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Entropy of a Taub-bolt-AdS spacetime from an improved action principle

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In this article the study of four-dimensional spaces with NUT charges and their ground states is continued. In particular, through the introduction of an improved action principle containing topological densities, it is shown that these spaces have simple and sound thermodynamical relations. As an example, the entropy of the Taub-bolt-AdS solution is computed in terms of a Noether charge.

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I. INTRODUCTION

In general terms, within a family of solutions characterized by a set of parameters, a ground state can be identified with a solution in the family whose number of symmetries is larger due to the fact that a subset of those parameters vanish, or satisfy a particular condition that reduces the dimension of the space of parameters. For gravity this usually leads to constant-curvature spaces or identified constant-curvature spaces without fixed points [1]. Now, it is tempting to try to identify those parameters with conserved charges; however, this is not general as that requires being able to define conservation in the space. For that, the space must have (at least asymptotically) a time-like symmetry.

Once conserved charges can be defined it becomes necessary to identify them. Among the different approaches to identify the conserved charges the Noether method definitively has a prominent role. It is worth recalling that the Noether charges are constructed out of symmetries of the solutions, or equivalently, from the subset of the symmetries of the theory with parameters satisfying some *Killing conditions*. In the case of gravity the symmetry of interest is the invariance under diffeomorphisms, and the *Killing* conditions define the *Killing* vectors.

It is usually overlooked that the soundness of the Noether charges relies on the action principle considered, and thus some pathologies might arise if the action principle is not adequate for the problem. For instance, the Noether charges computed out of the plain Einstein-Hilbert action, sometimes called Komar charges, have two major drawbacks. Different normalizations are required to compute the mass and the angular momenta of the solutions. They also diverge for asymptotically anti–de Sitter (AdS) spaces. In the context of the AdS/CFT conjecture this is certainly something that needs to be addressed. For instance, the holographic renormalization method [3] handles those divergences by the addition of boundary terms that preserve Dirichlet boundary conditions for the metric.

Finally, in the context of this work, it is worth mentioning that the first law of thermodynamics for stationary black holes can also be obtained in terms of Noether charges as well as an expression for their entropies [4].

The other version of the problem of charges corresponds to the Hamiltonian approach. In 1974, it was shown in Ref. [5] that the Hamiltonian generators of diffeomorphisms, besides having a bulk density (which actually vanishes on shell), must have as well a boundary part. It is precisely this boundary part that is responsible for the Hamiltonian charges, as its on-shell value evaluated on the different Killing vectors of the solution yields the mass or the angular momenta. Remarkably, the problem of different normalizations for the mass and angular momenta is absent in this approach. This discrepancy between the Komar charges and their Hamiltonian counterpart was clarified in Ref. [6] using phase-space techniques. The divergences mentioned above for asymptotically AdS spaces can be tamed at the expense of the introduction of an *ah hoc* background. In Ref. [7], a different solution based on the exterior curvature of the *infinity* was proposed.

In Ref. [8], a different approach to the problem of charges and the regularization of the action principle was proposed. This approach is based on the addition of topological densities and provides a regularized and well-defined action principle for asymptotically locally AdS spaces and simultaneously sound Noether charges. Moreover, these Noether charges are in one-to-one correspondence with their Hamiltonian (charges) counterpart [9], unlike those constructed out of the Komar potentials. Finally, the Noether charges (constructed out of this new action principle) coincide [10] with the (generalization of) Brown-York boundary charges defined in Ref. [11]. Remarkably, there is a natural way to extend this approach to any higher even dimension [12]. This was extended even further in Refs. [13,14], where its equivalence with the holographic renormalization method for all known cases was also shown.

A. Topological charges as Noether charges

To fix some ideas one can recall the case of electromagnetism. The action principle $\int F \wedge^* F$, through the gauge symmetry $A \rightarrow A + d\lambda$, gives rise to

$$Q(\lambda) = \int_{\partial\Sigma_\infty} \lambda^* F,$$

where $\partial\Sigma_\infty$ is the radial infinity of a constant time slice of the space. To identify the electric charge as a Noether charge the parameter of the gauge transformation λ must satisfy $d\hat{\lambda} = 0 \leftrightarrow \hat{\lambda} = \text{const}$, which corresponds to the *Killing* condition for $U(1)$. The electric charge is given by $q_e = Q(\hat{\lambda})/\hat{\lambda}$.

In four dimensions the presence of a nontrivial $U(1)$ fiber bundle determines the presence of magnetic charges whose value can be obtained from

$$q_m = \int_{\Sigma_\infty} F.$$

Remarkably, this charge can be incorporated into a Noether charge provided the Pontryagin density, $\int F \wedge F$, is added to the action principle. Notice that this does not change the equations of motion. By considering

$$I = \int F \wedge {}^*F + \beta \int F \wedge F,$$

the integral of the Noether current is given by

$$Q(\lambda) = \int_{\Sigma_\infty} \lambda({}^*F + \beta F),$$

and therefore $Q(\hat{\lambda})/\hat{\lambda} = q_e + \beta q_m$.

A ground state can be defined by the vanishing of both the electric and magnetic charges, i.e., $q_e = 0$ and $q_m = 0$. However, this is not the only possibility. It can be noticed that by fixing $\beta = \pm 1$ the action can be rewritten as

$$I = \pm \frac{1}{2} \int ({}^*F \pm F) \wedge ({}^*F \pm F),$$

and therefore for the (anti-)self-dual case ${}^*F = \pm F$ the action vanishes. Furthermore, considering the Euclidean version of the action principle, this has its minimum value in this case. This also defines $Q(\hat{\lambda}) = 0$. With this in mind, any (anti-)self-dual solution can be cast as a *ground state* as well.

