damping factor is inserted in the sum over spins. The damping factor regularizes the spin sum by suppressing the contribution from spins far away from a given spin configuration J_0. The damping is turned off together with the large J_0 limit. The regularization procedure indeed produces a small window of nonvanishing deficit angle. Then the authors are able to show that the effective action at J_0 from spin-foam amplitude approximates the Einstein-Hilbert action, when J_0 corresponds to a set of geometrical triangle areas on the triangulation.

In this paper we propose an improved regularization scheme in Sec. III, which is more suitable in analyzing the sum over contributions from different spin configurations. It is based on the following observations: The spin-foam asymptotics (with fixed spins) reproduce Regge geometries and the Regge action when the fixed spins are Regge-like, i.e., the spins $\vec{J}(\ell)$ that can be expressed as triangle areas in terms of a set of edge lengths $\{\ell\}$ on the triangulation (the spins only need to be close to Regge-like in order to produce the Regge geometry and the Regge action). Regge-like spins locate in a submanifold $\mathcal{M}_{\text{Regge}}$ in the space of all spin configurations. Motivated by this property, we decompose the sum over spins in the spin-foam amplitude into a sum over Regge-like spins along $\mathcal{M}_{\text{Regge}}$ and a sum along transverse directions that contains non-Regge-like spins. As an equivalent way to understand the flatness, its origin is the fact that non-Regge-like spins in transverse directions contribute nontrivially to the amplitude in the large spin asymptotics. Based on the above observations, we propose to only regularize the spin sum in transverse directions instead of the regularization in all directions as in [20]. The regularization is made by inserting a Gaussian distribution with width $\delta^{-1/2}$ in the transverse spin sum. The Gaussian produces the damping at the infinity in transverse directions. The regulator is removed by $\delta \to 0$ in the end together with the continuum limit.

The regularized sum in transverse directions can be computed explicitly, which produces a Gaussian of width $\delta^{1/2}$ peaked at a submanifold in the space of spin-foam variables. After carrying out the transverse spin sum, we are only left with the sum over Regge-like spins. Schematically the spin-foam amplitude reduces to be the following type,

$$Z = \sum_{J(\ell)} \int \mathrm{d}\mu(X) e^{S[J(\ell),X]} D_\delta(\ell, X), \tag{1}$$

where X labels spin-foam variables in addition to spins in the integral representation of Z. S is the spin-foam action used in the asymptotical analysis. D_δ contains the Gaussian of width $\delta^{1/2}$ mentioned above.

The action S in Eq. (1) only involves Regge-like spins. So the results of large spin asymptotics can be immediately applied to the semiclassical analysis in Sec. IV. We consider the spin-foam state sum in the semiclassical regime. Namely, we focus on a neighborhood $\mathcal{N}_{\text{Regge}} \subset \mathcal{M}_{\text{Regge}}$ such that the spins within $\mathcal{N}_{\text{Regge}}$ are uniformly large. We introduce a parameter $\lambda \gg 1$ as a typical value of spin in $\mathcal{N}_{\text{Regge}}$. The spin sum in Eq. (1) is performed in $\mathcal{N}_{\text{Regge}}$. Then the entire domain of the spin sum including transverse directions is denoted by $\mathcal{N}$. The spin-foam amplitude is denoted by $Z_{\mathcal{N},\delta}(\mathcal{K})$ depending on three types of parameters: the spin sum domain $\mathcal{N}$ of large spins $J \sim \lambda$, the regulator δ, and the triangulation $\mathcal{K}$. An interesting regime where $Z_{\mathcal{N},\delta}(\mathcal{K})$ exhibits desired semiclassical behavior is

$$\lambda \gg \delta^{-1} \gg 1. \tag{2}$$

In this regime, $Z_{\mathcal{N},\delta}(\mathcal{K})$ is dominated by the critical points of $S[J(\ell), X]$, which has been extensively studied in the literature [6–8,24,25]. With respect to $\int \mathrm{d}\mu(X)$, the critical points give Regge geometries on $\mathcal{K}$, taking into account that $\sum_{J(\ell)}$ reduce the critical points to the ones corresponding to geometries satisfying the Regge equation (the equation of motion of the Regge action). Because of Eq. (2), the leading contributions are computed by evaluating D_δ at the critical points. Then the Gaussian in D_δ together with the Regge equation constrains the deficit angles ε_f to be small (but nonvanishing)

$$|\gamma\varepsilon_f| \le \delta^{1/2}. \tag{3}$$

γ is a fixed $O(1)$ parameter throughout our discussion. Note that there exist some discrete ambiguities of the above constraint, due to the periodicity of the integrand in Eq. (1). But the ambiguities can be removed by suitably choosing $\mathcal{N}_{\text{Regge}}$. The regime where the Regge equation and the constraint Eq. (3) emerge from the spin-foam amplitude is referred to as the Einstein-Regge (ER) regime in Sec. V.

As promised, the regularization of the spin sum opens a small window for nontrivial ε_f. Small ε_f relates to the continuum limit of Regge geometries, because $|\varepsilon_f| \sim \ell^2/\rho^2$ [22] where ρ is the typical curvature radius of the smooth geometry approximated by the Regge geometry. $|\varepsilon_f| \ll 1$ relates to $\ell \ll \rho.\delta$ behaves as the bound of error in approximating smooth geometries by Regge geometries. The emerging smooth geometries have nontrivial curvatures.

In the ER regime, the configurations contributing dominantly the spin-foam amplitude contain the Regge geometries satisfying the Regge equation, and approximating (curved) smooth geometries. Regge geometries failing to approximate any smooth geometry are suppressed by the amplitude.

Equation (3) indicates that the regulator δ relates to the continuum limit. The window of nontrivial ε_f allows us to couple $\varepsilon_f \to 0$ to the refinement limit of the triangulation. The continuum limit at the semiclassical level is discussed in Sec. VI. We consider an infinite sequence of triangulations given by the refinement, such that all vertices of triangulations form a dense set in the 4-manifold where triangulations are embedded. A sequence of spin-foam amplitudes $Z_{\mathcal{N},\delta}(\mathcal{K})$ is defined on the sequence of triangulations. We let the limit $\delta \to 0$ couple to the refinement; i.e., $\delta \to 0$ is taken together with the continuum limit.

On the other hand, the typical spin value λ has to increase in refining the triangulation. Refining the triangulation increases the number of degrees of freedom in the spin-foam amplitude. It then requires a larger λ to suppress the quantum correction, so that the semiclassical behavior stands out as the leading order (see Sec. VI).

The semiclassical continuum limit involves taking simultaneously three limits: the triangulation refinement limit, $\lambda \to \infty$, and $\delta \to 0$. The limits are implemented to the sequence of $Z_{\mathcal{N},\delta}(\mathcal{K})$. At each $Z_{\mathcal{N},\delta}(\mathcal{K})$ in the sequence, Eq. (2) has to be satisfied, in order to keep a nontrivial ER regime. As a result, we obtain sequences of Regge geometries approaching smooth geometries in the limit. Each Regge geometry in each sequence (a) satisfies the Regge equation, (b) satisfies small deficit angle constraint Eq. (3), and (c) contributes dominantly to the corresponding $Z_{\mathcal{N},\delta}(\mathcal{K})$. We are able to achieve (a)–(c) because each Regge geometry in each sequence is inside the ER regime of the corresponding $Z_{\mathcal{N},\delta}(\mathcal{K})$.

At first sight, $\lambda \to \infty$ might seem to contradict the continuum limit, by the LQG relation $\mathbf{a} = \gamma\lambda\ell_P^2$ for the triangle areas. There is no contradiction because $\mathbf{a}$ is a dimensionful quantity, and the continuum limit corresponds to zoom out to a larger length unit, such that the numerical value of ℓ_P^2 measured by the unit shrinks at a faster rate than $\lambda \to \infty$. This observation motivates us to associate each triangulation and $Z_{\mathcal{N},\delta}(\mathcal{K})$ a mass scale μ whose μ^{-1} is a length unit. The refinement limit is labeled by the infrared (IR) limit $\mu \to 0$. All parameters of $Z_{\mathcal{N},\delta}(\mathcal{K})$ have nontrivial running with μ, i.e.,

$$\mathcal{K} = \mathcal{K}_\mu, \qquad \lambda = \lambda(\mu), \qquad \delta = \delta(\mu), \qquad Z_{\mathcal{N},\delta}(\mathcal{K}) = Z_{\mathcal{N}(\mu),\delta(\mu)}(\mathcal{K}_\mu). \tag{4}$$

Here $\lambda(\mu)$ increase monotonically as $\mu \to 0$ while $\delta(\mu)$ decrease monotonically. Equation (2) is satisfied at each μ. The dependence of λ on μ displays that the semiclassical limit is coupled to the continuum limit. Given the running scale μ, on each $\mathcal{K}_\mu$, the area is expressed as

$$\mathbf{a}(\mu) = \gamma\lambda(\mu)\ell_P^2 = a(\mu)\mu^{-2}. \tag{5}$$

The area in the μ^{-2} unit, $a(\mu)$, shrinks and approaches 0 in the IR limit $\mu \to 0$. In Regge geometries, the value of typical edge length $a(\mu)^{1/2}$ in the μ^{-1} unit approaches 0 as the refinement limit, which orders the sequence of Regge geometries to approach the smooth geometry at the IR. Smooth geometries living at the IR are associated with the largest length unit $\mu^{-1} \to \infty$.

The above discussion exhibits how scales and a renormalization-group-like behavior emerge from the spin-foam formulation that originally is scale independent. Possible ways of associating scales μ to triangulations $\mathcal{K}_\mu$ are classified in Sec. VII.

We have obtained from the spin-foam amplitude sequences of Regge geometries solving Regge equations, which converge to smooth geometries in the semiclassical continuum limit. Generically the resulting smooth geometries are solutions of the continuum Einstein equation. Although the general mathematical proof for the convergence of Regge solutions to Einstein equation solutions is not available in the literature, extensive studies of the Regge calculus provide many analytical and numerical results, which all support the convergence, and demonstrate the Regge calculus as a useful tool in numerical relativity (see, e.g., [26,27] for reviews). Among the results, there has been a rigorous proof of the convergence in the linearized Regge calculus and linearized Einstein equation [28–30]. Results in the nonlinear regime include, e.g., the Kasner universe, Brill waves, binary black holes, Friedmann-Lemaître-Robertson-Walker (FLRW) universe, etc. [27,31–34]. There has also been the convergence result by a certain average of Regge equations [35].

A key observation in all convergence results is that the deviation of the Regge calculus from general relativity is essentially the noncommutativity of rotations in the discrete theory, while the error from the noncommutativity is of higher order in edge lengths [36].[3]

We conclude that for any sequence of Regge solutions converging to the solution of the Einstein equation, the Regge solutions can be produced from the sequence of spin-foam amplitudes $Z_{\mathcal{N}(\mu),\delta(\mu)}(\mathcal{K}_\mu)$ as dominant configurations in the semiclassical approximation. The solution of the continuum Einstein equation lives at the IR limit $\mu \to 0$. The convergence to gravitational waves of the linearized Einstein equation in [28] leads to a mathematically rigorous example for the emergence of Einstein equation from the spin-foam amplitude.

There is a different argument for the emergence of the Einstein equation from the spin-foam amplitude, by the

[3] The author thanks Warner Miller for pointing this out.

convergence of effective actions (see Sec. VI). The analysis in this paper proposes a different regularization scheme from the one in [20]. However the results of the effective action in [20] and [37–39] can be reproduced here. The effective action relates to $S[J(\ell),X]$ in Eq. (1) evaluated at critical points of $\int d\mu(X)$ as $\lambda \gg 1$ (before carrying out $\sum_{J(\ell)}$). $S[J(\ell),X]$ at critical points gives Regge actions evaluated at Regge geometries with small ε_f by Eq. (3), when we consider the sequence $Z_{\mathcal{N}(\mu),\delta(\mu)}(\mathcal{K}_\mu)$ and take the semiclassical continuum limit. Regge actions converge to the Einstein-Hilbert action on the continuum, when Regge geometries converge to the smooth geometry [23,40]. Translating the known convergence result to our context uses the length unit μ^{-1}. We apply Eq. (5) to the Regge action $\frac{1}{\ell_P^2}\sum_f \mathbf{a}_f(\mu)\varepsilon_f(\mu)$ from $S[J(\ell),X]$ in $Z_{\mathcal{N}(\mu),\delta(\mu)}(\mathcal{K}_\mu)$,[4]

$$\frac{1}{\mu^2\ell_P^2}\sum_f a_f(\mu)\varepsilon_f(\mu) \to \frac{1}{\mu^2\ell_P^2}\int d^4x\sqrt{-g}R, \tag{6}$$

where the convergence happens as the edge length $a(\mu)^{1/2} \to 0$ at the IR.[5] Smooth geometries and $\int d^4x\sqrt{-g}R$ live in the IR limit $\mu \to 0$. $\sum_{J(\ell)}$ (or $\sum_\ell$) in Eq. (1) sums all convergence sequences of Regge geometries, and thus equivalently sums all smooth geometries in the limit. The spin-foam amplitude becomes a functional integral of Einstein-Hilbert action in the continuum (see Sec. VI for details). Then $\mu \to 0$ in Eq. (6) leads to the continuum vacuum Einstein equation

$$R_{\mu\nu} = 0 \tag{7}$$

by the variational principle.

