General post-Minkowskian expansion and application of the phase function

Cheng-Gang Qin and Cheng-Gang Shao*
Department of Physics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China
(Received 15 March 2017; published 10 July 2017)

The phase function is a useful tool to study all observations of space missions, since it can give all the information about light propagation in a gravitational field. For the extreme accuracy of the modern space missions, a precise relativistic modeling of observations is required. So, we develop a recursive procedure enabling us to expand the phase function into a perturbative series of ascending powers of the Newtonian gravitational constant. Any nth-order perturbation of the phase function can be determined by the integral along the straight line connecting two point events. To illustrate the result, we carry out the calculation of the phase function outside a static, spherically symmetric body up to the order of G^2. Then, we develop a precise relativistic model that is able to calculate the phase function and the derivatives of the phase function in the gravitational field of rotating and uniformly moving bodies. This model allows the computing of the Doppler, radio science, and astrometric observables of the space missions in the Solar System. With the development of space technology, the relativistic corrections due to the motion of a planet's spin must be considered in the high-precision space missions in the near future. As an example, we give the estimates of the relativistic corrections on the observables about the space missions TianQin and BEACON.

DOI: 10.1103/PhysRevD.96.024003

I. INTRODUCTION

With the development of space technology over the past decade, a series of space projects was proposed to test general relativity with high precision. For example, the Cassini spacecraft has reached an accuracy of 1 m on the range and 3×10^{-6} m/s on the Doppler [1]. The JUNO mission reaches an accuracy of the order of 10 cm for the range and 10^{-6} m/s for the Doppler [2,3], whose primary goals are the understanding of the composition of the planet, the gravity, and the magnetic field of Jupiter [4–6]. Similar accuracies are expected for BepiColombo [7]. The GRACE-Follow-On is expected to provide an accuracy of 1 nm for the range and corresponding estimates for the range rate [8,9]. Also, ACES obtains an accuracy of the order of 10^{-16} in fractional frequency [10].

In order to obtain the observables of general relativity, it is crucial to study light propagation in the gravitational field. There is a standard method to get all information about light propagation between two point events by solving null geodesic equations [11–13] or the eikonal equation [14]. Many solutions have been proposed in the post-Newtonian and in the post-Minkowskian (PM) approximations when dealing with the bending effects of light influenced by the gravitational field [15–18]. Besides, a different method is also available, initially based on the Synge world function [19] and then on the time transfer functions (TTFs) and frequency shift [20–24]. The TTFs in the field of a static, spherically symmetric body up to the second post-Minkowskian approximation have been determined by Ref. [19]. Then, the general post-Minkowskian expansion of the time transfer function has been given by Ref. [24].

The laser ranging interferometer (LRI) has an extreme precision in measuring distances. It will be adopted by future space missions to measure relativistic effects. The observable of the LRI originally comes from the interference of the Gaussian beams (GBs), which is the phase difference essentially. The phase of the Gaussian beam can give all information about light propagation in the gravitational field, such as the time transfer function, the vector of light, the frequency shift, and so on. It is well known that the Gaussian beam is not a strict plane wave. So we should consider the diffusion of a Gaussian beam in actual experiments. The general signal of interference comes from the phase of the GB, and the signal of the phase is averaged over the finite detector surface, which demonstrates pointing errors and the wave front distortion will cause errors. The phase function is a very useful tool to study the aforementioned noise, since it describes the phase of a light. It has been determined at the first post-Minkowskian (1PM) approximation in the field of the extended body [8].

As we know, the gravitational field in the Solar System is generated from rotating and moving planets (such as Jupiter and Earth). So a more complete relativistic model describing the Solar System should contain some rotating and moving bodies, which was neglected before. In modern times, the accuracy of measuring a variation of the distance has reached 1 pm on the ground, which demonstrates that contributions due to the motion of a rotating and moving body must be considered in the near future. In order to

*cgshao@mail.hust.edu.cn

compute astrometric observables of space missions in the Solar System, we have to develop a model that determines the phase function in the gravitational field of a rotating and uniformly moving body. This relativistic model is more suitable for high-precision space missions in the future. By solving the eikonal equation, we develop a relativistic model to determine the general post-Minkowskian expansion of the phase function. The general time transfer function and its derivatives can easily be determined by the phase function.

The present paper is organized as follows: In Sec. II, we discuss the phase function; in Sec. III, we recall the information about the eikonal equation and the expansion of the space-time metric. In Sec. IV, we show how to deduce the general post-Minkowskian expansion of the phase function from the eikonal function. Using this result, in Sec. V, we focus on the case of a static, spherically symmetric body within the second post-Minkowskian (2PM) approximation. We determine the explicit time transfer function up to the order of G^2 and the frequency shift up to the order of c^{-3}. Then, we also determine explicit expressions of the phase function and of its derivatives in the gravitational field of a rotating and uniformly moving body. Finally, in Sec. VI, we give our conclusions and remarks.

In this paper, c is the speed of the light in vacuum, and G is the Newtonian gravitational constant. The signature of the Lorentzian metric g of space-time V_4 is given as $\{+---\}$. We suppose that the space-time is covered by one global coordinate system $(x^\mu) = (x^0, \mathbf{x})$, where $x^0 = ct$, with t being a time coordinate and $\mathbf{x} = (x^i)$. Greek indices run from 0 to 3, and Latin indices run from 1 to 3. We assume that the curves of equations $x^i = \text{const}$ are timelike, which means that $g_{00} > 0$ anywhere. We employ the vector notation $\mathbf{a}$ in order to denote $(a^1, a^2, a^3) = (a^i)$. The Einstein convention on repeat indices is used here for the expressions like $a^i b^i$ and $A^\mu B_\mu$. The quantity $|\mathbf{a}|$ stands for the ordinary Euclidean norm of $\mathbf{a}$. For any quantity $f(x^\mu)$, $\partial_\nu f$ denotes the partial derivative of f with respect to x^ν.

II. PHASE FUNCTION

The LRI is widely used in many fields because of its extreme accuracy. The LRI observables—time series data—ultimately come from the continuous changes in the phase difference between the local laser beam and receiving laser beam. Since the measured phase signal is averaged over the detector surface, wave front distortion and diffraction will influence the phase signal, leading to noises [25]. The gravitational field is a source of wave front distortion, which means that high-precision space missions are necessary to consider this noise. The general TTF is not convenient to study the wave front distortion, but the phase function is possible to study that. So we introduce the phase function of the Gaussian beam.

Besides, the pointing error is also able to cause noise. For example, the power spectral density of the measurement noise 4.5 nm$/\text{Hz}^{1/2}$ constrains the pointing error to a $\delta\theta$ in our previous work about the Beyond Einstein Advanced Coherent Optical Network (BEACON) mission [26]. On the practical experiments, we must consider the diffusion of a Gaussian beam. Many previous papers studying the propagation of light in the gravitational field just consider the case of the light propagation between two point events, which neglects the diffusion of light. The phase function is a useful tool, which is possible to study the diffusion of light and pointing errors for some more precise space missions. In this paper, we study the phase function about light propagation between two point events for a brief consideration. We will discuss the complicated cases in the future.