In Ref. [15], the ideas above were extended to gravity in terms of the separation of the Weyl tensor into its electric and magnetic parts. There, the idea that spaces whose associate Weyl tensors are nontrivially (anti-)self-dual can be cast as proper ground states was proposed and justified. For this an improved action principle is introduced. This action is regularized for asymptotically locally AdS spaces with topological defects and the associated Noether charges vanish for spaces whose Weyl tensor is (anti-)self-dual.

In four dimensions the simplest nontrivial examples whose Weyl tensors are (anti-)self-dual are the Taub-NUT and Taub-NUT-AdS solutions. These solutions contain what is usually called a NUT charge (see, for instance, Ref. [16]), but they are still asymptotically locally flat and asymptotically locally AdS, respectively. For the AdS/CFT conjecture, this opened the possibility to address conformal field theories defined on conformal manifolds with topological nontrivial defects. For a discussion about this, see Ref. [17].

This article aims to proceed with the analysis of the action principle presented in Ref. [15]. Here it is shown that its Noether charges are in one-to-one correspondence with their Hamiltonian counterpart. As an example, we also compute the entropy of the Taub-bolt-AdS solution in terms of a Noether charge.

Before we proceed it is worth stressing that the spaces we consider in this work are Euclidean and have a well-defined asymptotically locally AdS region. In general, these spaces will be considered to have a well-defined split as $\mathcal{M} = S^1 \times \Sigma$, where Σ corresponds to a three-dimensional spacelike hypersurface and $\mathbb{R}$ stands for the time direction. In addition, $\partial\Sigma$ will be considered as the union of an exterior and an interior surface; thus, $\partial\Sigma = \partial\Sigma_\infty \oplus \partial\Sigma_H$.

To keep the notation as close as possible to a gauge theory, the computation will be expressed in the first-order formalism. However, changing to the second-order formalism is simple.

II. ADDITION OF TOPOLOGICAL INVARIANT DENSITIES AND CHARGES

Let us start by reviewing the original proposal in four dimensions [8]. This corresponds to adding the Euler density to the Einstein-Hilbert action (plus a negative cosmological constant). This allows us to express the action principle, either at first or second order, as

$$I_{\text{reg}} = \frac{l^2}{64\pi G} \int \bar{R}^{ab} \bar{R}^{cd} \varepsilon_{abcd} = \frac{l^2}{64\pi G} \int \delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4} \bar{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} \bar{R}^{\nu_3\nu_4}{}_{\mu_3\mu_4} \sqrt{g} d^4x, \quad (1)$$

where

$$\bar{R}^{\nu_1\nu_2}{}_{\mu_1\mu_2} = R^{\nu_1\nu_2}{}_{\mu_1\mu_2} + \frac{1}{l^2} \delta^{\mu_1\mu_2}_{\nu_1\nu_2}, \quad (2)$$

where $R^{\nu_1\nu_2}{}_{\mu_1\mu_2}$ is the Riemann tensor. The cosmological constant is given by $\Lambda = -3l^{-2}$. In Eq. (1),

$$R^{ab} = \frac{1}{2} e^a_{\nu_1} e^b_{\nu_2} R^{\nu_1\nu_2}{}_{\mu_1\mu_2} dx^{\mu_1} \wedge dx^{\mu_2}$$

is called the curvature two-form. Here $\{e^a{}_\nu\}$ is a orthonormal basis of four-dimensional (co)vectors of the cotangent space of the manifold $\mathcal{M}$. This defines a vielbein, $e^a = e^a{}_\nu dx^\nu$.

To continue with the discussion one can observe that the action principle (1) vanishes for any locally AdS space. In this way, this action principle is tailored such that any locally AdS spaces can be cast as ground states. Moreover, this is finite for any asymptotically locally AdS solution.

One remarkable fact about the action principle (1) is that on shell it can be cast as

$$I_{\mathrm{reg}}\big|_{\text{on shell}} = \frac{l^2}{64\pi G}\int C^{\nu_1\nu_2}{}_{\mu_1\mu_2} C^{\mu_1\mu_2}{}_{\nu_1\nu_2}\sqrt{g}d^4x, \tag{3}$$

where $C^{\nu_1\nu_2}{}_{\mu_1\mu_2}$ is the Weyl tensor. This connects this action principle with Weyl gravity in the same fashion as in Ref. [18]. In this context it is worth mentioning Ref. [14], where the connection between Eq. (3) and the standard electromagnetic action, $\int F^*F$, was originally discussed in these terms.

Here it is worth mentioning that the values of the masses of Taub-Nut-AdS and Taub-bolt-AdS solutions were discussed in Ref. [8]. This was done in the context of the previous results by Hawking [19] where the mass of Taub-bolt-AdS was computed with respect to the Taub-Nut-AdS solution. The result in Ref. [8] showed that both masses can be computed separately and that actually their difference reproduces the result in Ref. [19]. Because of this, the mass computed with the action principle (1) can be cast as the electric *mass* of the solution (see Ref. [15]).

