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Nonlinear stability of a brane wormhole

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We analytically study the nonlinear stability of a spherically symmetric wormhole supported by an infinitesimally thin brane of negative tension, which has been devised by Barcelo and Visser. We consider a situation in which a thin spherical shell composed of dust falls into an initially static wormhole; the dust shell plays the role of the nonlinear disturbance. The self-gravity of the falling dust shell is completely taken into account through Israel's formalism of the metric junction. When the dust shell goes through the wormhole, it necessarily collides with the brane supporting the wormhole. We assume the interaction between these shells is only gravity and show the condition under which the wormhole stably persists after the dust shell goes through it.

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I. INTRODUCTION

A wormhole is a fascinating spacetime structure by which shortcut trips or travels to disconnected worlds are possible. Active theoretical studies of this subject began with influential papers written by Morris, Thorne, and Yurtsever [1] and Morris and Thorne [2]. The earlier works were shown in the book written by Visser [3] and review paper by Lobo [4].

We should note that it is not a trivial task to define a wormhole in a mathematically rigorous and physically reasonable manner, although we may easily find a wormhole structure in each individual case. Hayward gave an elegant definition of the wormhole as an extension of the "black hole" defined by using a trapping horizon [5,6]. Recently, a more sophisticated definition has been proposed by Tomikawa, Izumi, and Shiromizu, and showed that the violation of the null energy condition is a necessary condition for the existence of a traversable stationary wormhole in the framework of general relativity, where the null energy condition means that $T_{\mu\nu}k^\mu k^\nu \geq 0$ holds for any null vector k^μ [7].[1]

But where does the exotic matter violating the null energy condition appear? In Refs. [1,2], the possibilities of quantum effects were discussed. Alternatively, such an exotic matter is often discussed in the context of cosmology. The phantom energy—whose pressure p is given through the equation of state $p = w\rho$ with $w < -1$ and positive energy density, $\rho > 0$—does not satisfy the null energy condition, and a few researchers showed the possibility of a wormhole supported by phantom-like matter [10–12]. Recently, theoretical studies from an observational point of view of a compact object made of exotic matter (possibly wormholes) were also reported [13–17], and observational constraints were reported by Takahashi and Asada [18].

It is important to study the stability of wormholes in order to determine whether they are really traversable. The stability against linear perturbations is a necessary condition for a traversable wormhole, but it is insufficient. The investigation of a nonlinear dynamical situation is necessary, and there have been a few studies in this direction [19–22]. In this paper, we also study the nonlinear stability of a wormhole in a similar way as in Ref. [22].

In Ref. [22], the wormhole was assumed to be spherically symmetric and supported by an infinitesimally thin spherical shell. The largest merit of a spherical thin shell wormhole is its finite number of dynamical degrees of freedom; hence, we can analyze this model analytically even in highly dynamical cases. The thin shell wormhole was first devised by Visser [23], and its stability against linear perturbations was investigated by Poisson and Visser [24]. Recently, the linear stability of the thin shell wormhole in a more general situation was investigated by Garcia, Lobo, and Visser [25].

We assume that the spherical shell supporting the wormhole is a brane whose equation of state is $P = -\sigma$, where P is the tangential pressure and σ is the energy per unit area. Furthermore, we assume the existence of a spherically symmetric electric field. This wormhole model was devised by Barcelo and Visser [26], and its higher-dimensional extension was studied by Kokubu and Harada [27]. The brane wormhole has a positive gravitational mass; this is an important difference between the present study and that in Ref. [22] in which the gravitational mass of the wormhole was negative. The sign of the mass will be significant for the stability, since a positive mass may cause gravitational collapse, leading to the formation of a black

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[1] Several researchers have pointed out the intriguing fact that stationary wormhole solutions exist even without the violation of the null energy condition, if they have a nonvanishing NUT charge which causes closed timelike curves [8,9].

hole. It is worth noticing that the positivity of the mass avoids the observational constraint given in Ref. [18]. Then, as in Ref. [22], we consider a situation in which an infinitesimally thin spherical dust shell concentric with the wormhole falls into the wormhole, or in other words, plays the role of a nonlinear disturbance in the wormhole spacetime. These spherical shells are treated by Israel's formulation of the metric junction [28]. When the dust shell goes through the wormhole, it necessarily collides with the brane supporting the wormhole. The collision between thin shells has already been studied by several researchers [29–31], and we follow them. Then, we show the condition under which the wormhole persists after the passage of a spherical shell.

This paper is organized as follows. In Sec. II, we derive the equations of motion for the brane supporting the wormhole and the spherical dust shell falling into the wormhole, in accordance with Israel's formalism of the metric junction. In Sec. III, we derive a static solution of the wormhole supported by the brane, which is the initial condition. In Sec. IV, we investigate the condition that a dust shell freely falls from infinity and reaches the wormhole throat. In Sec. V, we study the motion of the shells and the change in the gravitational mass of the wormhole after collision. In Sec. VI, we show the condition under which the wormhole persists after the dust shell goes through it. Some complicated manipulations and discussions on this subject are given separately in the Appendix. Section VII is devoted to a summary and discussion.

In this paper, we adopt the geometrized units in which the speed of light and Newton's gravitational constant are one. However, if necessary, they will be recovered.

II. EQUATIONS OF MOTION FOR SPHERICAL SHELLS

We consider two concentric spherical shells which are infinitesimally thin. As mentioned in the previous section, one is the brane supporting the wormhole and the other is composed of the dust which will cause a nonlinear perturbation for the wormhole.

The trajectories of these shells in the spacetime are timelike hypersurfaces: the one formed by the brane is denoted by Σ_1, and the other formed by the dust shell is denoted by Σ_2. These hypersurfaces divide the spacetime into three domains denoted by D_1, D_2, and D_3, respectively; Σ_1 divides the spacetime into D_1 and D_2, whereas Σ_2 divides the spacetime into D_2 and D_3. We also call Σ_1 and Σ_2 shell-1 and shell-2, respectively. This configuration is depicted in Fig. 1.

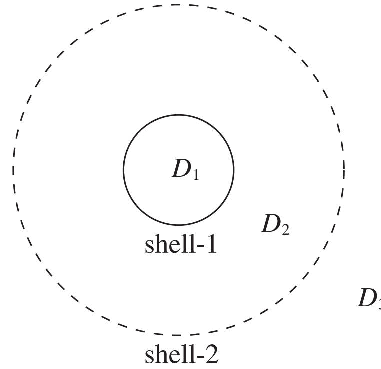

FIG. 1. The initial configuration is depicted.

The geometry of the domain D_i ($i = 1, 2, 3$) is assumed to be described by the Reissner-Nordström solution: the infinitesimal world interval is given by

$$ds^2 = -f_i(r)dt_i^2 + \frac{1}{f_i(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

with

$$f_i(r) = 1 - \frac{2M_i}{r} + \frac{Q_i^2}{r^2}, \quad (2)$$

where M_i and Q_i are the mass parameter and the charge parameter, respectively, whereas the gauge 1-form is given by

$$A_\mu = \left(-\frac{Q_i}{r}, 0, 0, 0\right). \quad (3)$$

We should note that the coordinate t_i is not continuous at the shells, whereas r, θ, and ϕ are everywhere continuous.

If $M_i > |Q_i|$ holds, two horizons can exist, and their locations are given by real roots of the algebraic equation $f_i(r) = 0$:

$$r = r_{i\pm} := M_i \pm \sqrt{M_i^2 - Q_i^2}. \quad (4)$$

If $M_i = |Q_i|$, there can be one degenerate horizon at $r = M_i$. If $M_i < |Q_i|$ holds, the roots of $f_i(r) = 0$ are complex or real negative roots, and hence there is no horizon.

Since finite energy and finite momentum are concentrated on the infinitesimally thin domains, the stress-energy tensor diverges on these shells. This means that these shells are categorized into the so-called curvature polynomial singularity through the Einstein equations [32]. Even though Σ_A ($A = 1, 2$) are spacetime singularities, we can derive an equation of motion for each spherical shell that is consistent with the Einstein equations using the so-called Israel's formalism, since each of these singularities is so weak that its intrinsic metric exists and the extrinsic curvature defined on each side of Σ_A is finite.

We cover the neighborhood of the singular hypersurface Σ_A by a Gaussian normal coordinate λ, where $\partial/\partial\lambda$ is a unit vector normal to Σ_A and is directed from D_A to D_{A+1}. Then, the sufficient condition to apply Israel's formalism is that the stress-energy tensor is written in the form

$$T_{\mu\nu} = S_{\mu\nu}\delta(\lambda - \lambda_A), \quad (5)$$

where Σ_A is located at $\lambda = \lambda_A$, $\delta(x)$ is Dirac's delta function, and $S_{\mu\nu}$ is the surface stress-energy tensor on Σ_A.

The junction condition of the metric tensor is obtained as follows. We impose that the metric tensor $g_{\mu\nu}$ is continuous even at Σ_A. Hereafter, n^μ denotes the unit normal vector of Σ_A, instead of $\partial/\partial\lambda$. The intrinsic metric of Σ_A is given by

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \tag{6}$$

and the extrinsic curvature is defined as

$$K^{(i)}_{\mu\nu} = -h^\alpha{}_\mu h^\beta{}_\nu \nabla^{(i)}_\alpha n_\beta, \tag{7}$$

where $\nabla^{(i)}_\alpha$ is the covariant derivative with respect to the metric in the domain D_i. This extrinsic curvature describes how Σ_A is embedded into the domain D_i. In accordance with Israel's formalism, the Einstein equations lead to

$$K^{(A+1)}_{\mu\nu} - K^{(A)}_{\mu\nu} = 8\pi\left(S_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\mathrm{tr}S\right), \tag{8}$$

where tr$S$ is the trace of $S_{\mu\nu}$. Equation (8) gives us the condition of the metric junction.

