Addendum to "Violating the string winding number maximally in anti–de Sitter space"

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We revisit the computation of maximally winding violating string amplitudes in three-dimensional anti–de Sitter space discussed in Ref. G. Giribet, Violating the string winding number maximally in anti-de Sitter space, Phys. Rev. D **84**, 024045 (2011). Here, we give an alternative derivation of these observables. This derivation, the simplest to the best of our knowledge, follows from identities between spectrally flowed representations of $\hat{sl}(2)_k$ Kac-Moody algebra and, in contrast to G. Giribet, Violating the string winding number maximally in anti-de Sitter space, Phys. Rev. D **84**, 024045 (2011), it does not resort to conjugate representations with auxiliary fields, but it rather involves the standard Wakimoto free field representation.

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I. INTRODUCTION

We reconsider the problem of computing tree-level scattering amplitudes of winding string states on Lorentzian three-dimensional anti–de Sitter (AdS_3) spacetime, focusing on processes that involve short strings and in which the total winding number conservation is violated maximally; that is, processes of n strings in which the total winding number $|\sum_{i=1}^{n}\omega_i|$ equals $n-2$ [1].

These amplitudes are given by n-point correlation functions of vertex operators that correspond to spectrally flowed representations of Kac-Moody algebra [2]. It was conjectured in [3] that these amplitudes are in correspondence with n-point correlation functions of Liouville theory, and a free field derivation of such correspondence was given in [4]. Here, we provided an alternative derivation. The advantage of the computation done here is that it is notably more succinct than the one of [4] as it does not resort to any conjugate representation of the vertex algebra; it rather involves the standard Wakimoto representation [5] with no need of auxiliary fields.

II. KAC-MOODY CURRENTS

The string σ-model on AdS_3 is given by the $SL(2,\mathbb{R})$ Wess-Zumino-Witten (WZW) action at level $k=R^2/\alpha'$, where R is the radius of AdS_3. The Hilbert space of string theory consists of Virasoro primary states organized in unitary representations of the $\hat{sl}(2)_k+\hat{sl}(2)_k$ Kac-Moody algebra that generates the affine symmetry of the WZW theory. More precisely, first one has to consider the universal covering of the $SL(2,\mathbb{R})$ discrete series $\mathcal{D}_j^{\pm}$, usually labeled by $j\in\mathbb{R}$ and $m=\pm j,\pm j\mp 1,\pm j\mp 2,\ldots$, together with the principal continuous series $\mathcal{C}_j^{\alpha}$ with $j=-\frac{1}{2}+i\lambda$ with $\lambda\in\mathbb{R}$, $0\leq\alpha\leq 1$ and $m=\alpha,\alpha\pm 1,\alpha\pm 2,\ldots$. Then, one has to include the spectrally flowed extension of these representations, $\mathcal{D}_j^{\pm,\omega}$ and $\mathcal{C}_j^{\alpha,\omega}$, which are needed to fully describe the string spectrum on the NS-NS AdS_3 background [2]. Here, we focus on the states of discrete representations, the so-called short strings.

The generators of the Kac-Moody $\hat{sl}(2)_k$ algebra satisfy the Lie products

$$[J_n^3,J_m^{\pm}]=\pm J_{n+m}^{\pm},$$
$$[J_n^3,J_m^3]=\frac{k}{2}m\delta_{n+m,0},$$
$$[J_n^+,J_m^-]=-2J_{n+m}^3+kn\delta_{n=m,0} \tag{1}$$

with $a=3,\pm$. These brackets, and its complex conjugate counterpart, are realized by defining the local currents

$$J^a(z)=\sum_{n\in\mathbb{Z}}J_n z^{-1-n},$$
$$\bar{J}^a(\bar{z})=\sum_{n\in\mathbb{Z}}\bar{J}_n\bar{z}^{-1-n} \tag{2}$$

and computing the operator product expansion (OPE) among them. A useful representation of these local currents has been given by Wakimoto [5], who proposed

$$J^+(z)=\beta(z), \tag{3}$$

$$J^3(z)=-\beta(z)\gamma(z)-\sqrt{\frac{k-2}{2}}\partial\phi(z), \tag{4}$$

$$J^-(z)=\beta(z)\gamma^2(z)+\sqrt{2k-4}\gamma(z)\partial\phi(z)+k\partial\gamma(z), \tag{5}$$

with the free field propagators

$$\langle\phi(z)\phi(w)\rangle=-\log(z-w),$$
$$\langle\beta(z)\gamma(w)\rangle=\frac{1}{(z-w)}; \tag{6}$$

and analogously for the anti-holomorphic contributions.