A. On gravitational solitons

In Ref. [15] the previous analysis was continued for spaces whose Weyl tensor is (anti-)self-dual. In this section these ideas are reexpressed in the first-order formalism. The Einstein-Hilbert action is not only regularized for asymptotically locally AdS spaces, by the addition of the Euler density, but it is also complemented by the addition of the Pontryagin density such that

$$I_{GP} = I_{\mathrm{reg}} + \alpha\int R^{ab}R_{ab}, \tag{4}$$

where α is a constant. This term can be identified with $\int F\wedge F$ for electromagnetism.

As discussed in detail in Ref. [15], the addition of the Pontryagin density—which in principle alters the value of the action principle—does not modify the finiteness of the action principle. Moreover, it must be stressed that most of the renowned analytic solutions (such as Schwarzschild or Kerr AdS) have a vanishing Pontryagin density. Therefore, any ground state of these solutions can be cast as a ground state of the action principle (4). This is in complete analogy to electromagnetism where, in the absence of magnetic charges, the ground state is defined by the vanishing of the electric charge.

It must be noticed that, due to the fact that the solutions are Einstein manifolds,

$$R^{ab}R_{ab} = \bar{R}^{ab}\bar{R}_{ab}.$$

It is worthwhile to notice the presence of an AdS pedigree in this case. Provided one defines an AdS_4 connection in terms of the vielbein and the spin connection as $W^{AB} = (\omega^{ab}, e^a/l)$ (see, for instance, Ref. [20]), the AdS_4 Pontryagin density $\mathcal{F}^{AB}\mathcal{F}_{AB}$ can be expressed as

$$\mathcal{F}^{AB}\mathcal{F}_{AB} = \bar{R}^{ab}\bar{R}_{ab} - 2T^a,$$

where T^a is the torsion two-form. In the case at hand, $T^a = 0$ and therefore one has the identity

$$\mathcal{F}^{AB}\mathcal{F}_{AB} = \bar{R}^{ab}\bar{R}_{ab}.$$

Now one can return to address how to define a ground state for solutions with a nonvanishing Pontryagin $\bar{R}^{ab}\bar{R}_{ab}$ density. This arises by observing that

$$\begin{aligned}\int R^{ab}R_{ab}\Big|_{\text{on shell}} &= \int \bar{R}^{ab}\bar{R}_{ab}\Big|_{\text{on shell}}\\ &= 4\int (C)^{\nu_1\nu_2}{}_{\mu_1\mu_2}(C^*)^{\mu_1\mu_2}{}_{\nu_1\nu_2}\sqrt{g}d^4x,\end{aligned}$$

where $(C^*)^{\mu_1\mu_2}{}_{\nu_1\nu_2} = \frac{1}{2}C_{\nu_1\nu_2\alpha\beta}\varepsilon^{\alpha\beta\mu_1\mu_2}$ can be identified with the dual of the Weyl tensor. The dual of $\bar{R}^{ab}$ is given by $(\bar{R}_{ab})^* = \frac{1}{2}\varepsilon_{abcd}\bar{R}_{cd}$.

With this in mind, the constant α in Eq. (4) can be fixed by requiring that the action principle satisfies (on shell)

$$I_{GP}\big|_{\text{on shell}} \backsim \int (C\pm C^*)^2.$$

This is done in order to identify any solution with a (anti-)self-dual Weyl tensor as the ground state.

With $\alpha = \pm 1$ the solutions with a (anti-)self-dual Weyl tensor can be considered, in a broad sense, as *instantons* of a conformal theory of gravity. On the other hand, this choice of α gives rise to a natural extension of the previous action principle (1). Finally, in the first-order formalism, the action principle is given by

$$I_{GP} = I_{\mathrm{reg}} \pm \frac{l^2}{32\pi G}\int \bar{R}^{ab}\bar{R}_{ab}. \tag{5}$$

To finish this section, one can notice that the two-form $F^{ab} = \bar{R}^{ab} \pm \frac{1}{2}\eta^{aa'}\eta^{bb'}\varepsilon_{a'b'cd}\bar{R}^{cd}$ can be defined such that

$$\begin{aligned}F^{ab}F_{ab} &= (\bar{R}^{ab}\pm(\bar{R}^{ab})^*)(\bar{R}_{ab}\pm(\bar{R}_{ab})^*)\\ &= (\bar{R}^{ab}\bar{R}^{cd}\varepsilon_{abcd}\pm 2\bar{R}^{ab}\bar{R}_{ab}),\end{aligned}$$

which allows to rewrite the action principle as

$$I_{GP} = \frac{l^2}{64\pi G}\int F^{ab}F_{ab}.$$

III. HAMILTONIAN CHARGES VERSUS NOETHER CHARGES

In Ref. [15] it was shown that the Noether charges associated with the Killing vectors are given by

$$Q(\xi) = \frac{l^2}{32\pi G}\int_{\partial\Sigma} I_\xi \omega^{ab}(\bar{R}^{cd}\varepsilon_{abcd} \pm 2\bar{R}_{ab}). \tag{6}$$

As mentioned above, the Noether charges of the action principle (1) are exactly the Hamiltonian charges [9]. In order to analyze this for the Noether charge (6) [defined from the action principle (5)], it is enough to follow Ref. [6]. It is easy to demonstrate that the variation along the parameters—say, $\hat{\delta}$ [6]—of the Hamiltonian generator associated with a diffeomorphism defined by $x \to x + \xi$ is given on shell by

$$\hat{\delta}H(\xi)|_{\text{on shell}} = \hat{\delta}G(\xi) = \hat{\delta}(Q(\xi)) + \int_{\partial\Sigma_\infty \oplus \partial\Sigma_{\mathcal{H}}} I_\xi(\hat{\delta}\omega^{ab}(\bar{R}^{cd}\varepsilon_{abcd} \pm \bar{R}_{ab})). \tag{7}$$