We suppose that a Gaussian beam is propagating along the z axis. The phase function $\varphi(x,y,z)$ is a function of the quasi-Cartesian coordinate (x^i), which describes the distribution of the GB phase. The equiphase surface equation satisfies [27]

$$\varphi(x,y,z) = \varphi(0,0,z_0), \tag{1}$$

where z_0 is the intersection of the equiphase surface and z axis. The unperturbed equiphase surface is a spherical surface without considering general relativity.

III. EIKONAL EQUATION

We consider a phase function within the framework of general relativity. In electromagnetism, the phase function of an electromagnetic wave is a scalar function which is invariant under a set of general coordinate transformations. As a direct consequence of Maxwell's equations, the phase function φ satisfies the eikonal equation [14]:

$$g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 0. \tag{2}$$

This equation describes the wave front of an electromagnetic wave propagating in the curved space-time. The $g^{\mu\nu}$ describes a gravitational field which is derived as the solution of Einstein's field equation. Generally, the set of quantities $g^{\mu\nu}$ is represented at any point x by a series in ascending powers of Newtonian gravitational constant G:

$$g_{\mu\nu}(x,G) = g^{(0)}_{\mu\nu} + \sum_{n=1}^{\infty} G^n g^{(n)}_{\mu\nu}(x), \tag{3}$$

where $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, with $\eta_{\mu\nu} = \text{diag}(+1,-1,-1,-1)$. Then, the concomitant expansion of the contravariant components $g^{\mu\nu}$ is given by

$$g^{\mu\nu}(x,G) = g^{\mu\nu}_{(0)} + \sum_{n=1}^{\infty} G^n g^{\mu\nu}_{(n)}(x), \tag{4}$$

where $g^{\mu\nu}_{(0)} = \eta^{\mu\nu}$ and the set of quantities $g^{\mu\nu}_{(n)}$ can be recursively determined by using the relations

$$g^{\mu\nu}_{(1)} = -\eta^{\mu\gamma}\eta^{\nu\delta}g^{(1)}_{\gamma\delta}, \tag{5}$$

$$g^{\mu\nu}_{(n)} = -\eta^{\mu\gamma}\eta^{\nu\delta}g^{(n)}_{\gamma\delta} - \sum_{p=1}^{n-1}\eta^{\mu\gamma}g^{(p)}_{\gamma\delta}g^{\nu\delta}_{(n-p)}. \tag{6}$$

In order to solve the eikonal equation, we introduce the covector $K_\mu = \partial_\mu\varphi$ which describes the covector of an electromagnetic wave front in the curved space-time. In the case of light, the vector $K^\mu = g^{\mu\nu}\partial_\nu\varphi$ is tangent to the light ray. If we know the specific expression of the phase function $\varphi(t,\mathbf{x})$, it is obvious that we can get a lot of information about the propagation of light from φ. To find a solution of Eq. (2), we expand the phase function φ by a series in ascending powers of the gravitational constant G. Assuming that the unperturbed solution is a plane wave, the expansion of the phase function may be given as

$$\varphi(t,\mathbf{x}) = \varphi_0 + \int k_\mu dx^\mu + \sum_{n=1}^{\infty} G^n\varphi^{(n)}(t,\mathbf{x}), \tag{7}$$

where φ_0 is an integration constant. k_μ $[k^\mu = k_0(1,\mathbf{k})]$ is a constant null vector along the direction of propagation of the unperturbed electromagnetic plane wave, which satisfies the relation $\eta_{\mu\nu}k^\mu k^\nu = 0$. The vector $\mathbf{k}$ is the unit vector along the propagation of the light ($|\mathbf{k}| = 1$), which gives the wave direction. $k_0 = \omega/c$, where ω is the constant angular frequency of the unperturbed electromagnetic wave. $\varphi^{(n)}$ is the perturbation of the phase function of the nth order in G. As the consequence of the definition of K^μ and Eq. (7), it is easy to see that the wave vector $K^\mu(t,\mathbf{x})$ of the electromagnetic wave in the curved space-time can be expanded as

$$K^\mu(t,\mathbf{x}) = g^{\mu\nu}\partial_\nu\varphi = k^\mu + \sum_{n=1}^{\infty} k^\mu_{(n)}(t,\mathbf{x}), \tag{8}$$

where $k^\mu_{(n)}$ is the nth perturbation of the wave vector with respect to Newtonian constant G.

IV. GENERAL POST-MINKOWSKIAN EXPANSION OF PHASE FUNCTION

Let us define $x_A = (ct_A, \mathbf{x}_A)$ as the event of emission A and $x_B = (ct_B, \mathbf{x}_B)$ as the event of reception B of a light signal. Let x_A and x_B be connected by a curve Γ_{AB}, which is the trajectory of a light signal. The curve Γ_{AB} can be defined by the parametric equation

$$x^\mu(\zeta) = \zeta(x^\mu_B - x^\mu_A) + x^\mu_A, \tag{9}$$

where $0 \le \zeta \le 1$, $x(0) = x_A$, and $x(1) = x_B$.

Here, we represent the light ray's trajectory, correct to Newtonian order, as

$$\{x^\mu\} \equiv \{x^\mu_{(0)}(\lambda)\} + o(G), \tag{10}$$

$$x^\mu_{(0)} = \lambda(x^\mu_B - x^\mu_A) + x^\mu_A, \tag{11}$$

where $x^\mu_{(0)}$ is the 0th-order geodesic path connecting x_A and x_B, and we define curve $\Gamma^{(0)}_{AB}$ by $x_{(0)}$.

According to Eq. (2), this expression at any point x on the curve Γ_{AB} is given by

$$g^{\mu\nu}(x)\partial_\mu\varphi(x_A,x)\partial_\nu\varphi(x_A,x) = 0. \tag{12}$$

We set that $\varphi(x_A,x) = \varphi(x)$. Inserting the expansions of Eqs. (4) and (7) into Eq. (12), it is easily seen that Eq. (12) splits up into an infinite set of partial differential equations as follows:

$$k^\mu\partial_\mu\varphi^{(n)}(x) = \phi^{(n)}(x), \tag{13}$$

where $\eta^{\mu\nu}k_\nu\partial_\mu\varphi^n(x) = k^\mu\partial_\mu\varphi(x)$, since it is safe to lower or raise indices using $\eta_{\mu\nu}$ or $\eta^{\mu\nu}$. $\phi^{(n)}(x)$ is given by

$$\phi^{(1)}(x) = -\frac{1}{2}g^{\mu\nu}_{(1)}(x)k_\mu k_\nu, \tag{14}$$

$$\begin{aligned}\phi^{(n)}(x) = &-\frac{1}{2}g^{\mu\nu}_{(n)}(x)k_\mu k_\nu \\ &-\frac{1}{2}\sum_{p=1}^{n-1}\Big[\eta^{\mu\nu}\partial_\mu\varphi^{(p)}(x)\partial_\nu\varphi^{(n-p)}(x) \\ &+ 2g^{\mu\nu}_{(p)}(x)k_\mu\partial_\nu\varphi^{(n-p)}(x) + g^{\mu\nu}_{(p)}(x) \\ &\times\sum_{q=1}^{n-p-1}\partial_\mu\varphi^{(q)}(x)\partial_\nu\varphi^{(n-p-q)}(x)\Big],\end{aligned} \tag{15}$$

where $n \ge 2$.