By the spherical symmetry of the system, the surface stress-energy tensors of the shells should be of the perfect-fluid type:

$$S_{\mu\nu} = \sigma_A u_\mu u_\nu + P_A(h_{\mu\nu} + u_\mu u_\nu), \tag{9}$$

where σ_A and P_A are the energy per unit area and the pressure on Σ_A, respectively, and u^μ is the 4-velocity.

Due to the spherical symmetry, the motion of shell-A is of the form $t_i = T_{A,i}(\tau)$ and $r = R_A(\tau)$, where $i = A$ or $i = A+1$, that is to say, i represents one of two domains divided by shell-A, and τ is the proper time of the shell. The 4-velocity is given by

$$u^\mu = (\dot{T}_{A,i}, \dot{R}_A, 0, 0), \tag{10}$$

where a dot denotes a derivative with respect to τ. Then, n_μ is given by

$$n_\mu = (-\dot{R}_A, \dot{T}_{A,i}, 0, 0). \tag{11}$$

Together with u^μ and n^μ, the following unit vectors form an orthonormal frame:

$$\hat{\theta}^\mu = \left(0, 0, \frac{1}{r}, 0\right), \tag{12}$$

$$\hat{\phi}^\mu = \left(0, 0, 0, \frac{1}{r\sin\theta}\right). \tag{13}$$

The extrinsic curvature is obtained as

$$K^{(i)}_{\mu\nu} u^\mu u^\nu = \frac{1}{f_i \dot{T}_{A,i}}\left(\ddot{R}_A + \frac{f'_i(R_A)}{2}\right), \tag{14}$$

$$K^{(i)}_{\mu\nu}\hat{\theta}^\mu\hat{\theta}^\nu = K^{(i)}_{\mu\nu}\hat{\phi}^\mu\hat{\phi}^\nu = -n^\mu\partial_\mu \ln r|_{D_i} = -\frac{f_i(R_A)}{R_A}\dot{T}_{A,i} \tag{15}$$

and the other components vanish, where a prime denotes a derivative with respect to its argument. By the normalization condition $u^\mu u_\mu = -1$, we have

$$\dot{T}_{A,i} = \pm\frac{1}{f_i(R_A)}\sqrt{\dot{R}_A^2 + f_i(R_A)}. \tag{16}$$

Substituting the above equation into Eq. (15), we have

$$K^{(i)}_{\mu\nu}\hat{\theta}^\mu\hat{\theta}^\nu = \mp\frac{1}{R_A}\sqrt{\dot{R}_A^2 + f_i(R_A)}. \tag{17}$$

From the $u-u$ component of Eq. (8), we obtain the following relations:

$$\frac{d(\sigma_A R_A^2)}{d\tau} + P_A\frac{dR_A^2}{d\tau} = 0. \tag{18}$$

In the case of the equation of state

$$P_A = w_A\sigma_A, \tag{19}$$

where w_A is constant, by substituting Eq. (19) into Eq. (18), we obtain

$$\sigma_A \propto R_A^{-2(w_A+1)}. \tag{20}$$

A. Shell-1: The brane

As mentioned, we assume that shell-1 is a brane, i.e.,

$$w_1 = -1.$$

Without loss of generality, we assume $Q_2 \geq 0$. Furthermore, we focus on the case of

$$Q_2 = |Q_1| = Q \geq 0.$$

Since the electric charge of shell-1 is equal to $Q_2 - Q_1$, the electric charge of shell-1 is zero in the case of $Q_2 = Q_1$, whereas the electric charge of shell-1 may not vanish in the case of $Q_2 = -Q_1$. As will be shown later, the results in both cases are identical to each other.

By this assumption, the union of the domains D_1 and D_2 should have a wormhole structure due to shell-1. This means that $n^a\partial_a \ln r|_{D_1} < 0$ and $n^a\partial_a \ln r|_{D_2} > 0$ (see Fig. 2), and we have

$$K^{(1)}_{\mu\nu}\hat{\theta}^\mu\hat{\theta}^\nu = +\frac{1}{R_1}\sqrt{\dot{R}_1^2 + f_1} \quad \text{and}$$
$$K^{(2)}_{\mu\nu}\hat{\theta}^\mu\hat{\theta}^\nu = -\frac{1}{R_1}\sqrt{\dot{R}_1^2 + f_2}. \tag{21}$$

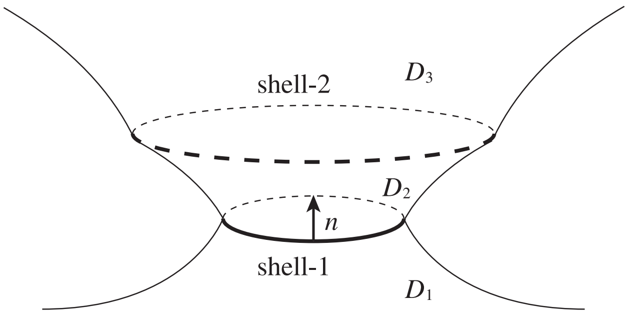

FIG. 2. Shell-1 forms the wormhole structure.

Here, note that Eq. (21) implies $\dot{T}_{1,1}$ is negative, whereas $\dot{T}_{1,2}$ is positive. Hence, the direction of the time coordinate basis vector in D_1 is opposite that in D_2.

From the $\theta-\theta$ component of Eq. (8), we obtain the following relations:

$$\sqrt{\dot{R}_1^2+f_2(R_1)}+\sqrt{\dot{R}_1^2+f_1(R_1)}=-4\pi\sigma_1 R_1. \tag{22}$$

Equation (22) is satisfied only if σ_1 is negative, and hence we assume that this is the case. From Eq. (20), we have

$$\sigma_1=-\frac{\mu}{4\pi}, \tag{23}$$

where μ is a positive constant, which hereafter we call the stress constant.

Let us rewrite Eq. (22) in the form of the energy equation for shell-1. First, we write it in the form

$$\sqrt{\dot{R}_1^2+f_2(R_1)}=-\sqrt{\dot{R}_1^2+f_1(R_1)}+\mu R_1, \tag{24}$$

and then take the square of both sides of the above equation to obtain

$$\sqrt{\dot{R}_1^2+f_1(R_1)}=\frac{1}{2\mu R_1}[f_1(R_1)-f_2(R_1)+(\mu R_1)^2]. \tag{25}$$

By taking the square of both sides of the above equation, we have

$$\dot{R}_1^2+V_1(R_1)=0, \tag{26}$$

where

$$V_1(r)=1-\frac{1}{r^4}\left(\frac{M_2-M_1}{\mu}\right)^2-\frac{M_1+M_2}{r}+\frac{Q^2}{r^2}-\left(\frac{\mu}{2}\right)^2 r^2. \tag{27}$$

Equation (26) is regarded as the energy equation for shell-1. The function V_1 corresponds to the effective potential. In the allowed domain for the motion of shell-1, the inequality $V_1\leq 0$ should hold, but this inequality is not a sufficient condition for the allowed region.

The left-hand side of Eq. (24) is non-negative, and hence the right-hand side of it should also be non-negative. Then, substituting Eq. (25) into the right-hand side of Eq. (24), we have

$$0\leq-\sqrt{\dot{R}_1^2+f_1(R_1)}+\mu R_1=\frac{\mu R_1}{2}-\frac{M_2-M_1}{\mu R_1^2}. \tag{28}$$

Further manipulation leads to

$$R_1^3\geq\frac{2}{\mu^2}(M_2-M_1). \tag{29}$$

By a similar argument, we obtain

$$-\sqrt{\dot{R}_1^2+f_2(R_1)}+\mu R_1\geq 0. \tag{30}$$

Then, by a similar procedure, we have

$$R_1^3\geq\frac{2}{\mu^2}(M_1-M_2). \tag{31}$$

Hence, we have the following constraint:

$$R_1\geq\left(\frac{2|M_1-M_2|}{\mu^2}\right)^{\frac{1}{3}}. \tag{32}$$

In order to find the allowed domain for the motion of shell-1, we need to take into account the constraint (32) in addition to the condition $V_1\leq 0$.

B. Shell-2: The dust shell

As mentioned, we assume that shell-2 is composed of nonexotic dust, i.e., $w_2=0$ and $\sigma_2>0$. The proper mass of shell-2 is defined as

$$m_2\equiv 4\pi\sigma_2 R_2^2. \tag{33}$$

We find that m_2 is constant by Eq. (20) and positive by $\sigma_2>0$. We also assume

$$Q_3=Q_2=Q.$$

This assumption means that shell-2 is electrically neutral.