One can recognize that the contribution from $\partial\Sigma_\infty$ of the second term vanishes for any space whose Weyl tensor asymptotically becomes (anti-)self-dual. To address the internal boundary $\partial\Sigma_{\mathcal{H}}$, it is necessary to impose boundary conditions. As discussed in Ref. [9], $\delta\omega^{ab}|_{\partial\Sigma_{\mathcal{H}}} = 0$ is indeed a proper boundary condition and therefore the Noether charges of the action (5) can be fixed as

$$G(\xi) = Q(\xi),$$

proving that the Noether charges (6) are indeed equivalent to the Hamiltonian charges.

It must be noticed that the boundary condition $\delta\omega^{ab}|_{\partial\Sigma_{\mathcal{H}}} = 0$ fixes the temperature in the case that $\partial\Sigma_{\mathcal{H}}$ is connected with the presence of a Killing horizon defined by ξ. To observe this, one can notice that the temperature of the Killing horizon can be read from the relation [21]

$$I_\xi \omega^a{}_b \xi^b|_{\mathbb{R}\times\partial\Sigma_H} = \kappa\xi^b, \tag{8}$$

where κ is the surface gravity. The temperature is given by $T = \kappa/4\pi$.

IV. ENTROPY IN A NOETHER CHARGE

To obtain the thermodynamics defined by the action principle in Eq. (5), one can follow the ideas in Ref. [4]. As a detailed discussion about this in the first-order formalism can be found in Ref. [10], only the highlights will be discussed here. To obtain the first law it is enough to notice (see Ref. [6] for the general expression) that the conservation of the flux along trajectories defined by the variations of parameters of the solutions, within the space configurations, implies that

$$\hat{\delta}\frac{l^2}{32\pi G}\int_{\partial\Sigma_{\mathcal{H}}} I_\xi\omega^{ab}(\bar{R}^{cd}\varepsilon_{abcd} \pm 2\bar{R}_{ab}) = \hat{\delta}\frac{l^2}{32\pi G}\int_{\partial\Sigma_\infty} I_\xi\omega^{ab}(\bar{R}^{cd}\varepsilon_{abcd} \pm 2\bar{R}_{ab}).$$

Next, one can observe that the rhs corresponds to the variation of the Noether charges evaluated in $\partial\Sigma_\infty$, such as the mass and the rest of the conserved charges, i.e.,

$$\hat{\delta}\left(\frac{l^2}{32\pi G}\int_{\partial\Sigma_{\mathcal{H}}} I_\xi\omega^{ab}(\bar{R}^{cd}\varepsilon_{abcd} \pm 2\bar{R}_{ab})\right) = \hat{\delta}M + \cdots, \tag{9}$$

which allows to identify, due to the fact that $T\hat{\delta}S = \hat{\delta}M + \cdots$,

$$T\hat{\delta}S = \hat{\delta}\left(\frac{l^2}{32\pi G}\int_{\partial\Sigma_{\mathcal{H}}} I_\xi\omega^{ab}(\bar{R}^{cd}\varepsilon_{abcd} \pm 2\bar{R}_{ab})\right).$$

Now, due to the boundary condition at the horizon [Eq. (8)], it is possible to single out the period [4,10]. This yields a similar expression for the entropy,

$$S = \hat{S} + \frac{\beta l^2}{32\pi G}\int_{\partial\Sigma_{\mathcal{H}}} I_\xi\omega^{ab}(\bar{R}^{cd}\varepsilon_{abcd} \pm 2\bar{R}_{ab}), \tag{10}$$

where β^{-1} is the inverse of the period and S_0 is an extensive constant independent of the parameters of the solution. As $\hat{S}$ cannot depend on the parameters of the solution (such as the mass) but must depend on l, it can be argued that $\hat{S} \sim P_0 V_0^{\text{reg}}$. In this case the mass must be identified with the enthalpy of the system instead of the energy. For a discussion, see Ref. [22].

V. GEOMETRY OF TAUB-NUT-ADS AND TAUB-BOLT-ADS SOLUTIONS

The four-dimensional Taub-bolt-AdS and Taub-NUT-AdS solutions are known to be described by

$$ds^2 = f(r)^2(d\tau + 2n\cos(\theta)d\varphi)^2 + \frac{dr^2}{f(r)^2} + (r^2 - n^2)(d\theta^2 + \sin(\theta)^2 d\varphi^2), \tag{11}$$

where

$$f(r)^2 = \frac{(r^2 + n^2) - 2mr + l^{-2}(r^4 - 6n^2r^2 - 3n^4)}{r^2 - n^2}.$$

It is easy to notice that ∂_φ and ∂_τ are two Killing vectors of the geometry.

The different between Taub-NUT-AdS and Taub-bolt-AdS solutions relies on the structure of the function $f(r)^2$, the norm of the Killing vector ∂_τ. Indeed, depending on the form of $f(r)^2$ its largest zero, say $r = r_+$, either corresponds to a point or a two-dimensional surface in the geometry. This latter case is called the bolt of the solution and is the case for a typical stationary black hole.

