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Characteristic formulation for metric $f(R)$ gravity

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In recent years, the characteristic formulation of numerical relativity has found increasing use in the extraction of gravitational radiation from numerically generated spacetimes. In this paper, we formulate the characteristic initial value problem for $f(R)$ gravity. We consider, in particular, the vacuum field equations of Metric $f(R)$ gravity in the Jordan frame, without utilizing the dynamical equivalence with scalar-tensor theories. We present the full hierarchy of nonlinear hypersurface and evolution equations necessary for numerical implementation in both tensorial and *eth* forms. Furthermore, we specialize the resulting equations to situations where the spacetime is almost Minkowski and almost Schwarszchild using standard linearization techniques. We obtain analytic solutions for the dominant $\ell = 2$ mode and show that they satisfy the concomitant constraints. These results are ideally suited as testbed solutions for numerical codes. Finally, we point out that the characteristic formulation can be used as a complementary analytic tool to the $1 + 1 + 2$ semitetrad formulation.

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I. INTRODUCTION

Initial value formulations have a long and eventful history in numerical relativity, dating back to the seminal works of [1–5]. This topic has been a subject of several review articles; see, for example, [6] and references therein. For the purposes of fixing context, we recall that relativistic initial value formulations generally come in different flavors, among which, those that are based on a $3 + 1$ foliation of spacetime are the most popular. The other formulations are generalized harmonic, characteristic and hyperboloidal. The generalized harmonic formulation is based on a harmonic decomposition of the Ricci tensor, resulting in evolution equations for the 4-metric in some harmonic coordinates [7,8]. The characteristic approach [2,4] is based on foliations of spacetime on outgoing null hypersurfaces while the hyperboloidal formulation is based on spacetime foliations by spacelike hypersurfaces that smoothly intersect null infinity $\mathcal{I}^+$ [3,9]. In this work, we are interested in setting up a characteristic formulation of the field equations of metric $f(R)$ gravity.

Geometrically, foliating spacetime with null hypersurfaces presents a natural approach to study gravitational radiation since these represent the characteristic surfaces of the field equations. Indeed, a characteristic formulation of the field equations presents a gauge invariant and unambiguous description of gravitational waves in a nonlinear setting, where the perturbative methods of $3 + 1$ formulations are not adequate. However, one of the major challenges of characteristic evolutions is the possible development of caustics during evolution. These are coordinate singularities that arise due to the focusing of light rays generating the null hypersurfaces. Algorithms to handle this undesirable feature have been proposed [10,11], but there has, apparently, not been a numerical implementation in wide use. Nevertheless, caustic formation is only an issue in standalone evolutions of nonlinear spacetimes by characteristic methods. More recent applications of characteristic formulations are in Cauchy characteristic extraction (CCE) and Cauchy characteristic matching (CCM) methods. In CCE, one takes metric data on some inner timelike worldtube Γ, computed from a $3 + 1$ Cauchy code, and propagates it to future null infinity $\mathcal{I}^+$ via a characteristic code, thus enabling wave-form extraction at $\mathcal{I}^+$ [12,13]. This scheme represents a special case of the more general CCM [14,15] which, in turn, uses data from the characteristic code as exact boundary conditions for the metric functions of the $3 + 1$ Cauchy code.

Within the numerical relativity community, there are now a number of characteristic codes being used, with differing levels of sophistication. For instance, some codes employ second order finite difference schemes [16], others use higher order schemes [85] while others have adopted spectral methods [17]. Another point of distinction among different codes is the coordinate system used to cover the sphere labeling the null directions of the light cones. Common choices range from a stereographic coordinate system [18] to multipatch coordinate systems [16,19]. There have also been efforts to introduce adaptive mesh refinement schemes to characteristic evolution codes [20,21]. Overall, these codes have made it possible to demonstrate the versatility of characteristic methods in numerical relativity and have found extensive applications in, for example, binary black hole mergers [18,22–24], stellar core collapse [25–27], Einstein-Klein-Gordon systems [28–30], observational cosmology [31–33], etc. These systems represent potential astrophysical laboratories for testing general relativity in the nonlinear regime.

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Over the years, the theory of general relativity has been subjected to a wide range of experimental tests and has no doubt emerged as one of the most successful theories in Physics. However, there has been considerable interest in the literature to study gravity theories whose Lagrangians contain higher order curvature invariants such as R^2, $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, $R\Box R$, $R\Box^k R$ [34–36]. The motivation for these alternative theories of gravity stems from a variety of grounds, most notably from within the dark sector in cosmology [37]. Moreover, the inflationary paradigm arises naturally in alternative theories of gravity without postulating additional inflaton fields [34,36,38]. These higher order corrections also arise in the effective action of quantum gravity. For example, in the low energy limit of string theory or when considering compactifications of extra dimensions in M theory [39]. In this work, we restrict our attention to the fourth order metric $f(R)$ gravity. Although simpler than most other alternative theories, general predictions in the theory demand a numerical treatment, especially when considering strong field sources as in numerical relativity.

We derive the full set of nonlinear equations necessary for a numerical implementation. We further present linearized solutions about some fixed background spacetimes that may aid in code development in the form of testbed solutions. These solutions are based on a linearization of the exact equations on Minkowski and Schwarzschild backgrounds using standard techniques. In principle, one could consider other background solutions about which to linearize. However, one must be able to analytically cast the metric of such background solutions in Bondi-Sachs form, which is a nontrivial task for most known solutions [40]. For example, a Bondi-Sachs representation of the Kerr solution involves elliptic integrals, which require numerical evaluation [41]. The existence and stability conditions for both Minkowski and Schwarzschild spacetimes in the context of $f(R)$ gravity have been studied by several authors; see [42–45]. Within the Bondi-Sachs framework, linearized perturbations, in the manner considered here, have been studied in general relativity by [46–49], and have been used as testbed solutions and in analytic descriptions of binary black holes in circular [46,50] and eccentric orbits [51]. Different approaches on the subject can be found in [12,52,53].

This paper is structured as follows: We review the field equations of metric $f(R)$ and its equivalence to scalar-tensor theories in § II. In § III, we present the Bondi-Sachs coordinates. The decomposed field equations in tensorial form are given in IV A, and in § V we present them in the complementary *eth* formalism which is commonly used in numerical codes. We present linearized equations in § VI and their solutions when linearized about Minkowski background in § VI C 1 and Schwarzschild background in § VI C 2. Finally, we conclude in § VII. For convenience, we provide the Christoffel symbols for the Bondi-Sachs metric in the Appendix. Throughout this paper, we use geometrized units $G = c = 1$ and metric signature $(- + + +)$.