The wormhole structure does not exist around shell-2 due to $\sigma_2>0$. Hence, the extrinsic curvature of shell-2 is given by

$$K^{(2)}_{\mu\nu}\hat{\theta}^\mu\hat{\theta}^\nu=-\frac{1}{R_2}\sqrt{\dot{R}_2^2+f_2(R_2)}\quad\text{and}$$
$$K^{(3)}_{\mu\nu}\hat{\theta}^\mu\hat{\theta}^\nu=-\frac{1}{R_2}\sqrt{\dot{R}_2^2+f_3(R_2)}. \tag{34}$$

By using the above result, the θ-θ component of the junction condition leads to

$$\sqrt{\dot{R}_2^2+f_3(R_2)}-\sqrt{\dot{R}_2^2+f_2(R_2)}=-\frac{m_2}{R_2}. \quad (35)$$

Since m_2 is positive, we find from the above equation that $f_2(R_2)>f_3(R_2)$, or equivalently, $M_3>M_2$. From Eq. (35), we have

$$\sqrt{\dot{R}_2^2+f_3(R_2)}=\sqrt{\dot{R}_2^2+f_2(R_2)}-\frac{m_2}{R_2}. \quad (36)$$

By taking the square of both sides of Eq. (36), we have

$$\sqrt{\dot{R}_2^2+f_2(R_2)}=\frac{M_3-M_2}{m_2}+\frac{m_2}{2R_2}. \quad (37)$$

By taking the square of both sides of Eq. (37), we obtain an energy equation for shell-2,

$$\dot{R}_2^2+V_2(R_2)=0, \quad (38)$$

where

$$V_2(r)=1-E^2-\frac{2M_{\rm d}}{r}+\frac{Q_1^2}{r^2}-\left(\frac{m_2}{2r}\right)^2, \quad (39)$$

with

$$E\equiv\frac{M_3-M_2}{m_2} \quad \text{and} \quad M_{\rm d}\equiv\frac{1}{2}(M_2+M_3). \quad (40)$$

Note that E is a constant which corresponds to the specific energy of shell-2.

In the allowed domain for the motion of shell-2, the effective potential V_2 should be nonpositive. But, as in the case of shell-1, it is not a sufficient condition for the allowed domain. Since the left-hand side of Eq. (36) is non-negative, the following inequality should be satisfied:

$$\sqrt{\dot{R}_2^2+f_2(R_2)}-\frac{m_2}{R_2}\geq 0. \quad (41)$$

Substituting Eq. (37) into the left-hand side of Eq. (41), we have

$$R_2\geq R_{\rm b}\coloneqq\frac{m_2^2}{2(M_3-M_2)}. \quad (42)$$

The above inequality should also be taken into account as a condition for the allowed domain.

As mentioned, in the case of $M_3\geq Q$, the horizon may appear in the domain D_3; when the radius R_2 of shell-2 becomes smaller than or equal to

$$R_{\rm H}\coloneqq r_{3+}=M_3+\sqrt{M_3^2-Q^2}, \quad (43)$$

a black hole including both the wormhole and shell-2 forms. Here it should be noted that Eq. (42) is derived by assuming that u_2^t is positive, but u_2^t can change its sign within the black hole ($R_2<R_{\rm H}$). Hence, if $R_{\rm b}$ is smaller than $R_{\rm H}$, Eq. (42) looses its validity, and thus the allowed domain for the motion of shell-2 is solely determined by the condition $V_2\leq 0$. The allowed domain for the motion of shell-2 satisfies $V_2\leq 0$ as well as Eq. (42) only if $R_{\rm b}\geq R_{\rm H}$.

III. STATIC WORMHOLE SOLUTION

We consider a situation in which the brane supporting the wormhole is initially static and located at $r=a$. Furthermore, we assume that the wormhole is initially mirror symmetric with respect to $r=a$, i.e., $f_1(r)=f_2(r)=f(r)$, or equivalently, $M_1=M_2=M_{\rm w}$. In order for shell-1 to be in a static configuration, its areal radius $R_1=a$ should satisfy $V_1(a)=0=V_1'(a)$. Furthermore, $V_1''(a)>0$ should hold so that this structure is stable.

The condition $V_1(a)=0$ leads to the following relation between the stress constant μ and the areal radius a:

$$\mu^2=\frac{4}{a^2}f(a). \quad (44)$$

Together with the above condition, the condition $V_1'(a)=0$ leads to

$$a^2-3M_{\rm w}a+2Q^2=0. \quad (45)$$

The roots of the above equation are given by

$$a=a_\pm\coloneqq\frac{1}{2}\left(3M_{\rm w}\pm\sqrt{9M_{\rm w}^2-8Q^2}\right).$$

The following inequality should hold so that a is real and positive:

$$M_{\rm w}\geq\frac{2\sqrt{2}}{3}Q. \quad (46)$$

Equation (46) implies that $M_{\rm w}$ is non-negative.

Together with Eqs. (44) and (45), the condition $V_1''(a)>0$ leads to

$$a<\sqrt{2}Q. \quad (47)$$

The above condition implies that the charge parameter Q_i cannot vanish so that the areal radius a is positive. Since we have

$$a_\pm-\sqrt{2}Q=\frac{1}{2}\sqrt{3M_{\rm w}-2\sqrt{2}Q}\left(\sqrt{3M_{\rm w}-2\sqrt{2}Q}\pm\sqrt{3M_{\rm w}+2\sqrt{2}Q}\right),$$

$a=a_+$ does not satisfy Eq. (47), but $a=a_-$ does.

Since μ^2 should be positive, Eq. (44) implies that $f(a_-)>0$ should be satisfied. By using Eq. (45), the condition $f(a_-)>0$ leads to

$$3M_{\rm w}^2 - 2Q^2 > M_{\rm w}\sqrt{9M_{\rm w}^2 - 8Q^2}.$$

By taking the square of both sides of the above inequality, we obtain $M_{\rm w} < Q$.

To summarize this section, the areal radius a and the stress constant μ of the static wormhole are given as functions of $M_{\rm w}$ and Q;

$$a = \frac{1}{2}(3M_{\rm w} - \sqrt{9M_{\rm w}^2 - 8Q^2}), \tag{48}$$

$$\mu = \frac{a}{2}\sqrt{1 - \frac{2M_{\rm w}}{a} + \frac{Q^2}{a^2}}, \tag{49}$$

with the constraint

$$M_{\rm w} < Q < \frac{3}{2\sqrt{2}}M_{\rm w}. \tag{50}$$

Equations (48) and (50) lead to

$$M_{\rm w} < a < \frac{3}{2}M_{\rm w}. \tag{51}$$

IV. CAN SHELL-2 REACH THE WORMHOLE THROAT?

We consider the condition that shell-2 enters the wormhole supported by shell-1. The allowed domain for the motion of shell-2 is determined by the conditions (42) and $V_2 \le 0$. Shell-2 is assumed to come from spatial infinity. By this assumption, $E \ge 1$ should be satisfied so that $V_2(r) < 0$ for sufficiently large r.

A. The case of $Q \le m_2/2$

In this case, $V_2(r)$ is negative for $r \ge a$. It should be noted that, in this case,

$$M_3 = M_2 + Em_2 > M_{\rm w} + 2EQ > Q$$

is satisfied, and hence $R_{\rm H}$ is real and positive. As explained in the paragraph including Eq. (43), since we have

$$R_{\rm H} - R_{\rm b} = M_3 + \sqrt{M_3^2 - Q^2} - \frac{m_2^2}{2(M_3 - M_2)}$$
$$= M_{\rm w} + \frac{m_2(2E^2 - 1)}{2E} + \sqrt{M_3^2 - Q^2} > 0,$$

the allowed domain for the motion of shell-2 is only determined by the condition $V_2 < 0$, and hence shell-2 can reach the wormhole throat $r = a$ in this case.

B. The case of $Q > m_2/2$

We consider the cases of $E = 1$ and $E > 1$ separately.

1. The case of $E = 1$

In this case, the positive real root of $V_2(R_{\rm z}) = 0$ is given by

$$R_{\rm z} = \frac{4Q^2 - m_2^2}{4(2M_{\rm w} + m_2)}.$$

The allowed domain for the motion of shell-2 is $R_2 \ge R_{\rm z}$. We have

$$a - R_{\rm z} = \frac{1}{2}(3M_{\rm w} - \sqrt{9M_{\rm w}^2 - 8Q^2}) - \frac{4Q^2 - m_2^2}{4(2M_{\rm w} + m_2)}$$
$$= \frac{1}{2M_{\rm w} + m_2}\left[\left(\sqrt{9M_{\rm w}^2 - 8Q^2} - \frac{2M_{\rm w} + m_2}{4}\right)^2 - \frac{25}{4}M_{\rm w}^2 + 7Q^2 + \frac{3}{16}m_2^2 + \frac{5}{4}M_{\rm w}m_2\right]$$
$$> \frac{1}{2M_{\rm w} + m_2}\left[\left(\sqrt{9M_{\rm w}^2 - 8Q^2} - \frac{2M_{\rm w} + m_2}{4}\right)^2 + \frac{3}{4}Q^2 + \frac{3}{16}m_2^2 + \frac{5}{4}M_{\rm w}m_2\right] > 0, \tag{52}$$

where we have used $M_{\rm w} < Q$ in Eq. (50). The above inequality implies that shell-2 can reach the wormhole throat $r = a$.