The quantum behavior of spin foams near a classical curvature singularity derived in [20] can be reproduced in the present regularization scheme. Large J and Eq. (3) show that the semiclassical approximation is valid only in the regime that ($\ell^2 \sim \mathbf{a}_f$)

$$\ell_P \ll \ell \ll \rho. \tag{8}$$

However a large curvature may violate $\ell_P \ll \rho$, and lead to the incompatibility between $\ell \ll \rho$ and large J. Therefore the semiclassical analysis in this paper is not valid near the curvature singularity. Similar to [20], spin foams near the singularity are of small spins, in order that the amplitudes are not suppressed. It shows that the classical singularity corresponds to the quantum regime of spin foams, where the theory is well defined but with large quantum fluctuations.

As a key ingredient in the argument, Eq. (3) comes from the regularized flatness. It shows that the flatness is a good property of the spin-foam amplitude, which guarantees that spin foams behave correctly near a classical singularity.

We remark that the presentation in this paper uses the spin-foam models of Engle-Pereira-Rovelli-Livine/Freidel-Krasnov (EPRL/FK), both in Lorentzian and Euclidean signatures [41,42]. But the discussion and results are valid for any other spin-foam models that have both the correct large spin asymptotics, and the flatness (e.g., the model with timelike tetrahedra [43] and its recent asymptotical analysis [10]).

The architecture of this paper is as follows: Sec. II provides a review on the recent development of the spin-foam large spin asymptotics. Section III discusses the regularization of the spin sum along directions transverse to the submanifold $\mathcal{M}_{\text{Regge}}$ of Regge-like spins. Section IV analyzes the semiclassical approximation of the regularized spin-foam amplitude, which gives the Regge equation and small deficit angle constraint Eq. (3). Section IV defines the Einstein-Regge regime of the spin-foam amplitude, in which the amplitude exhibits the desired semiclassical property. Section VI discusses the semiclassical continuum limit of sequences of spin-foam amplitudes, which approaches the continuum Einstein equation. Section VII classifies possible runnings of scales μ associated to triangulations.

II. LARGE-J ASYMPTOTICS OF SPIN-FOAM AMPLITUDE

We consider the EPRL/FK spin-foam amplitude $Z(\mathcal{K})$ defined on a triangulation $\mathcal{K}$. $Z(\mathcal{K})$ has the following integral representation [25]:

$$\begin{aligned} Z(\mathcal{K}) &= \sum_{J_f}\prod_f \dim(J_f)A_{J_f}(\mathcal{K}) \\ &= \sum_{J_f}\prod_f \dim(J_f)\int_{\mathrm{SL}(2,\mathbb{C})}\prod_{(v,e)} dg_{ve} \\ &\quad\times \int_{\mathbb{CP}^1}\prod_{v\in\partial f} dz_{vf}\, e^{S[J_f,g_{ve},z_{vf}]}. \end{aligned} \tag{9}$$

v, e, and f label the 4-simplices, tetrahedra, and triangles. They equivalently label the vertices, dual edges, and faces in the dual complex $\mathcal{K}^*$. $J_f \in \mathbb{Z}_+/2$ are SU(2) spins associated to triangles f. $g_{ve} \in \mathrm{SL}(2,\mathbb{C})$ are associated to half-edges (v,e) in $\mathcal{K}^*$ where v is an end point of e. z_{vf} are 2-spinors modulo complex rescaling. The spin-foam action $S[J_f,g_{ve},z_{vf}]$ reads

[4] $\mathbf{a}_f(\mu) = \gamma J_f(\mu)\ell_P^2 = a_f(\mu)\mu^{-2}$.

[5] The convergence requires the fatness of simplices to be bounded away from 0 in addition to shrinking edge lengths; see [23,40] for details.

$$S[J_f, g_{ve}, z_{vf}] = \sum_f J_f F_f[g_{ve}, z_{vf}],$$

$$F_f[g_{ve}, z_{vf}] = \ln \prod_{e\subset\partial f} \frac{\langle g^\dagger_{ve} z_{vf}, g^\dagger_{v'e} z_{v'f}\rangle^2}{\langle g^\dagger_{ve} z_{vf}, g^\dagger_{ve} z_{vf}\rangle \langle g^\dagger_{v'e} z_{v'f}, g^\dagger_{v'e} z_{v'f}\rangle} + i\gamma \ln \prod_{e\subset\partial f} \frac{\langle g^\dagger_{ve} z_{vf}, g^\dagger_{ve} z_{vf}\rangle}{\langle g^\dagger_{v'e} z_{v'f}, g^\dagger_{v'e} z_{v'f}\rangle}. \quad (10)$$

Here $\langle , \rangle$ is an SU(2) invariant Hermitian inner product between 2-spinors. S is defined modulo $2\pi i\mathbb{Z}$ because of $J \in \mathbb{Z}/2$, while F_f is defined modulo $4\pi i\mathbb{Z}$. The Barbero-Immirzi parameter $\gamma \in \mathbb{R}$ is treated as a constant of $O(1)$ in this paper. It is straightforward to show that the real part of F_f is nonpositive $\mathrm{Re}F_f \le 0$ by using Cauchy-Schwarz inequality [25].

$Z(\mathcal{K})$ is the spin-foam amplitude in Lorentzian signature. The amplitude in Euclidean signature is written in a similar manner. Differences from Eq. (9) contain that integrals over $\mathrm{SL}(2,\mathbb{C})$ are replaced by integrals over $(g^+_{ve}, g^-_{ve}) \in \mathrm{SO}(4)$, and integrals over z_{vf} are replaced by integrals over 2-spinors ξ_{ef} [one for each pair (e,f) with $e \subset f$ in $\mathcal{K}^*$], where ξ_{ef} is normalized by the Hermitian inner product on $\mathbb{C}^2$. F_f for Euclidean amplitude reads [6,24,44]

$$F_f[g^\pm_{ve}, \xi_{ef}] = \sum_\pm \sum_{v\in f} \frac{1\pm\gamma}{2} j_f \ln \langle \xi_{ef} | (g^\pm_{ve})^{-1} g^\pm_{ve'} | \xi_{e'f}\rangle. \quad (11)$$

The above presents the expression of the Euclidean amplitude with $\gamma < 1$. The expression for $\gamma > 1$ can be found in [44].

In the following we often present the analysis in the notation of Lorentzian amplitude. The same analysis can be applied to Euclidean amplitude. The result is valid for both signatures.

The asymptotical analysis of the partial amplitude $A_{J_f}(\mathcal{K})$ as J_f uniformly large has been well developed by the recent progress [6–8,24,25,38]. Since S is linear to J_f, as J_f is uniformly large, $A_{J_f}(\mathcal{K})$ is dominated by contributions from the critical points of the action $S[J_f, g_{ve}, z_{vf}]$, i.e., configurations $(\mathring{J}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ satisfying $\mathrm{Re}S = 0$ and $\partial_g S = \partial_z S = 0$. Importantly, the critical points can be interpreted as simplicial geometries (Regge geometries) on the four-dimensional triangulation. The spins $\mathring{J}_f$ are interpreted as triangle areas $\mathring{\mathbf{a}}_f = \gamma \mathring{J}_f \ell_P^2$. When the triangulation is sufficiently refined, the critical points can approximate arbitrary geometries on a four-dimensional manifold.

It is shown in [8,25] that at a critical point $(\mathring{J}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ corresponding to a nondegenerate Regge geometry with global orientation and global time orientation, its leading contribution to $A_{J_f}(\mathcal{K})$ gives the Regge action

$$A_{J_f}(\mathcal{K}) \sim \exp\left(\frac{i}{\ell_P^2}\sum_f \mathring{\mathbf{a}}_f \mathring{\varepsilon}_f + \frac{i}{\ell_P^2}\sum_{f\subset\partial\mathcal{K}} \mathring{\mathbf{a}}_f \mathring{\Theta}_f + \cdots\right), \quad (12)$$

where $\mathring{\varepsilon}_f, \mathring{\Theta}_f$ are the bulk deficit angle and boundary dihedral angle from the geometrical interpretation of $(\mathring{J}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$. The asymptotic formula of $A_{J_f}(\mathcal{K})$ is given by a sum over critical points weighted by the contribution from each critical point.

Note that it is possible to have time-nonoriented geometries from critical points. In this case, $\mathring{\varepsilon}_f$ is replaced by $\mathring{\varepsilon}_f \pm \gamma^{-1}\pi$ in Eq. (12). See [8] for details.

Equation (12) holds for Regge-like spins J_f. Namely, it requires that spins $\mathring{J}_f$ can be expressed as areas in terms of edge lengths ℓ from a Regge geometry on the triangulation.

$$\gamma \tilde{J}_f(\ell) = \frac{1}{4}\sqrt{2(\ell_{ij}^2\ell_{jk}^2 + \ell_{ik}^2\ell_{jk}^2 + \ell_{ij}^2\ell_{ik}^2) - \ell_{ij}^4 - \ell_{ik}^4 - \ell_{jk}^4}, \quad (13)$$

where ℓ's are the edge lengths (in Planck units) of the triangle f. Regge-like spins span a subspace in the space of all spins.[6]

The situation of non-Regge-like spins is subtle. Non-Regge-like spins J_f do not lead to any solution to the critical equations $\mathrm{Re}S = \partial S = 0$. Especially $\mathrm{Re}S < 0$ for any solution to $\partial S = 0$[7] with non-Regge-like J_f. Although critical equations are not satisfied, the contribution to the spin-foam spin sum is non-negligible [18,20,38]. Indeed, by the stationary phase approximation (see theorems 7.7.5 and 7.7.1 in [45]), in case there is no critical point in the region of integral $\int_K e^{\lambda S} \mathrm{d}\mu$,

$$\left| \int_K e^{\lambda S(x)} \mathrm{d}\mu(x) \right| \le C \left(\frac{1}{\lambda}\right)^k \sup_K \frac{1}{(|S'|^2 + \mathrm{Re}(S))^k}, \quad (14)$$

the integral decays faster than $(1/\lambda)^k$ for all $k \in \mathbb{Z}_+$, provided that $\sup([|S'|^2 + \mathrm{Re}(S)]^{-k})$ is finite [i.e., does not cancel the $(1/\lambda)^k$ behavior in front]. But for the non-Regge-like J_f, the corresponding $A_{J_f}(\mathcal{K})$ may not decay faster than $(1/\lambda)^k$ for all $k \in \mathbb{Z}_+$. It happens for non-Regge-like spins close to Regge-like $J_f = \lambda j_f$ $(\lambda \gg 1)$ with the small gap $\Delta j_f \sim \frac{1}{2\lambda}$. In this case, $\sup([|S'|^2 + \mathrm{Re}(S)]^{-k})$ is likely to be large and cancel the $(1/\lambda)^k$ behavior. Therefore, the non-Regge-like spins have nontrivial contribution to the spin-foam spin sum.

[6]In general for nondegenerate simplicial four-dimensional manifolds the number of triangles is greater than the number of edges.

[7]To study the asymptotics with non-Regge-like spins, the equation of motion should be replaced by $\partial\mathcal{S} = 0$, where $\mathcal{S}$ is the analytic continuation of S. See [37,38] for detail.

III. REGULARIZING THE NON-REGGE-LIKE SPIN SUM

In order to understand the contribution from non-Regge-like spins, we split the spin sum into a sum over Regge-like spins and a sum over non-Regge-like spins in the following analysis. Then the non-Regge-like spin sum is carried out explicitly, with a regulator inserted, while the Regge-like spin-sum is treated by the usual stationary phase approximation.

The space of internal spins J_f, $\mathfrak{L}_J$, is a cubic lattice in the smooth space $\mathcal{M}_J \simeq \mathbb{R}^{N_f}$ (J_f at different f can be regarded as independent in the spin sum; see Appendix A for an explanation). We define the submanifold $\mathcal{M}_{\text{Regge}}$ to be the image of the smooth embedding in Eq. (13) from the space of edge lengths $\mathcal{M}_\ell$ into $\mathcal{M}_J$. We denote by $\tilde{J}_f(\ell)$ the image of the embedding from a given $\{\ell\}$. $\tilde{J}_f(\ell)$ is a smooth function defined by Eq. (13), and may not be a half-integer.