The mass of this solution [15] is given by

$$\begin{aligned} M = Q(\partial_\tau) &= \frac{l^2}{32\pi G}\int_{\partial\Sigma_\infty} I_{\partial_\tau}\omega^{ab}(\bar{R}^{cd}\varepsilon_{abcd} \pm 2\bar{R}_{ab}) \\ &= m - |n|\left(1 - \frac{4n^2}{l^2}\right) \\ &= \frac{(r_+ - |n|)^2}{2r_+ l^2}((r_+ + |n|)^2 - (l^2 - 4n^2)). \end{aligned} \quad (12)$$

A. Asymptotical geometry and squashed S^3

It is easy to confirm that this solution is asymptotically locally AdS, and for $m = 0$ it corresponds to a locally AdS solution, as expected. This motivated the identification of m as a mass parameter in Ref. [19]. The presence of the NUT charge can be noticed on the geometry of the asymptotical transverse section to the radial direction which tends to a squashed S^3 (see, for instance, ref. [23]).

After redefining $\tau = 2n\psi$ in Eq. (11) a suitable vierbein can be defined in terms of the dreibein $\tilde{e}^i$ for S^3 depicted in the Appendix. This vierbein is given by

$$e^a = \left(\frac{dr}{f(r)}, g^1(r)\tilde{e}^1, g^2(r)\tilde{e}^2, g^3(r)\tilde{e}^3\right), \quad (13)$$

with $a = 0, 1, 2, 3$. Here,

$$g^3 = 4nf(r),$$
$$g^1 = g^2 = 2\sqrt{r^2 - n^2}.$$

For simplicity, one can use the convention

$$g^i(r)\tilde{e}^i = (g^1(r)\tilde{e}^1, g^2(r)\tilde{e}^2, g^3(r)\tilde{e}^3),$$

where the repetition of i indices does not imply summation. For the rest of this article the same convention will be used. In terms of this vielbein, it is easy to observe that

$$\lim_{r\to\infty} ds^2 \approx \frac{l^2 dr^2}{r^2} + r^2\left(\left(\frac{2n}{l}\right)^2 (e^3)^2 + (\tilde{e}^1)^2 + (\tilde{e}^2)^2\right),$$

making explicit the presence of the NUT in this solution. Indeed, for $n = \pm l/2$ this condition disappears and the transverse section becomes a sphere.

The torsion-free connection of this vielbein is given by

$$\begin{aligned} \omega^{0i} &= -f(r)\frac{dg^i}{dr}\tilde{e}^i, \\ \omega^{ij} &= C^{ij}{}_k\tilde{e}^k, \end{aligned} \quad (14)$$

where

$$C^{ij}{}_k = \frac{(g^i)^2 + (g^j)^2 - (g^k)^2}{g^i g^j}\varepsilon_{ijk}.$$

Finally, $\bar{R}^{ab}$ is given by

$$\begin{aligned} \bar{R}^{0i} &= \left(-\frac{d}{dr}\left(f\frac{dg^i}{dr}\right)\frac{f}{g^i} + \frac{1}{l^2}\right)e^0 e^i \\ &\quad + f\left(C^i{}_{kl}\frac{dg^k}{dr} - \varepsilon^i{}_{kl}\frac{dg^i}{dr}\right)\tilde{e}^k\tilde{e}^l, \\ \bar{R}^{ij} &= \left(\frac{g^i g^j}{l^2} - f^2\frac{dg^i}{dr}\frac{dg^j}{dr}\right)\tilde{e}^i\tilde{e}^j + (C^{ij}{}_k\varepsilon^k{}_{lm} + C^i{}_{kl}C^{kj}{}_m)\tilde{e}^l\tilde{e}^m \\ &\quad + \frac{d}{dr}C^{ij}{}_k dr\tilde{e}^k. \end{aligned} \quad (15)$$

VI. ENTROPY

As mentioned above, the asymptotical conserved charges of the Taub-bolt-AdS solution were discussed in Ref. [15]. Following Ref. [4], in a manner of speaking, it only remains to confirm that the entropy can be obtained as the Noether charge associated with the Killing vector ∂_τ on $\partial\Sigma_{\mathcal{H}}$.

Before we proceed, a general geometrical consideration can be made concerning these solutions. Following the standard approach, the periods are to be fixed to avoid a conical singularity in the plane $r\psi$ at $r = r_+$. As expressed in Eq. (13), the period of ψ is fixed such that $0 \le \psi < 4\pi$, and therefore, independently of the particular value of r_+, the following must be satisfied:

$$\frac{d}{dr}(f^2(r))\Big|_{r=r_+} = \frac{1}{2n}. \quad (16)$$

With this in mind, one must stress that the difference between both geometries is that while for the NUT solution $r_+ = |n|$, for the bolt solution $r_+ = r_b > |n|$.

For the solution above it is easy to demonstrate that

$$\begin{aligned} I_{\partial_\psi}\omega^{ab} &= -f(r)\frac{dg^3}{dr}\delta^{ab}_{01} + \frac{1}{2}\delta^{ab}_{ij}C^{ij}{}_3 \\ &= -f(r)\frac{dg^3}{dr}\delta^{ab}_{01} + \left(2 - \frac{(g^3)^2}{(g^2)^2}\right)\delta^{ab}_{12} \\ &= -2n\frac{df(r)^2}{dr}\delta^{ab}_{01} + 2\delta^{ab}_{12}\left(1 - 2n^2\frac{f^2(r)}{r^2 - n^2}\right). \end{aligned} \quad (17)$$

To study the behavior near $r = r_+$, it is necessary to separate the NUT and bolt cases.