We assume now that x moves along the curve $\Gamma^{(0)}_{AB}$. That means that x varies as a function of λ according to Eq. (11). Combining Eq. (11) and the definition of $\mathbf{k}$, the total derivative of $\varphi^{(n)}$ along $\Gamma^{(0)}_{AB}$ is given by

$$\frac{d\varphi^{(n)}(x_{(0)}(\lambda))}{d\lambda} = \frac{R_{AB}}{k_0}\phi^{(n)}(x_{(0)}(\lambda)), \tag{16}$$

where $R_{AB} = |\mathbf{x}_B - \mathbf{x}_A|$ is the Euclidean distance between x_A and x_B along the curve $\Gamma^{(0)}_{AB}$.

It is easy to show that Eq. (16) has one and only one solution satisfying boundary condition $\varphi(x_A) = \varphi_0$, that is,

$$\varphi^{(n)}(x_{(0)}(\lambda)) = \frac{R_{(0)}}{k_0}\int_0^\lambda \phi^{(n)}(x_{(0)}(\lambda'))d\lambda', \quad (17)$$

where $0 \le \lambda' \le \lambda$ and $R_{(0)} = |\mathbf{x}_{(0)}(\lambda) - \mathbf{x}_A|$. And $\phi^{(n)}(x_{(0)}(\lambda))$ is a continuous function of λ in the range of [0, 1]. As a consequence of Eqs. (13)–(15) and (17), the solution of the eikonal equation is given by

$$\varphi^{(1)}(x_A, x_B) = -\frac{R_{AB}}{2k_0}\int_0^1 g^{\mu\nu}_{(1)}(x_{(0)}(\lambda))k_\mu k_\nu d\lambda, \quad (18)$$

$$\varphi^{(n)}(x_A, x_B) = -\frac{R_{AB}}{2k_0}\int_0^1 g^{\mu\nu}_{(n)}(x_{(0)}(\lambda))k_\mu k_\nu d\lambda - \frac{R_{AB}}{2k_0}\int_0^1 \sum_{p=1}^{n-1}\Big[\eta^{\mu\nu}\partial_\mu\varphi^{(p)}(x_{(0)}(\lambda))\partial_\nu\varphi^{(n-p)}(x_{(0)}(\lambda)) + g^{\mu\nu}_{(p)}(x_{(0)}(\lambda))\sum_{q=1}^{n-p-1}\partial_\mu\varphi^{(q)}(x_{(0)}(\lambda))\partial_\nu\varphi^{n-p-q}(x_{(0)}(\lambda)) + 2g^{\mu\nu}_{(p)}(x_{(0)}(\lambda))k_\mu\partial_\nu\varphi^{(n-p)}(x_{(0)}(\lambda))\Big]d\lambda, \quad (19)$$

where $n \ge 2$. All integrals are calculated along the straight line $\Gamma^{(0)}_{AB}$. Using these expressions, we can calculate the phase function up to any order.

Now, we briefly consider the phase function up to the order G^2, that is, $n = 2$. The phase function is given by

$$\varphi(x_A, x_B) = \varphi_0 + \int k_\mu dx^\mu(\lambda) - \frac{GR_{AB}}{2k_0}\int_0^1 g^{\mu\nu}_{(1)}(x_{(0)}(\lambda))k_\mu k_\nu d\lambda + \frac{G^2R_{AB}}{k_0}\int_0^1\Big\{-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi^{(1)}(x_{(0)}(\lambda))\partial_\nu\varphi^{(1)}(x_{(0)}(\lambda)) - g^{\mu\nu}_{(1)}(x_{(0)}(\lambda))k_\mu\partial_\nu\varphi^{(1)}(x_{(0)}(\lambda)) - \frac{1}{2}g^{\mu\nu}_{(2)}(x_{(0)}(\lambda))k_\mu k_\nu\Big\}d\lambda + o(G^3). \quad (20)$$

On the other hand, we also can define the curve Γ_{AB} by another parametric equation:

$$x^\mu(\varsigma) = \varsigma(x^\mu_A - x^\mu_B) + x^\mu_B. \quad (21)$$

Using a similar method, the phase function is given by

$$\varphi(x_B, x_A) = \varphi_0 + \int k_\mu dx^\mu(\varsigma) + \frac{GR_{AB}}{2k_0}\int_0^1 g^{\mu\nu}_{(1)}(x_{(0)}(\varsigma))k_\mu k_\nu d\varsigma - \frac{G^2R_{AB}}{k_0}\int_0^1\Big\{-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi^{(1)}(x_{(0)}(\varsigma))\partial_\nu\varphi^{(1)}(x_{(0)}(\varsigma)) + g^{\mu\nu}_{(1)}(x_{(0)}(\varsigma))k_\mu\partial_\nu\varphi^{(1)}(x_{(0)}(\varsigma)) - \frac{1}{2}g^{\mu\nu}_{(2)}(x_{(0)}(\varsigma))k_\mu k_\nu\Big\}d\varsigma + o(G^3). \quad (22)$$

From Eqs. (20) and (22), it is convenient to obtain the general post-Minkowskian expansion of emission time transfer function $T_e(t_A, \mathbf{x}_A, \mathbf{x}_B)$ and reception time transfer function $T_r(\mathbf{x}_A, t_B, \mathbf{x}_B)$, respectively.

V. APPLICATION OF THE PHASE FUNCTION

The phase function is a convenient tool to compute the radio science and astrometric observables, which contains all the information about light propagation between two point events. In this section, we will apply the phase function to give some useful results. First, we determine the TTF and frequency shift in the case of a static and spherically symmetric body from the phase function and its derivatives. Then we determine the phase function and the derivatives of the phase function in the case of a rotating, uniformly moving, and spherically symmetric body, which is a more precise relativistic model of the space missions in the Solar System. From our estimates, the contribution of the motion of spin of the body will be measurable in future space missions.

A. Time transfer function

In order to illustrate the previous results, let us determine the phase function and time transfer function in the case of the gravitational field outside a static and spherically symmetric body of mass M. Choosing spatial isotropic coordinates and putting $|\mathbf{x}_{(0)}| = r$, we suppose that the space-time metric components may be written as

$$g^{(1)}_{00} = -\frac{2M}{c^2r}, \qquad g^{(2)}_{00} = \frac{2M^2\beta}{c^4r^2},$$
$$g^{(1)}_{0i} = 0, \qquad g^{(2)}_{00} = 0,$$
$$g^{(1)}_{ij} = -\frac{2M\gamma}{c^2r}\delta_{ij}, \qquad g^{(2)}_{ij} = -\frac{3M^2\delta}{2c^4r^2}\delta_{ij}, \quad (23)$$

where β and γ are usual post-Newtonian parameters and δ is the post-post-Newtonian parameter ($\beta = \gamma = \delta = 1$ in general relativity). We suppose that the points x_A and x_B are outside the body. Here, we have $r_A = |\mathbf{x}_A|$ and $r_B = |\mathbf{x}_B|$. Inserting Eqs. (5), (6), and (23) into Eq. (20), we obtain the expressions of $\varphi^{(1)}(x_A, x_B)$ and $\varphi^{(2)}(x_A, x_B)$. By a simple integral, $\varphi^{(1)}(x_A, x_B)$ is given by

$$\varphi^{(1)}(x_A, x_B) = -(1+\gamma)\frac{MR_{AB}}{c^2}k_0\int_0^1 \frac{1}{|\mathbf{x}_{(0)}|}d\lambda = -(1+\gamma)\frac{Mk_0}{c^2}\ln\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}}. \tag{24}$$