Given a compact neighborhood $\mathcal{N}_{\text{Regge}}$ in $\mathcal{M}_{\text{Regge}}$ that contains $\tilde{J}_f(\ell)$ all satisfying $\tilde{J}_f(\ell) \gg 1$,[8] we define local coordinates $(\ell, \tilde{t})$ in $\mathcal{M}_J$, where edge lengths ℓ are coordinates in $\mathcal{M}_{\text{Regge}}$; $\{\tilde{t}_i\}_{i=1}^M$ are transverse coordinates to $\mathcal{M}_{\text{Regge}}$. We denote the coordinate basis for $\tilde{t}_i$ by $\hat{e}^i = ((\hat{e}^i)_f)_f$, and choose $\mathcal{N}$ to be the coordinate chart. $\hat{e}^i$ $(i = 1, \ldots, M)$ may be assumed as constant vectors in $\mathbb{R}^{N_f}$, so that the coordinate axes of t_i are straight lines in $\mathbb{R}^{N_f}$. The transverse submanifolds coordinatized by t_i are parallel planes $\mathbb{R}^M \hookrightarrow \mathbb{R}^{N_f}$. This assumption can always be achieved locally in a compact neighborhood $\mathcal{N}_{\text{Regge}}$. The transverse plane located at $\{\ell\}$ is denoted by $\mathcal{M}_{\text{NR}}(\ell) \simeq \mathbb{R}^M$.

For any set of internal spins $\vec{J} \in \mathcal{N}$, it is expressed in the $(\ell, \tilde{t})$ coordinate, in which ℓ's give a unique $\vec{\tilde{J}}(\ell) \in \mathcal{N}_{\text{Regge}}$. So $\vec{J}$ is written as

$$\vec{J} = \vec{\tilde{J}}(\ell) + \sum_{i=1}^{M} \tilde{t}_i \hat{e}^i, \quad \text{with} \quad J_f(\ell) \gg 1. \tag{15}$$

Recall that $\vec{\tilde{J}}(\ell)$ are in general not spins. We define $\vec{J}(\ell)$ to be a set of spins in the transverse plane $\mathcal{M}_{\text{NR}}(\ell)$, at the same $\{\ell\}$ as the ones determining $\vec{\tilde{J}}(\ell)$, and require that $\vec{J}(\ell)$ has the shortest distance to $\vec{\tilde{J}}(\ell)$ measured in $\mathbb{R}^{N_f}$. $\vec{J}(\ell)$ defined in this way might not be unique. But when there are multiple choices, we make an arbitrary choice of $\vec{J}(\ell)$. The resulting $\vec{J}(\ell)$ is a representative of $\vec{\tilde{J}}(\ell) \in \mathcal{N}_{\text{Regge}}$. Obviously the spins $\vec{J}$ can also be written as $\vec{J} = \vec{J}(\ell) + \sum_{i=1}^M \tilde{t}_i \hat{e}^i$ using the representative. Given that both $\vec{J}, \vec{J}(\ell)$ are spins, then $\sum_{i=1}^M t_i \hat{e}^i$ are half-integers, so that $\vec{J}(\ell) + n \sum_{i=1}^M \tilde{t}_i \hat{e}^i$ are also spins when $n \in \mathbb{Z}$. Spins in $\mathcal{M}_{\text{NR}}(\ell)$ form an M-dimensional periodic lattice $\mathfrak{L}_{\text{NR}}(\ell)$, whose lattice basis is denoted by $\{\hat{e}^i(\ell)\}_{i=1}^M$. Therefore, any internal spins $\vec{J} \in \mathcal{N}$ can be expressed as

$$\vec{J} = \vec{J}(\ell) + \sum_{i=1}^{M} t_i \hat{e}^i(\ell), \quad \text{with} \quad J_f(\ell) \gg 1, \tag{16}$$

where $t_i \in \mathbb{Z}$.

That $\mathfrak{L}_{\text{NR}}(\ell)$ is a periodic lattice equivalent to the existence of parallel M-dimensional lattice planes in $\mathfrak{L}_J$ intersecting $\mathcal{N}_{\text{Regge}}$ transversely, which is always true locally (see Appendix B for an explanation). The local property is sufficient for the present discussion.

$\vec{J}(\ell)$ in Eq. (16) is a representative of Regge-like spins, although it might not be precisely located at $\mathcal{N}_{\text{Regge}}$. Its distance to $\mathcal{N}_{\text{Regge}}$ is at most of $O(1)$.[9] The large-J asymptotics of $A_{J(\ell)}$ is the same as the situation of Regge-like spins in Eq. (12) by the argument at the end of the last section (see also [38]). Non-Regge-like spins with $t_i \neq 0$ in each $\mathfrak{L}_{\text{NR}}(\ell)$ is going to be summed explicitly under certain regularization, before the stationary phase approximation.

If we denote by $\langle , \rangle$ the Euclidean inner product in $\mathbb{R}^{N_f}$, the spin-foam action is written as

$$\sum_f J_f F_f \equiv \langle \vec{J}, \vec{F} \rangle = \langle \vec{J}(\ell), \vec{F} \rangle + \sum_i t_i \langle \hat{e}^i(\ell), \vec{F} \rangle. \tag{17}$$

We define the spin-foam state sum in the coordinate chart $\mathcal{N}$ by restricting the spin sum in $\mathcal{N}$,

$$\begin{aligned} Z_{\mathcal{N}}(\mathcal{K}) &= \sum_{\vec{J} \in \mathcal{N}} \prod_f \dim(J_f) \int \mathrm{d}g_{ve} \mathrm{d}z_{vf} e^{\langle \vec{J}, \vec{F} \rangle} \\ &= \sum_{\vec{J}(\ell)} \sum_{t_i \in \mathbb{Z}} \mu(\ell, t) \int \mathrm{d}g_{ve} \mathrm{d}z_{vf} e^{\langle \vec{J}(\ell), \vec{F} \rangle + \sum_i t_i \langle \hat{e}^i(\ell), \vec{F} \rangle}, \end{aligned} \tag{18}$$

where $\mu(\ell, t) \equiv 2^{N_f} \prod_f (J_f(\ell) + \sum_{i=1}^M t_i (\hat{e}^i)_f(\ell))$. The spin sum only involves spins in the bulk. Boundary spins are set to be Regge-like $J_f = J_f(\ell)$, $f \in \partial\mathcal{K}$, as the boundary condition.

We perform a regularization (or deformation) of $\sum_{t_i \in \mathbb{Z}}$ by inserting a Gaussian weight

$$\sum_{t_i \in \mathbb{Z}} \to \sum_{t_i \in \mathbb{Z}} e^{-\frac{\delta}{4} \sum_{i=1}^M t_i t_i}. \tag{19}$$

The regulators $\delta \ll 1$, which are turned off appropriately by $\delta \to 0$ in the end. The amplitude with the insertion $e^{-\frac{\delta}{4} \sum_{i=1}^M t_i t_i}$ is denoted by $Z_{\mathcal{N},\delta}(\mathcal{K})$, which is a deformation

[8] $\mathcal{M}_{\text{Regge}}$ may have self-intersections, but $\mathcal{N}_{\text{Regge}}$ is always obtained as the smooth image of a neighborhood of ℓ's in the space of edge lengths.

[9] $\vec{J}(\ell)$ generically satisfy the triangle inequality everywhere on $\mathcal{K}$ since $\vec{\tilde{J}}(\ell)$ do.

from the original amplitude. When $\delta\to 0$, $Z_{\mathcal{N},\delta}(\mathcal{K})$ returns to the spin-foam amplitude restricted to the domain $\mathcal{N}$ of spins. The deformation turns out to be crucial in opening a small window of nontrivial curvature. The exponentially damping behavior of $e^{-\frac{\delta}{4}\sum_{i=1}^{M}t_it_i}$ at $t\to\infty$ also justifies the Poisson resummation in the following.

We treat the sum over t_i via the Poisson resummation (see Appendix C for some discussions about the sum),

$$\sum_{t_i\in\mathbb{Z}}\mu(\ell,t)e^{-\frac{\delta}{4}\sum_{i=1}^{M}t_it_i+\sum_i t_i\langle\hat{e}^i(\ell),\vec{F}\rangle}$$
$$=\sum_{k^j\in\mathbb{Z}}\int \mathrm{d}t_i\mu(\ell,t)e^{-\frac{\delta}{4}\sum_{i=1}^{M}t_it_i+\sum_i t_i\langle\hat{e}^i(\ell),\vec{F}+2\pi i\sum_j k^j\hat{e}^*_j(\ell)\rangle},\tag{20}$$

where $\hat{e}^*_j(\ell)$ is the lattice vector of the lattice $\mathfrak{L}^*_{\mathrm{NR}}(\ell)$ dual to $\mathfrak{L}_{\mathrm{NR}}(\ell)$, satisfying $\langle\hat{e}^i(\ell),\hat{e}^*_j(\ell)\rangle=\delta^i_j$.

We make a shorthand notation by

$$\left\langle\hat{e}^i,\vec{F}+2\pi i\sum_j k^j\hat{e}^*_j\right\rangle\equiv\Phi^i_{(k)}\equiv i\psi^i_{(k)}e^{i\phi^i_{(k)}},\tag{21}$$

where $\psi^i_{(k)}\in\mathbb{R},\phi^i_{(k)}\in[0,2\pi)$. The quantities $\Phi^i_{(k)},\psi^i_{(k)},\phi^i_{(k)}$ depend on ℓ,g_{ve},z_{vf}. We perform the Gaussian integral of t,

$$\int \mathrm{d}t_i\mu(\ell,t)e^{-\frac{\delta}{4}\sum_{i=1}^{M}t_it_i+\sum_{i=1}^{M}t_i\Phi^i_{(k)}}$$
$$=2^{N_f}\left(\frac{4\pi}{\delta}\right)^{\frac{M}{2}}\prod_f\left(J_f(\ell)+\sum_{i=1}^{M}(\hat{e}^i)_f\frac{\partial}{\partial\Phi^i_{(k)}}\right)e^{\sum_{i=1}^{M}\frac{1}{\delta}\Phi^i_{(k)}\Phi^i_{(k)}}$$
$$=2^{N_f}\left(\frac{4\pi}{\delta}\right)^{\frac{M}{2}}\prod_f\left(J_f(\ell)+\sum_{i=1}^{M}\frac{2}{\delta}\Phi^i_{(k)}(\hat{e}^i)_f\right)e^{\sum_{i=1}^{M}\frac{1}{\delta}\Phi^i_{(k)}\Phi^i_{(k)}}$$
$$\equiv D^{(k)}_\delta(\ell,g_{ve},z_{vf}).\tag{22}$$

The spin-foam amplitude now reads

$$Z_{\mathcal{N},\delta}=\sum_{\vec{J}(\ell)}\int \mathrm{d}g_{ve}\mathrm{d}z_{vf}e^{\langle\vec{J}(\ell),\vec{F}\rangle}\sum_{\{k^j\}\in\mathbb{Z}^M}D^{(k)}_\delta(\ell,g_{ve},z_{vf}).\tag{23}$$

The regulator δ defines a deformation from the original spin-foam amplitude $Z_{\mathcal{N}}$.

As it becomes clear in the next section, when F_f is restricted to be purely imaginary, $\Phi^i_{(k)}=i\psi^i_{(k)}\in i\mathbb{R}$. Then $D^{(k)}_\delta$ reduces to

$$D^{(k)}_\delta(\ell,g_{ve},z_{vf})=\left(\frac{4\pi}{\delta}\right)^{\frac{M}{2}}e^{-\frac{1}{\delta}\sum_{i=1}^{M}\psi^i_{(k)}\psi^i_{(k)}}$$
$$\times 2^{N_f}\prod_f\left(J_f(\ell)+\frac{2i}{\delta}\sum_{i=1}^{M}\psi^i_{(k)}(\hat{e}^i)_f\right).\tag{24}$$

As $\delta\to 0$, $D^{(k)}_\delta$ contains a Gaussian peaked at $\psi^i_{(k)}=0$ with width $\sqrt{\delta}$. Its center $\psi^i_{(k)}=0$ means

$$\left\langle\hat{e}^i,\vec{F}+2\pi i\sum_j k^j\hat{e}^*_j\right\rangle=\langle\hat{e}^i,\vec{F}\rangle+2\pi ik^i=0.\tag{25}$$

The sum over $\{k^j\}\in\mathbb{Z}^M$ in Eq. (23) reflects that $Z_{\mathcal{N}}$ is periodic in $F_f\to F_f+4\pi i$. The above peakedness of $D^{(k)}_\delta$ and the sum over $\{k^j\}$ is a consequence of the periodicity.

IV. REGGE EQUATION AND SMALL DEFICIT ANGLE

The amplitude $Z_{\mathcal{N},\delta}$ depends on two independent scales (λ,δ), where (1) λ is the mean value of $\tilde{J}_f\equiv\lambda j_f$ in $\mathcal{N}_{\mathrm{Regge}}\subset\mathcal{N}$, and (2) δ is the regulator in $D^{(k)}_\delta$ for regulating the transverse $\vec{t}$ sum of non-Regge-like spins. Here $\lambda\gg 1$ since we are interested in the large-J regime, while $\delta\ll 1$ since the regulator should be turned off in the end. However we may let two scaling limits $\lambda\to\infty$ and $\delta\to 0$ compete, to find a physically interesting regime.