A. Taub-NUT

As mentioned above the Taub-Nut-AdS solution has a (anti-)self-dual Weyl tensor. To satisfy such a condition the value of m in Eq. (11) must be given by

$$m = |n|\left(1 - 4\frac{n^2}{l^2}\right). \tag{18}$$

It is worth mentioning that the case $m = 0 \leftrightarrow n = \pm l/2$ corresponds to a (locally) AdS space [17]. In this case,

$$f^2(r)_{\mathrm{NUT}} = \left(\frac{r - |n|}{r + |n|}\right)\left(1 + \frac{(r - |n|)(r + 3|n|)}{l^2}\right), \tag{19}$$

where one can notice that $r_+ = |n|$ explicitly. This determines that near $r = r_+ = |n|$ the geometry is given by

$$\lim_{r\to n} ds^2 \approx 2|n|\left(\frac{dr^2}{r - |n|} + 4(r - |n|)ds^2_{S^3}\right) \tag{20}$$

$$\approx d\rho^2 + \rho^2 ds^2_{S^3}. \tag{21}$$

One can readily notice that near $r = r_+ = |n|$ the geometry acquires a $SO(4)$ symmetry. Indeed, it can be noticed that $r = \pm n$ defines a point in this geometry and thus no horizon is presented in this case. Furthermore, every plane between the orbit of a SO(4) symmetry and ρ becomes flat as $r \to n$. In fact, one can take any of the generators of these orbits of symmetry and compute the period through Eq. (8) and the result is 4π, as expected. This only confirms the existence of nontrivial periods on the S^3 as it shrinks to a point.

B. Taub-bolt

For Taub-bolt-AdS, as mentioned above, the form of the metric is similar. However, the value of $r_+ > n$. In fact,

$$\begin{aligned} f(r)^2_{\mathrm{bolt}} = \frac{1}{r^2 - n^2}\Bigg[4\frac{r^4}{n^2 l^2} + \frac{r^2}{n^2}\left(4 - 24\frac{n^2}{l^2}\right) \\ + \frac{r}{n}\left[-4\frac{n^2 s^3}{l^2} + \left(24\frac{n^2}{l^2} - 4\right)s \\ + \frac{1}{s}\left(12\frac{n^2}{l^2} - 4\right) + 4 - 12\frac{n^2}{l^2}\right]\Bigg], \end{aligned} \tag{22}$$

where s is a parameter (see, for instance, Ref. [24]). It is worth stressing that the value of m in Eq. (11) in this case is given by

$$m = \frac{1}{2}\frac{(s^4 - 6s^2 - 3)n^3}{l^2 s} + \frac{1}{2}\frac{(s^2 + 1)n}{s}, \tag{23}$$

which vanishes for $\frac{n^2}{l^2} = \frac{s^2+1}{6s^2+3-s^4}$. In this case the solution is a locally AdS space with squashing $\frac{4(s^2+1)}{6s^2+3-s^4}$. Conversely, it must be noticed for $n = \pm l/2$ (the case where the squashing disappears at the asymptotical region) that $m = \frac{1}{16}\frac{(s^2-1)^2}{s}$. Obviously, for $s = \pm 1$ in this case the AdS space case is recovered.

Because $r_+ > n$ the near-horizon geometry is described by

$$\begin{aligned} \lim_{r\to r_+} ds^2 \approx d\rho^2 + \frac{\rho^2}{4}(d\psi + \cos(\theta)d\varphi)^2 \\ + (r_+^2 - n^2)(d\theta^2 + \sin^2(\theta)d\varphi^2), \end{aligned}$$

which is the generalization of a typical stationary black hole.

Computing the temperature from Eq. (8) in this case is unnecessary as this is fixed by $f^2(r_+)' = 1/(2n)$ for any value of $r_+ > |n|$. However, there is another aspect. After a direct computation for Taub-bolt-AdS,

$$I_{\partial_\psi}\omega^{ab}|_{\partial\Sigma_{\mathcal{H}}} = -\delta^{ab}_{03} + 2\delta^{ab}_{12}, \tag{24}$$

which differs in δ^{ab}_{12} from the analogous computation for Taub-NUT-AdS. Since the eigenvalues are identical, this difference indicates that the eigenvectors differ. This is due to the fact that for Taub-bolt-AdS the orbits of the symmetries on S^2 do not shrink to points as $r \to r_+$, showing the presence of a genuine Killing horizon at r_+.

C. The entropy of Taub-bolt

Given the previous considerations, it is possible to evaluate Eq. (10) with $\beta^{-1} = 8\pi n$ (the temperature), yielding

$$S_{TB} = \frac{3}{G}(r_+^2 - n^2)\pi = \frac{3}{4G}A_{\mathcal{H}}, \tag{25}$$

where $A_{\mathcal{H}}$ is the area of the horizon. The difference with the usual area 1/4 law is due to the presence of the NUT and has been noticed previously; see, for instance, Refs. [23,25–27]. It can be noticed that for $r_+ \to |n|$ this expression vanishes. This is expected as this corresponds to the Taub-NUT-AdS case. It is interesting to compare this computation with the discussion in Refs. [19,28], where a careful discussion was necessary to address the presence of the NUT charge.