II. METRIC $f(R)$ GRAVITY

A. Field equations

The gravitational field equations of metric $f(R)$ theories can be derived starting from a simple generalization of the Einstein-Hilbert action

$$S = \frac{1}{16\pi}\int dx^4[\sqrt{-g}f(R) + 16\pi\mathcal{L}_{\rm mat}], \tag{1}$$

where $f(R)$ is a general function of the Ricci scalar R, g is the determinant of the spacetime metric g_{ab} and $\mathcal{L}_{\rm mat}$ is the Lagrangian of matter fields. Varying the action (1) with respect to the metric g_{ab} and assuming that the connection is the Levi-Civita connection,[1] one obtains the equation of motion

$$\Sigma_{ab} = 8\pi T_{ab} \tag{2}$$

where T_{ab} is the energy momentum tensor of standard matter fields, given in terms of the variational derivative of $\mathcal{L}_{\rm mat}$ as

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm mat})}{\delta g^{ab}}. \tag{3}$$

The symmetric tensor Σ_{ab} is given by

$$\begin{aligned}\Sigma_{ab} &= f'R_{ab} - \frac{1}{2}fg_{ab} - \nabla_a\nabla_b f' + g_{ab}\Box f' \\ &= f'R_{ab} - \frac{1}{2}fg_{ab} - f''\nabla_a\nabla_b R - f'''\nabla_b R\nabla_a R \\ &\quad + g_{ab}(f'''\nabla^c R\nabla_c R + f''\Box R),\end{aligned} \tag{4}$$

where $\Box = \nabla_c\nabla^c$ is the d'Alembertian operator and we use $'$ to denote differentiation with respect to the Ricci scalar R. Interestingly, Σ_{ab} contains terms involving second derivatives of the Ricci scalar R which translates to fourth derivatives of the metric, hence the characterization as "fourth order gravity." Unlike in general relativity, the relation between the Ricci scalar R and the trace T of the energy momentum tensor is no longer algebraic ($R = -8\pi T$) but differential, given as

$$3\Box f' - 2f + f'R = 8\pi T. \tag{5}$$

Equation (5) governs the dynamics of the scalar degree of freedom inherent in the theory. As in the 3 + 1 formulation

[1] Relaxing this assumption, such that the affine connection $\Gamma^a{}_{bc}$ is independent of the metric g_{ab}, is the basis of Palatini $f(R)$ and leads to field equations that are different from those of metric $f(R)$ considered here.

[54], it is convenient to use the equivalent form for the field equations

$$E_{ab} \equiv \Sigma_{ab} - \kappa^2 T_{ab} - \frac{1}{3} g_{ab}(\Sigma - \kappa^2 T) = 0. \quad (6)$$

where we have introduced the notation E_{ab} for later convenience. Finally, we note that in the limit of constant scalar curvature $R = R_0$, the trace Eq. (5) reduces to an algebraic relation $-2f + f'R = 8\pi T$ and the field Eqs. (4) become

$$R_{ab} - \frac{1}{2} g_{ab} R + \lambda g_{ab} = 8\pi T_{ab} \quad (7)$$

where $\lambda = R_0/4$ is an effective cosmological constant Λ.

B. Equivalence with scalar-tensor theories

It has long been known that metric $f(R)$ gravity theories are dynamically equivalent to special cases of Brans-Dicke scalar-tensor theories [55–58]. We briefly review this equivalence in the following. Starting from the action (1), one can introduce a new field χ and recast (1) into the equivalent form

$$S = \frac{1}{16\pi} \int dx^4 [\sqrt{-g} f(\chi) + f'(\chi)(R - \chi)] + \int d^4x \mathcal{L}_{\mathrm{mat}}, \quad (8)$$

Varying the new action (8) with respect to χ leads to

$$f''(\chi)(R - \chi) = 0. \quad (9)$$

Then, provided that $f''(\chi) \neq 0$, the above implies $\chi = R$, and consequently, the action (8) becomes (1). If we further define an auxiliary field ϕ

$$\phi = f'(\chi) \quad (10)$$

and supposing that the relation is invertible, then the action (8) can be expressed as

$$S = \frac{1}{16\pi} \int dx^4 \sqrt{-g} [\chi(\phi) R - V(\phi)] + \int d^4x \mathcal{L}_{\mathrm{mat}}, \quad (11)$$

where the potential $V(\phi)$ is given by

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)). \quad (12)$$

The action (11) corresponds to the Jordan frame representation of a Brans-Dicke scalar-tensor theory without a kinetic term for the scalar field, i.e., with Brans-Dicke parameter $\omega_{BD} = 0$. By transforming to the Einstein frame, one can proceed to show that this is conformally equivalent to the Einstein-Hilbert action with a scalar field that couples minimally to the Ricci scalar [59]. This equivalence can be a convenient tool when studying various modified gravity theories. However, one should exercise caution when interpreting results; see, for example, [60–66].

III. THE BONDI-SACHS METRIC

For the characteristic initial value problem, we employ coordinates (u, r, x^A) based on a family of outgoing null hypersurfaces emanating from an inner worldtube Γ denoting the inner boundary of the characteristic domain. Within this system, $u = r - t$ is a retarded time coordinate labeling the hypersurfaces, r is a surface area coordinate and $x^A (A = 2, 3)$ are labels for the null rays.[2] Then, the Bondi-Sachs metric takes the form

$$ds^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \quad (13)$$

$$= -\left(e^{2\beta} \frac{V}{r} - r^2 h_{AB} U^A U^B\right) \mathrm{d}u^2 - 2e^{2\beta} \mathrm{d}u \mathrm{d}r - 2r^2 h_{AB} U^B \mathrm{d}u \mathrm{d}x^A + r^2 h_{AB} \mathrm{d}x^A \mathrm{d}x^B. \quad (14)$$

It is straightforward to compute the contravariant components of the Bondi-Sachs metric. The nonzero components are

$$g^{rr} = e^{-2\beta} \frac{V}{r}, \qquad g^{rA} = -e^{-2\beta} U^A,$$
$$g^{ru} = -e^{-2\beta}, \qquad g^{AB} = r^{-2} h^{AB}. \quad (15)$$

The Christoffel symbols for the above metric are given in the Appendix. We note that it is sometimes convenient to use W instead of the more usual Bondi-Sachs variable V, where $W \coloneqq V - r$. The 2-tensor h_{AB}, with $h^{AB} h_{BC} = \delta^A{}_C$, satisfies the determinant condition

$$\det(h_{AB}) = \det(q_{AB}) \quad (16)$$

where q_{AB} is the unit 2-sphere metric, so that h_{AB} has only two degrees of freedom. By considering the metric of $r = \mathrm{const}$ surfaces,[3] one identifies h_{AB} as the conformal 2-metric of surfaces of constant u which foliate the worldtube, and $e^{2\beta} V/r$ corresponds to the square of the lapse function while $-U^A$ represents the shift vector. In total, the metric (14) contains only six free variables h_{AB}, β, V and U^A, which are in general, a function of the coordinates. Evolution equations for these Bondi-Sachs variables are derived from the field equations of gravity.

IV. THE FIELD EQUATIONS

In analogy with the $3 + 1$ formulation, the field equations within the Bondi-Sachs formalism can be classified into main and constraint equations. In the following sections, we present these in turn.

[2] Here, and in the following, we will generally use uppercase indices for the angular directions. These will run from 2 to 3.

[3] This can be obtained from (14) by setting $dr = 0$.