2. The case of $E > 1$

In this case, the positive real root of $V_2(R_{\rm z}) = 0$ is given by

$$R_{\rm z} = \frac{1}{E^2 - 1}\left[-M_{\rm w} - \frac{m_2 E}{2} + \sqrt{\left(M_{\rm w} + \frac{m_2 E}{2}\right)^2 + (E^2 - 1)\left(Q^2 - \frac{m_2^2}{4}\right)}\right].$$

The allowed domain for the motion of shell-2 is $R \ge R_{\rm z}$. We can easily see that $R_{\rm z} \to 0$ and so $a > R_{\rm z}$, in the limit of $E \to \infty$. The derivative of $R_{\rm z}$ with respect to E with $M_{\rm w}$, m_2, and Q fixed is given by

$$\frac{\partial R_{\rm z}}{\partial E} = \frac{X - Y}{(E^2 - 1)^2\sqrt{(M_{\rm w} + \frac{1}{2}m_2 E)^2 + (E^2 - 1)(Q^2 - \frac{m_2^2}{4})}}, \tag{53}$$

where

$$X = \left(\frac{1}{2}m_2E^2 + \frac{1}{2}m_2 + 2M_wE\right)\sqrt{\left(M_w + \frac{1}{2}m_2E\right)^2 + (E^2-1)\left(Q^2 - \frac{m_2^2}{4}\right)}, \tag{54}$$

$$Y = Q^2E^3 + \frac{3}{2}M_wm_2E^2 - \left(Q^2 - \frac{1}{2}m_2^2 - 2M_w^2\right)E + \frac{1}{2}M_wm_2. \tag{55}$$

It is not so difficult to see that Y is positive for $E \geq 1$, whereas X is trivially positive. Since we have

$$Y^2 - X^2 = \left(Q^2 - \frac{1}{4}m_2^2\right) \times \left(Q^2E^2 + Mm_2E + \frac{1}{4}m_2^2\right)(E^2-1)^2 > 0,$$

we find

$$\frac{\partial R_z}{\partial E} < 0$$

for $E > 1$. As a result, since (as already shown) $a > R_z$ holds for both $E = 1$ and $E \to \infty$, we have $a > R_z$ even for $E > 1$.

In the case of $M_3 \geq Q$, as already shown in the case of $Q < m_2/2$, since $R_b < R_H$ holds, the allowed domain for the motion of shell-2 is only determined by the condition $V_2 \leq 0$. Hence, shell-2 can reach the wormhole throat $r = a$.

In the case of $M_3 < Q$, or equivalently, $M_w < Q - m_2E$, no horizon forms in D_3, and hence we need to study whether R_z is larger than R_b. In the case of $E = 1$, we have

$$R_z - R_b = \frac{4Q^2 - 4m_2M_w - 3m_2^2}{4(2M_w + m_2)} > \frac{(2Q - m_2)^2}{4(2M_w + m_2)} > 0.$$

In the case of $E > 1$, we have

$$\begin{aligned} R_z - R_b &= \frac{1}{2E(E^2-1)}\left[-2ME + m_2 - 2m_2E^2 + 2E\sqrt{\left(M_w + \frac{m_2E}{2}\right)^2 + (E^2-1)\left(Q^2 - \frac{m_2^2}{4}\right)}\right] \\ &> \frac{1}{2E(E^2-1)}\left[-2ME + m_2 - 2m_2E^2 + 2E\sqrt{\left(M_w + \frac{m_2E}{2}\right)^2 + (E^2-1)\left\{(M_w + m_2E)^2 - \frac{m_2^2}{4}\right\}}\right] \\ &= \frac{1}{2E(E^2-1)}[2ME(E-1) + m_2(2E^2-1)(E-1)] > 0. \end{aligned} \tag{56}$$

Since now we have $R_b < R_z$ for $E \geq 1$, Eq. (42) gives no additional constraint on the allowed domain for the motion of shell-2. As a result, shell-2 can also reach the wormhole throat $r = a$ in $M_3 < Q$.

To summarize this section, shell-2 reaches the wormhole throat $r = a$ from infinity if it moves inward initially. This result is different from the case of the wormhole with a negative mass studied in Ref. [22]: in the negative-mass case, E should be larger than unity, or in other words, a larger initial ingoing velocity than the present positive-mass case is necessary for shell-2 to reach the wormhole throat, since the gravitational force produced by a wormhole with a negative mass is repulsive.

V. COLLISION BETWEEN THE SHELLS

When shell-2 goes through the wormhole, it necessarily collides with shell-1 located at the wormhole throat $r = a$. This situation is shown in Fig. 3. In this section, we show how the mass parameter in the domain between the shells changes due to the collision.

We assume that the interaction between these shells is gravity only, or in other words, these shells merely go through each other: Both the 4-velocity and the proper mass $4\pi\sigma_A R_A^2$ of each shell are continuous at the collision event.

In the domain D_2, we may introduce two kinds of orthonormal frames $(u_A^\alpha, n_A^\alpha, \hat{\theta}^\alpha, \hat{\phi}^\alpha)$ at the collision event, where $A = 1, 2$. We can express the 4-velocity u_1^α of shell-1 by using the orthonormal frame $(u_2^\alpha, n_2^\alpha, \theta^\alpha, \phi^\alpha)$, and the converse is also possible:

$$\begin{aligned} u_1^\alpha &= [-u_2^\alpha u_{2\beta} + n_2^\alpha n_{2\beta} + \hat{\theta}^\alpha\hat{\theta}_\beta + \hat{\phi}^\alpha\hat{\phi}_\beta] \\ u_1^\beta &= -(u_1^\beta u_{2\beta})u_2^\alpha + (u_1^\beta n_{2\beta})n_2^\alpha, \end{aligned} \tag{57}$$

$$\begin{aligned} u_2^\alpha &= [-u_1^\alpha u_{1\beta} + n_1^\alpha n_{1\beta} + \hat{\theta}^\alpha\hat{\theta}_\beta + \hat{\phi}^\alpha\phi_\beta] \\ u_2^\beta &= -(u_2^\beta u_{1\beta})u_1^\alpha + (u_2^\beta n_{1\beta})n_1^\alpha. \end{aligned} \tag{58}$$

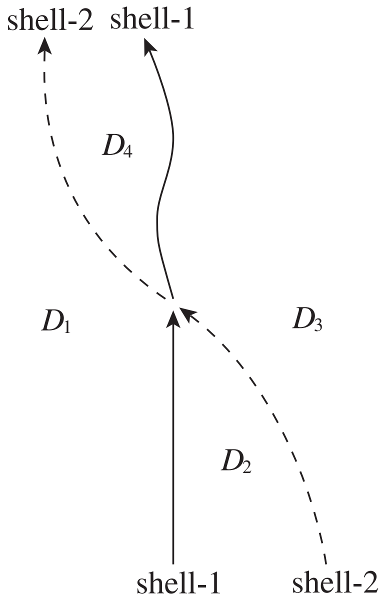

FIG. 3. Shell-1 supporting the wormhole is initially static. Shell-2 falls into the wormhole and collides with shell-1. The interaction between these shells is assumed to be gravity only: the shells merely go through each other.

The components of u_A^α and n_A^α with respect to the coordinate basis in D_2 are given by

$$u_1^\alpha = \left(\frac{1}{\sqrt{f}}, 0, 0, 0\right), \tag{59}$$

$$n_1^\alpha = (0, \sqrt{f}, 0, 0), \tag{60}$$

$$u_2^\alpha = \left(\frac{1}{f}\sqrt{\dot{R}_2^2 + f}, \dot{R}_2, 0, 0\right), \tag{61}$$

$$n_2^\alpha = \left(\frac{\dot{R}_2}{f}, \sqrt{\dot{R}_2^2 + f}, 0, 0\right), \tag{62}$$

where $f = f(a)$. Hence, we have

$$u_1^\beta u_{2\beta} = u_2^\beta u_{1\beta} = -\sqrt{\frac{\dot{R}_2^2}{f} + 1}, \tag{63}$$

$$u_1^\beta n_{2\beta} = -\frac{\dot{R}_2}{\sqrt{f}}, \tag{64}$$

$$u_2^\beta n_{1\beta} = \frac{\dot{R}_2}{\sqrt{f}}. \tag{65}$$

A. Shell-1 after the collision

The orthonormal frame $(u_2^\alpha, n_2^\alpha, \hat{\theta}^\alpha, \hat{\phi}^\alpha)$ at the collision event is also available in the domain D_3. The components of u_2^α and n_2^α with respect to the coordinate basis in D_3 are given by

$$u_2^\alpha = \left(\frac{1}{f_3}\sqrt{\dot{R}_2^2 + f_3}, \dot{R}_2, 0, 0\right), \tag{66}$$

$$n_2^\alpha = \left(\frac{\dot{R}_2}{f_3}, \sqrt{\dot{R}_2^2 + f_3}, 0, 0\right), \tag{67}$$

where $f_3 = f_3(a)$. By using the above equations, we obtain the components of u_1^α with respect to the coordinate basis in D_3 as

$$\begin{aligned} u_1^{t_3} &= -(u_1^\beta u_{2\beta})u_2^{t_3} + (u_1^\beta n_{2\beta})n_2^{t_3} \\ &= -(u_1^\beta u_{2\beta})\frac{1}{f_3}\sqrt{\dot{R}_2^2 + f_3} + (u_1^\beta n_{2\beta})\frac{\dot{R}_2}{f_3} \\ &= \frac{1}{f_3\sqrt{f}}\left[\sqrt{(\dot{R}_2^2 + f)(\dot{R}_2^2 + f_3)} - \dot{R}_2^2\right], \end{aligned} \tag{68}$$

$$\begin{aligned} u_1^r &= -(u_1^\beta u_{2\beta})u_2^r + (u_1^\beta n_{2\beta})n_2^r \\ &= -(u_1^\beta u_{2\beta})\dot{R}_2 + (u_1^\beta n_{2\beta})\sqrt{\dot{R}_2^2 + f_3} \\ &= \frac{\dot{R}_2}{\sqrt{f}}\left(\sqrt{\dot{R}_2^2 + f} - \sqrt{\dot{R}_2^2 + f_3}\right), \end{aligned} \tag{69}$$

$$u_1^\theta = u_1^\phi = 0. \tag{70}$$

The above components are regarded as those of the 4-velocity of shell-1 just after the collision event. By using Eqs. (35) and (69), we have

$$u_1^r = \frac{m_2 \dot{R}_2}{a\sqrt{f}}. \tag{71}$$

By taking the square of Eq. (35) and using Eq. (38), we have

$$\begin{aligned} \sqrt{(\dot{R}_2^2 + f)(\dot{R}_2^2 + f_3)} &= \dot{R}_2^2 + \frac{f + f_3}{2} - \frac{1}{2}\left(\frac{m_2}{a}\right)^2 \\ &= E^2 - \left(\frac{m_2}{2a}\right)^2. \end{aligned} \tag{72}$$

The above equation implies

$$E^2 > \left(\frac{m_2}{2a}\right)^2. \tag{73}$$

Then, we have

$$u_1^{t_3} = \frac{1}{f_3\sqrt{f}}\left[1 - \frac{2M_{\rm d}}{a} + \frac{Q^2}{a^2} - \frac{1}{2}\left(\frac{m_2}{a}\right)^2\right]. \tag{74}$$

We can check that the normalization condition $-f_3(u_1^{t_3})^2 + f_3^{-1}(u_1^r)^2 = -1$ is satisfied.