For the term of $\varphi^{(2)}(x_A, x_B)$, we deduce from Eq. (24) that

$$g^{\mu\nu}_{(1)}k_\mu \partial_\nu \varphi^{(1)}(x_{(0)}) = \frac{2\gamma(\gamma+1)M^2k_0^2}{c^4}\frac{1}{r^2}. \tag{25}$$

Then,

$$\frac{-R_{AB}}{k_0}\int_0^1 \left\{g^{\mu\nu}_{(1)}(x_{(0)})k_\mu\partial_\nu\varphi^{(1)}(x_{(0)}) + \frac{1}{2}g^{\mu\nu}_{(2)}(x_{(0)})k_\mu k_\nu\right\}d\lambda = \left(\beta - 2 - 2\gamma - \frac{3}{4}\delta\right)\frac{M^2k_0R_{AB}}{c^4}\int_0^1 \frac{1}{r^2}d\lambda, \tag{26}$$

where

$$\int_0^1 \frac{1}{r^2}d\lambda = \frac{\arccos(\mathbf{n}_A\cdot\mathbf{n}_B)}{r_Ar_B\sqrt{1-(\mathbf{n}_A\cdot\mathbf{n}_B)^2}}, \tag{27}$$

with

$$\mathbf{n}_A = \frac{\mathbf{x}_A}{r_A}, \qquad \mathbf{n}_B = \frac{\mathbf{x}_B}{r_B}. \tag{28}$$

Another term of the second-order phase function can be calculated by this relation:

$$-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi^{(1)}(x_{(0)})\partial_\nu\varphi^{(1)}(x_{(0)}) = \frac{(1+\gamma)^2M^2k_0^2}{c^4}\frac{r_A^2(1+\mathbf{n}_A\cdot\mathbf{n})}{r^2r_A^2(1+\mathbf{n}_A\cdot\mathbf{n})^2}. \tag{29}$$

It is easy to prove that

$$\frac{r_A^2(1+\mathbf{n}_A\cdot\mathbf{n})}{r^2r_A^2(1+\mathbf{n}_A\cdot\mathbf{n})^2} = \frac{d}{d\lambda'}\left[\frac{\lambda'}{r_Ar(1+\mathbf{n}_A\cdot\mathbf{n})}\right]. \tag{30}$$

Combining Eqs. (29) and (30), we obtain

$$\frac{R_{AB}}{k_0}\int_0^1 -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi^{(1)}(x_{(0)})\partial_\nu\varphi^{(1)}(x_{(0)})d\lambda = (1+\gamma)^2\frac{M^2k_0}{c^4}\frac{R_{AB}}{r_Ar_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)}. \tag{31}$$

Inserting Eqs. (24), (26), and (31) into the phase function $\varphi(x_A, x_B)$, the final expression of the phase function is

$$\varphi(x_A, x_B) = \varphi_0 + k_0(cT_{AB} - R_{AB}) - (1+\gamma)\frac{GMk_0}{c^2}\ln\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} + \frac{G^2M^2k_0R_{AB}}{c^4}\left\{\left(\beta - 2 - 2\gamma - \frac{3}{4}\delta\right)\frac{\arccos(\mathbf{n}_A\cdot\mathbf{n}_B)}{r_Ar_B\sqrt{1-(\mathbf{n}_A\cdot\mathbf{n}_B)^2}} + \frac{(1+\gamma)^2}{(r_Ar_B + \mathbf{x}_A\cdot\mathbf{x}_B)}\right\} + o(G^3). \tag{32}$$

This expression allows us to calculate effects of the second order in G, which must be taken into account in the near future. For example, for the Juno solar conjunction—at which the spacecraft, the Sun, and Earth are almost aligned—the contribution of the mass of the Sun on the range between Juno and Earth is $\sim 10^4$ m for the first-order effect. And the second contribution of that may reach $\sim 10^{-1}$ m, which must be taken into account in the Juno mission.

Next, we show that the time transfer function is deduced from the phase function by a simple procedure. Supposing that a light connects the point x_A and point x_B along the curve Γ_{AB}, this coherent process of the light signal implies that the phase satisfies the relation $\varphi(x_A) = \varphi(x_B)$. With the original condition $\varphi(x_A) = \varphi_0$, the phase function at point x_B satisfies

$$\varphi(x_B) = \varphi_0. \tag{33}$$

It is clear that the two time transfer functions are equivalent, so we have the relation

$$T_r(\mathbf{x}_A, t_B, \mathbf{x}_B) = T_e(t_A, \mathbf{x}_A, \mathbf{x}_B) = T(\mathbf{x}_A, \mathbf{x}_B). \tag{34}$$

According to Eqs. (32) and (33), the expression of the time transfer function is easily given as follows:

$$T_{AB} = \frac{R_{AB}}{c} + (1+\gamma)\frac{GM}{c^3}\ln\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} - \frac{G^2M^2R_{AB}}{c^5}\left\{\left(\beta - 2 - 2\gamma - \frac{3}{4}\delta\right)\frac{\arccos(\mathbf{n}_A\cdot\mathbf{n}_B)}{r_Ar_B\sqrt{1-(\mathbf{n}_A\cdot\mathbf{n}_B)^2}} + \frac{(1+\gamma)^2}{(r_Ar_B + \mathbf{x}_A\cdot\mathbf{x}_B)}\right\} + o(G^3), \tag{35}$$

which is equivalent to the expression of TTF about the second-order post-Minkowskian approximation obtained by Refs. [19,24]. The second term of the T_{AB} is the well-known Shapiro time delay [28].

B. Frequency shift

In the case of a static and spherically symmetric body, we consider that a spacecraft S_A and a spacecraft S_B are moving on their orbit with the coordinate velocities $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively. First, we use the most primitive method to determine the frequency, which is shown in Fig. 1. We suppose that S_A emits the Nth crest of a light at the coordinate time t_A, while t_A connects with the proper time τ_A of S_A. Then, spacecraft S_A emits $(N+1)$th crest of the same light at coordinate time t'_A, while t'_A connects with proper time τ'_A of S_A. The Nth and $(N+1)$th crests are received by S_B at the coordinate times t_B and t'_B, which connects the proper time τ_B and τ'_B of spacecraft S_B, respectively. We further suppose that ν_A is the proper frequency of light as measured by S_A at the instant of emission and ν_B is the proper frequency of the same light as measured by S_B at the instant of receipt. The light frequency transfer from S_B to S_A is characterized by the ratio ν_B/ν_A, which may be rewritten as $(\Delta\varphi_B/\Delta\tau_B)/(\Delta\varphi_A/\Delta\tau_A)$, and there has the relation $\Delta\varphi_A = \Delta\varphi_B$ for the Nth and $(N+1)$th crests of light. $\Delta\tau_A$ and $\Delta\tau_B$ are the intervals of proper time with respect to $\Delta\varphi_A$ and $\Delta\varphi_B$, respectively.