A. Main equations

The main equations are further classified into hypersurface and evolution equations. The hypersurface equations form a hierarchical set of equations for the Bondi-Sachs variables β, U^A and V to be integrated radially once h_{AB} and R are given on some $u = \text{const}$ slice. These are derived from the $R^u{}_\alpha$ components of the field equations, giving

$$\beta_{,r}\left(1+\frac{r}{2}\frac{f''}{f'}R_{,r}\right) = \frac{r}{16}h^{AC}h^{BD}h_{AB,r}h_{CD,r} + \frac{r}{4f'}\left(f''R_{,rr}+f'''R_{,r}R_{,r}\right) \tag{17}$$

$$(r^2Q_A)_{,r} = 2r^4(r^{-2}\beta_{,A})_{,r} - r^2h^{BC}\mathrm{D}_Ch_{AB,r} + \frac{2r^2}{f'}\left\{r^2f''(r^{-2}R_{,A})_{,r}+f'''R_{,A}R_{,r}+f''\beta_{,A}R_{,r} - \frac{r^2}{2}f''h_{AB}U^B{}_{,r}R_{,r} - \frac{1}{2}f''h^{DC}h_{AC,r}R_{,D}\right\} \tag{18}$$

$$2e^{-2\beta}V_{,r} = \mathcal{R} - 2\mathrm{D}^A\mathrm{D}_A\beta - 2\mathrm{D}^A\beta\mathrm{D}_A\beta + r^{-2}e^{-2\beta}\mathrm{D}_A(r^4U^A)_{,r} - \frac{r^4}{2}e^{-4\beta}h_{AB}U^A{}_{,r}U^B{}_{,r} + \frac{e^{-2\beta}}{f'}\left\{e^{2\beta}f''\mathrm{D}^A\mathrm{D}_AR + f'''e^{2\beta}\mathrm{D}^AR\mathrm{D}_AR - 2f''R_{,u} + 2Vf''R_{,r} - r^2f''R_{,r}\mathrm{D}_CU^C - 2rU^Cf''R_{,C} - \frac{r^2}{3}e^{2\beta}f + \frac{2r^2}{3}e^{2\beta}f'R\right\} \tag{19}$$

where in (18) we have used the auxiliary quantity Q_A,

$$Q_A = r^2e^{-2\beta}h_{AB}U^B{}_{,r}. \tag{20}$$

To obtain the evolution equation for h_{AB}, it suffices to consider the trace-free symmetric part of the angular components of the field equations,

$$m^Am^B\Big[r(rh_{AB,u})_{,r} - \frac{1}{2}(rVh_{AB,r})_{,r} - 2e^\beta\mathrm{D}_A\mathrm{D}_Be^\beta - \frac{1}{2}r^4e^{-2\beta}h_{BD}h_{AC}U^C{}_{,r}U^D{}_{,r} + U^Cr^2\mathrm{D}_Ch_{AB,r} + h_{AC}\mathrm{D}_B(r^2U^C)_{,r} + \frac{1}{2}r^2h_{AB,r}\mathrm{D}_CU^C - r^2h_{BE}h_{AC,r}(\mathrm{D}^CU^E - \mathrm{D}^EU^C) - \frac{1}{f'}\Big\{f''e^{2\beta}\mathrm{D}_A\mathrm{D}_BR + f'''e^{2\beta}R_{,A}R_{,B} - \frac{r^2}{2}f''h_{AB,r}R_{,u} + \frac{r}{2}Vf''h_{AB,r}R_{,r} - r^2f''R_{,r}\mathrm{D}_AU_B - \frac{r^2}{2}f''h_{AB,u}R_{,r} - \frac{r^2}{2}f''U^Ch_{AB,r}R_{,C}\Big\}\Big] = 0 \tag{21}$$

where m^A is a complex dyad such that $h^{AB} = m^{(A}\bar{m}^{B)}$. The trace Eq. (5) gives the following evolution equation for the quantity f':

$$-\frac{2}{r}\partial_u\partial_r(rf') + \frac{V}{r}\partial_r\partial_rf' + \frac{1}{r}\partial_rV\partial_rf' + \frac{V}{r^2}\partial_rf' - 2U^A\partial_A\partial_rf' - \partial_rf'\mathrm{D}_AU^A - \frac{2}{r}U^A\partial_Af' - \partial_Af'\partial_rU^A + r^{-2}e^{2\beta}\mathrm{D}_A\mathrm{D}^Af' + 2e^{2\beta}r^{-2}h^{AC}\partial_Cf'\partial_A\beta = \frac{2}{3}e^{2\beta}f - \frac{1}{3}e^{2\beta}f'R. \tag{22}$$

To turn this into an equation for the Ricci scalar R, one uses the fact that $f' = df(R)/dR$, and proceed via the chain rule such that

$$f'_{,x} = f''R_{,x} \tag{23a}$$

$$f'_{,xy} = f''R_{,xy} + f'''R_{,x}R_{,y}. \tag{23b}$$

B. Conservation conditions

Up to this point, we have only focused on the main equations. The remaining components of the field equations R^r_α, split into the trivial equation

$$E_{ur} = 0 \tag{24}$$

and supplementary equations

$$E_{uu} = 0 \quad \text{and} \quad E_{Au} = 0, \tag{25}$$

where we have used the notation [cf. Eq. (6)]

$$E_{ab} \equiv \Sigma_{ab} - \kappa^2T_{ab} - \frac{1}{3}g_{ab}(\Sigma - \kappa^2T) = 0. \tag{26}$$

Along with the main equations, these make up the full set of components for the field equations. Because of the Bianchi identities, and assuming that the main equations are satisfied, the trivial equation is satisfied identically, while the supplementary equations need only be satisfied on a single spherical cross section of the worldtube as was shown in the general relativity case by [2,4].

Clearly, a key to this conservation property is the Bianchi identities. In $f(R)$ gravity, the divergence of the field equations takes the form [67]

$$\nabla^a\Sigma_{ab} = \nabla^a\left(f'R_{ab} - \frac{1}{2}fg_{ab} - \nabla_a\nabla_b f' + g_{ab}\Box f'\right) = 0$$
$$= R_{ab}\nabla^a f' + f'\nabla^a R_{ab} - \frac{1}{2}g_{ab}\nabla^a f - \nabla^a\nabla_a\nabla_b f' + g_{ab}\nabla^a\nabla^c\nabla_c f' \quad (27)$$

$$= R_{ab}\nabla^a f' + f'\nabla^a R_{ab} - \frac{1}{2}g_{ab}f'\nabla^a R - (\nabla^a\nabla_b\nabla_a - \nabla_b\nabla^c\nabla_c)f'. \quad (28)$$

Then, the generalized Bianchi identities $\nabla^a\Sigma_{ab} = 0$ follow geometrically because

$$\nabla^a\left(R_{ab} - \frac{1}{2}g_{ab}R\right) = 0 \quad \text{and}$$
$$(\nabla^a\nabla_b\nabla_a - \nabla_b\nabla^c\nabla_c)f' = R_{ab}\nabla^a f' \quad (29)$$

as a result of the standard Bianchi and Ricci identities.

General expressions for (24) and (25) are lengthy and are not required in most numerical applications. We give, instead, linearized expressions in § VI.