The above result implies that just after the collision, the derivative of the areal radius of shell-1 with respect to its proper time becomes

$$\dot{R}_1|_{\rm after} = \frac{m_2 \dot{R}_2}{a\sqrt{f}}. \tag{75}$$

Since shell-2 falls into the wormhole just before the collision, $\dot{R}_2$ is negative. This fact implies that shell-1 or equivalently the radius of the wormhole throat begins shrinking just after the collision since m_2 is assumed to be positive.

The domain between shell-1 and shell-2 after the collision is called D_4. From the junction condition between D_3 and D_4, shell-2 obeys the following equation just after the collision:

$$\dot{R}_1^2|_{\rm after} = -1 + \left(\frac{M_3 - M_4}{\mu R_1^2}\right)^2 + \frac{M_3 + M_4}{R_1} - \frac{Q^2}{R_1^2} + \left(\frac{\mu R_1}{2}\right)^2. \tag{76}$$

From the above equation and Eq. (75), we obtain

$$\dot{R}_2^2 = -f\left(\frac{a}{m_2}\right)^2 \left[1 - \left(\frac{M_3 - M_4}{\mu a^2}\right)^2 - \frac{M_3 + M_4}{a} + \frac{Q^2}{a^2} - \left(\frac{\mu a}{2}\right)^2\right]. \tag{77}$$

Here note that $\dot{R}_2$ is the value of shell-2 just before the collision.

B. Shell-2 after the collision

Since the orthonormal frame $(u_1^\alpha, n_1^\alpha, \hat{\theta}^\alpha, \hat{\phi}^\alpha)$ is also available in the domain D_1, by using Eqs. (15), (16), and (21) the components of u_1^α and n_1^α with respect to the coordinate basis in D_1 are given by

$$u_1^\alpha = \left(-\frac{1}{\sqrt{f}}, 0, 0, 0\right), \tag{78}$$

$$n_1^\alpha = (0, -\sqrt{f}, 0, 0). \tag{79}$$

As already noted just below Eq. (21), the time component of u_1^α with respect to the coordinate basis in D_1 is negative.

By using the above equations, we obtain the components of u_2^α with respect to the coordinate basis in D_1 as

$$\begin{aligned} u_2^{t_1} &= -(u_2^\beta u_{1\beta}) u_1^{t_1} + (u_2^\beta n_{1\beta}) n_1^{t_1} \\ &= (u_2^\beta u_{1\beta}) \frac{1}{\sqrt{f}} = -\frac{1}{f}\sqrt{\dot{R}_2^2 + f}, \end{aligned} \tag{80}$$

$$u_2^r = -(u_2^\beta u_{1\beta}) u_1^r + (u_2^\beta n_{1\beta}) n_1^r = -(u_2^\beta n_{1\beta})\sqrt{f} = -\dot{R}_2, \tag{81}$$

$$u_2^\theta = u_2^\phi = 0. \tag{82}$$

Since $\dot{R}_2$ is negative, shell-2 begins expanding after the collision. This is a reasonable result because of the wormhole structure.

Due to the spherical symmetry, D_4 is also described by the Reissner-Nordström geometry with the mass parameter M_4 and the unchanged charge parameter Q. From the junction condition between D_1 and D_4, we have

$$\dot{R}_2^2|_{\rm after} = -1 + \left(\frac{M_1 - M_4}{m_2}\right)^2 + \frac{M_1 + M_4}{R_2} - \frac{Q^2}{R_2^2} + \left(\frac{m_2}{2R_2}\right)^2. \tag{83}$$

From Eq. (81), since $\dot{R}_2^2$ is unchanged by the collision, we have

$$\dot{R}_2^2 = -1 + \left(\frac{M_1 - M_4}{m_2}\right)^2 + \frac{M_1 + M_4}{a} - \frac{Q^2}{a^2} + \left(\frac{m_2}{2a}\right)^2. \tag{84}$$

Here again, note that $\dot{R}_2$ is the value of shell-2 just before the collision.

C. The mass parameter M_4 in D_4

From Eqs. (38) and (39), we can write $\dot{R}_2^2$ just before the collision in the form

$$\dot{R}_2^2 = -1 + \left(\frac{M_3 - M_2}{m_2}\right)^2 + \frac{M_2 + M_3}{a} - \frac{Q^2}{a^2} + \left(\frac{m_2}{2a}\right)^2. \tag{85}$$

Then, Eqs. (77), (84), and (85) determine the unknown parameter M_4.

Since $M_1 = M_2 = M_{\rm w}$, Eqs. (84) and (85) lead to

$$\left(\frac{M_{\rm w} - M_3}{m_2}\right)^2 + \frac{M_{\rm w} + M_3}{a} = \left(\frac{M_{\rm w} - M_4}{m_2}\right)^2 + \frac{M_{\rm w} + M_4}{a}. \tag{86}$$

By solving the above equation with respect to M_4, we obtain two roots: $M_4 = M_3$ and $M_4 = 2M_{\rm w} - M_3 - m_2^2/a$. By using Eqs. (44) and (77), we find that the latter one, i.e.,

$$M_4 = 2M_{\rm w} - M_3 - \frac{m_2^2}{a}, \tag{87}$$

is the solution we need, where we have used $M_3 = M_{\rm w} + m_2 E$. Hence, after the collision, the wormhole does not have mirror symmetry with respect to $r = a$.

VI. THE CONDITION THAT THE WORMHOLE PERSISTS

In this section, we consider the condition that the wormhole stably exists after the passage of shell-2. From Eq. (76), the effective potential of shell-1 after the collision is given by

$$V_{1|\mathrm{after}}(r) = \frac{1}{r^4}\left[-\frac{\mu^2}{4}r^6 + r^4 - (M_3 + M_4)r^3 + Q^2r^2 - \left(\frac{M_4 - M_3}{\mu}\right)^2\right]. \quad (88)$$

By using Eqs. (44), (45), and (87), we have

$$\mu^2 = \frac{2(a - M_\mathrm{w})}{a^3}, \quad (89)$$

$$Q^2 = \frac{a}{2}(3M_\mathrm{w} - a), \quad (90)$$

$$M_3 + M_4 = 2M_\mathrm{w} - \frac{m_2^2}{a}, \quad (91)$$

$$M_3 - M_4 = 2m_2E + \frac{m_2^2}{a}. \quad (92)$$

Equations (89)–(92) imply that the effective potential $V_{1|\mathrm{after}}$ is characterized by four parameters: M_w, a, m_2, and E. By regarding M_w as a parameter to determine the unit of length, the motion of the wormhole after the passage of shell-2 is characterized by three parameters: a, m_2, and E.

A. No black hole formation

First of all, $a > R_\mathrm{H}$ must be satisfied in the case of $M_3 \geq Q$. If not, the wormhole is enclosed by an event horizon after shell-2 enters the domain $r \leq R_\mathrm{H}$, and hence the wormhole cannot stably persist.

The inequality $M_3 \geq Q$ leads to

$$m_2 \geq \frac{1}{E}\left(\sqrt{\frac{a(3M_\mathrm{w} - a)}{2}} - M_\mathrm{w}\right), \quad (93)$$

whereas the inequality $a > R_\mathrm{H}$ leads to

$$m_2 < \frac{a - M_\mathrm{w}}{4E}. \quad (94)$$

If Eq. (93) holds, Eq. (94) should be satisfied. It is not so difficult to see that

$$\frac{a - M_\mathrm{w}}{4} > \sqrt{\frac{a(3M_\mathrm{w} - a)}{2}} - M_\mathrm{w},$$

and hence both of Eqs. (93) and (94) can hold simultaneously. In the case of

$$m_2 < \frac{1}{E}\left(\sqrt{\frac{a(3M_\mathrm{w} - a)}{2}} - M_\mathrm{w}\right),$$

$M_3 < Q$ holds, and hence no horizon appears in D_3 even if shell-2 enters the wormhole. As a result, the event horizon does not form due to the passage of shell-2 only if the inequality (94) holds.