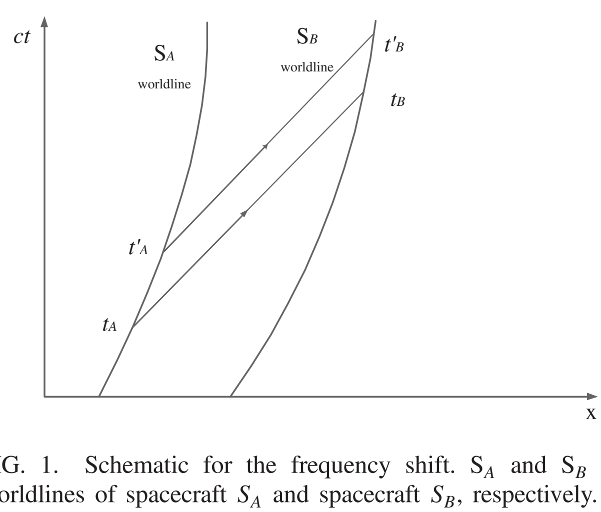

FIG. 1. Schematic for the frequency shift. S_A and S_B are worldlines of spacecraft S_A and spacecraft S_B, respectively. At time t_A, S_A emits the Nth crest of light to S_B, which is received by S_B at time t_B. At time t'_A, S_A emits the $(N+1)$th crest of light to S_B, which is received by S_B at time t'_B.

First, the relation between proper time and coordinate time is deduced from $(cd\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu$. The expansion of Eq. (23) demonstrates that this relation can be written as

$$\frac{d\tau}{dt} = \chi\sqrt{g_{00}(x)}, \tag{36}$$

where

$$\chi = \left[1 + \frac{g_{ij}\mathbf{v}^i\mathbf{v}^j}{g_{00}c^2}\right]^{1/2}. \tag{37}$$

It is obvious that the Lorentz contraction factor in special relativity is a part of χ. When we neglect the gravitational field, $\chi = \sqrt{1-\mathbf{v}^2/c^2}$.

Finally, ν_B/ν_A is rewritten as follows:

$$\frac{\nu_B}{\nu_A} = \frac{\Delta\tau_A}{\Delta\tau_B} = \frac{\chi_A g_{00}^{1/2}(\mathbf{x}_A)\Delta t_A}{\chi_B g_{00}^{1/2}(\mathbf{x}_B)\Delta t_B}, \tag{38}$$

where $\sqrt{g_{00}(\mathbf{x}_A)}/\sqrt{g_{00}(\mathbf{x}_B)}$ is the gravitational redshift effect, which comes from the difference of gravitational potentials [29]. Δt_A and Δt_B are intervals of coordinate time.

From Fig. 1, the time intervals are defined by

$$\Delta t_B = t'_B - t_B, \tag{39}$$

$$\Delta t_A = t'_A - t_A, \tag{40}$$

and more,

$$t_B = t_A + T_{AB}, \tag{41}$$

$$t'_B = t'_A + T_{AB}{}', \tag{42}$$

where the time T_{AB} and $T_{AB}{}'$ is the time transfer function of the Nth and $(N+1)$th crest of light, respectively. Inserting Eqs. (40)–(42) into Eq. (39), the time interval Δt_B is rewritten as a function of Δt_A:

$$\Delta t_B = \Delta t_A + T'_{AB} - T_{AB}. \tag{43}$$

In the limit of small velocities, we can expand the following equations with respect to $\mathbf{v}_A$ and $\mathbf{v}_B$:

$$\ln\frac{r_A+r_B+R_{AB}}{r_A+r_B-R_{AB}}-\ln\frac{r'_A+r'_B+R_{AB}{}'}{r'_A+r'_B-R_{AB}{}'}=-\left[\frac{(r_A+r_B)\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)}{r_Ar_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)}-\frac{R_{AB}(\mathbf{n}_B\cdot\mathbf{v}_B+\mathbf{n}_A\cdot\mathbf{v}_A)}{r_Ar_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)}+o(c^{-1})\right]\Delta t_B \quad (44)$$

and

$$\frac{R_{AB}}{c}-\frac{R'_{AB}}{c}=-\left[\frac{\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)}{c}+\frac{\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)}{c}\frac{\mathbf{k}\cdot\mathbf{v}_A}{c}+\frac{\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)}{c}\left(\frac{\mathbf{k}\cdot\mathbf{v}_A}{c}\right)^2+o(c^{-4})\right]\Delta t_B, \quad (45)$$

where $R_{AB}{}'\equiv|\mathbf{x}'_B-\mathbf{x}'_A|=R_{AB}$ in the case of $\Delta t_B=0$.

Combining Eqs. (43)–(45) and (35), the relation is easily given:

$$\frac{\Delta t_A}{\Delta t_B}=1-\mathbf{k}\cdot\left(\frac{\mathbf{v}_B-\mathbf{v}_A}{c}\right)-\mathbf{k}\cdot\left(\frac{\mathbf{v}_B-\mathbf{v}_A}{c}\right)\mathbf{k}\cdot\left(\frac{\mathbf{v}_A}{c}\right)-\frac{\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)}{c}\left(\frac{\mathbf{k}\cdot\mathbf{v}_A}{c}\right)^2 -\frac{(1+\gamma)GM}{c^3}\left[\frac{(r_A+r_B)\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)}{r_Ar_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)}-\frac{R_{AB}(\mathbf{n}_B\cdot\mathbf{v}_B+\mathbf{n}_A\cdot\mathbf{v}_A)}{r_Ar_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)}\right]+o(c^{-4}). \quad (46)$$

Considering the approximation of the weak field and the limit of small velocities, those approximations are reasonable:

$$\frac{\chi_A}{\chi_B}=1+\frac{\mathbf{v}_B^2}{2c^2}-\frac{\mathbf{v}_A^2}{2c^2}+o(c^{-4}), \quad (47)$$

$$\left(\frac{g_{00}(\mathbf{x}_A)}{g_{00}(\mathbf{x}_B)}\right)^{\frac{1}{2}}=1-\frac{GM}{c^2r_A}+\frac{GM}{c^2r_B}+O(c^{-4}). \quad (48)$$

It follows from Eqs. (46)–(48) that the expression of the frequency shift of the order of c^{-3} is given by

$$\frac{\Delta\nu}{\nu}=\frac{\nu_B}{\nu_A}-1=\left(\frac{\Delta\nu}{\nu}\right)_s+\left(\frac{\Delta\nu}{\nu}\right)_g, \quad (49)$$

where

$$\left(\frac{\Delta\nu}{\nu}\right)_s=-\frac{1}{c}\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)-\frac{1}{c^2}\left(\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)(\mathbf{k}\cdot\mathbf{v}_A)-\frac{\mathbf{v}_B^2-\mathbf{v}_A^2}{2}\right) -\frac{1}{c^3}\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)\left[\left(\frac{\mathbf{v}_B^2-\mathbf{v}_A^2}{2}\right)+(\mathbf{k}\cdot\mathbf{v}_A)^2\right]+o(c^{-4}), \quad (50)$$

$$\left(\frac{\Delta\nu}{\nu}\right)_g=\frac{1}{c^2}\left(\frac{GM}{r_B}-\frac{GM}{r_A}\right)-\frac{1}{c^3}\left(\frac{GM}{r_B}-\frac{GM}{r_A}\right)(\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)) -\frac{(1+\gamma)GM}{c^3}\left[\frac{(r_A+r_B)\mathbf{k}\cdot(\mathbf{v}_B-\mathbf{v}_A)}{r_Ar_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)}-\frac{R_{AB}(\mathbf{n}_B\cdot\mathbf{v}_B+\mathbf{n}_A\cdot\mathbf{v}_A)}{r_Ar_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)}\right]+o(c^{-4}). \quad (51)$$

$(\Delta\nu/\nu)_s$ is the special relativity effect, and $(\Delta\nu/\nu)_g$ is the general relativity effect.