V. SPIN-WEIGHTED AND *ETH* FORMALISM

Within the spin-weighted formalism, the unit sphere metric q_{AB} is expressed in terms of a dyadic product $q_{AB} = q_{(A}\bar{q}_{B)}$, where the dyad q^A is a complex[4] basis 2-vector satisfying $q^A q_A = 0$, $q^A\bar{q}_A = 2$ and $q_A = q_{AB}q^B$ [68,69]. We note that the basis vectors are not unique, up to a phase transformation. For a given q_A, one can construct an alternative basis $\hat{q}_A = e^{i\alpha}q_A$, where the phase α is real. Using the dyad vectors q^A, rank-n tensor fields $T_{A_1A_2\ldots A_n}$ on the sphere can be conveniently represented by scalar fields,

$$T = q^{A_1}\ldots q^{A_m}\bar{q}^{A_{m+1}}\ldots\bar{q}^{A_n}T_{A_1\ldots A_n}. \quad (30)$$

The spin weight s of such scalar fields depends on the rank n of the tensor field and is given by $s = 2m - n$, where m is the number of q^A factors and $n - m$ represents the number of $\bar{q}^A$ factors appearing in (30). In general, the scalars (30) will have the transformation property $T \to e^{i\alpha s}T$. With this in mind, the three spin-weighted scalars

$$J = \frac{1}{2}q^A q^B h_{AB}, \quad \bar{J} = \frac{1}{2}\bar{q}^A\bar{q}^B h_{AB} \quad \text{and} \quad K = \frac{1}{2}q^A\bar{q}^B h_{AB} \quad (31)$$

with respective spin weights $+2$, -2 and 0, contain all the degrees of freedom of the 2-tensor h_{AB}. Using (31), h_{AB} is irreducibly decomposed as

$$2h_{AB} = \bar{J}q_A q_B + J\bar{q}_A\bar{q}_B + K(q_A\bar{q}_B + \bar{q}_A q_B), \quad (32)$$

with the inverse 2-metric h^{AB} given by

$$2h^{AB} = -\bar{J}q^A q^B - J\bar{q}^A\bar{q}^B + K(q^A\bar{q}^B + \bar{q}^A q^B). \quad (33)$$

Furthermore, the determinant condition (16) implies the relation

$$K^2 = 1 + J\bar{J}. \quad (34)$$

Consequently, the scalar K contains no additional information, and h_{AB} is uniquely determined by J, for an arbitrary Bondi-Sachs metric. Similarly, U^A and Q^A are decomposed into the spin-weighted fields

$$U = U^A q_A, \qquad \bar{U} = U^A\bar{q}_A,$$
$$Q = Q^A q_A \qquad \bar{Q} = Q^A\bar{q}_A \quad (35)$$

with respective spins of $+1$, -1, $+1$ and -1. We note that within this spin-weighted formalism, the scalar quantities β, V and R are spin-0 fields.

In addition to the spin-weighted scalars, it is convenient to define complex differential *eth* operators ð and $\bar{ð}$ whose action on a quantity X of spin weight s is given as

$$ðX = q^A\partial_A X + s\Upsilon X, \qquad \bar{ð}X = \bar{q}^A\partial_A X - s\bar{\Upsilon}X \quad (36)$$

where

$$\Upsilon = -\frac{1}{2}q^A\bar{q}^B\nabla_A q_B. \quad (37)$$

The resulting quantities $ðX$ and $\bar{ð}X$ have spin weights $s + 1$ and $s - 1$, respectively. More generally, the operator ð ($\bar{ð}$) acting on a spin-weighted scalar has the effect of raising (lowering) the spin weight by 1.

For the stereographic coordinate system $x^A = (q, p)$, which we adopt in this work, the unit sphere metric q_{AB} is given as

$$q_{AB}dx^A dx^B = \frac{4}{q^2 + p^2 + 1}(dq^2 + dp^2). \quad (38)$$

The dyad vectors then become

$$q^A = \frac{q^2 + p^2 + 1}{2}(1, i) \quad \text{and} \quad q_A = \frac{2}{q^2 + p^2 + 1}(1, i). \quad (39)$$

With this choice, (37) becomes $\Upsilon = q + ip$.

Using the above formalism, the hypersurface equations become

[4] We will generally use an overbar on a complex quantity to denote complex conjugation.

$$\beta_{,r}\left(1+\frac{r}{2}\frac{f''}{f'}R_{,r}\right)=N_\beta+M_\beta \tag{40a}$$

$$U_{,r}=r^{-2}e^{2\beta}Q+N_U \tag{40b}$$

$$(r^2Q)_{,r}=-r^2(\bar{\eth}J+\eth K)_{,r}+2r^4\eth(r^{-2}\beta)_{,r}+N_Q+M_Q \tag{40c}$$

$$W_{,r}=\frac{1}{2}e^{2\beta}\mathcal{R}-1-e^{\beta}\eth\bar{\eth}e^{\beta}+\frac{1}{4}r^{-2}[r^4(\eth\bar{U}+\bar{\eth}U)]_{,r}+N_W+\frac{1}{2}e^{2\beta}M_W \tag{40d}$$

where the 2-Ricci scalar $\mathcal{R}$ is given by

$$\mathcal{R}=2K-\eth\bar{\eth}K+\frac{1}{2}(\bar{\eth}^2J+\eth^2\bar{J})+\frac{1}{4K}(\bar{\eth}\bar{J}\eth J-\bar{\eth}J\eth\bar{J}). \tag{41}$$

The evolution equations become

$$2(rJ)_{,ur}=[r^{-1}V(rJ)_{,r}]_{,r}-r^{-1}(r^2\eth U)_{,r}+2r^{-1}e^{\beta}\eth^2e^{\beta}-J(r^{-1}W)_{,r}+N_J+r^{-1}M_J, \tag{42}$$

$$2(rf')_{,ur}=[r^{-1}V(rf')_{,r}]_{,r}-f'(r^{-1}W)_{,r}-U\bar{\eth}f'-\bar{U}\eth f' \tag{43}$$

$$+\frac{r^{-1}e^{2\beta}}{K}[J(\eth\bar{J}\bar{\eth}f'+\bar{\eth}\bar{J}\eth f')+\bar{J}(\bar{\eth}J\eth f'+\eth J\bar{\eth}f')]+Ke^{2\beta}r^{-1}(\eth\bar{\eth}f'+\bar{\eth}f'\eth\beta+\eth f'\bar{\eth}\beta)-\frac{r}{2}[\bar{\eth}f'U_{,r}+\eth f'\bar{U}_{,r}+f'_{,r}(\eth\bar{U}+\bar{\eth}U)+2(\bar{U}\eth f'_{,r}-U\bar{\eth}f'_{,r})]-\frac{r^{-1}e^{-2\beta}}{2}\times[\bar{J}\eth^2f'+J\bar{\eth}^2f'+\bar{\eth}J\bar{\eth}f'+\eth\bar{J}\eth f'+2(J\bar{\eth}f'\bar{\eth}\beta+\bar{J}\eth f'\eth\beta)]+\frac{re^{2\beta}}{3}(2f-f'R) \tag{44}$$

where, again, one is to use the chain rule (23) to obtain an evolution equation for the Ricci scalar R. The terms N_β, N_U, N_Q, N_W and N_J are nonlinear aspherical terms whose representation in terms of spin-weighted variables is given in [18]. The terms M_β, M_Q, M_W and M_J are modified gravity terms arising from the $f(R)$ corrections. These can be computed as

$$M_\beta=\frac{r}{4f'}\left(f''R_{,rr}+f'''R_{,r}R_{,r}\right), \tag{45}$$

$$f'M_Q=r(f''r^{-1}\eth R)_{,r}-\frac{1}{2}r^2e^{-2\beta}f''R_{,r}(KU_{,r}+J\bar{U}_{,r})-f''R_{,r}\eth\beta-\frac{1}{2}f''K(K_{,r}\eth R+J_{,r}\bar{\eth}R)+\frac{1}{2}f''(\bar{J}J_{,r}\eth R+JK_{,r}\bar{\eth}R), \tag{46}$$