λ relates to the length scale where the semiclassical expansion of the spin-foam amplitude is defined, since the typical lattice spacing is $\ell\sim(\lambda\gamma\ell_P^2)^{1/2}$ for geometries in $\mathcal{N}$. It turns out that the other parameter δ relates to the continuum limit in refining the lattice. δ provides a bound to ensure the lattice spacing ℓ is always much smaller than the typical curvature radius ρ in all geometries emergent from the spin-foam amplitude. It guarantees that the simplicial geometries approach the continuum in the lattice refinement.

It turns out that an interesting way of arranging limits is to first take $\lambda\to\infty$, then $\delta\to 0$. In other words, the interesting regime is that $\lambda\gg 1/\delta\gg 1$

When we first take the asymptotical limit $\lambda\to\infty$, D_δ does not oscillate or suppress, and thus does not affect critical equations from $\langle\vec{J}(\ell),\vec{F}\rangle$. When $\vec{J}(\ell)=\lambda\vec{j}(\ell)$ represents Regge-like spins, there always exist solutions to critical equations,

$$\mathrm{Re}\vec{F}=\partial_g\langle\vec{j}(\ell),\vec{F}\rangle=\partial_z\langle\vec{j}(\ell),\vec{F}\rangle=0.\tag{26}$$

Solutions $(j_f(\ell),g_{ve}(\ell),z_{vf}(\ell))$ correspond to nondegenerate Regge geometries on $\mathcal{K}$, parametrized by the edge lengths ℓ, which relate $\vec{J}$ by Eq. (13). There may not be a unique set of ℓ corresponding to a given Regge-like $\vec{J}$. If it happens, critical solutions contain different Regge geometries with different sets of edge lengths.

Note that when $\vec{J}(\ell)$ is a representative away from $\mathcal{N}_{\mathrm{Regge}}$ with $O(1)$ distance, $(j_f(\ell),g_{ve}(\ell),z_{vf}(\ell))$ are approximate solutions to the critical equations with $O(1/\lambda)$ errors.

Given a set of edge lengths $\vec{\ell}$ of a nondegenerate Regge geometry, in principle, it corresponds to 2^{N_σ} critical solutions (N_σ is the number of 4-simplices), which have indefinite local four-dimensional orientations at each 4-simplex σ [7,8].[10] Within 2^{N_σ} solutions, there are two solutions corresponding to two different global orientations. Here we are only concerned about the sector of critical solutions corresponding to globally oriented Regge geometries. Small perturbations do not flip the 4-simplex orientation, and thus do not relate solutions from different sectors.[11] We are going to determine whether the critical solutions in the sector give dominant contribution to the spin-foam amplitude in the regime $\lambda \gg 1/\delta \gg 1$. It turns out that a subset of critical solutions indeed gives the leading contribution to the amplitude. As is shown in the following, among critical solutions in this sector, the dominant contribution of spin-foam amplitude comes from the critical solutions whose corresponding Regge geometries are of small deficit angle $\varepsilon_f \ll 1$ and satisfying the Regge equation.

At critical solutions with global orientation, the asymptotical limit $\lambda \to \infty$ gives[12]

$$Z_{\mathcal{N},\delta} \sim \sum_{\ell} e^{\frac{i}{\ell_P^2} S_{\rm Regge}[\ell] + \cdots} \sum_{\{k^j\}\in\mathbb{Z}^M} D_\delta^{(k)}(\ell, g_{ve}(\ell), z_{vf}(\ell)). \quad (28)$$

We have replaced $\sum_{\vec{J}(\ell)}$ by $\sum_\ell$, since critical solutions contain all possible ℓ relating to $\vec{J}$. $S_{\rm Regge}$ is the Regge action

$$S_{\rm Regge}[\ell] = \sum_f \mathbf{a}_f \varepsilon_f + \sum_{f\subset\partial\mathcal{K}} \mathbf{a}_f \Theta_f, \qquad \mathbf{a}_f = \gamma \tilde{J}_f(\ell)\ell_P^2, \quad (29)$$

where $\tilde{J}_f(\ell) \in \mathcal{N}_{\rm Regge}$ has been represented by its nearest neighbor $J_f(\ell)$. Here $\cdots$ stands for the subleading corrections in large J.

In the above asymptotical behavior, $S_{\rm Regge}[\ell]$ is obtained by evaluation of $\langle \vec{\tilde{J}}(\ell), \vec{F}\rangle$ at the critical solution corresponding to the Regge geometry $\{\ell\}$. F_f evaluated at the critical solution gives $i\gamma\varepsilon_f$ at each internal f and gives $i\gamma\Theta_f$ at each boundary f, where ε_f and Θ_f are the bulk deficit angle and boundary dihedral angle in the Regge geometry. See [8] for the detailed derivation.

At the leading order, $D_\delta^{(k)}$ takes a value at the critical solution $g_{ve}(\ell), z_{vf}(\ell)$. At each critical point, ${\rm Re}\vec{F} = 0$, and $F_f = i\gamma\varepsilon_f$ for each internal f. Thus $\Phi^i_{(k)} \in i\mathbb{R}$, and

$$D_\delta^{(k)}(\ell, g_{ve}(\ell), z_{vf}(\ell)) = \left(\frac{4\pi}{\delta}\right)^{\frac{M}{2}} e^{-\frac{1}{\delta}\sum_{i=1}^M \psi^i_{(k)}(\ell)\psi^i_{(k)}(\ell)} \times 2^{2N_f} \prod_f \left(J_f(\ell) + \frac{2i}{\delta}\sum_{i=1}^M \psi^i_{(k)}(\ell)(\hat{e}^i)_f(\ell)\right), \quad (30)$$

where

$$\psi^i_{(k)}(\ell) = \gamma\langle \hat{e}^i, \vec{\varepsilon}\rangle + 2\pi k^i. \quad (31)$$

Because of the Gaussian $e^{-\frac{1}{\delta}\sum_{i=1}^M \psi^i_{(k)}\psi^i_{(k)}}$ with small δ, each $D_\delta^{(k)}$ is essentially supported within a small neighborhood of size $\sqrt{\delta}$ at $\psi^i_{(k)} = 0$. As $\delta \ll 1$, each D_δ effectively suppresses the contributions from configurations with large $\psi^i_{(k)}$, and picks out the configurations with small $\psi^i_{(k)}$.

As the large-J limit $\lambda \to \infty$ gives $\ell_P^2 \ll \mathbf{a}_f$, from the variational principle (see Appendix C), the leading contribution of Eq. (28) is given by the $\{\ell\}$ configurations satisfying Regge equation

$$\sum_f \frac{\partial \mathbf{a}_f}{\partial \ell}\varepsilon_f = 0, \quad \text{or} \quad \gamma\left\langle \frac{\partial \vec{J}}{\partial \ell}, \vec{\varepsilon}\right\rangle = 0. \quad (32)$$

Each solution of the Regge equation gives the leading order contribution to $Z_{\mathcal{N},\delta}$, which is proportional to

[10] This result is valid for the Lorentzian spin-foam amplitude. The Euclidean amplitude gives 4^{N_σ} critical solutions instead of 2^{N_σ}. There are four solutions (g_{ve}, g'_{ve}), (g'_{ve}, g_{ve}), (g_{ve}, g_{ve}), (g'_{ve}, g'_{ve}) in each 4-simplex. But different critical solutions are still understood as belonging to different well-separated sectors, as in the Lorentzian case. Again we only consider the sector of $g^+_{ve} \neq g^-_{ve}$ with a global orientation.

[11] The 4-simplex orientation only takes discrete values ± 1 [8]. Small deformations among critical solutions do not affect the value of orientation.

[12] Note that at each $\{\ell\}$ in $\sum_\ell$ in Eq. (28), the critical solutions beyond the above sector may contribute some exponentials in addition to $e^{iS_{\rm Regge}[\ell]/\ell_P^2 + \cdots}$. If we denote by σ all possible assignment of orientations to simplices (σ also includes the solutions with $g^+_{ve} = g^-_{ve}$ in the Euclidean amplitude), the asymptotical behavior Eq. (28) of $Z_{\mathcal{N},\delta}$ may be more properly written as

$$\sum_\sigma \sum_\ell e^{\frac{i}{\ell_P^2}S_\sigma[\ell] + \cdots} \sum_{\{k^j\}\in\mathbb{Z}^M} D^{(k)}_{\delta,\sigma}(\ell, g_{ve}(\ell), z_{vf}(\ell)). \quad (27)$$

Each $iS_\sigma[\ell]/\ell_P^2$ is the spin-foam action evaluated at the critical solution with orientations σ in simplices. Equation (28) corresponds to the term where σ endows $\mathcal{K}$ with a global orientation. The leading contributions to $Z_{\mathcal{N},\delta}$ in Eq. (27) have been organized into disjoint sectors associated to different σ. Each sector σ has its own partition function $\sum_\ell e^{iS_\sigma/\ell_P^2 + \cdots} \sum_{k^j} D^{(k)}_{\delta,\sigma}$. Small perturbations do not relate critical solutions from different sectors. In other words, those critical solutions without global orientation only give nonperturbative corrections to Eq. (28). In this paper, we focus on the sector in Eq. (27) with a global orientation, and study the geometries making leading contributions to the amplitude.

$$e^{\frac{i}{\ell_P^2}\sum_{f\subset\partial\mathcal{K}}\mathbf{a}_f\Theta_f}\sum_{\{k^j\}\in\mathbb{Z}^M} e^{-\frac{1}{\delta}\sum_{i=1}^M \psi^i_{(k)}(\ell)\psi^i_{(k)}(\ell)}(\cdots). \quad (33)$$

Note that the bulk terms in $S_{\mathrm{Regge}}[\ell]$ vanish at each solution of the Regge equation. Now we take $\delta \ll 1$, the Gaussian $e^{-\frac{1}{\delta}\sum_{i=1}^M \psi^i_{(k)}\psi^i_{(k)}}$ suppresses the amplitude contributed by the solutions $\{\ell\}$, which have relatively large $\psi^i_{(k)}(\ell) = \gamma\langle\hat{e}^i, \vec{\varepsilon}\rangle + 2\pi k^i$; i.e., the essential contribution of the spin-foam amplitude $Z^{(\vec{k}=0)}_{\mathcal{N},\delta}$ comes from the solutions $\{\ell\}$ satisfying

$$|\gamma\langle\hat{e}^i, \vec{\varepsilon}\rangle| \leq \delta^{1/2} \ll 1 \quad \mathrm{mod}\ 2\pi k^i. \quad (34)$$

Let us temporarily ignore the terms with $k^j \neq 0$ in Eq. (33). $\partial\vec{J}/\partial\ell$ are tangent vectors on the submanifold $\mathcal{M}_{\mathrm{Regge}}$ of Regge-like spins. Thus $\partial\vec{J}/\partial\ell$ and $\hat{e}^i$ form a complete basis in $\mathcal{N}$. The Regge equation Eq. (32) and the requirement Eq. (34) at $k^j = 0$ combine and give that all deficit angles have to be small,

$$|\gamma\varepsilon_f| \leq \delta^{1/2} \ll 1. \quad (35)$$

Namely, given a solution $\{\ell\}$ to the Regge equation, all its deficit angles ε_f have to be small in order to provide a nonsuppressed contribution to the spin-foam amplitude at $k^j = 0$. The Barbero-Immirzi parameter γ is a fixed $O(1)$ parameter in our discussion. If γ was not fixed and sent to 0 combining the semiclassical limit, Eq. (35) would allow large deficit angle in the semiclassical Regge geometries, which reproduced the result in [39,46].

When the simplicial triangulation is refined, given a Regge geometry $\{\ell\}$ that approximates a smooth geometry,[13] the deficit angle relates to the typical lattice spacing ℓ of the Regge geometry and the typical curvature radius ρ of the smooth geometry by [22][14]

$$\varepsilon \sim \frac{\ell^2}{\rho^2}\left[1 + O\left(\frac{\ell^2}{\rho^2}\right)\right]. \quad (36)$$

The Regge geometry has to satisfy $\ell^2 \ll \rho^2$ in order to approximate the smooth geometry, since the ratio between ℓ and a geodesic length ℓ_s of the smooth geometry is $\ell/\ell_s = 1 + O(\ell^2/\rho^2)$. Note that the smooth limit of Regge geometry also requires that the fatness of simplices is bounded away from 0, to avoid any degenerate simplex. See, e.g., [23,40,47] for details.