VII. FREE ENERGY

In order to compute the value of the action principle it is natural to reexpress the action principle (5) in terms of the fields

$$F^{0i} = \bar{R}^{0i} \pm \frac{1}{2}\varepsilon^{0ijk}\bar{R}_{jk} \quad \text{and} \quad F^{jk} = \bar{R}^{jk} \pm \varepsilon^{jk0i}\bar{R}_{0i}.$$

In fact, this significantly simplifies the computations, as $F^{0i}F_{0i} = F^{ij}F_{ij}$ and thus the action principle actually can be merely rewritten as

$$I = \frac{l^2}{16\pi}\int F^{0i}F_{0i}. \tag{26}$$

The next step corresponds to analyzing the projection of $F^{0i}F_{0i}$ in terms of the vielbein. Through the identity

$$F^{0i}F_{0i} = 2\tilde{e}^3 I_{\partial_\psi}(F^{0i})F_{0i},$$

where

$$I_{\partial_\psi}F^{0i} = -dr\frac{d}{dr}\left(-f\frac{d}{dr}g^3 \pm C_3^{12}\right),$$

the action principle can be written as

$$I = \frac{l^2}{8\pi}\int -\tilde{e}^3 \wedge dr\frac{d}{dr} \times \left[-f\frac{d}{dr}g^3 \pm C_3^{12}\right]_{r_+}^{\infty} \wedge (\tilde{e}^1 \wedge \tilde{e}^2 I_{\tilde{E}_1\wedge\tilde{E}_2}F_{0i}), \tag{27}$$

where it must be noticed that

$$\int_{r_+}^{\infty} dr\frac{d}{dr}\left[-f\frac{d}{dr}g^3 \pm C_3^{12}\right] = \left[-f\frac{d}{dr}g^3 \pm C_3^{12}\right]_{r_+}^{\infty}.$$

Moreover, it can be recognized that the expression for the Noether charges [see Eq. (6)] can be identified. Therefore,

$$I = 8n\pi(Q(\partial_\tau)|_{\partial\Sigma_\infty} - Q(\partial_\tau)|_{\partial\Sigma_{\mathcal{H}}}), \tag{28}$$

where $8n\pi$ is the inverse of the temperature. Finally, this yields

$$I = \frac{1}{T}M - (S - S_0) = \frac{F}{T}, \tag{29}$$

where F can be cast as the free energy of the system.

VIII. CONCLUSIONS AND COMMENTS

In this article the discussion of the ground sates with nut charges was continued. In particular, the discussion was centered around the Taub-NUT-AdS solution. As already noticed, this solution only has vanishing charges [15]. However, the presence of the NUT charge makes the geometry differ significantly from the behavior of the ground states of more renowned black holes. For instance, Taub-NUT-AdS is not a locally AdS space and the number of global symmetries is not increased with respect to the Taub-bolt-AdS solution. Nonetheless, the Taub-Nut-AdS solution has no horizon and is certainly smooth at $r = |n|$. In order to understand this one has to put it in the same context as the presence of magnetic charges in electromagnetism, where the presence of electric charges is also necessary to define the ground state.

Here it was shown that the improved action principle presented in Ref. [15] is suitable for considering spaces with a nontrivial (anti-)self-dual Weyl tensor as ground states. In fact, it was shown that the action principle (5) provides a well-defined thermodynamics in spite of the presence of a NUT charge in the space. Moreover, it is also noteworthy that an entropy can be recovered from the expression in Eq. (25).

The generalization of these results to the known extensions of the Taub-bolt-AdS spaces with electric/magnetic charge and angular momentum is clear. Furthermore, the extension of these results to the topological black holes with NUT charge (roughly speaking, solutions whose transverse sections tend to squashed H_3) is also clear.

Unfortunately the generalization to higher dimensions is not obvious. The condition of (anti-)self-duality of the Weyl tensor is a property of four dimensions and cannot be extended univocally to dimensions higher than four.

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APPENDIX: A USEFUL AND NONTRIVIAL VIELBEIN ON S^3

In order to construct a suitable vielbein for S^3, defined by the quadratic form $V^2 + X^2 + Y^2 + Z^2 = 1$, one can use the Euler angles such that

$$X + iY = \cos\left(\frac{\theta}{2}\right)e^{\frac{i}{2}(\psi+\varphi)},$$
$$Z + iV = \sin\left(\frac{\theta}{2}\right)e^{\frac{i}{2}(\psi-\varphi)},$$

where $0 \le \theta < \pi$, $0 \le \varphi < 2\pi$, and $0 \le \psi < 4\pi$. This defines the vielbein [29]

$$\tilde{e}^1 = \frac{1}{2}(\sin(\psi)d\theta - \sin(\theta)\cos(\psi))d\varphi,$$
$$\tilde{e}^2 = \frac{1}{2}(\cos(\psi)d\theta + \sin(\theta)\sin(\psi)d\varphi),$$
$$\tilde{e}^3 = \frac{1}{2}(d\psi + \cos(\theta)d\varphi), \tag{A1}$$

which satisfies

$$d\tilde{e}^i = \varepsilon^{ijk}\tilde{e}^j\tilde{e}^k, \tag{A2}$$

where ε^{ijk} is the three-dimensional Levi-Civita symbol. Therefore,

$$ds^2_{S^3} = \frac{1}{4}((d\psi + \cos(\theta)d\varphi)^2 + d\theta^2 + \sin(\theta)^2 d\varphi^2).$$

These last results show that this construction was made to manifestly express S^3 as a S^1 fiber bundle on a S^2.