Hereafter, we focus on the following bounded domain in the parameter space (a, m_2):

$$\mathcal{D} = \left\{(a, m_2) | M_\mathrm{w} < a < \frac{3}{2}M_\mathrm{w} \text{ and } 0 < m_2 < \frac{a - M_\mathrm{w}}{4E}\right\}. \quad (95)$$

B. Allowed domain for the motion of shell-1

The allowed domain for the motion of shell-1 after the collision should be restricted to $r > 0$ and bounded so that the wormhole stably persists. We introduce the function

$$W(r) \coloneqq r^4 V_{1|\mathrm{after}}(r) = -\frac{\mu^2}{4}r^6 + r^4 - (M_3 + M_4)r^3 + Q^2r^2 - \left(\frac{M_4 - M_3}{\mu}\right)^2. \quad (96)$$

It is easy to see that the function $W(r)$ has a negative minimum at $r = 0$. Since $W(r)$ has at most five extrema, $W(r)$ should have two non-negative maxima and one negative minimum in $r > 0$ and one maximum in $r < 0$, so that there is a bounded domain of $V_{1|\mathrm{after}} < 0$ in $r > 0$.

We introduce the function $w(r)$ defined as

$$\frac{dW(r)}{dr} = -\frac{3\mu^2}{2}rw(r) \coloneqq -\frac{3\mu^2}{2}r\left[r^4 - \frac{8}{3\mu^2}r^2 + \frac{2(M_3 + M_4)}{\mu^2}r - \frac{4Q^2}{3\mu^2}\right]. \quad (97)$$

The quartic equation $w(r) = 0$ should have three positive real roots and one negative real root, so that there is a bounded domain of $V_{1|\mathrm{after}} < 0$ in $r > 0$. In the Appendix, we see that this is the case as long as the parameters a and m_2 are restricted to the domain $\mathcal{D}$. Thus, $W(r)$ has two maxima and one minimum in $r > 0$ and one maximum in $r < 0$. The radial coordinates of the extrema of $W(r)$ other than $r = 0$, i.e., the roots of $w(r) = 0$ are denoted by r_A, r_B, r_C, and r_D, all of which are functions of not E, but rather a and m_2; the explicit forms of r_A, r_B, r_C, and r_D are given through Ferrari's formula for the roots of a quartic equation, but we will not show them here since the expressions of the roots are too complicated to get any information from them. We assume $r_\mathrm{A} < 0 < r_\mathrm{B} < r_\mathrm{C} < r_\mathrm{D}$, and hence $W(r)$ has

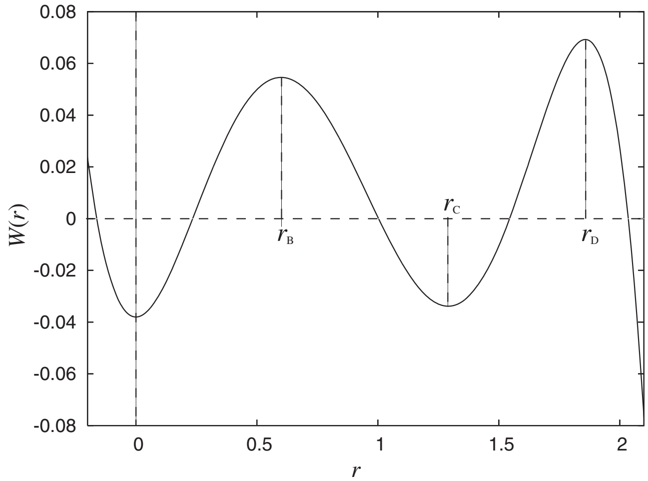

FIG. 4. Adopting the units $M_\mathrm{w} = 1$, the function $W(r)$ with $(a, m_2, E) = (1.3, 0.05, 1)$ is depicted. There is a maximum of $W(r)$ at $r = r_\mathrm{A} < 0$. However, it is very large compared with extrema in $r \geq 0$, and hence we do not show it in this figure.

maxima at $r = r_\mathrm{A}$, $r = r_\mathrm{B}$, and $r = r_\mathrm{D}$, whereas it has minima at $r = 0$ and $r = r_\mathrm{C}$ (see Fig. 4).

In the case of $m_2 = 0$, since $V_{1|\mathrm{after}}(r)$ is equal to $V_1(r)$, we have $r_\mathrm{C} = a$ and $W(r_\mathrm{C}) = 0$, and both $W(r_\mathrm{B})$ and $W(r_\mathrm{D})$ are positive (see Fig. 5). By contrast, in the case of nonvanishing m_2, we have

$$W(a) = -\frac{m_2^2 a^2}{2(a - M_\mathrm{w})} \times \left[2(2E^2 - 1)a + 2M_\mathrm{w} + 4Em_2 + \frac{m_2^2}{a}\right] < 0,$$

and hence $W(r_\mathrm{C})$ must be negative by the continuous dependence of $W(r)$ on the parameter m_2.

Since shell-1 shrinks just after the passage of shell-2 [see Eq. (75)], if $W(r_\mathrm{B})$ is negative, shell-1 (or equivalently, the wormhole) collapses to form a black hole. If $W(r_\mathrm{B})$ vanishes, shell-1 asymptotically approaches $r = r_\mathrm{B}$ and thus the size of the wormhole remains finite. If $W(r_\mathrm{B})$ is positive, shell-1 bounces off the potential barrier, and then R_1 increases. In this case, $W(r_\mathrm{D})$ should be equal to or larger than zero so that the wormhole persists with a finite size. The domain in (a, m_2) space with E fixed in which the wormhole persists after the passage of the shell-2 is a curve $W(r_\mathrm{B}) = 0$ and a domain restricted by $W(r_\mathrm{B}) > 0$ and $W(r_\mathrm{D}) \geq 0$. Hence the critical curves in (a, m_2) space with E fixed are given by the condition

$$W(r_\mathrm{B}) = 0 \quad \text{and} \quad W(r_\mathrm{D}) = 0.$$

In Fig. 6, we depict the domain in (a, m_2) space with $E = 1$ in which the wormhole persists after the passage of shell-2 as an unshaded region. Figure 7 is the same as Fig. 6 but zoomed into the neighborhood of the intersections of the curves $W(r_\mathrm{B}) = 0$, $W(r_\mathrm{D}) = 0$, and $a - M = 4m_2$, i.e., the upper bound of the domain $\mathcal{D}$. The mass of shell-2, m_2, is bounded from above by $0.0785026M_\mathrm{w}$ at which the initial radius of the wormhole throat, a, equals $1.31581M_\mathrm{w}$. This result shows another physically significant difference from the case of the wormhole with a negative mass investigated in Ref. [22]: the upper bound on m_2 is of the order $|M_\mathrm{w}|$ in the negative-mass case, since the gravitational collapse needed to form a black hole is prevented by the negative mass of the wormhole.

Here it should be noted that E appears only in the last term on the right-hand side of Eq. (96) [see Eq. (92)], and the inclination of $W(r)$ does not depend on E. The area of the domain in (a, m_2) space in which the wormhole persists

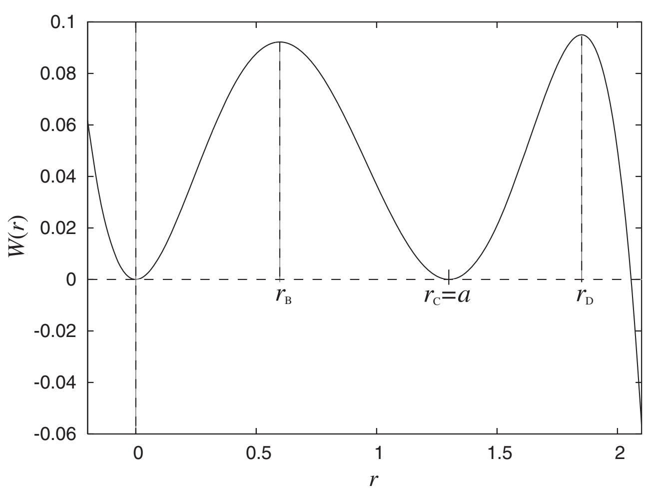

FIG. 5. The same as Fig. 4 but with $m_2 = 0$.

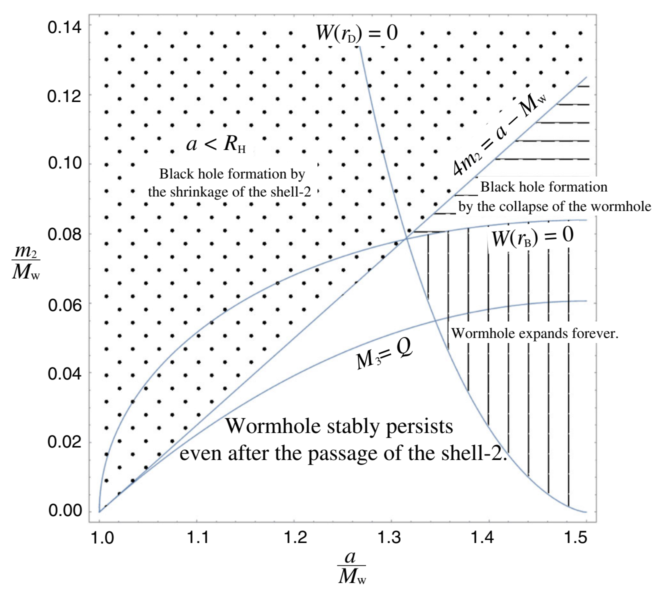

FIG. 6. The (a, m_2) space with $E = 1$ is depicted. The domain in which the wormhole stably persists is shown as the unshaded region.