When the lattice is sufficiently refined, and when δ is sent to be small, Regge geometries sufficiently approximating smooth geometries all satisfy Eq. (35) and survive as dominant contribution to $Z_{\mathcal{N},\delta}$ at $k^j = 0$. Regge geometries suppressed by D_δ are the ones that fail to approximate any smooth geometry. The regulator δ behaves similarly as the bound of error in the piecewise linear approximation of smooth metric

$$|\ell/\ell_s - 1| \simeq O(\ell^2/\rho^2) \leq \delta^{1/2}. \quad (37)$$

The leading contribution to the semiclassical spin-foam amplitude must satisfy both Regge equation (32) and Eq. (35). Therefore, the solutions of the Regge equation that approximate smooth geometries all give dominant contributions to the spin-foam amplitude.

The terms with $k^j \neq 0$ add discrete ambiguities to the constraint Eq. (35). However, different k^j correspond to disjoint sectors of discrete geometries satisfying Eq. (34). Geometries in sectors of $k^j \neq 0$ do not approximate any smooth geometry. Small perturbations cannot relate two geometries satisfying Eq. (34) with different k^j.

The geometries in sectors with $k^j \neq 0$ may have nonsuppressed contributions to the semiclassical spin-foam amplitude (as has been pointed out in [37,38]). However the sectors are sensitive to the choice of the neighborhood $\mathcal{N}_{\mathrm{Regge}}$ in defining $Z_{\mathcal{N},\delta}$. For example, we assume the neighborhood $\mathcal{N}_{\mathrm{Regge}}$, which contains the physical Regge geometries only with relatively small deficit angles; i.e., $\gamma\langle\hat{e}^i, \vec{\varepsilon}\rangle$ is not close to any $2\pi k^i$ with $k^i \neq 0$. Then the terms with $k^j \neq 0$ in Eq. (33) only have negligible contribution to $Z_{\mathcal{N},\delta}$. The dominant contribution to $Z_{\mathcal{N},\delta}$ comes from the geometries with small deficit angles. The $k^j = 0$ sector is physically most relevant because it is the only sector containing discrete geometries approaching the continuum as the simplicial lattice being refined.

It is mentioned in Sec. II that critical points in the spin-foam action contain time-nonoriented geometries [25], which gives $F_g = i(\gamma\varepsilon_f \pm \pi)$. Within this type of critical point, the equation of motion Eq. (32), the constraint Eq. (34) or Eq. (35), is modified by the replacement $\gamma\varepsilon_f \to \gamma\varepsilon_f \pm \pi$. The constraint then leads to $\gamma\varepsilon_f$ being close to $\pm\pi$. These critical points form two disjoint sectors away from the ones discussed above. Geometries in this sector do not approximate any smooth geometry, and can be treated in the same way as the $k^j \neq 0$ sectors. Some

[13]If we embed the Regge geometry in $\mathbb{R}^N$, $N > 4$, the corresponding smooth geometry is a smooth enveloping surface $\mathcal{S}$ of the Regge geometry, where all vertices (end points of ℓ's) in the Regge geometry are located in $\mathcal{S}$. $\mathcal{S}$ is required to satisfy $\rho \gg \ell$ everywhere. Once a $\mathcal{S}$ is chosen, the Regge geometry is a piecewise linear approximation to $\mathcal{S}$ satisfying $|\ell/\ell_s - 1| \simeq O(\ell^2/\rho^2)$ where ℓ_s is the geodesic length connecting the end points of ℓ [22].

[14]Given a small 2-face f embedded in a smooth geometry, the loop holonomy of spin connection along ∂f gives $e^{\varepsilon\hat{X}}$, where $\hat{X}$ is the bivector tangent to f. As f is small, the holonomy gives $1 + \int_f F \simeq 1 + \varepsilon\hat{X}$, which implies $\varepsilon \simeq \ell^2/\rho^2$ since F is the curvature 2-form of the spin connection. Typical spacings of $\mathcal{K}$ and $\mathcal{K}^*$ are of similar scales.

discussion of the Euclidean amplitude is given in Appendix D.

V. EINSTEIN-REGGE REGIME

We refer to the regime of the spin-foam model, where the Regge equation emerges together with the constraint $\gamma\varepsilon_f \leq \delta^{1/2}$, as the ER regime. The ER regime is defined by considering the deformed spin-foam amplitude $Z_{\mathcal{N},\delta}(\mathcal{K})$, and imposing the following requirements on the parameters $\mathcal{K}$, $\mathcal{N}$, δ:

(i) The neighborhood $\mathcal{N}$ contains a submanifold $\mathcal{N}_{\mathrm{Regge}} \subset \mathcal{N}$. All $\tilde{J}_f(\ell)$ in $\mathcal{N}_{\mathrm{Regge}}$ are large $\tilde{J}_f(\ell) \gg 1$. The mean value of $\tilde{J}_f(\ell)$ in $\mathcal{N}_{\mathrm{Regge}}$ is denoted by λ. Parameters λ and δ satisfy $\lambda \gg \delta^{-1} \gg 1$.

(ii) The neighborhood $\mathcal{N}$ of the spin-foam spin sum has to be compatible with the triangulation $\mathcal{K}$. Namely, Regge geometries $\{\ell\}$ in the neighborhood $\mathcal{N}_{\mathrm{Regge}} \subset \mathcal{N}$ all have relatively small deficit angles ε_f (e.g., requiring $\gamma\varepsilon_f < \pi$). $\mathcal{N}_{\mathrm{Regge}}$ should contain Regge geometries that approximate smooth geometries.

In the ER regime specified by the above requirements, the spin-foam amplitude obtains dominant contributions from Regge geometries in $\mathcal{N}$, which satisfy both the Regge equation (32) and the bound $\varepsilon_f \leq \gamma^{-1}\delta^{1/2}$. These Regge geometries contain the ones approximating smooth geometries by Eq. (37). They satisfy the following (approximate) bound by Eq. (36):

$$\rho^2 \geq \frac{\gamma\lambda\ell_P^2}{\sqrt{\delta}} \gg \ell^2 \gg \ell_P^2. \tag{38}$$

The inequality $\ell_P^2 \ll \ell^2 \ll \rho^2$, satisfied by the dominant configurations, is the condition that the discrete geometry is semiclassical ($\ell^2 \gg \ell_P^2$), as well as approaching the continuum limit ($\ell^2 \ll \rho^2$) [20,37,48].

It is anticipated that geometries both satisfying the Regge equation and approximating the continuum should approximate the smooth solution to the *continuum* Einstein equation. We come back to this point in the next section.

Note that in this work, we limit ourselves to understanding the dominance in the spin-foam amplitude from classical geometries with a global orientation. As it has been mentioned in the last section, geometries without global orientation live in other well-separated sectors. They may provide nonpertrubative corrections to the contribution studied above, although they do not affect the perturbative expansion at any classical geometry.

VI. SEMICLASSICAL CONTINUUM LIMIT

So far the discussion is based on a fixed triangulation. We may change our viewpoint and consider a sequence of triangulations $\mathcal{K}_n$, where each $\mathcal{K}_{n+1}$ is a refinement of $\mathcal{K}_n$. The vertices of all $\mathcal{K}_n$'s are a dense set in the manifold where the triangulations are embedded. The sequence of $\mathcal{K}_n$ defines a sequence of spin-foam amplitudes $Z_{\mathcal{N},\delta}(\mathcal{K}_n)$. The smooth geometry can be understood as the limit of a sequence of discrete geometries $\{\ell_n\}$ on the sequence of triangulations $\mathcal{K}_n$, where the discrete geometries approach $\ell^2/\rho^2 \to 0$. When each of the discrete geometries $\{\ell_n\}$ in the sequence satisfies the Regge equation on $\mathcal{K}_n$, it gives the nonsuppressed contribution to the spin-foam amplitude $Z_{\mathcal{N},\delta}$ on $\mathcal{K}_n$.

Let us describe more detailed behavior of geometries $\{\ell_n\}$ and amplitudes $Z_{\mathcal{N},\delta}$ on the sequence of triangulations $\mathcal{K}_n$. Generically on a more refined triangulation, the large system size requires a larger λ to obtain the semiclassical behavior as the leading order in the spin-foam amplitude. Indeed in the $1/\lambda$ quantum correction of the amplitude, the coefficient of $1/\lambda^s$ is a sum over all g_{ve}, z_{vf} degrees of freedom on the triangulation (see, e.g., [25]).

$$i^{-s}\sum_{l-m=s}\sum_{2l\geq 3m}\frac{2^{-l}}{l!m!}\left[\sum\nolimits_{a,b}H^{-1}_{ab}(x_0)\frac{\partial^2}{\partial x_a\partial x_b}\right]^l g_{x_0}(x_0)^m, \tag{39}$$

where x_0 is a critical point, $H(x) = S''(x)$ denotes the Hessian matrix, and $g_{x_0}(x)$ is given by

$$g_{x_0}(x) = S(x) - S(x_0) - \frac{1}{2}H^{ab}(x_0)(x-x_0)_a(x-x_0)_b. \tag{40}$$

Here a, b label all degrees of freedom on the triangulation. A refined triangulation carries a larger number of degrees of freedom, thus generically producing a larger coefficient. It requires a smaller $1/\lambda$ to suppress the quantum correction and let the semiclassical behavior stand out. Therefore, the discrete geometry $\{\ell_n\}$ on $\mathcal{K}_n$ has larger and larger λ as $\mathcal{K}_n$ becomes more and more refined. Even if it happens that the above generic behavior is violated in a certain situation, i.e., the coefficient of $1/\lambda$ does not increase in refining the lattice, tuning λ larger still suppresses the quantum correction. So λ can in general be set to be monotonically increasing in refining the lattice.

Naively it might sound unexpected to have λ be larger in the refinement since the triangle area $\ell^2 \sim \mathbf{a} = \gamma\lambda\ell_P^2$. However, the continuum limit is controlled by the ratio ℓ^2/ρ^2. The ratio becomes smaller when the curvature radius ρ in Planck unit increases at a faster rate than λ, or equivalently, when we zoom out to larger length units such that the value of ℓ_P decreases at a faster rate. Zooming out to larger length units is required by the semiclassical limit.

Formally we associate each triangulation $\mathcal{K}_n$ with a mass scale μ_n whose inverse μ_n^{-1} is a length unit. n becoming larger is the refinement of $\mathcal{K}_n$, while μ_n becomes smaller.

The length unit μ_n^{-1} increases as a refinement of the triangulation. Given the 1-to-1 correspondence between $\mathcal{K}_n$ and μ_n, we may simply label the triangulation and discrete geometry as $\mathcal{K}_\mu$ and $\{\ell_\mu\}$ by its associated scale μ.$\mathcal{K}_\mu$ is refined as μ going to the IR. On each $\mathcal{K}_\mu$, the discrete geometry gives the triangle area $\mathbf{a}(\mu)$

$$\mathbf{a}(\mu) = \gamma\lambda(\mu)\ell_P^2 = a(\mu)\mu^{-2}. \tag{41}$$

Here the running of ℓ_P is not considered since we are in the semiclassical limit. $\lambda(\mu)$ increases monotonically in the refinement $\mu \to 0$ as discussed above. However we can assign the scale μ to $\mathcal{K}_\mu$ such that $a(\mu) \to 0$ as $\mu \to 0$.[15] Using the dimensionless length $a(\mu)$, we can define the convergence of the sequence of geometries $\{\ell_\mu\}$ [where $\ell_\mu \sim a(\mu)^{1/2}\mu^{-1}$ for each geometry on $\mathcal{K}_\mu$ converge to a smooth geometry] by requiring $\lim_{\mu\to 0} a(\mu) = 0$ and the fatness bounded away from 0. The target smooth geometry has the dimensionless curvature radius denoted by L, which is the curvature evaluated at the IR unit $\mu \to 0$; i.e., the dimensionful curvature radius is

$$\rho(\mu) = L\mu^{-1}. \tag{42}$$

The sequence of discrete geometries $\{\ell_\mu\}$ approaches the smooth geometry because

$$\frac{\mathbf{a}(\mu)}{\rho(\mu)^2} = \frac{a(\mu)}{L^2} \to 0, \quad \text{as} \quad \mu \to 0. \tag{43}$$

Note that since μ is of mass dimension, $\mu \to 0$ may be understood more appropriately as $\mu\ell_P \to 0$.

The dependence of λ on μ shows that the semiclassical limit is taken at the same time as the lattice refinement limit. Possible assignments of scales μ to triangulations $\mathcal{K}_\mu$ are classified in Sec. VII.

As an illustration of the above idea, let us consider a smooth sphere with a unit curvature radius $L = 1$. It is standard to define discrete geometries on a sequence of refined triangulations of the sphere, which approaches the smooth sphere in the continuum limit. We assign a mass scale μ to label the triangulation $\mathcal{K}_\mu$ such that the refinement relates to $\mu \to 0$. On each $\mathcal{K}_\mu$, edge lengths are $\sqrt{a(\mu)}$ satisfying $\lim_{\mu\to 0} a(\mu) = 0$. $\sqrt{a(\mu)}$ are understood as edge lengths in the unit μ^{-1}. The scale μ should be chosen such that $a(\mu)\mu^{-2}/\ell_P^2 \to \infty$ as $\mu \to 0$, in order to have $\lambda(\mu)$ increasing in the refinement. Geometries in the sequence now associate with different scales μ. The smooth sphere lives at the IR limit whose curvature radius $L = 1$ is measured at the IR unit $\mu^{-1} \to \infty$.