The associated torsion-free connection is given by $\tilde{\omega}^{ij} = \varepsilon^{ijk}\tilde{e}^k$ and thus

$$\tilde{R}^{ij} = d\tilde{\omega}^{ij} + \tilde{\omega}^i_k\tilde{\omega}^{kj} = \tilde{e}^i\tilde{e}^j, \tag{A3}$$

proving that $\tilde{e}^i$ is indeed a sound vielbein for S^3. Finally, it is worth mentioning that this construction yields

$$\tilde{e}^1 \wedge \tilde{e}^2 = \frac{1}{4}\sin(\theta)d\theta \wedge d\varphi \tag{A4}$$

and

$$(\tilde{e}^1)^2 + (\tilde{e}^2)^2 = \frac{1}{4}(d\theta^2 + \sin(\theta)^2 d\varphi^2).$$

[1] See Ref. [2] for a definition of these spaces.
[2] J. A. Wolf, *Spaces of Constant Curvature*, 5th ed. (American Mathematical Society, Providence, 1984), p. 69.
[3] K. Skenderis, Lecture notes on holographic renormalization, Classical Quantum Gravity **19**, 5849 (2002).
[4] R. M. Wald, Black hole entropy in Noether charge, Phys. Rev. D **48**, R3427 (1993).
[5] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y.) **88**, 286 (1974).
[6] J. Lee and R. M. Wald, Local symmetries and constraints, J. Math. Phys. **31**, 725 (1990).
[7] S. W. Hawking and D. N. Page, Thermodynamics of black holes in anti–de Sitter space, Commun. Math. Phys. **87**, 577 (1983).
[8] R. Aros, M. Contreras, R. Olea, R. Troncoso, and J. Zanelli, Conserved charges for gravity with locally AdS asymptotics, Phys. Rev. Lett. **84**, 1647 (2000).
[9] R. Aros, Boundary conditions in first order gravity: Hamiltonian and ensemble, Phys. Rev. D **73**, 024004 (2006).
[10] R. Aros, The horizon and first order gravity, J. High Energy Phys. 04 (2003) 024.
[11] J. D. Brown and J. York, Quasilocal energy and conserved charges derived from the gravitational action, Phys. Rev. D **47**, 1407 (1993).
[12] R. Aros, M. Contreras, R. Olea, R. Troncoso, and J. Zanelli, Conserved charges for even dimensional asymptotically AdS gravity theories, Phys. Rev. D **62**, 044002 (2000).
[13] R. Olea, Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes, J. High Energy Phys. 06 (2005) 023.
[14] O. Miskovic and R. Olea, Topological regularization and self-duality in four-dimensional anti–de Sitter gravity, Phys. Rev. D **79**, 124020 (2009).
[15] R. Araneda, R. Aros, O. Miskovic, and R. Olea, Magnetic mass in 4D AdS gravity, Phys. Rev. D **93**, 084022 (2016).
[16] C. J. Hunter, Action of instantons with nut charge, Phys. Rev. D **59**, 024009 (1998).
[17] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Large N phases, gravitational instantons, and the nuts and bolts of AdS holography, Phys. Rev. D **59**, 064010 (1999).
[18] J. Maldacena, Einstein gravity from conformal gravity, arXiv:1105.5632.
[19] S. W. Hawking, C. J. Hunter, and D. N. Page, Nut charge, anti–de Sitter space and entropy, Phys. Rev. D **59**, 044033 (1999).
[20] J. Zanelli, (Super)-gravities beyond 4 dimensions, arXiv: hep-th/0206169.
[21] It is easy to demonstrate that Eq. (8) is equivalent to the relation $\xi^\mu\nabla_\mu(\xi^\nu)|_{\mathbb{R}\times\partial\Sigma_H} = \kappa\xi^\nu$ discussed in Ref. [4].
[22] D. Kastor, S. Ray, and J. Traschen, Enthalpy and the mechanics of AdS black holes, Classical Quantum Gravity **26**, 195011 (2009).
[23] C. V. Johnson, Thermodynamic volumes for AdS-Taub-NUT and AdS-Taub-Bolt, Classical Quantum Gravity **31**, 235003 (2014).
[24] D. N. Page and C. N. Pope, Einstein metrics on quaternionic line bundles, Classical Quantum Gravity **3**, 249 (1986).
[25] R. B. Mann and C. Stelea, On the thermodynamics of NUT charged spaces, Phys. Rev. D **72**, 084032 (2005).
[26] D. Astefanesei, R. B. Mann, and E. Radu, Breakdown of the entropy/area relationship for NUT-charged spacetimes, Phys. Lett. B **620**, 1 (2005).
[27] C. V. Johnson, The extended thermodynamic phase structure of Taub-NUT and Taub-Bolt, Classical Quantum Gravity **31**, 225005 (2014).
[28] L. Fatibene, M. Ferraris, M. Francaviglia, and M. Raiteri, Noether charges, Brown-York quasilocal energy and related topics, J. Math. Phys. **42**, 1173 (2001).
[29] T. Eguchi, P. B. Gilkey, and A. J. Hanson, Gravitation, gauge theories and differential geometry, Phys. Rep. **66**, 213 (1980).