$$f'M_W=-\frac{1}{2}r^2f''e^{-2\beta}R_{,r}(\eth\bar{U}+\bar{\eth}U)-rf''e^{-2\beta}(\bar{U}\eth R+U\bar{\eth}R)-\frac{1}{2}f''[\bar{\eth}(J\bar{\eth}R)+\eth(\bar{J}\eth R)]-\frac{1}{2}f'''[\bar{J}(\eth R)^2-2K\bar{\eth}R\eth R+J(\bar{\eth}R)^2]+2f''e^{-2\beta}(R_{,r}V-rR_{,u})+\frac{1}{2}f''(\eth R\bar{\eth}K+\bar{\eth}R\eth K)+f''K\bar{\eth}\eth R-\frac{r^2}{3}(f-2f'R), \tag{47}$$

$$f'M_J=f''\eth\eth R+f'''(\eth R)^2-\frac{1}{2}f''(J\eth\bar{J}\eth R+J\bar{\eth}J\bar{\eth}R+K\eth J\bar{\eth}R-K\bar{\eth}J\eth R-2J\eth K\bar{\eth}R)-\frac{1}{2}e^{-2\beta}f''(r^2J)_{,r}[\bar{U}\eth R+U\bar{\eth}R]+f''e^{-2\beta}R_{,r}Vr^{-1}(r^2J)_{,r}-\frac{1}{2}f''e^{-2\beta}r^2R_{,r}(2K\eth U+2J\eth\bar{U}+U\bar{\eth}J+\bar{U}\eth J)-f''e^{-2\beta}R_{,u}(r^2J)_{,r}-r^2f''e^{-2\beta}R_{,r}J_{,u}. \tag{48}$$

As in the 3 + 1 case, it may be necessary to define $\psi=R_{,u}$ so that the hypersurface equations contain no u derivatives.

VI. LINEARIZED PERTURBATIONS

In the following, we specialize the above nonlinear equations to situations where the spacetime is almost Schwarzschild and almost Minkowski. In outgoing null coordinates, the Schwarzschild metric takes the Eddington-Finkelstein form

$$ds^2=-\left(1-\frac{2M}{r}\right)\mathrm{d}u^2-2\mathrm{d}u\mathrm{d}r+r^2q_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{49}$$

where it is to be understood that $M = 0$ corresponds to Minkowski space. The existence and stability of both Schwarzschild and Minkowski spacetimes in $f(R)$ gravity can be found in, for example, [42–45]. The line element (49) corresponds to $J = U = \beta = 0$ and $W = -2M$. We therefore designate the following quantities and their derivatives as first order:

$$J, \bar{J}, U, \bar{U}, w, \beta = \mathcal{O}(\epsilon) \tag{50}$$

with $W = -2M + w$. We note that the scalar K is unity to linear order because of the determinant condition (34). The linearization procedure proceeds by discarding terms of order $\mathcal{O}(\epsilon^2)$ and higher, i.e., terms involving products of the first order quantities (50). We note that the Ricci scalar R vanishes for the background metric (49). In order to deal with the $f(R)$ corrections, we therefore perform a Taylor expansion about the background such that, to linear order,[5]

$$f(R) = f'_{(0)} R. \tag{51}$$

where $f'_{(0)}$ is a background quantity and $R = \mathcal{O}(\epsilon)$. To avoid having to write prefactors $f'_{(0)}$ and $f''_{(0)}$, we note that one can define an effective mass for the scalaron field as

$$m^2 = \frac{1}{3}\frac{f'}{f''}. \tag{52}$$

The linearized main Eqs. (40) then become

$$\beta_{,r} - \frac{1}{3m^2} R_{,rr} = 0, \tag{53a}$$

$$r^3 U_{,rr} + 4r^2 U_{,r} + r\bar{\eth} J_{,r} + 4\eth\beta - 2r\eth\beta_{,r} + \frac{2}{3m^2} R - \frac{2r}{3m^2} R_{,r} = 0, \tag{53b}$$

$$4\beta - 2\eth\bar{\eth}\beta + \frac{1}{2}(\bar{\eth}^2 J + \eth^2 J) + \frac{1}{2r^2}[r^4(\eth\bar{U} + \bar{\eth}U)]_{,r} - 2w_{,r} - \frac{r^2}{3} R - \frac{2r}{3m^2}\left(1 - \frac{2M}{r}\right) R_{,r} + \frac{2}{3m^2} R_{,u} = 0, \tag{53c}$$

$$2r(rJ)_{,ur} - 2\eth^2\beta + 2r\eth U + r^2 \eth U_{,r} - 2(r - M) J_{,r} - r^2\left(1 - \frac{2M}{r}\right) J_{,rr} - \frac{1}{3m^2}\eth\eth R = 0, \tag{53d}$$

$$\left(1 - \frac{2M}{r}\right) R_{,rr} - \frac{2}{r}(R_{,u} + rR_{,ur}) + \frac{2}{r}\left(1 - \frac{M}{r}\right) R_{,r} + r^{-2}\eth\bar{\eth} R - m^2 R = 0. \tag{53e}$$

A noteworthy feature of the above equations is that the $f(R)$ terms have prefactors of $1/m^2$. Therefore, as $m \to \infty$, the equations will resemble those of general relativity. This is the basic principle behind screening mechanisms that allow modified gravity to behave like general relativity in certain environments by suitably altering the mass of the scalaron field.

The trivial Eq. (24) simplifies to

$$\frac{1}{r^2}\left[2(r - M)\beta_{,r} + r^2\left(1 - \frac{2M}{r}\right)\beta_{,rr} + \frac{1}{2} r w_{,rr} + \eth\bar{\eth}\beta - 2r^2\beta_{,ru} - \frac{1}{4}[r^2(\eth\bar{U} + \bar{\eth}U)]_{,r}\right] = \frac{1}{3m^2}\left(R_{,ur} - \frac{M}{r^2} R_{,r}\right) - \frac{1}{6} R, \tag{54}$$

while the constraints (25) respectively become

$$\frac{1}{4r^2}[-4r^2\eth\beta_{,u} + 2r^2\bar{\eth}J_{,u} - 2r^4 U_{,ur} + 4r^2 U + 2r\eth w_{,r} - 2\eth w + r^2(\eth\bar{\eth}U - \eth\eth\bar{U}) + 2r^2(r - 2M)(4U_{,r} + rU_{,rr})] = \frac{1}{3m^2}\eth R_{,u} \tag{55}$$

and

$$\frac{1}{2r^3}[-4r(r - 2M)\beta_{,u} + 2(r - 2M)\eth\bar{\eth}\beta + r(r - 2M) w_{,rr} + \eth\bar{\eth} w + 2r w_{,u} - Mr(\eth\bar{U} + \bar{\eth}U) - r^3(\eth\bar{U} + \bar{\eth}U)_{,u} - 4r^2(r - 2M)\beta_{,ru} + 2r(r - 2M)^2\beta_{,rr} + 4(r - 2M)(r - M)\beta_{,r}] = \frac{1}{3m^2}\left[R_{,uu} + \frac{M}{r^2} R_{,u} - \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) R_{,r}\right] - \frac{1}{6}\left(1 - \frac{2M}{r}\right) R. \tag{56}$$

[5]We use the fact that $f(0) = 0$, which is one of the conditions for the stability of the Schwarzschild solution in $f(R)$ gravity [44].