On the other hand, there is a straightforward method to give the frequency shift from the result of the phase function, Eq. (32). Section III demonstrates that the frequency can be represented as

$$\nu=\frac{c}{2\pi}\partial_0\varphi. \quad (52)$$

So the frequency shift is rewritten as follows:

$$\frac{\nu_B}{\nu_A}=\left(\frac{d\varphi_B}{dc\tau_B}\right)\Big/\left(\frac{d\varphi_A}{dc\tau_A}\right), \quad (53)$$

where $d\tau_A = \sqrt{g_{00}(\mathbf{x}_A)}\chi_A dt_A$ and $d\tau_B = \sqrt{g_{00}(\mathbf{x}_B)}\chi_B dt_B$. From Eqs. (22) and (32), the derivative of phase function φ_A is given by

$$\frac{d\varphi_A}{dct_A} = k_0\left\{1 - \mathbf{k}\cdot\frac{\mathbf{v}_A}{c} + \frac{(1+\gamma)GM}{c^3} \times\left[-\frac{(r_A+r_B)\mathbf{k}\cdot\mathbf{v}_A}{r_A r_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)} - \frac{R_{AB}\mathbf{n}_A\cdot\mathbf{v}_A}{r_A r_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)}\right]\right\} + o(c^{-5}). \tag{54}$$

From Eqs. (20) and (32), the derivative of phase function φ_B is given by

$$\frac{d\varphi_B}{dct_B} = k_0\left\{1 - \mathbf{k}\cdot\frac{\mathbf{v}_B}{c} - \frac{(1+\gamma)GM}{c^3} \times\left[\frac{(r_A+r_B)\mathbf{k}\cdot\mathbf{v}_B}{r_A r_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)} - \frac{R_{AB}\mathbf{n}_B\cdot\mathbf{v}_B}{r_A r_B(1+\mathbf{n}_A\cdot\mathbf{n}_B)}\right]\right\} + o(c^{-5}). \tag{55}$$

Inserting Eqs. (54) and (55) into Eq. (53), we can obtain the frequency shift up to the order of c^{-3}, which is identical to Eq. (49). In this way, the physical meaning of it is clear and definite. It is easy to determine the frequency shift up to the order of c^{-5} in this method.

We perform numerical estimates of the frequency shifts for our previous work of BEACON [26]; we suppose now that S_A is spacecraft 4 with $r_A = (8 \pm 0.016)\times 10^7$ m and S_B is spacecraft 1 with $r_B = 8\times 10^7$ m. We use the difference of distances $r_A - r_B = 0.002 r_B$. We take the velocities of spacecrafts $|\mathbf{v}_A| = |\mathbf{v}_B| = 2.2\times 10^3$ m/s. Then, the other useful parameters concerning Earth are as follows: $GM = 3.987\times 10^{14}$ m^3/s^2, and the effective Earth radius is 6.5×10^6 m. From these data, it is easy to give the following upper bounds: $|\mathbf{k}\cdot(\mathbf{v}_A/c)| \le 7.3\times 10^{-6}$ for spacecraft 1 and spacecraft 4. The first-order Doppler effect is $|\mathbf{k}\cdot(\mathbf{v}_B - \mathbf{v}_A)/c| \le 1.46\times 10^{-5}$. The contribution of the first-term general relativity effect is $GM(1/r_B c^2 - 1/r_A c^2) \le 1.11\times 10^{-13}$. And the contribution of the third-term general relativity effect is bounded by 4×10^{-14} for $\gamma = 1$.

C. The case of a rotating, uniformly moving, and spherically symmetric body

Let us suppose that the gravitational field is generated by a spherically symmetric body. We are interested in calculating the contributions of the mass, spin, and the motion of the body on light propagation. First, we consider the metric describing such a space-time. The metric for this body at 1PM order in its own local reference system is given by $H_{\mu\nu} = \eta_{\mu\nu} + GH^{(1)}_{\mu\nu} + o(G^2)$, where $H^{(1)}_{\mu\nu}$ is given by

$$H^{(1)}_{00} = -\frac{2w(\mathbf{y})}{c^2},\quad H^{(1)}_{0i} = \delta_{ik}\frac{4w^k(\mathbf{y})}{c^3},\quad H^{(1)}_{ij} = -\frac{2w(\mathbf{y})}{c^2}\delta_{ij}, \tag{56}$$

where w is the scalar potential, which depends on the local coordinate $\mathbf{y}$. w^i is the vector potential, which also depends on the local coordinate $\mathbf{y}$. The local coordinate reference system is denoted by $y^\mu = (cT, \mathbf{y})$.

As shown in Fig. 2, body b is moving with coordinate velocity $\mathbf{v}_b$, whose spin is denoted by $\mathbf{K}_s$. The global coordinate is denoted by $x^\mu = (ct, \mathbf{x})$, whose origin is point GC. A light connects points A and B. By performing a Poincaré transformation, we can obtain the metric in the case of a uniformly moving and spherically symmetric body. The coordinate transformation is given as follows:

$$x^\mu = a^\mu + \Lambda^\mu_\nu y^\nu, \tag{57}$$

where $a^\mu = (ct_0, \mathbf{a}(t_0))$ is a constant four vector which specifies the origin of the coordinate system: It points from the origin of the global reference system to the origin of the comoving frame at $T = 0$. And the trajectory of the moving body in the global reference system is given by

$$\mathbf{x}_b = \mathbf{a} + c\mathbf{p}(t - t_0). \tag{58}$$

Λ^μ_ν is given by

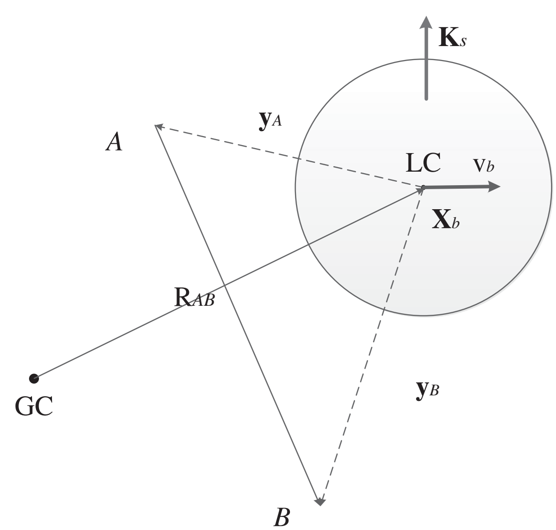

FIG. 2. Representative geometry of the case of a rotating and uniformly moving body. GC is the origin of the global coordinate, and LC is the center of the body b, whose coordinate velocity is $\mathbf{v}_b$. $\mathbf{K}_s$ is the unit vector of the body's spin. The local coordinate of points A and B is given by $\mathbf{y}_A$ and $\mathbf{y}_B$, respectively. $\mathbf{R}_{AB}$ is the vector connecting A and B.