III. SPECTRAL FLOW

Algebra (1) is invariant under the spectral flow operation

$$J_n^3 \to J_n^3 + \frac{k}{2}\omega\delta_{n,0}, \qquad J_n^\pm \to J_{n\pm\omega}^\pm. \tag{7}$$

This generates a whole family of new representations, usually denoted $\mathcal{C}_j^{\alpha,\omega}$ and $\mathcal{D}_j^{\pm,\omega}$, and hereafter called spectrally flowed representations. More precisely, for $|\omega| > 1$, automorphism (7) does generate new representations; however, the cases $\omega = \pm 1$ are special in the sense that the highest (and lowest) weight representations of the sector $\omega = 0$ coincide with lowest (resp highest) weight representations of the sector $\omega = -1$ (resp. $\omega = +1$). This results in the identification of the discrete representations

$$\mathcal{D}_j^{\pm,\omega=0} \leftrightarrow \mathcal{D}_{-\frac{k}{2}-j}^{\mp,\omega=\pm 1}, \tag{8}$$

which will be crucial for the argument herein.

The new Kac-Moody primaries $|j,m,\omega\rangle$, which are annihilated by the positive modes of the new (spectrally flowed) currents, are essential to construct the string spectrum [2]. Algebraically, these states are defined as those that obey

$$J_0^3|j,m,\omega\rangle = \left(m + \frac{k}{2}\omega\right)|j,m,\omega\rangle,$$
$$\bar{J}_0^3|j,\bar{m},\omega\rangle = \left(\bar{m} + \frac{k}{2}\omega\right)|j,\bar{m},\omega\rangle, \tag{9}$$

together with

$$J_{n>\mp\omega}^\pm|j,m,\omega\rangle = 0, \qquad \bar{J}_{n>\mp\omega}^\pm|j,\bar{m},\omega\rangle = 0. \tag{10}$$

In the case of long strings, corresponding to states of the continuous representations $\mathcal{C}_j^{\alpha,\omega}$, the parameter ω of the spectral flow transformation is interpreted as the winding number of the asymptotic states, associated to the presence of a nonvanishing B-field in the background. In the case short strings, those described by states of the discrete representations $\mathcal{D}_j^{\pm,\omega}$, the geometrical interpretation of ω is less clear, but it still contributes to the mass-shell condition in a way that resembles a winding number.

Due to the duality among different representations (8), it will be enough for our purpose to consider the spectral flow sector $\omega = 0$. In terms of the Wakimoto fields, the vertex operators that create the states of this sector take the form

$$\Phi_{j,m,\bar{m}}^{\omega=0}(z) = c_0\gamma^{j-m}(z)\bar{\gamma}^{j-\bar{m}}(\bar{z})e^{\sqrt{\frac{2}{k-2}}j\phi(z,\bar{z})}, \tag{11}$$

where c_0 is a normalization constant that here we will set to 1 for convention. It can be easily checked that operators (11) of the spectral flow sector $\omega = 0$ have the following OPE with the $sl(2)_k$ Kac-Moody currents

$$J^3(z)\Phi_{j,m,\bar{m}}^0(w) \simeq \frac{m}{(z-w)}\Phi_{j,m,\bar{m}}^0(w) + \cdots \tag{12}$$

$$J^\pm(z)\Phi_{j,m,\bar{m}}^0(w) \simeq \frac{(\pm j - m)}{(z-w)}\Phi_{j,m\pm 1,\bar{m}}^0(w) + \cdots \tag{13}$$

which realize (9)–(10) for $\omega = 0$.

Vertex operators $\Phi_{j,m,\bar{m}}^\omega$ are the objects that create the states $|j,m,\omega\rangle \times |j,\bar{m},\omega\rangle$ out of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ invariant vacuum $|0\rangle$; namely

$$\lim_{z\to 0}\Phi_{j,m,\bar{m}}^\omega(z,\bar{z})|0\rangle = |j,m,\omega\rangle \times |j,\bar{m},\omega\rangle \tag{14}$$

The conformal dimension of these states is given by

$$h_{j,m}^\omega = -\frac{j(j+1)}{k-2} - m\omega - \frac{k}{4}\omega^2,$$
$$h_{j,\bar{m}}^\omega = -\frac{j(j+1)}{k-2} - \bar{m}\omega - \frac{k}{4}\omega^2 \tag{15}$$

Notice that, indeed, the states $|j,\pm j,0\rangle$ and $|-k/2-j,\pm k/2 \pm j,\mp 1\rangle$ have the same quantum numbers, in accordance to (8). That is, they have the same eigenvalue under J_0^3, namely $\pm j$, and the same conformal weight $h_{j,\pm j}^0 = h_{-k/2-j,\pm k/2\pm j}^{\mp 1}$. The formula for the conformal weight (15) also remains unchanged under the Weyl reflection $j \to -1-j$. That is, the states $|j,m,\omega\rangle$ and $|-1-j,m,\omega\rangle$ have the same quantum numbers, and in particular $h_{j,m}^\omega = h_{-1-j,\pm m}^{\pm\omega}$.