Finally, one can derive an expression for the linearized Ricci scalar R from the metric variables. One is free to do so since in metric $f(R)$ gravity, one assumes that the Christoffel symbols are related to derivatives of the metric in the usual way, unlike in Palatini $f(R)$ gravity. Therefore, from $R = g^{ab}R_{ab}$ one obtains

$$R = -\frac{4}{r^2}\eth\bar{\eth}\beta - \frac{4M}{r^2}\beta_{,r} + \frac{4}{r^2}\beta - 2\left(1-\frac{2M}{r}\right)\beta_{,rr} + 4\beta_{,ur} - \frac{1}{r^3}(r^2 w_{,r})_{,r} + \frac{1}{2r^2}(\bar{\eth}^2 J + \eth^2\bar{J}) + \frac{1}{r^3}(r^3\eth\bar{U} + r^3\bar{\eth}U)_{,r}. \tag{57}$$

This expression can be used as a consistency check with the result obtained by integrating the trace Eq. (53e).

A. Eigenfunction decomposition

It is convenient to write the metric quantities in terms of eigenfunctions of the $\eth$ and $\bar{\eth}$ operators. Without loss of generality, we assume that the linearized variables can be written as [47]

$$R = R_0(r)\mathrm{Re}(e^{i\nu u})Z_{\ell m}, \tag{58a}$$

$$\beta = \beta_0(r)\mathrm{Re}(e^{i\nu u})Z_{\ell m}, \tag{58b}$$

$$w = w_0(r)\mathrm{Re}(e^{i\nu u})Z_{\ell m}, \tag{58c}$$

$$U = U_0(r)\mathrm{Re}(e^{i\nu u})\eth Z_{\ell m}, \tag{58d}$$

$$J = J_0(r)\mathrm{Re}(e^{i\nu u})\eth^2 Z_{\ell m}. \tag{58e}$$

A more consistent representation would be in terms of a multipolar series involving sums over ℓ and m as is done in, for example, [50]. The above corresponds to having these quantities fixed, which is sufficient for our purposes. In (58) the ${}_sZ_{\ell m}$ are orthonormal real-valued spin s spherical harmonics defined as [68]

$${}_sZ_{\ell m} = \begin{cases} \frac{i}{\sqrt{2}}[(-1)^m {}_sY_{\ell m} + {}_sY_{\ell -m}] & \text{for } m < 0 \\ {}_sY_{\ell m} & \text{for } m = 0 \\ \frac{1}{\sqrt{2}}[(-1)^m {}_sY_{\ell m} + {}_sY_{\ell -m}] & \text{for } m > 0. \end{cases} \tag{59}$$

The ${}_sY_{\ell m}$ are the standard spin-weighted spherical harmonics

$${}_sY_{\ell m} = \begin{cases} \sqrt{\frac{(\ell-s)!}{(\ell+s)!}}\eth^s Y_{\ell m} & \text{for } s \geq 0 \\ (-1)^s\sqrt{\frac{(\ell-s)!}{(\ell+s)!}}\eth^{-s} Y_{\ell m} & \text{for } s < 0. \end{cases} \tag{60}$$

B. Master equation

Using the ansatz (58), we are able to reduce the linearized Eqs. (53) into a set of linear ordinary differential equations in r, for the quantities β_0, U_0, w_0, J_0 and R_0. For brevity, we shall henceforth drop the zero subscript on these quantities. In the following, we restrict our attention to the particular case of $\ell = 2$. We emphasize that this choice is motivated by simplicity; it is possible to consider other ℓ values. We further make the change of variable $r = 1/x$. With these simplifications, the linearized equations become

$$4x\beta_{,x} + \frac{x^2}{3m^2}R_{,xx} + \frac{2x}{3m^2}R_{,x} = 0, \tag{61a}$$

$$4\beta + 2x\beta_{,x} + xU_{,xx} - 2U_{,x} + 4xJ_{,x} + \frac{2}{3m^2}R + \frac{2x}{3m^2}R_{,x} = 0, \tag{61b}$$

$$16x\beta + 24xJ - 24U + 6xU_{,x} + 2x^3 w_{,x} - \frac{x^{-2}}{3}R + \frac{2}{3m^2}(i\nu + 3x)R + \frac{2x^2}{3m^2}(1-2xM)R_{,x} = 0, \tag{61c}$$

$$-4x\beta + 4U - 2xU_{,x} + 4x^3MJ_{,x} - 2x^3(1-2xM)J_{,xx} + 4i\nu J - 4xi\nu J_{,x} - \frac{2x}{3m^2}R = 0, \tag{61d}$$

$$x^4(1-2xM)R_{,xx} - 2x^2(x^2M - i\nu)R_{,x} - (2xi\nu - 6x^2 + m^2)R = 0. \tag{61e}$$

Using standard techniques, it is possible to derive a master equation for the Bondi-Sachs variable J. Interestingly, this takes the same form as that obtained in the general relativity case [47],

$$x^3(1-2xM)J_{,xxxx} + (4x^2 + 2i\nu x - 14x^3M)J_{,xxx} - (4x + 16Mx^2 + 2i\nu)J_{,xx} = 0. \tag{62}$$

This master equation can be further simplified by defining an auxiliary variable $J_{,xx} = J_2$ [47]. Then, J_2 obeys

$$x^3(1-2xM)J_{2,xx} + (4x^2 + 2i\nu x - 14x^3M)J_{2,x} - (4x + 16Mx^2 + 2i\nu)J_2 = 0. \tag{63}$$

We are now in a position to solve the above linearized ordinary differential equations for the various metric quantities.

C. Solutions

The solution procedure proceeds in a hierarchical order, mirroring that of a numerical scheme. First, we obtain solutions for J and R from Eqs. (63) and (61e). Having obtained R, Eq. (61a) can be solved for β. Having β, R and J, Eq. (61b) can be solved for U, and finally Eq. (61c) is solved for w. In the following sections, we consider separately the cases of Minkowski ($M=0$) and Schwarzschild ($M\neq 0$) backgrounds. In all cases, we verify that the R obtained by solving (61e) is consistent with that reconstructed from (57). We also evaluate the constraints by plugging in the obtained solutions.