$$\Lambda^0_0 = \gamma,$$
$$\Lambda^0_i = \Lambda^i_0 = \gamma p^i,$$
$$\Lambda^i_j = \delta_{ij} + \frac{\gamma^2}{1+\gamma} p^i p^j, \tag{59}$$

where $\mathbf{p} = \mathbf{v}_b/c$ and $\gamma = 1/\sqrt{1-\mathbf{p}^2}$ (in this subsection, γ is not the post-Newtonian parameter). The metric transformation is given by

$$g^{\mu\nu} = \eta^{\mu\nu} + G g^{\mu\nu}_{(1)} + o(G^2) = \Lambda^\mu_\alpha \Lambda^\nu_\beta H^{\alpha\beta}, \tag{60}$$

where the spin transformation in the metric is neglected, since it is too small. It is easy to obtain that

$$g^{\mu\nu}_{(1)} = \Lambda^\mu_\alpha \Lambda^\nu_\beta H^{\alpha\beta}_{(1)}. \tag{61}$$

The set of expressions of $g^{\mu\nu}_{(1)}$ is given by inserting Eq. (59) into Eq. (61), which leads to

$$g^{00}_{(1)} = \frac{2w}{c^2}\gamma^2(1+p^2) + \frac{8\mathbf{w}\cdot\mathbf{p}}{c^3}\gamma^2,$$
$$g^{0i}_{(1)} = \frac{4w^i\gamma}{c^3} + \frac{4w\gamma^2 p^i}{c^2} + \frac{4\mathbf{w}\cdot\mathbf{p}}{c^3}\frac{2\gamma^3+\gamma^2}{1+\gamma}p^i,$$
$$g^{ij}_{(1)} = \frac{2w}{c^2}(\delta_{ij} + 2\gamma^2 p^i p^j) + \frac{4\gamma}{c^3}(w^i p^j + w^j p^i) + \frac{4\mathbf{w}\cdot\mathbf{p}}{c^3}\frac{2\gamma^3}{1+\gamma}p^i p^j, \tag{62}$$

where $w = w(y^\mu)$, $w^i = w^i(y^\mu)$, and $\mathbf{w}\cdot\mathbf{p} = \sum^{i=1,2,3} w^i p^i$. This metric can describe such a gravitational field which is generated by a rotating, uniformly moving, and spherically symmetric body. If the metric describes the geometry due to N uniformly moving and spherically symmetric bodies at 1PM, we can express the metric with a linear summation for every body.

In this case, the one-order phase function is given by

$$\varphi^{(1)}(x_A, x_B) = -\frac{R_{AB}}{2k_0}\int_0^1 g^{\mu\nu}_{(1)} k_\mu k_\nu d\lambda. \tag{63}$$

Replacing the expression of the metric (62) in (63) gives

$$\varphi^{(1)}(x_A, x_B) = -\frac{R_{AB}k_0}{2}\int_0^1 \bigg\{\frac{4w}{c^2}\gamma^2(1-\mathbf{k}\cdot\mathbf{p})^2 - \frac{8\mathbf{k}\cdot\mathbf{w}}{c^3}\gamma(1-\mathbf{k}\cdot\mathbf{p}) + \frac{8\mathbf{p}\cdot\mathbf{w}}{c^3}\frac{\gamma}{(1+\gamma)} \times [\gamma^2(1-\mathbf{k}\cdot\mathbf{p})^2 + \gamma(1-\mathbf{k}\cdot\mathbf{p})]\bigg\} d\lambda. \tag{64}$$

It is useful to rewrite the expressions of the scalar potential and vector potential, respectively, as

$$w(y^i) = w(\Lambda'^i_\mu(x^\mu - a^\mu)), \tag{65}$$

$$\mathbf{w}(y^i) = \mathbf{w}(\Lambda'^i_\mu(x^\mu - a^\mu)), \tag{66}$$

where Λ'^i_μ is the inverse of Λ^μ_i. It is convenient to set $a^\mu = x^\mu_{b0}$. We express $\Lambda'^i_\mu(x^\mu - a^\mu)$ in the form as

$$\Lambda'^i_\mu(x^\mu - a^\mu) = X^i_{bB} - \lambda G^i_{AB}, \tag{67}$$

where

$$\mathbf{X}_{bA/B} = \mathbf{x}_{A/B} + \frac{\gamma^2}{1+\gamma}\mathbf{p}[\mathbf{p}\cdot(\mathbf{x}_{A/B} - \mathbf{x}_{b0})] - \mathbf{x}_{b0} - \gamma\mathbf{v}_b(t_B - t_0), \tag{68}$$

$$\mathbf{G}_{AB} = R_{AB}\mathbf{g}_{AB}, \tag{69}$$

and

$$\mathbf{g}_{AB} = \left[\mathbf{k} - \gamma\mathbf{p} + \frac{\gamma^2}{1+\gamma}\mathbf{p}(\mathbf{p}\cdot\mathbf{k})\right], \tag{70}$$

$$g_{AB} = |\mathbf{g}_{AB}| = \gamma(1-\mathbf{k}\cdot\mathbf{p}). \tag{71}$$

Let us denote by I and $\mathbf{S}$ the integrals appearing in the phase function expression (64) in the case where the body is static. They are given, respectively, by

$$I(\mathbf{x}_{bA}, \mathbf{x}_{bB}) = I'(\mathbf{R}_{AB}, \mathbf{x}_{bB}) = \int_0^1 w(\mathbf{x}_{bB} - \lambda\mathbf{R}_{AB})d\lambda, \tag{72}$$

$$S^i(\mathbf{x}_{bA}, \mathbf{x}_{bB}) = S'^i(\mathbf{R}_{AB}, \mathbf{x}_{bB}) = \int_0^1 w^i(\mathbf{x}_{bB} - \lambda\mathbf{R}_{AB})d\lambda, \tag{73}$$

where $\mathbf{x}_{bA/B} = \mathbf{x}_{A/B} - \mathbf{x}_b$. The solutions of integrals depend on $(\mathbf{x}_{bA}, \mathbf{x}_{bB})$ or $(\mathbf{x}_{bB}, \mathbf{R}_{AB})$ because $\mathbf{x}_{bA} = \mathbf{x}_{bB} - \mathbf{R}_{AB}$. In order to apply in the moving case, the integrals have to be replaced by

$$I'(\mathbf{G}_{AB}, \mathbf{X}_{bB}) = \int_0^1 w(\mathbf{X}_{bB} - \lambda\mathbf{G}_{AB})d\lambda, \tag{74}$$

$$\mathbf{S}'(\mathbf{G}_{AB}, \mathbf{X}_{bB}) = \int_0^1 \mathbf{w}(\mathbf{X}_{bB} - \lambda\mathbf{G}_{AB})d\lambda, \tag{75}$$

where the two variables are defined by Eqs. (68)–(71). This method is similar to what was proposed in Refs. [2,30].