IV. CORRELATION FUNCTIONS

Consider the maximally winding violating correlation function

$$X_n^{2-n} = \left\langle \prod_{a=1}^{2}\Phi_{-1-j_a,m_a,m_a}^0(z_a,\bar{z}_a)\prod_{i=3}^{n}\Phi_{j_i,j_i,j_i}^{+1}(z_i,\bar{z}_i)\right\rangle \tag{16}$$

which involves $n-2$ highest-weight states of the spectral flow sector $\omega = 1$ (the argument works for $\omega = -1$ as well). In virtue of (8) we can equal this correlator to the following one

$$X_n^{2-n} = c^{2-n}\left\langle \prod_{a=1}^{2}\Phi_{-1-j_a,m_a,m_a}^0(z_a,\bar{z}_a)\right.$$
$$\left.\times\prod_{i=3}^{n}\Phi_{-\frac{k}{2}-j_i,\frac{k}{2}+j_i,\frac{k}{2}+j_i}^0(z_i,\bar{z}_i)\right\rangle \tag{17}$$

which only involves vertices (11), of the sector $\omega = 0$. This is different to the computation done in [4], where the presence of vertices with $\omega \neq 0$ demands the inclusion of extra fields. The factor c^{2-n} in (17) stands for the relative normalization between the operators $\Phi_{j,\pm j,\pm j}^0$ and $\Phi_{-\frac{k}{2}-j,\pm\frac{k}{2}\pm j,\pm\frac{k}{2}\pm j}^{\mp 1}$, which remains unspecified [3].

Involving only operators of the unflowed sector, correlator (17) admits to be computed by using the Wakimoto representation (11), straightforwardly applying the techniques described in reference [6], with no need of auxiliary fields cf. [7,8]. This results in

$$X_n^{2-n} = c^{2-n} \prod_{a=1}^{2} \frac{\Gamma(-j_a - m_a)}{\Gamma(1+j_a+m_a)} \prod_{i<j}^{n} |z_i - z_j|^{-\frac{4}{k-2}(j_i+\frac{k}{2})(j_j+\frac{k}{2})+2b_{ij}} \times \Gamma(-s) \int \prod_{r=1}^{s} d^2 w_r \prod_{r=1}^{s} \prod_{i=1}^{n} |z_i - w_r|^{-\frac{4}{k-2}(j_i+\frac{k}{2})} \prod_{r<t}^{s} |w_t - w_r|^{-\frac{4}{k-2}}, \tag{18}$$

where

$$s = -1 - \sum_{i=1}^{n} j_i - \frac{k}{2}(n-2) \tag{19}$$

and $b_{ij} = 0$ for $i, j > 3, 4, \ldots n$; $b_{ai} = j_a$ for $a = 1, 2$ and $i > 2$; $b_{ab} = j_a + j_b - 2b^2$ for a, $b = 1, 2$.

The integrand in (18) follows from the operator product expansions

$$\gamma(z_a)^{j_a - m_a} \bar{\gamma}(\bar{z}_a)^{j_a - m_a} \prod_{r=1}^{s} \beta(w_r) \bar{\beta}(\bar{w}_r) \simeq \frac{\Gamma^2(1+j_a+m_a+s)}{\Gamma^2(1+j_a+m_a)} \prod_{r=1}^{s} |z_a - w_r|^{-2} + \cdots$$

and

$$e^{-\sqrt{\frac{2}{k-2}}(j_a+1)\phi(z_a,\bar{z}_a)} \prod_{r=1}^{s} e^{-\sqrt{\frac{2}{k-2}}\phi(w_r,\bar{w}_r)} \simeq \prod_{r=1}^{s} |z_a - w_r|^{-\frac{4}{k-2}(j_a+1)} + \cdots,$$

$$e^{-\sqrt{\frac{2}{k-2}}(j_i+\frac{k}{2})\phi(z_i,\bar{z}_i)} \prod_{r=1}^{s} e^{-\sqrt{\frac{2}{k-2}}\phi(w_r,\bar{w}_r)} \simeq \prod_{r=1}^{s} |z_i - w_r|^{-\frac{4}{k-2}(j_i+\frac{k}{2})} + \cdots,$$