Let us turn to the semiclassical behavior of $Z_{\mathcal{N},\delta}$ on the sequence of $\mathcal{K}_\mu$. Here $\mathcal{N}$ depends on μ since λ does. We take $\mathcal{N}(\mu)$'s to satisfy the requirement of the ER regime. Then $\mathcal{N}(\mu)$'s contain sequences of Regge geometries that converge to smooth geometries, since $a(\mu) \to 0$ as $\mu \to 0$, and moreover, since $\lambda(\mu)$ increases as $\mu \to 0$. The existence of the ER regime $\lambda(\mu) \gg \delta^{-1} \gg 1$ can be achieved by smaller δ, if we make $\delta = \delta(\mu)$ run with the scale. Namely, we can make $\delta(\mu) \to 0$ as $\mu \to 0$, while $\lambda(\mu) \gg \delta(\mu)^{-1} \gg 1$ is satisfied. For sequences of discrete geometries $\{a(\mu)^{1/2}\}$ converging to smooth geometries at the IR, they give dominant contributions to $Z_{\mathcal{N}(\mu),\delta(\mu)}$ at each μ, if they satisfy the Regge equation on each $\mathcal{K}_\mu$ and

$$\gamma\frac{a(\mu)}{L^2} \leq \delta(\mu)^{\frac{1}{2}}. \tag{44}$$

We may choose decreasing rates of $\delta(\mu)^{1/2}$ and $a(\mu)$ to be the same, to keep all converging geometries contributing dominantly. $\delta(\mu) \to 0$ as $\mu \to 0$ means that the regulator δ is removed in the continuum limit, where $Z_{\mathcal{N},\delta}$ goes back to its original definition Eq. (9).

Spin-foam amplitudes give sequences of Regge geometries converging to smooth geometries, where each geometry satisfies the Regge equation on its lattice. It is thus expected that each smooth geometry as the limit is a solution of the continuum Einstein equation. However, due to complexities of both the Regge equation and Einstein equation, a general mathematical proof is unfortunately not available in the literature as far as we know. However, there have been extensive studies on the continuum limit of the Regge calculus, which gives many analytic and numeric examples (see [26,27] for summaries). In all the examples, solutions of the Regge equation always converge to smooth solutions to the Einstein equation. Among the examples, there have been constructions of solutions of linearized Regge equations in the Euclidean signature, which converge to solutions to the linearized Einstein equation [28–30]. In the nonlinear regime, there have been numerical simulations of time evolutions in the Regge calculus in the Lorentzian signature, as a tool of numerical relativity. Nontrivial results include, e.g., the Kasner universe, Brill waves, binary black holes, and the FLRW universe [27,31–34]. A key observation in the convergence results is that the deviation of the Regge calculus from general relativity is the noncommutativity of rotations in the discrete theory, while the error from the noncommutativity is of higher order in edge lengths [36]. There is also the convergence result by a certain average of Regge equations [35]. The existing results all demonstrate that the Regge calculus is a consistent second order accurate discretization of general relativity.

[15] Considering the gap $\Delta J_f = \frac{1}{2}$, $\Delta a_f(\mu) = \gamma\Delta J_f(\mu)\mu^2\ell_P^2 = \frac{1}{2}\gamma\mu^2\ell_P^2 \to 0$ as $\mu \to 0$.

Given any sequence of solutions to the Regge equation that converges to a solution to the continuum Einstein equation, our analysis shows that each solution gives the dominant contribution to the spin-foam amplitude on $\mathcal{K}_\mu$ in the semiclassical limit. The smooth solution to the Einstein equation is the limit of a sequence of dominant configurations from the spin-foam amplitude.

As an example, Euclidean spin-foam amplitudes on $\mathcal{K}_\mu$ can give a sequence of solutions to linearized Regge equations, which coincide with the ones constructed in [28]. Edge lengths used there should be identified with $\sqrt{a(\mu)}$ [more precisely, relate to $a(\mu)$ by Eq. (13)]. The sequence of geometries provides dominant contribution to spin-foam amplitudes, and converges in the IR limit $\mu\to 0$ to smooth gravitational waves satisfying the linearized Einstein equation.

There is another way to obtain the continuum Einstein equation from the convergence of Regge actions. Let us come back to Eq. (28) and consider the sequence $Z_{\mathcal{N}(\mu),\delta(\mu)}(\mathcal{K}_\mu)$. For each sequence of Regge geometries converging to the smooth geometry as $\mu\to 0$, Regge actions converge to the Einstein-Hilbert action on the continuum, when Regge geometries converge to the smooth geometry (the convergence again requires the fatness of simplices to be bounded from 0 in addition to shrinking edge lengths; see [23,40] for details). Translating the known convergence result to our context uses the length unit μ^{-1}. We apply Eq. (41) to the Regge action $\frac{1}{\ell_P^2}\sum_f \mathbf{a}_f(\mu)\varepsilon_f(\mu)$,

$$\frac{1}{\mu^2\ell_P^2}\sum_f a_f(\mu)\varepsilon_f(\mu)=\frac{1}{\mu^2\ell_P^2}\int \mathrm{d}^4x\sqrt{-g}R[1+\epsilon(\mu)], \quad (45)$$

where $\epsilon(\mu)$ satisfies $\lim_{\mu\to 0}\epsilon(\mu)=0$. The convergence happens as the edge length $a(\mu)^{1/2}\to 0$ at the IR. Smooth geometries and $\int \mathrm{d}^4x\sqrt{-g}R$ live at the IR limit $\mu\to 0$. $\mu^2\ell_P^2$ is the numerical value of ℓ_P^2 in the unit μ^{-2}. $\mu^2\ell_P^2$ tends to 0 when we zoom out to the larger unit.

Given a Regge geometry $\{\ell\}$ approximating the smooth geometry, there is a smooth enveloping surface $\mathcal{S}$ whose curvature satisfies $\rho\gg\ell$ everywhere, and $|\ell/\ell_s-1|\simeq O(\ell^2/\rho^2)$, as well as the fatness bounded away from 0. Small perturbations at $\{\ell\}$ generically do not break the above properties, so only lead to Regge geometries still approximating smooth geometries.

Indeed, consider a small perturbation of both the Regge geometry and correspondingly, its smooth enveloping surface $\mathcal{S}'$, i.e., $|l'-l|\le\delta_1$ and $|l'_s-l_s|\le\delta_2$ with $0<\delta_{1,2}<l^2<l/2$ (l denotes the edge length in unit μ^{-1}). In [23,40], the rigorous approximation criterion is $|l-l_s|\le Cl^2$, which gives $|l'-l'_s|\le Cl^2+\delta_1+\delta_2<(C+2)l^2\le C'(l-\delta_1)^2\le C'l'^2$ for $C'=4(C+2)>\frac{C+2}{(1-\delta_1/l)^2}$. So the perturbed Regge geometry still satisfies the approximation criterion.

The vicinity of a Regge geometry approximating the smooth geometry only covers Regge geometries that approximate smooth geometries, so Eq. (45) is valid in the vicinity. Considering the vicinity is sufficient for the variational principle. The partition function Eq. (28) within the vicinity (of each approximated smooth geometry) behaves as

$$Z_{\mathcal{N}(\mu),\delta(\mu)}(\mathcal{K}_\mu)\simeq\int[Dg_{\mu\nu}]e^{\frac{i}{\mu^2\ell_P^2}\int \mathrm{d}^4x\sqrt{-g}R[1+\epsilon(\mu)]}. \quad (46)$$

Moreover, Eq. (46) manifests that the IR limit $\mu\to 0$ leads to the stationary phase approximation in Eq. (46), whose variational principle gives the continuum vacuum Einstein equation $R_{\mu\nu}=0$.

The above argument shows that the spin-foam amplitude reduces to a partition function of Einstein-Hilbert action in the semiclassical continuum limit.

We remark that in the above analysis, the regulator δ plays an interesting role by opening a window to allow small nonvanishing deficit angles ε_f for Regge geometries approximating the continuum. Given a sequence of Regge geometries approaching a smooth geometry with nontrivial curvature, the small window of ε_f allows each Regge geometry in the sequence to have dominant contribution in their corresponding (regularized) spin-foam amplitudes $Z_{\mathcal{N},\delta}$.

The above result is achieved by taking an appropriate limit combining $\lambda\to\infty$ and $\delta\to 0$ with respect to the requirement $\lambda\gg\delta^{-1}\gg 1$ of the ER regime. However, if the requirement was violated by sending $\delta\to 0$ before $\lambda\to\infty$, we would lose the window of nonvanishing curvature for each Regge geometry in the sequence. Then there would be no smooth curved geometry as the limit from spin-foam amplitudes. This behavior was the flatness observed in [17,18].

VII. RUNNING SCALE

In this section we classify all possible assignments of scales μ to triangulations $\mathcal{K}_\mu$. In the above discussion, there are two requirements relevant to assigning scales μ to triangulations $\mathcal{K}_\mu$.

(i) $\lambda(\mu)$ always suppresses the growth of the coefficient in (39) at arbitrary order s.

(ii) $\lambda(\mu)\mu^2\propto a(\mu)$ monotonically decreases as $\mu\to 0$.

We denote the coefficient (39) at the order λ^{-s} by $f_s(\mu)$, exhibiting its dependence on triangulation $\mathcal{K}_\mu$. It is required that $|f_s(\mu)|/\lambda(\mu)^s$ should not blow up as $\mu\to 0$ for all s,

$$0\le\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{|f_s(\mu)|}{\lambda(\mu)^s}\right)=-\frac{s|f_s|}{\lambda^{s+1}}\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}+\frac{1}{\lambda^s}\frac{\mathrm{d}|f_s|}{\mathrm{d}\mu}, \quad (47)$$

which gives

$$\frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \leq \frac{1}{s|f_s|}\frac{\mathrm{d}|f_s|}{\mathrm{d}\mu}. \tag{48}$$

On the other hand, monotonically decreasing $\lambda(\mu)\mu^2 \propto a(\mu)$ as $\mu \to 0$ implies

$$0 < \frac{\mathrm{d}}{\mathrm{d}\mu}(\lambda(\mu)\mu^2) = \mu^2\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} + 2\mu\lambda, \tag{49}$$

which gives

$$0 > \frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} > -\frac{2}{\mu}. \tag{50}$$

Combining Eq. (48) gives

$$\frac{1}{s|f_s|}\frac{\mathrm{d}|f_s|}{\mathrm{d}\mu} > -\frac{2}{\mu}. \tag{51}$$

Recall that μ is assigned to a sequence of triangulations $\mathcal{K}_n \equiv \mathcal{K}_{\mu_n} \equiv \mathcal{K}_\mu$. The variable $\mu \equiv \mu_n$ is actually discrete. $|f_s(\mu)|$ and $\lambda(\mu)$ have been assumed to be a differentiable function that continues $|f_s(\mu_n)|$ and $\lambda(\mu_n)$.

Integrating Eq. (51),

$$\int_{\mu_n}^{\mu_{n-1}} \frac{1}{s|f_s|}\frac{\mathrm{d}|f_s|}{\mathrm{d}\mu}\mathrm{d}\mu > -\int_{\mu_n}^{\mu_{n-1}}\frac{2}{\mu}\mathrm{d}\mu, \tag{52}$$

which gives

$$\frac{\mu_{n-1}}{\mu_n} > \left|\frac{f_s(\mu_n)}{f_s(\mu_{n-1})}\right|^{\frac{1}{2s}}. \tag{53}$$

Thus the assignment of μ depends on the behavior of coefficients $f_s(\mu_n)$ for all s. All possibilities are classified as follows:

(1) The simplest situation is that all $|f_s(\mu)|$ stop increasing at finite $\mu_s > \mu_* > 0$; then Eq. (53) does not impose any constraint to μ when $\mu < \mu_*$, since μ_{n-1}/μ_n is always greater than 1. It is easy to find $\lambda(\mu)$ to satisfy Eq. (50).

(2) If there are finitely many $s \geq 1$ whose $|f_s(\mu)|$ increase monotonically as $\mu \to 0$, finitely many $|\frac{f_s(\mu_n)}{f_s(\mu_{n-1})}| > 1$ impose a nontrivial lower bound to μ_{n-1}/μ_n. Because the number of increasing $|f_s(\mu)|$ is finite, there is a bounded B_n at each n,

$$\left|\frac{f_s(\mu_n)}{f_s(\mu_{n-1})}\right|^{\frac{1}{2s}} \leq \max_{s\geq 1}\left|\frac{f_s(\mu_n)}{f_s(\mu_{n-1})}\right|^{\frac{1}{2s}} \equiv B_n. \tag{54}$$

We can choose $\frac{\mu_{n-1}}{\mu_n} > B_n$ at each n, so that Eq. (53) is satisfied uniformly to all orders s.