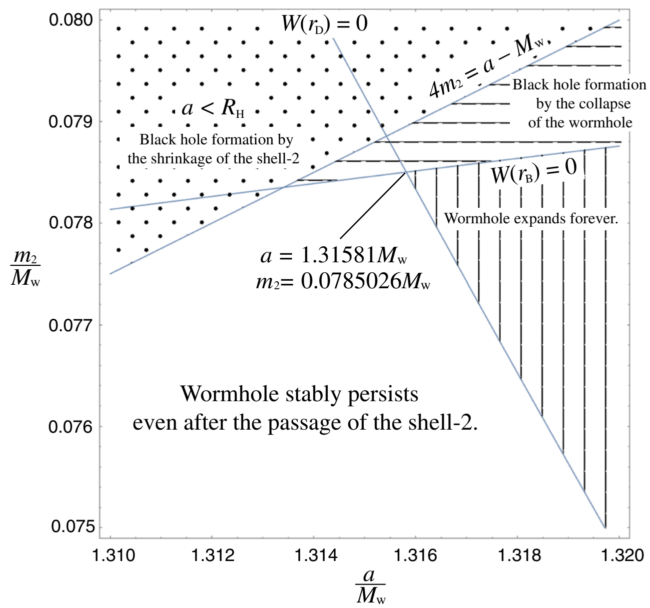

FIG. 7. A close-up of the neighborhood of the intersections of the curves $W(r_B) = 0$, $W(r_D) = 0$, and $4m_2 = a - M_w$.

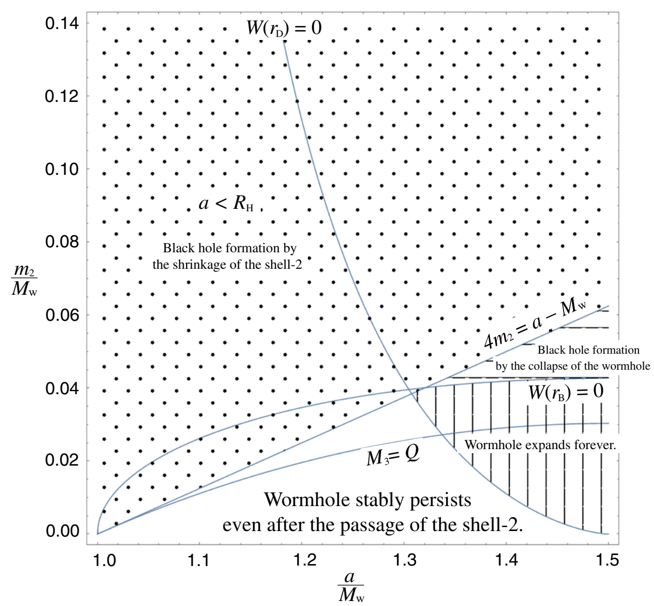

FIG. 8. The same as Fig. 6, but $E = 2$.

decreases as E increases. However, there is a domain in which the wormhole persists for any E larger than unity. Figure 8 is the same as Fig. 6 but with $E = 2$.

VII. SUMMARY AND DISCUSSION

We analytically studied the nonlinear stability of a wormhole supported by an infinitesimally thin spherical brane, i.e., a thin spherical shell whose tangential pressure is equal to its energy per unit area with an opposite sign. We considered a situation in which a thin spherical shell composed of dust concentric with the brane goes through the initially static wormhole in order to play the role of a nonlinear disturbance. We took into account the self-gravities of both the brane and the dust shell through Israel's formalism of the metric junction. The wormhole was assumed to have a mirror symmetry with respect to the brane supporting it. As Barcelo and Visser have shown, in such a situation the gravitational mass of the static wormhole should be positive, and the static electric field should exist in order for the wormhole to be stable against linear perturbations. Then, we studied the condition that the wormhole persists after the dust shell goes through it. We assumed that the interaction between the brane and the dust shell is only gravity, or in other words, the 4-velocities of these shells were assumed to be continuous at the collision event. In this model, there are three free parameters—the initial areal radius a of the wormhole, the conserved specific energy E, and the proper mass m_2 of the dust shell—if we regard the initial gravitational mass M_w of the wormhole as a unit of length. Then, we showed that there is a domain of nonzero measure in (a, m_2) space for $E \geq 1$ in which the wormhole persists after the dust shell goes through it. In the case of $E = 1$, the maximum mass of the dust shell m_2 is almost equal to $0.08M_w$.

Assuming $a \simeq Q \simeq M_w$, through the geodesic deviation equations, the tidal acceleration $A_{\rm tidal}$ felt by a spacecraft at the throat of the wormhole $r = a$ is roughly estimated as

$$A_{\rm tidal} = \frac{2M_w\ell}{a^3}\left(\frac{3Q^2}{2M_w a} - 1\right) \simeq \frac{c^6\ell}{G^2M_w^2} = 10\left(\frac{M_w}{4\times10^5\ M_\odot}\right)^{-2}\left(\frac{\ell}{40\ \mathrm{m}}\right)\ \mathrm{m/s^2},$$

where ℓ is the length of the spacecraft and $M_\odot$ is the solar mass (2×10^{30} kg). The area of the wormhole throat with $M_w = 4\times10^5\ M_\odot$ is about $4\pi M_w^2 \simeq 4.5\times10^{12}$ km^2. Here, let us imagine 10^{12} spacecrafts placed with almost equal spacing on a sphere concentric with a spherical brane wormhole with $M_w = 4\times10^5\ M_\odot$. Together, they can be regarded as a dust shell if they almost freely fall into the wormhole along the radial direction. If the size of a spacecraft is about 40 m, the tidal acceleration on each spacecraft is of the order of 10 m/s^2 even at the throat of the wormhole. Then, since the average separation between adjacent spacecrafts is of the order of 1 km, they can safely go through the wormhole. Since the mass of each spacecraft will be about 2×10^6 kg, the total mass of the shell composed of these spacecrafts is $2\times10^{18}\,\mathrm{kg}\simeq10^{-12}M_\odot$. The present result suggests that the wormhole supported by the negative tension brane stably persists even after the passage of these spacecrafts.

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APPENDIX: ON THE ROOTS OF THE QUARTIC EQUATION $w(r) = 0$

In this appendix, we show that if the parameters a and m_2 are in the domain $\mathcal{D}$ of $E \geq 1$, the quartic equation $w(r) = 0$ has three positive real roots and one negative real root.

In accordance with Eqs. (89)–(92), we regard μ and Q as functions of a, $M_3 + M_4$ as a function of a, and m_2 and $M_3 - M_4$ as functions of a, m_2, and E.

First, we show that $M_3 + M_4$ is bounded below by a positive value. Because of Eq. (94), by using Eq. (91) we have

$$M_3 + M_4 = 2M_\mathrm{w} - \frac{m_2^2}{a} > N(a, M_\mathrm{w}, E),$$

where

$$N(a, M_\mathrm{w}, E) \coloneqq \frac{(16E^2 + 1)M_\mathrm{w}}{8E^2} - \frac{1}{16E^2}\left(a + \frac{M_\mathrm{w}^2}{a}\right).$$

Because of Eq. (51),

$$\frac{\partial N}{\partial a} = \frac{M_\mathrm{w}^2 - a^2}{16E^2 a^2} < 0$$

is satisfied, and hence we have

$$N(a, M_\mathrm{w}, E) > N\left(\frac{3}{2}M_\mathrm{w}, M_\mathrm{w}, E\right) = \frac{1}{96}\left(192 - \frac{1}{E^2}\right)M_\mathrm{w} \geq \frac{191}{96}M_\mathrm{w},$$

where we have used $E \geq 1$. As a result, we obtain

$$M_3 + M_4 > \frac{191}{96}M_\mathrm{w}. \tag{A1}$$

The derivative of $w(r)$ is given by

$$\frac{dw(r)}{dr} = 4r^3 - \frac{16}{3\mu^2}r + \frac{2(M_3 + M_4)}{\mu^2}. \tag{A2}$$

If the inequality

$$(M_3 + M_4)\mu < \frac{32}{27} \tag{A3}$$

holds, the cubic equation $dw(r)/dr = 0$ has three real roots. We show that Eq. (A3) necessarily holds in the domain $\mathcal{D}$ of $E \geq 1$. Because of Eq. (51), we have

$$\frac{d\mu}{da} = \frac{3M_\mathrm{w} - 2a}{a^3}\sqrt{\frac{a}{2(a - M_\mathrm{w})}} > 0, \tag{A4}$$

and hence

$$\mu < \mu|_{a=\frac{3}{2}M_\mathrm{w}} = \frac{1}{M_\mathrm{w}}\sqrt{\frac{8}{27}} \tag{A5}$$

holds. Equation (A5) leads to Eq. (A3) as follows:

$$(M_3 + M_4)\mu = \left(2M_\mathrm{w} - \frac{m_2^2}{a}\right)\mu < 2M_\mathrm{w}\mu < \sqrt{\frac{32}{27}} < \frac{32}{27}.$$

By virtue of Eqs. (A1) and (A3), we find that $w(r)$ has one minimum in $r < 0$ and one maximum and one minimum in $r > 0$.