1. Minkowski background

Following the above procedure, we first consider the static case, $\nu=0$, obtaining the solutions

$$R = C_1 x e^{m/x}(m^2-3xm+3x^2) + C_2 x e^{-m/x}(m^2+3xm+3x^2), \tag{64}$$

$$\beta = \frac{C_1}{12m^2}e^{\frac{m}{x}}(12x^2m-5xm^2-12x^3+m^3) - \frac{C_2}{12m^2}e^{-\frac{m}{x}}(12x^2m+5xm^2+12x^3+m^3)+C_3, \tag{65}$$

$$J = C_4+\frac{C_5}{x^2}+C_6x+C_7x^3, \tag{66}$$

$$U = \frac{x}{6m^2}R+\frac{2C_5}{x}+2x^2C_6+2xC_3-3x^4C_7, \tag{67}$$

$$w = -\frac{C_1}{6m^2x}e^{\frac{m}{x}}(6x^2m-6x^3+m^3-3m^2x) - 6x^2C_7-\frac{10}{x}C_3+\frac{12}{x}C_4 + \frac{C_2}{6m^2x}e^{-\frac{m}{x}}(6x^2m+6x^3+m^3+3m^2x) - \frac{6}{x^3}C_5+C_8. \tag{68}$$

As expected, the trivial Eq. (54) is identically satisfied. The constraints (55) and (56) respectively lead to

$$C_8=0, \tag{69}$$

$$4(2C_3-3C_4)-C_8x=0. \tag{70}$$

For the dynamic case, $\nu\neq 0$, we obtain

$$R = iC_1x\exp\left(\frac{i\nu-\sqrt{m^2-\nu^2}}{x}\right)\times(m^2-\nu^2+3x\sqrt{m^2-\nu^2}+3x^2) + iC_2x\exp\left(\frac{i\nu+\sqrt{m^2-\nu^2}}{x}\right)\times(m^2-\nu^2-3x\sqrt{m^2-\nu^2}+3x^2), \tag{71}$$

$$\beta = -\frac{C_1}{12m^2}\exp\left(\frac{i\nu-\sqrt{m^2-\nu^2}}{x}\right)[5ix(m^2-\nu^2)+3x(4ix+\nu)(x+\sqrt{m^2-\nu^2}) + (m^2-\nu^2)(\nu+i\sqrt{m^2-\nu^2})] - \frac{C_2}{12m^2}\exp\left(\frac{i\nu+\sqrt{m^2-\nu^2}}{x}\right)[5ix(m^2-\nu^2) + 3x(4ix+\nu)(x-\sqrt{m^2-\nu^2})+(m^2-\nu^2)(\nu+i\sqrt{m^2-\nu^2})]+C_3, \tag{72}$$

$$J = C_4+C_5x+\frac{C_6x^3}{6}+\frac{C_7}{2}\exp\left(\frac{2i\nu}{x}\right)(x-i\nu)^2x, \tag{73}$$

$$U = \frac{x}{6m^2}R-i\nu C_4+2C_5x^2+2xC_3-\frac{C_6x^3}{6}(4i\nu+3x)+\frac{C_7x^3}{2}\exp\left(\frac{2i\nu}{x}\right)(2i\nu-3x), \tag{74}$$

$$w = -\frac{C_1}{6m^2x}\exp\left(\frac{i\nu-\sqrt{m^2-\nu^2}}{x}\right)\left[(m^2-\nu^2)\left(\nu-3ix-i\sqrt{m^2-\nu^2}\right) + 3x(2ix-\nu)\left(\sqrt{m^2-\nu^2}+x\right)\right] - \frac{C_2}{6m^2x}\exp\left(\frac{i\nu+\sqrt{m^2-\nu^2}}{x}\right) \times\left[(m^2-\nu^2)\left(\nu-3ix+i\sqrt{m^2-\nu^2}\right)+3x(2ix-\nu)\left(\sqrt{m^2-\nu^2}-x\right)\right] + \frac{6C_4}{x^2}(i\nu+2x)-C_6(2i\nu+x)-\frac{10C_3}{x}-C_7x^2\exp\left(\frac{2i\nu}{x}\right)+C_8. \tag{75}$$

Again, the trivial Eq. (54) is identically satisfied. The constraints (55) and (56) lead to

$$C_8 - 2\nu^2 C_6 = 0, \tag{76}$$

$$12i\nu C_5 + 6x\nu^2 C_6 + 12(2C_3 - 3C_4) - (3x - i\nu)C_8. \tag{77}$$

We note that one can recover the static solutions by simply setting $\nu = 0$ in the dynamical solution. With this in mind, we will only consider the dynamic case in the next section.

2. Schwarzschild background

When the background is Schwarzschild, we are not able to find analytical solutions in closed form. This is true even in general relativity for the case $M \neq 0$ and $\nu \neq 0$ [47,49]. In principle, one could write the solutions in terms of confluent hypergeometric functions or as a power series about the singular points of the concomitant ordinary differential equations (ODEs). Here, we opt for the latter. The singular points of the ODEs (61e) and (63) are as follows:

$$\text{Regular}: \; x = \infty \qquad x = \frac{1}{2M}, \tag{78}$$

$$\text{Irregular}: \; x = 0. \tag{79}$$

In the following we compute series solutions about the regular singular point $x = 1/2M$, corresponding to $r = 2M$. We write $z = x - 1/2M$ and expand the solutions about $z = 0$, obtaining

$$R = C_1\left[1 + \frac{4M(2i\nu M + 2m^2M^2 + 3)}{4i\nu M - 1} z + \mathcal{O}(z^2)\right], \tag{80}$$

$$\beta = C_2 - C_1\left[\frac{4M^2m^2 + 8i\nu M + 5}{12m^2(4i\nu M - 1)} + \frac{4M^4m^4 + 8M^3m^2 i\nu - 8\nu^2M^2 + 12m^2M^2 + 23i\nu M + 6}{3m^2(4i\nu M - 1)(2i\nu M - 1)} z + \mathcal{O}(z^2)\right], \tag{81}$$

$$J = C_3 + C_4 z + C_5 \frac{z^2}{2}\left[1 + \frac{8M(i\nu M + 3)}{3(4i\nu M - 3)} z + \mathcal{O}(z^2)\right], \tag{82}$$

$$\begin{aligned} U = {} & \frac{C_2(2Mz + 1)}{M} - i\nu C_3 - 3C_4(8M^3z^3 + 20z^2M^2 + 2i\nu M + 14Mz + 3) \\ & - \frac{C_5(2i\nu M - 1)}{8M^3}[1 + 4Mz + 8M^2z^2 + \mathcal{O}(z^3)] + \frac{2Mz + 1}{12Mm^2} R, \end{aligned} \tag{83}$$

$$\begin{aligned} w = {} & C_6 + \frac{40M^2 z C_2}{2Mz + 1} - C_3\left[\frac{2Mz + 1 + 2Mi\nu(Mz + 1)}{(2Mz + 1)^2}\right]48M^2 z + C_4\left[\frac{8i\nu M(Mz + 1) + 4Mz(Mz + 3) + 5}{(2Mz + 1)^2}\right]6Mz \\ & + C_1\left[\frac{2M^2(16M^3m^2 i\nu + 16\nu^2M^2 + 36i\nu M - 1)}{3m^2(4i\nu M - 1)} z + \mathcal{O}(z^2)\right] \\ & - C_5(2i\nu M - 1)[6z - 12Mz^2 + 32M^2z^3 + \mathcal{O}(z^4)]. \end{aligned} \tag{84}$$

This time, the trivial equation becomes a series in z, and is identically satisfied order by order. The constraints (55) and (56) respectively become

$$\begin{aligned} 3M^2m^2C_6 = {} & M^2(4Mi\nu + 1)C_1 - 36M^2m^2C_2 \\ & + 72M^2m^2(Mi\nu + 1)C_3 \\ & - 9Mm^2(2Mi\nu + 3)C_4 \\ & + 6m^2(3i\nu M + 2\nu^2M^2 - 1)C_5, \end{aligned} \tag{85}$$

$$\begin{aligned} 0 = {} & 48M^2C_2 - 72M^2C_3 + 12M(2Mi\nu + 3)C_4 \\ & + \frac{16i\nu^3M^3 - 24\nu^2M^2 - 38i\nu M + 15}{Mi\nu + 2} C_5. \end{aligned} \tag{86}$$

For the irregular singular point $x = 0$, it is still possible to obtain a series solution for J [47]. However, the same procedure does not work for R (61e); hence, a solution for the other quantities is not possible. In any case, standard methods for obtaining series solutions are not guaranteed to work for irregular singular points.