where the ellipses stand for subleading contributions with less Wick contractions. Recall that the β-dependent operators

$$\int d^2 w \beta(w) \bar{\beta}(\bar{w}) e^{-\sqrt{\frac{2}{k-2}}\phi(w,\bar{w})} \tag{20}$$

come from the interaction term of the $SL(2,\mathbb{R})$ WZW action when written in the Wakimoto representation [6] and, in the Coulomb gas approach, they act as certain amount (s) of screening operators needed to compensate the dilatonic background charge.

The integral on the right-hand side of (18) can also be identified as a correlation function in Liouville field theory. More precisely, the n-point function of exponential primary operators in Liouville theory takes the form [9]

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i, \bar{z}_i) \right\rangle_{\mathrm{L}} = \Gamma(-s) \prod_{i<j}^{n} |z_i - z_j|^{-4\alpha_i \alpha_j} \times \int \prod_{r=1}^{s} d^2 w_r \prod_{r=1}^{s} \prod_{i=1}^{n} |z_i - w_r|^{-4b\alpha_i} \prod_{r<t}^{s} |w_t - w_r|^{-4b^2} \tag{21}$$

with

$$bs + \sum_{i=1}^{n} \alpha_i = Q, \qquad Q = b + \frac{1}{b}. \tag{22}$$

In these variables, the Liouville central charge reads $c = 1 + 6Q^2$. Then, the dictionary between (18) and (21) is simple and is the one of [3]; namely

$$\alpha_i = b\left(j_i + \frac{b^2}{2} + 1\right), \qquad b^2 = \frac{1}{k-2}. \tag{23}$$

In conclusion, we arrive to the formula

$$X_n^{2-n} = c^{2-n} \prod_{i=1}^{2} \frac{\Gamma(-j_i - m_i)}{\Gamma(1+j_i+m_i)} \prod_{i<j}^{n} |z_i - z_j|^{2b_{ij}} \times \left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i, \bar{z}_i) \right\rangle_{\mathrm{L}}, \tag{24}$$

which is analogous to the one conjectured in [3] and what we actually wanted to prove.

V. CONCLUSIONS

We have derived formula (24), which expresses the n-point function of maximally winding violating processes in AdS_3 in terms of n-point correlation functions of Liouville field theory. This is analogous to the expressions proposed in [3,4], here obtained in a remarkably succinct way without resorting to nothing but well-known dualities among spectrally flowed representations and to the standard Wakimoto fields, with no need of auxiliary fields cf. [4,7,8].

Despite its simplicity, the derivation presented here has to be regarded as complementary to that in [4] and by no means as its generalization. This is because the one here has its limitations as well: it only involves states with winding numbers $\omega = 0, \pm 1$ and deals with the cases in which the $n-2$ states with $\omega \neq 0$ belong to the highest or lowest weight representations. Still, this is the simplest derivation of winding violating processes in AdS_3 and the simplest example of WZW-Liouville correspondence given so far.

[1] J. M. Maldacena and H. Ooguri, Strings in AdS_3 and the SL (2,R) WZW model. Part 3: Correlation functions, Phys. Rev. D **65**, 106006 (2002).

[2] J. M. Maldacena and H. Ooguri, Strings in AdS_3 and the SL (2,R) WZW model. Part 1: The spectrum, J. Math. Phys. (N.Y.) **42**, 2929 (2001).

[3] S. Ribault, Knizhnik-Zamolodchikov equations and spectral flow in AdS3 string theory, J. High Energy Phys. 09 (2005) 045.

[4] G. Giribet, Violating the string winding number maximally in anti-de Sitter space, Phys. Rev. D **84**, 024045 (2011).

[5] M. Wakimoto, Fock representations of the affine Lie algebra $A(1)^1$, Commun. Math. Phys. **104**, 605 (1986).

[6] K. Becker and M. Becker, Interactions in the SL(2,R)/U(1) black hole background, Nucl. Phys. **B418**, 206 (1994).

[7] G. Giribet and C. Núñez, Aspects of the free field description of string theory on AdS_3, J. High Energy Phys. 06 (2000) 033.

[8] G. Giribet and C. Núñez, Correlators in AdS_3 string theory, J. High Energy Phys. 06 (2001) 010.

[9] M. Goulian and M. Li, Correlation Functions in Liouville Theory, Phys. Rev. Lett. **66**, 2051 (1991).