(3) If there are infinitely many $|f_s(\mu)|$ increasing monotonically as $\mu \to 0$, and if the rate $|\frac{f_s(\mu_n)}{f_s(\mu_{n-1})}| \leq A_n e^{C_n s}$ (for certain constants A_n, $C_n > 0$) bounded by exponentially growing when going to higher orders s, then there is an upper bound B_n at each n ($A_n^{\frac{1}{2s}}$ is bounded in $s \geq 1$),

$$\left|\frac{f_s(\mu_n)}{f_s(\mu_{n-1})}\right|^{\frac{1}{2s}} \leq A_n^{\frac{1}{2s}} e^{C_n/2} \leq B_n. \tag{55}$$

We can again choose $\frac{\mu_{n-1}}{\mu_n} > B_n$ at each n, so that Eq. (53) is satisfied uniformly to all orders s.

(4) If $|\frac{f_s(\mu_n)}{f_s(\mu_{n-1})}|^{\frac{1}{2s}}$ is not bounded from above as $s \to \infty$ at any n, Eq. (53) can only be satisfied at any truncation of the λ^{-1} asymptotic expansion. At any truncation up to λ^{-s_0} order, $|\frac{f_s(\mu_n)}{f_s(\mu_{n-1})}|^{\frac{1}{2s}}$ at each n is bounded from above within finitely many $1 \leq s \leq s_0$. The bound changes for different s_0. Then the rate $\frac{\mu_{n-1}}{\mu_n}$ has to be justified order by order.

We conjecture that the third situation should be most relevant. f_s in quantum mechanics and quantum field theories have the following generic behavior as $s \to \infty$ (see, e.g., [49–51]),

$$|f_s| \simeq \eta s! s^\alpha \beta^s (1+\epsilon(s))^s, \qquad \lim_{s\to\infty}\epsilon(s) = 0, \tag{56}$$

where constants η, α, β may depend on different theories and different numbers of degrees of freedom. This behavior leads to

$$\left|\frac{f_s(\mu_n)}{f_s(\mu_{n-1})}\right|^{\frac{1}{2s}} \simeq \left(\frac{\eta_n}{\eta_{n-1}}\right)^{\frac{1}{2s}} s^{\frac{1}{2s}(\alpha_n-\alpha_{n-1})}\left(\frac{\beta_n}{\beta_{n-1}}\right)^{\frac{1}{2}}[1+\epsilon(s)]^{\frac{1}{2}}, \tag{57}$$

bounded from above for large s.

VIII. CONCLUSION AND OUTLOOK

The discussion of this paper explains the emergence of the Einstein equation from the spin-foam amplitudes in the semiclassical continuum limit. However, the spin-foam amplitude seems to contain more solutions than the Einstein equation does. The analysis here mainly focuses on the sector of critical points in the spin-foam amplitude that corresponds to nondegenerate geometries with a global orientation. Solutions to the Einstein equation emerge within this sector. There exist other well-separated sectors where the spin-foam amplitude gives the degenerate geometry and geometries without a global orientation [7,8]. Those geometries may not satisfy the Einstein equation, and their physical meaning remains open (see, e.g., [52] for some discussion). Note that there exists the spin-foam model (the proper vertex) whose asymptotics give a single orientation to each 4-simplex [53]. The discussion in this paper is also valid in this model.

A key step in the discussion is the regularization of the non-Regge-like spin sum in Eq. (19), which is a deformation of the spin-foam amplitude. We take the point of view that the spin-foam amplitude defined on a triangulation might be an effective theory from a complete LQG theory as the continuum limit of the spin foam. As the level of effective theory, the deformation has to be implemented in the spin-foam amplitude to reproduce the desired semiclassical limit. As is shown in Sec. VI, the deformation is turned off in the continuum limit. It suggests that the spin-foam amplitudes, with or without the deformation, should have the same continuum limit. The amplitude with the deformation is one effective description of the complete LQG theory, whose advantage is the correct semiclassical behavior.

Although the regularization includes a Gaussian damping factor in the non-Regge-like spin sum, it is not allowed to completely remove non-Regge-like spins in the spin sum. Removing all non-Regge-like spins would be an *ad hoc* modification of the model, which modified the continuum limit. The modification would remove the small-ε_f constraint (3) or (35) and break the desired behavior of the spin-foam amplitude near a classical curvature singularity in [20] (reviewed briefly at the end of Sec. I). In our opinion, the existence of non-Regge-like spins and its consequence, the flatness, are nice properties of spin-foam amplitude, when treated properly.

There has been recent progress on the spin-foam amplitude with cosmological constant [9,54–59]. Research is being undergone to apply the present analysis to the formalism with cosmological constant. Another possible future direction is to apply the analysis to the sum over triangulations in group field theory (GFT). The method developed in this work might be helpful to understand the emergence of classical geometries from GFT, and the relation to phase transitions. Our results on the spin-foam amplitude might also be applied to the tensor network approach in the bulk-boundary duality [60,61], by the relation between random tensor networks and spin-networks [48]. The recent work in [62] applies discrete three-dimensional bulk gravity to random tensor networks, and reproduces correctly the holographic Rényi entropy of two-dimensional CFT. The result here may be useful in the generalization to four bulk dimensions.

Finally, we mention that there have been earlier studies on the continuum limit in spin foams, e.g., [63–69]. There are also some recent results on emerging classical spacetimes from GFT, e.g., [70–72].

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APPENDIX A: SPIN SUM IN SPIN-FOAM AMPLITUDE

In this section, we show that $\sum_{J_f}$ in the spin-foam amplitude Eq. (9) can be understood as a free spin sum, where spins J_f from different f are independent.

The summand of $\sum_{J_f}$ can be written as [up to a factor of $\dim(J_f)$] [25]

$$\int \mathrm{d}g_{ve} \sum_{\{M_{ef}\}} \prod_{(v,f)} \langle J_f, \gamma J_f; J_f, M_{ef} | g_{ev} g_{ve'} | J_f, \gamma J_f; J_f, M_{e'f} \rangle. \tag{A1}$$

The inner product takes place in the $\mathrm{SL}(2,\mathbb{C})$ unitary irrep $\mathcal{H}^{(J,\gamma J)} \simeq \bigoplus_{k=J}^{\infty} V_k$, where V_k is the irrep of an SU(2) subgroup of $\mathrm{SL}(2,\mathbb{C})$. The canonical basis $|J,\gamma J, J, M\rangle$ is a state in the lowest-level $V_{k=J}$, where m is the magnetic quantum number. Each of the inner products associates to a triangle f and a vertex v of f. e, e' label the edges adjacent to v.

We pick a g_{ve} and make a change of variable $g_{ve} \to g_{ve} h_e$, $h_e \in \mathrm{SU}(2)$, followed by an integration $\int_{\mathrm{SU}(2)} \mathrm{d}h_e$. The operation does not change the value of Eq. (A1) because of the normalization of the Haar measure $\mathrm{d}h_e$ on SU(2). $\mathrm{d}(g_{ve}h_e) = \mathrm{d}g_{ve}$ because $\mathrm{d}g_{ve}$ is a Haar measure on $\mathrm{SL}(2,\mathbb{C})$. Thus the integral $\int_{\mathrm{SU}(2)} \mathrm{d}h_e$ operates as follows:

$$\int_{\mathrm{SU}(2)} \mathrm{d}h_e \prod_{f,e\subset f} h_e | J_f, \gamma J_f; J_f, M_{ef} \rangle. \tag{A2}$$

It only affects four states $|J_f,\gamma J_f; J_f, M_{ef}\rangle$ whose f contains the edge e. h_e leaves V_j invariant. $h_e|J_f,\gamma J_f; J_f, M_{ef}\rangle$ is essentially the same as $h_e|J_f, M_{ef}\rangle$. The integral $\int_{\mathrm{SU}(2)} \mathrm{d}h_e \prod_{f,e\subset f} h_e$ is a projector onto the invariant subspace of the tensor product $V_{J_1} \otimes \cdots \otimes V_{J_4}$. If four J_f's only give a trivial invariant subspace, the above integral vanishes identically for all M_{ef}. Indeed we consider the matrix element

$$\int_{\mathrm{SU}(2)} \mathrm{d}h \prod_{i=1}^{4} \langle J_i, N_i | h | J_i, M_i \rangle = \sum_{J=|J_1-J_2|}^{J_1+J_2} \sum_{K=|J_3-J_4|}^{J_3+J_4} C^{J_1,J_2;J}_{N_1,N_2;N_1+N_2} C^{J_1,J_2;J}_{M_1,M_2;M_1+M_2} C^{J_3,J_4;K}_{N_3,N_4;N_3+N_4} C^{J_3,J_4;K}_{M_3,M_4;M_3+M_4}$$
$$\times \int_{\mathrm{SU}(2)} \mathrm{d}h \langle J, N_1+N_2 | h | J, M_1+M_2 \rangle \langle K, N_3+N_4 | h | K, M_3+M_4 \rangle$$
$$= \sum_{J=|J_1-J_2|}^{J_1+J_2} \sum_{K=|J_3-J_4|}^{J_3+J_4} C^{J_1,J_2;J}_{N_1,N_2;N_1+N_2} C^{J_1,J_2;J}_{M_1,M_2;M_1+M_2} C^{J_3,J_4;K}_{N_3,N_4;N_3+N_4} C^{J_3,J_4;K}_{M_3,M_4;M_3+M_4}$$
$$\times \sum_{\tilde{J}=|J-K|}^{J+K} C^{J,K,\tilde{J}}_{N_1+N_2,N_3+N_4,\tilde{N}} C^{J,K,\tilde{J}}_{M_1+M_2,M_3+M_4,\tilde{M}} \int_{\mathrm{SU}(2)} \mathrm{d}h \langle \tilde{J}, \tilde{N} | h | \tilde{J}, \tilde{M} \rangle, \tag{A3}$$

where the last integral gives $\int_{\mathrm{SU}(2)} \mathrm{d}h \langle \tilde{J}, \tilde{N} | h | \tilde{J}, \tilde{M} \rangle = \delta_{\tilde{J},0} \delta_{\tilde{M},0} \delta_{\tilde{N},0}$. It constrains

$$J = K, \qquad N_1 + N_2 + N_3 + N_4 = 0, \qquad M_1 + M_2 + M_3 + M_4 = 0. \tag{A4}$$

(J_1, J_2, J), (J_3, J_4, K) satisfying triangle inequality and $J = K$ implies that there is a nontrivial invariant subspace. If $J \neq K$ the integral vanishes identically.

Note that in the above we have used the product formula of representation matrices,

$$\langle J_1, N_1 | h | J_1, M_1 \rangle \langle J_2, N_2 | h_e | J_2, M_2 \rangle = \sum_{J=|J_1-J_2|}^{J_1+J_2} C^{J_1,J_2;J}_{N_1,N_2;N_1+N_2} C^{J_1,J_2;J}_{M_1,M_2;M_1+M_2} \times \langle J, N_1+N_2 | h | J, M_1+M_2 \rangle,$$
$$\langle J_3, N_3 | h | J_3, M_3 \rangle \langle J_4, N_4 | h_e | J_4, M_4 \rangle = \sum_{K=|J_3-J_4|}^{J_3+J_4} C^{J_3,J_4;K}_{N_3,N_4;N_3+N_4} C^{J_3,J_4;K}_{M_3,M_4;M_3+M_4} \times \langle K, N_3+N_4 | h | K, M_3+M_4 \rangle,$$

where $C^{J_1,J_2;J}_{M_1,M_2;M_1+M_2}$ is the Clebsch-Gordan coefficient.

We can understand the spin sum $\sum_{J_f}$ as a sum over independent spins, while the integral in the summand imposes the constraint that J_f's should give nontrivial invariant subspace for four f's sharing the same edge e. For spins in $\sum_{J_f}$, which does not satisfy the constraint, their contributions vanish.

What we have done in the main text is simply interchange the spin sum and integral. Schematically,

$$\sum_J \dim(J) \int \mathrm{d}g \mathrm{d}z e^{S[J,g,z]} = \int \mathrm{d}g \mathrm{d}z \sum_J \dim(J) e^{S[J,g,z]}. \tag{A5}$$

This interchange can be justified by understanding $\sum_J$ as a finite sum, where a large-J cutoff is imposed. The cutoff may relate to the cosmological constant. As another independent justification of interchanging spin sum and integral, we focus on the compact neighborhood $\mathcal{N}_{\mathrm{Regge}}$ in the submanifold $\mathcal{M}_{\mathrm{Regge}}$ in the main discussion. $\mathcal{N}_{\mathrm{Regge}}$ only has finitely many spins (representatives). The spin sum in transverse directions has been regularized by a Gaussian weight with regulator δ, which exponentially decays at infinity as $\delta \neq 0$. It qualifies to interchange the transverse spin sum with the integral.