VII. CONCLUDING REMARKS

In this work, we have presented a characteristic formulation for metric $f(R)$ gravity. We have cast the full nonlinear system both in tensorial form using the language of [70,71] and also in the *eth* formalism [68,69] that is commonly used in numerical relativity codes. The nonlinear equations assume a simple structure as can be seen from § IV and § V, with $f(R)$ modifications encoded in the variables M_β, M_Q, M_W and M_J. This makes it straightforward to modify existing codes that were originally built for general relativity to include terms arising from $f(R)$ gravity.

A numerical implementation of the equations presented in this work will pave a way for Cauchy characteristic extraction methods in modified gravity. The recent detections of gravitational waves [72,73] has opened up the possibility of constraining modified theories of gravity with gravitational wave data. This topic has revived some interest in the characterization of gravitational radiation in $f(R)$ gravity theories [74,75]. On the mathematical side, we have not addressed the Well-posedness of the timelike-null cone problem, upon which CCE is based. Interestingly, this is still an open question, even in general relativity. However, there have been encouraging results [76,77].

The linearized solutions presented in §VI C 1 will serve as testbed solutions for validating numerical codes. Another interesting area of application is in the linearized description of the binary black hole problem [46,50,51]. A potential application for this scenario is in the context of wave-form extraction. Generally, one needs initial data on the null cone in some far field region exterior to a timelike worldtube. In this case, a linearized solution for the binary black hole problem presents a consistent approximation to the initial data [46]. On the other hand, the series solutions in § VI C 2 are of somewhat limited use as testbed solutions. This is largely due to their finite radius of convergence. However, they may still find analytical use in the study of gravitational wave scattering off a Schwarzschild black hole, which is a topic of broad interest; see [78] and references therein.

Finally, we note that, in principle, applications of the characteristic formulation of the field equations can go beyond numerical simulations. For example, one could use the formulation as an analytical tool to investigate various aspects of spherically symmetric solutions and their perturbations in $f(R)$ gravity [79–83]. Using the characteristic formulation in this way will allow for a transparent interpretation and generalization of analytical results by using ready-built characteristic codes. It would also be of interest to pursue comparisons with the covariant $1+1+2$ semitetrad formalism [84].

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APPENDIX: CHRISTOFFEL SYMBOLS

In the following we present the Christoffel symbols for the Bondi-Sachs metric (14):

$$\Gamma^u{}_{AB} = \frac{1}{2}e^{-2\beta}\partial_r(r^2 h_{AB}) \tag{A1}$$

$$\Gamma^r{}_{rr} = 2\partial_r\beta \tag{A2}$$

$$\Gamma^A{}_{rB} = \frac{1}{2}(r^{-2}h^{AC})\partial_r(r^2 h_{CB}) \tag{A3}$$

$$\Gamma^u{}_{uu} = -\frac{1}{2}e^{-2\beta}[2\partial_u(-e^{2\beta}) + \partial_r(r^{-1}Ve^{2\beta}) - \partial_r(r^2 h_{AB}U^A U^B)] \tag{A4}$$

$$\Gamma^r{}_{rA} = \frac{1}{2}e^{-2\beta}[\partial_A(e^{2\beta}) + r^2 h_{AB}\partial_r(U^B)] \tag{A5}$$

$$\Gamma^r{}_{ru} = -\frac{1}{2}e^{-2\beta}[\partial_r(-r^{-1}Ve^{2\beta} + r^2 h_{AB}U^A U^B)] - \frac{1}{2}e^{-2\beta}U^A[\partial_r(-r^2 h_{AB}U^B) + \partial_A(e^{2\beta})] \tag{A6}$$

$$\Gamma^u{}_{uA} = \frac{1}{2}e^{-2\beta}[\partial_A(e^{2\beta}) - U^B\partial_r(r^2 h_{AB}) - r^2 h_{AB}\partial_r(U^B)] \tag{A7}$$

$$\Gamma^r{}_{uu} = -\frac{1}{2}e^{-2\beta}\partial_u(-e^{2\beta}r^{-1}V + r^2 h_{AB}U^A U^B) - \frac{1}{2}e^{-2\beta}r^{-1}V[2\partial_u(e^{-2\beta}) + \partial_r(-e^{2\beta}r^{-1}V + r^2 h_{AB}U^A U^B)] + \frac{1}{2}e^{-2\beta}U^A[2\partial_u(r^2 h_{AB}U^B) + \partial_A(-e^{2\beta}r^{-1}V + r^2 h_{CD}U^C U^D)] \tag{A8}$$

$$\Gamma^r{}_{uA} = -\frac{1}{2}e^{-2\beta}[\partial_A(-e^{2\beta}r^{-1}V + r^2 h_{AB}U^A U^B)] - \frac{1}{2}r^{-1}Ve^{-2\beta}[\partial_A(e^{2\beta}) - \partial_r(r^2 h_{AB}U^B)] - \frac{1}{2}e^{-2\beta}U^B[\partial_u(r^2 h_{AB}) - \partial_A(r^2 h_{CB}U^C) + \partial_B(r^2 h_{AC}U^C)] \tag{A9}$$

$$\Gamma^r{}_{AB} = \frac{1}{2}e^{-2\beta}[2\partial_A(r^2 h_{BC}U^C) + \partial_u(r^2 h_{AB})] - \frac{1}{2}r^{-1}Ve^{-2\beta}[\partial_r(r^2 h_{AB})] - r^2 e^{-2\beta}U_D{}^{(2)}\Gamma^D{}_{AB} \tag{A10}$$

$$\Gamma^A{}_{uu} = \frac{1}{2}e^{-2\beta}U^A[2\partial_u(e^{2\beta}) + \partial_r(-e^{2\beta}r^{-1}V + r^2 h_{CD}U^C U^D)] - \frac{1}{2}r^{-2}h^{AB}[2\partial_u(r^2 h_{CB}U^C) + \partial_B(-e^{2\beta}r^{-1}V + r^2 h_{CD}U^C U^D)] \tag{A11}$$

$$\Gamma^A{}_{ur} = \frac{1}{2} r^{-2} h^{AC} [\partial_r(-r^2 h_{CD} U^D) + \partial_C(e^{2\beta})] \tag{A12}$$

$$\Gamma^A{}_{Bu} = \frac{1}{2} e^{-2\beta} U^A [\partial_B(e^{2\beta}) - \partial_r(r^2 h_{CB} U^C)] + \frac{1}{2} r^{-2} h^{AC} [\partial_B(-r^2 h_{CD} U^D) + \partial_u(r^2 h_{BC}) + \partial_C(r^2 h_{BD} U^D)] \tag{A13}$$

$$\Gamma^A{}_{BC} = \frac{1}{2} e^{-2\beta} U^A [\partial_r(r^2 h_{BC})] + {}^{(2)}\Gamma^A{}_{BC} \tag{A14}$$

In the above, ${}^{(2)}\Gamma^A{}_{BC}$ represents the Christoffel symbols of the 2-metric h_{AB}.

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