Hereafter, the three real roots of $dw(r)/dr = 0$ are denoted by $r = r_i$ ($i = 1, 2, 3$):

$$r_1 = \frac{4}{3\mu}\cos\left(\frac{\theta}{3}\right), \qquad r_2 = \frac{4}{3\mu}\cos\left(\frac{\theta + 2\pi}{3}\right) \quad \text{and} \quad r_3 = \frac{4}{3\mu}\cos\left(\frac{\theta + 4\pi}{3}\right), \tag{A6}$$

where

$$\theta = \arccos\left(-\frac{27}{32}(M_3 + M_4)\mu\right). \tag{A7}$$

Since

$$-1 < -\frac{27}{32}(M_3 + M_4)\mu < 0$$

is satisfied by virtue of Eq. (A3),

$$\frac{\pi}{2} < \theta < \pi \tag{A8}$$

holds. Equation (A8) leads to $r_1 > r_3 > 0 > r_2$.

We introduce the function

$$U(\rho) = -\frac{4}{3\mu^2}\left[\rho^2 - \frac{9(M_3 + M_4)}{8}\rho + Q^2\right].$$

Then, since r_i satisfies

$$r_i^4 = \frac{4}{3\mu^2}r_i^2 - \frac{M_3 + M_4}{2\mu^2}r_i,$$

we have

$$w(r_i) = U(r_i).$$

Because of Eqs. (50) and (A1), the quadratic equation $U(\rho) = 0$ has two real roots:

$$\rho = \rho_\pm \coloneqq \frac{9}{16}\left[M_3 + M_4 \pm \sqrt{(M_3 + M_4)^2 - \left(\frac{16Q}{9}\right)^2}\right].$$

If the inequalities

$$U(r_1) < 0, \qquad U(r_2) < 0 \quad \text{and} \quad U(r_3) > 0,$$

or equivalently,

$$r_1 > \rho_+, \qquad r_2 < \rho_- \quad \text{and} \quad \rho_- < r_3 < \rho_+ \qquad \text{(A9)}$$

are simultaneously satisfied, the quartic equation $w(r) = 0$ has four real roots. We will see below that Eq. (A9) holds.

Since both $\rho_\pm$ are positive, $r_2 < \rho_-$ is trivially satisfied because of $r_2 < 0$.

From Eq. (A7), we can see that

$$\frac{\partial\theta}{\partial m_2} = -\frac{27\mu m_2}{16a\sin\theta} < 0,$$

where we have used Eq. (A8) in the inequality. Thus, we see that

$$\frac{\partial r_1}{\partial m_2} = -\frac{4}{9\mu}\sin\left(\frac{\theta}{3}\right)\frac{\partial\theta}{\partial m_2} > 0,$$

and

$$r_1 > r_1|_{m_2=0} = \frac{4}{3\mu}\cos\left[\frac{1}{3}\arccos\left(-\frac{27}{16}M_\mathrm{w}\mu\right)\right] \qquad \text{(A10)}$$

It is not difficult to see that

$$\frac{\partial\rho_+}{\partial m_2} < 0$$

holds, and hence we have

$$\rho_+ < \rho_+|_{m_2=0}. \qquad \text{(A11)}$$

We depict $(r_1 - \rho_+)\mu$ for $m_2 = 0$ in Fig. 9. Since (as shown in Fig. 9) $r_1 > \rho_+$ holds for $m_2 = 0$, we have from Eqs. (A10) and (A11)

$$r_1 > \rho_+ \quad \text{for } m_2 > 0.$$

We can easily see that ρ_+ is an increasing function of $M_3 + M_4$, whereas ρ_- is a decreasing function of $M_3 + M_4$, in the domain $\mathcal{D}$ of $E \geq 1$. Then, Eq. (A1) implies

$$\begin{aligned}\rho_+ &> \rho_+|_{M_3+M_4=\frac{191}{96}M_\mathrm{w}} \\ &= \frac{9}{16}\left[\frac{191}{96} + \sqrt{\left(\frac{191}{96}\right)^2 - \left(\frac{16Q}{9M_\mathrm{w}}\right)^2}\right]M_\mathrm{w} \\ &> \frac{9}{16}\left[\frac{191}{96} + \sqrt{\left(\frac{191}{96}\right)^2 - \frac{32}{9}}\right]M_\mathrm{w} > M_\mathrm{w}. \qquad \text{(A12)}\end{aligned}$$

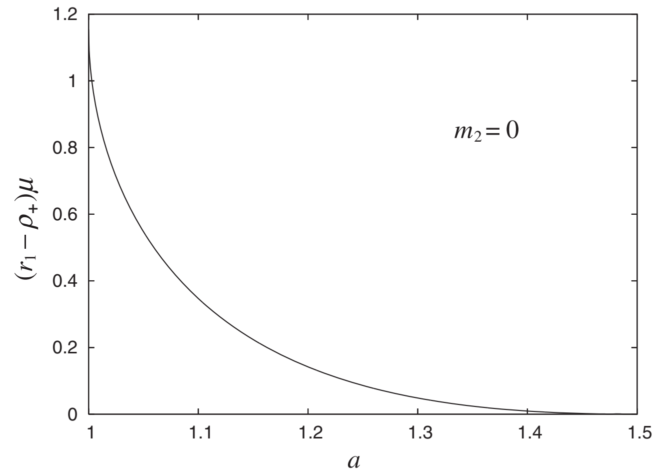

FIG. 9. We depict $(r_1 - \rho_+)\mu$ with $m_2 = 0$ as a function of a in units of $M_\mathrm{w} = 1$.

We have

$$\begin{aligned}\frac{1}{4}\frac{dw}{dr}\bigg|_{r=M_\mathrm{w}} &= \frac{1}{a^3\mu^2}\left(M_\mathrm{w}^3a^3\mu^2 - \frac{1}{3}M_\mathrm{w}a^3 - \frac{m_2^2a^2}{2}\right) \\ &< -\frac{M_\mathrm{w}}{3a^3\mu^2}f(a), \qquad \text{(A13)}\end{aligned}$$

where

$$f(a) = a^3 - 6M_\mathrm{w}^2a + 6M_\mathrm{w}^3.$$

It is easy to see that $f(a) > 0$ holds for $a > 0$. This result implies that $dw/dr|_{r=M_\mathrm{w}} < 0$ holds. As a result, we have

$$r_3 < M_\mathrm{w} < \rho_+,$$

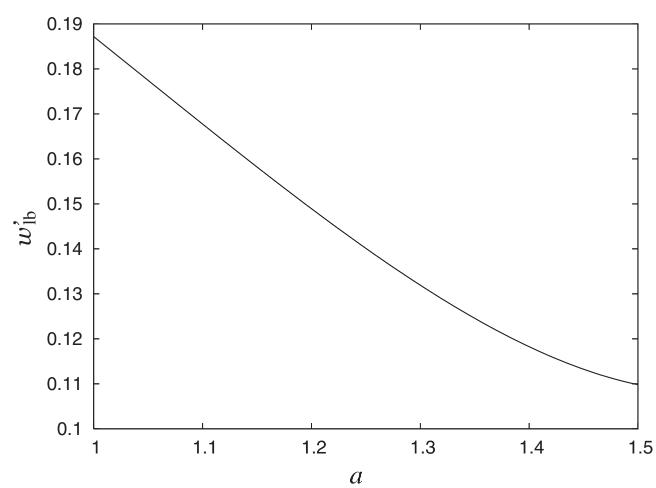

FIG. 10. We depict w'_{lb} as a function of a in units of $M_\mathrm{w} = 1$. It is positive in the domain of our interest.

since $r = r_3$ is the lower bound of the domain of $dw/dr < 0$ in $r > 0$.

It is easy to see that the following inequality holds in the domain $\mathcal{D}$ of $E \geq 1$:

$$\frac{\partial \rho_-}{\partial m_2} > 0,$$

and hence we have

$$\rho_- < \rho_{\rm ub}(a) \coloneqq \rho_-|_{m_2 = \frac{a - M_{\rm w}}{4}} = \frac{9}{8}\left[1 - \frac{(a - M_{\rm w})^2}{32 M_{\rm w} a} - \sqrt{\left(1 - \frac{(a - M_{\rm w})^2}{32 M_{\rm w} a}\right)^2 - \left(\frac{8Q}{9M_{\rm w}}\right)^2}\right] M_{\rm w} \tag{A14}$$

where we have used Eq. (94) and $E \geq 1$ in the inequality. We can see that

$$\frac{\mu^2}{4}\frac{dw}{dr}\bigg|_{r=\rho_{\rm ub}} = \mu^2 \rho_{\rm ub}^3 - \frac{4}{3}\rho_{\rm ub} + M_{\rm w} - \frac{m_2^2}{2a} > w'_{\rm lb}(a)$$
$$\coloneqq \mu^2 \rho_{\rm ub}^3 - \frac{4}{3}\rho_{\rm ub} + M_{\rm w} - \frac{(a - M_{\rm w})^2}{32a}, \tag{A15}$$

where we have used Eq. (94) and $E \geq 1$ in the inequality. In Fig. 10, we depict $w'_{\rm lb}$ as a function of a. From this figure, we find that $dw/dr > 0$ at $r = \rho_{\rm ub}$, and hence $r_3 > \rho_{\rm ub} > \rho_-$ holds for the same reason as that leading to $r_3 < M_{\rm w} < \rho_+$. As a result, we have $\rho_- < r_3 < \rho_+$.

The result obtained above implies that the quartic equation $w(r) = 0$ has four real roots. Here, we recall that the function $w(r)$ has one minimum in $r < 0$, whereas one maximum and one minimum exist in $r > 0$. Then, since $w < 0$ and $dw/dr > 0$ at $r = 0$, we find that one root of $w(r) = 0$ is negative and the other three are positive.

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