APPENDIX B: TRANSVERSE LATTICE PLANE

The lattice of all spins $\mathfrak{L}_J$ is isomorphic to $\mathbb{Z}^{N_f}$, where a lattice basis can be chosen to be $\vec{b}^I = (b^I_f)_{f=1}^{N_f}$ $(I = 1, \ldots, N_f)$, where $b^I_f = \delta^I_f$. We define a square matrix $B = (\vec{b}^1, \ldots, \vec{b}^{N_f})$ and denote $\mathfrak{L}_J \simeq \mathbb{Z}^{N_f} = \mathfrak{L}(B)$. Obviously B is an identity matrix.

A unimodular matrix is a matrix $U \in \mathbb{Z}^{N_f} \times \mathbb{Z}^{N_f}$ such that $\det U = 1$. Unimordular matrices relate equivalent lattice bases. Namely, columns of $B' = BU$ are a basis of $\mathbb{Z}^{N_f}$ equivalent to the standard basis $\vec{b}^I$. Here B' is simply U since B is an identity matrix. Thus columns of B' give a basis of $\mathbb{Z}^{N_f}$ if and only if it is unimodular.

The basis from B' is obtained from B via the following operations on columns (unimodular transformation): (1) adding the Ith column n times to the Jth column, (2) interchanging two columns, and (3) flipping the sign of a column.

The local neighborhood $\mathcal{N}_{\mathrm{Regge}} \subset \mathcal{M}_{\mathrm{Regge}}$ can be viewed approximately as an $(N_f - M)$-dimensional plane in $\mathbb{R}^{N_f}$. Among the original basis vectors $\vec{b}^I$, there should have been a set of vectors $\vec{b}^K$, say $K = 1, \ldots, M_0$, $M_0 \leq M$, that transverse nicely to $\mathcal{N}_{\mathrm{Regge}}$; i.e., $\vec{b}^K$ does not close to any tangent vector of $\mathcal{N}_{\mathrm{Regge}}$. If $M_0 < M$ and $\vec{b}^J$ is relatively close to a tangent vector of $\mathcal{N}_{\mathrm{Regge}}$, $\vec{b}^J$ can be improved by the unimodular transformation $\vec{b}^J \to \vec{b}^J + \sum_{K=1}^{M_0} n_K \vec{b}^K$,

$n_K \in \mathbb{Z}$, which gives a better transverse lattice vector. Iterating this procedure leads to the M transverse lattice vector, while the procedure corresponds to a unimodular matrix U, such that $B' = BU$ gives a new basis as its columns. The new basis contains M transverse basis vectors $\hat{e}^i$ that span $\mathfrak{L}_{\mathrm{NR}}$.

APPENDIX C: POISSON RESUMMATION AND EULER-MACLAURIN FORMULA

In the discussion of the spin sum in Sec. III, we have used the Poisson resummation formula to carry out the sum over t. The sum is of the following type,

$$\sum_{t\in\mathbb{Z}} e^{-\delta t^2 + t\Phi} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{-\delta t^2 + t(\Phi + 2\pi i k)} \mathrm{d}t, \tag{C1}$$

where the integral for each k is computed explicitly.

However, the sum can also be studied by the asymptotic expansion using the Euler-Maclaurin formula

$$\sum_{i=m}^{n} f(i) = \int_m^n f(x)\mathrm{d}x + \frac{f(n) + f(m)}{2} + \sum_{k=1}^{\lfloor p/2 \rfloor} \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(n) - f^{(2k-1)}(m)) + R, \tag{C2}$$

where B_{2k} is the kth Bernoulli number. The error term R depends on n, m, p, and f,

$$R = (-1)^{p+1} \int_m^n f^{(p)}(x) \frac{P_p(x)}{p!} \mathrm{d}x, \tag{C3}$$

where $P_p(x)$ is the periodic Bernoulli function. R satisfies the following bound:

$$|R| \leq \frac{2\zeta(p)}{(2\pi)^p} \int_m^n |f^{(p)}(x)|. \tag{C4}$$

Letting $f(t) = e^{-\delta t^2 + t\Phi}$ (exponentially decay at $t \to \pm\infty$), we obtain

$$\sum_{t\in\mathbb{Z}} e^{-\delta t^2 + t\Phi} = \int_{\mathbb{R}} e^{-\delta t^2 + t\Phi} \mathrm{d}t + R. \tag{C5}$$

The first term is the same as the $k = 0$ term in the Poisson resummation. However, since $f^{(p)} \sim \Phi^p e^{-\delta t^2 + t\Phi}$, the error term R is not negligible unless Φ is small. R essentially collects the sum of all $k \neq 0$ contributions in the Poisson resummation.

Viewing $\sum_{t\in\mathbb{Z}} e^{-\delta t^2 + t\Phi}$ as a function of Φ, it is clear that replacing the sum by the integral is only a local approximation of the function (the meaning of asymptotic expansion). $\sum_{t\in\mathbb{Z}} e^{-\delta t^2 + t\Phi}$ is periodic in $\Phi \to \Phi + 2\pi i$, while $\int e^{-\delta t^2 + t\Phi}\mathrm{d}t$ breaks the periodicity. The periodicity is not manifest in the Euler-Maclaurin expansion, but is manifest in the Poisson resummation formula.

The small Φ relates to the small $\gamma\varepsilon_f$ in Sec. IV. Thus the result with $k = 0$ in Sec. IV can be reproduced by using the Euler-Maclaurin expansion in the regime where R is negligible. The ER regime essentially requires that $\sum_{t\in\mathbb{Z}} e^{-\delta t^2 + t\Phi}$ can be approximated by $\int e^{-\delta t^2 + t\Phi}\mathrm{d}t$.

Similarly when one consider the large-J spin sum in spin-foam amplitude, one would like to rescale $J_f = \lambda j_f$ where $\Delta j_f = \frac{1}{2\lambda}$ ($\lambda \gg 1$) and understand the spin sum as the Riemann sum, i.e., schematically,

$$\sum_j e^{\lambda j F} = 2\lambda \sum_j \Delta j e^{\lambda j F} \sim 2\lambda \int \mathrm{d}j e^{\lambda j F} = 2 \int \mathrm{d}J e^{JF}.$$

However, because of the Euler-Maclaurin expansion Eq. (C2), we know that the above approximation may be valid only in the regime of small F. In general the error terms are not negligible. It can also be seen in the Euler-Maclaurin expansion of $\sum_j \Delta j f(j)$ where $f(j) = e^{\lambda j F}$. The λ^{-n} correction involves the nth derivative $f^{(n)}(j) = \lambda^n F^n e^{\lambda j F}$, which cancels λ^{-n}.

In the discussion of the variational principle of Regge action in Sec. IV, we have implicitly used the Euler-Maclaurin expansion for Eq. (28),

$$\sum_{\ell} e^{\frac{i}{\ell_P^2} S_{\mathrm{Regge}}[\ell] + \cdots} = \int \mathrm{d}\ell e^{\frac{i}{\ell_P^2} S_{\mathrm{Regge}}[\ell] + \cdots} + \text{error terms}. \tag{C6}$$

In general the error terms are not negligible as far as the full amplitude is concerned. However, as far as the equation of motion is concerned, the variational principle is applied to the first term, whose dominant contribution comes from solutions of the Regge equation.

APPENDIX D: ACTION AND ANGLES IN EUCLIDEAN EPRL AMPLITUDE

Considering an internal dual face f, at each large-J critical point (of a globally oriented nondegenerate geometry) in the Euclidean spin-foam amplitude, the loop holonomy along ∂f made by $g^{\pm}_{ve}$'s is written as

$$G^{\pm}_f(v) \equiv g^{\pm}_{ve} g^{\pm}_{ev_k} g^{\pm}_{v_k e_k} \cdots g^{\pm}_{e_1 v} = \exp(i\Phi^{\pm}_f \hat{X}^{\pm}_f(v)), \tag{D1}$$

where $\hat{X}_f(v) = (\hat{X}^+_f(v), \hat{X}^-_f(v))$ is the normalized bivector along the triangle f. $\Phi^{\pm}_f = \sum_v \phi^{\pm}_{eve'}$ where $\phi^{\pm}_{eve'}$ within each 4-simplex satisfies [24,44]

$$\phi^+_{eve'} - \phi^-_{eve'} = \mu(v)\Theta_f(v) \in [-\pi, \pi]. \tag{D2}$$

$\mu(v)$ relates to the orientation of the 4-simplex v, which we set to be globally $\mu(v) = -1$ for globally oriented spacetime geometries.

The action contributed by f evaluated at the critical point reads [6,24],

$$S_f = \sum_{\pm} 2iJ_f^{\pm}\Phi_f^{\pm} = iJ_f(\Phi_f^+ + \Phi_f^-) + i\gamma J_f(\Phi_f^+ - \Phi_f^-). \tag{D3}$$

Each $\Phi_f^{\pm}$ is defined modulo 2π: $\Phi_f^{\pm} \sim \Phi_f^{\pm} + 2\pi$. So $\Phi_f^+ \pm \Phi_f^- \sim \Phi_f^+ \pm \Phi_f^- + 4\pi$. However, simultaneous transformations $\Phi_f^+ \pm \Phi_f^- \to \Phi_f^+ \pm \Phi_f^- + 2\pi$ do not change e^{S_f} since $(1+\gamma)j_f \in \mathbb{Z}$. We can set the following range of angles:

$$\Phi_f^+ + \Phi_f^- \in [-2\pi, 2\pi], \qquad \Phi_f^+ - \Phi_f^- \in [-\pi, \pi]. \tag{D4}$$

Equation (D2) implies $\Phi_f^+ - \Phi_f^- = -\sum_{v\in f}\Theta_f(v)$ mod 4π. But simultaneous transformations can give

$$\Phi_f^+ - \Phi_f^- = 2\pi - \sum_{v\in f}\Theta_f(v) = \varepsilon_f, \tag{D5}$$

when we set $\varepsilon_f \in [-\pi, \pi]$ to include Regge geometries close to the continuum. $\varepsilon_f \in [-\pi, \pi]$ is made by choosing suitable $\mathcal{N}_{\mathrm{Regge}}$.

On the other hand, $G_f(e)$ represented in the vector representation $\hat{G}_f(e)$ reads [6,24]

$$\hat{G}_f(v) = \exp(*\hat{X}_f(v)\theta_f)\exp(\pi\eta_f\hat{X}_f(v)), \tag{D6}$$

where $\eta_f \in \{0,1\}$ labels two different types of critical points.

Lifting $\hat{G}_f(v) \in \mathrm{SO}(4)$ to $(G_f^+(v), G_f^-(v)) \in \mathrm{SU}(2)\times\mathrm{SU}(2)$ evaluates $\Phi_f^{\pm} = \frac{1}{2}(\eta_f\pi \pm \theta_f) - k_f\pi$, where $k_f \in \{0,1\}$ label lift ambiguities.

$$\varepsilon_f = \Phi_f^+ - \Phi_f^- = \theta_f, \quad \Phi_f^+ + \Phi_f^- = \pi\eta_f - 2k_f\pi. \tag{D7}$$

Equation (D4) implies

$$\eta_f - 2k_f \equiv n_f \in \{-1, 0, 1\}. \tag{D8}$$

There is canonical lift with $k_f = 0$ corresponding to the lift of SO(4) spin connection to $\mathrm{SU}(2)\times\mathrm{SU}(2)$. $\eta_f = k_f = 0$ indeed corresponds to a critical solution, which can be constructed by the Regge geometry with the canonical lift.[16] Other lifts $k^{\pm} \neq 0$ and $\eta_f \neq 0$ may correspond to different critical solutions.[17]

The action is expressed as

$$S_f = iJ_f[\gamma\varepsilon_f + n_f\pi]. \tag{D9}$$

Therefore, repeating the analysis in Sec. IV leads to the replacement

$$\gamma\varepsilon_f \to \gamma\varepsilon_f + n_f\pi \tag{D10}$$

in Eqs. (32) and (35). After the replacement Eq. (35) gives disjoint sectors of geometries whose $\gamma\varepsilon_f$ are close to $-n_f\pi$. The only sector having geometries approximating the continuum is the one with all $n_f = 0$. Other sectors are suppressed in the amplitude by suitably choosing $\mathcal{N}_{\mathrm{Regge}}$.

[16] The construction of critical solutions from arbitrary Regge geometries can be done locally in each 4-simplex as explained in [44]; see also [25] in the Lorentzian signature.

[17] One should also take into account the gauge invariance $g_{ve}^{\pm} \to \kappa_{ve} g_{ve}^{\pm}$ ($\kappa_{ve} = \pm 1$) of spin-foam action, which removes some lift ambiguities.

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