Then, all the results in the uniformly moving case can be derived from that in the static case by replacing $\mathbf{x}_{bB}$ by $\mathbf{X}_{bB}$ and $\mathbf{R}_{AB}$ by $\mathbf{G}_{AB}$. We can use those conversions in our phase function, where for each "static case" quantity on the left we give the "moving case" equivalent on the right:

$$\begin{aligned}
\mathbf{x}_{bB} &\to \mathbf{X}_{bB},\\
r_{bB} = |\mathbf{x}_{bB}| &\to X_{bB} = |\mathbf{X}_{bB}|,\\
\mathbf{n}_{bB} &\to \mathbf{N}_{bB} = \frac{\mathbf{X}_{bB}}{X_{bB}},\\
\mathbf{x}_{bA} = \mathbf{x}_{bB} - \mathbf{R}_{AB} &\to \mathbf{X}_{bB} - \mathbf{G}_{AB} = \mathbf{X}_{bA} + \gamma\mathbf{p}R_{AB},\\
r_{bA} = |\mathbf{x}_{bA}| &\to X_{bA} = |\mathbf{X}_{bA}|,\\
\mathbf{n}_{bA} &\to \mathbf{N}_{bA-G} = \frac{\mathbf{X}_{bA} + \gamma\mathbf{p}R_{AB}}{|\mathbf{X}_{bA} + \gamma\mathbf{p}R_{AB}|},\\
\mathbf{R}_{AB} &\to \mathbf{G}_{AB} = \mathbf{g}_{AB}R_{AB},\\
R_{AB} &\to R_{AB}|\mathbf{g}_{AB}|,\\
\mathbf{k} = \frac{\mathbf{R}_{AB}}{R_{AB}} &\to \frac{\mathbf{g}_{AB}}{|\mathbf{g}_{AB}|}.
\end{aligned} \tag{76}$$

With Eqs. (72)–(75) and the conversions (76), we can rewrite the one-order phase function in the field of a rotating and moving body as

$$\begin{aligned}
&\varphi^{(1)}(\mathbf{x}_A, t_B, \mathbf{x}_B)\\
&= -\frac{R_{AB}k_0}{2}\left\{\frac{4\gamma^2(1-\mathbf{k}\cdot\mathbf{p})^2}{c^2} I(\mathbf{X}_{bA} + \gamma\mathbf{p}R_{AB}, \mathbf{X}_{bB})\right.\\
&\quad - \frac{8\gamma(1-\mathbf{k}\cdot\mathbf{p})}{c^3}\mathbf{k}\cdot\mathbf{S}(\mathbf{X}_{bA} + \gamma\mathbf{p}R_{AB}, \mathbf{X}_{bB})\\
&\quad + \frac{8}{c^3}\frac{\gamma}{(1+\gamma)}[\gamma^2(1-\mathbf{k}\cdot\mathbf{p})^2\\
&\quad \left. + \gamma(1-\mathbf{k}\cdot\mathbf{p})]\mathbf{p}\cdot\mathbf{S}(\mathbf{X}_{bA} + \gamma\mathbf{p}R_{AB}, \mathbf{X}_{bB})\right\}.
\end{aligned} \tag{77}$$

Finally, this expression can be rewritten as

$$\begin{aligned}
\varphi^{(1)}(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \gamma(1-\mathbf{k}\cdot\mathbf{p})\varphi_I'^{(1)} + \mathbf{k}\cdot\varphi_S'^{(1)}\\
&\quad - \frac{\gamma}{1+\gamma}[\gamma(1-\mathbf{k}\cdot\mathbf{p}) + 1]\mathbf{p}\cdot\varphi_S'^{(1)},
\end{aligned} \tag{78}$$

where $\varphi_{I/S}'^{(1)}(\mathbf{x}_{bA}, \mathbf{x}_{bB})$ denotes the expression in the static case and $\varphi_{I/S}'^{(1)} = \varphi_{I/S}'^{(1)}(\mathbf{X}_{bA} + \gamma\mathbf{p}R_{AB}, \mathbf{X}_{bB})$ is given by

$$\begin{aligned}
\varphi_I'^{(1)} &= -\frac{2G_{AB}k_0}{c^2} I'(\mathbf{G}_{AB}, \mathbf{X}_{bB})\\
&= -\frac{2G_{AB}k_0}{c^2}\int_0^1 w(\mathbf{X}_{bB} - \lambda\mathbf{G}_{AB})d\lambda,
\end{aligned} \tag{79}$$

$$\begin{aligned}
\varphi_S'^{(1)} &= \frac{4G_{AB}k_0}{c^3}\mathbf{S}'(\mathbf{G}_{AB}, \mathbf{X}_{bB})\\
&= \frac{4G_{AB}k_0}{c^3}\int_0^1 \mathbf{w}(\mathbf{X}_{bB} - \lambda\mathbf{G}_{AB})d\lambda,
\end{aligned} \tag{80}$$

respectively. The expression (78) is very useful, since it allows us to determine the phase function or time transfer function in the case of a rotating and uniformly moving body from the corresponding static phase function or time transfer function. It will recover the expression of the static phase function when the coordinate velocity of the body is zero, $\mathbf{p} = 0$. If the phase function describes the case of N rotating, uniformly moving, and spherically symmetric bodies at 1PM, we can express it with a linear summation for every body.

The derivatives of the phase function can be computed from (78), which can be used to compute the frequency shift and direction. In the case of a rotating and uniformly moving body, their expressions are given by

$$\begin{aligned}
\partial_{x_A^i}\varphi^{(1)}(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \gamma(1-\mathbf{k}\cdot\mathbf{p})\partial_{Aj}\varphi_I'^{(1)}\left[\delta_{ij} + \frac{\gamma^2}{1+\gamma}p^ip^j - \gamma\mathbf{k}^ip^j\right] + \frac{\gamma(p^i - \mathbf{k}^i\mathbf{k}\cdot\mathbf{p})}{R_{AB}}\varphi_I'^{(1)}\\
&\quad + \mathbf{k}\cdot\partial_{Aj}\varphi_S'^{(1)}\left[\delta_{ij} + \frac{\gamma^2}{1+\gamma}p^ip^j - \gamma\mathbf{k}^ip^j\right]\\
&\quad - \frac{\gamma}{1+\gamma}[\gamma(1-\mathbf{k}\cdot\mathbf{p}) + 1]\mathbf{p}\cdot\partial_{Aj}\varphi_S'^{(1)}\left[\delta_{ij} + \frac{\gamma^2}{1+\gamma}p^ip^j - \gamma\mathbf{k}^ip^j\right] - \frac{\gamma^2}{1+\gamma}\mathbf{p}\cdot\varphi_S'^{(1)}\frac{p^i - \mathbf{k}^i\mathbf{k}\cdot\mathbf{p}}{R_{AB}},
\end{aligned} \tag{81}$$

$$\begin{aligned}
\partial_{x_B^i}\varphi^{(1)}(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \gamma(1-\mathbf{k}\cdot\mathbf{p})\partial_{Bj}\varphi_I'^{(1)}\left[\delta_{ij} + \frac{\gamma^2}{1+\gamma}p^ip^j\right] + \gamma(1-\mathbf{k}\cdot\mathbf{p})\partial_{Aj}\varphi_I'^{(1)}\gamma\mathbf{k}^ip^j\\
&\quad - \frac{\gamma(p^i - \mathbf{k}^i\mathbf{k}\cdot\mathbf{p})}{R_{AB}}\varphi_I'^{(1)} + \mathbf{k}\cdot\partial_{Bj}\varphi_S'^{(1)}\left[\delta_{ij} + \frac{\gamma^2}{1+\gamma}p^ip^j\right] + \mathbf{k}\cdot\partial_{Aj}\varphi_S'^{(1)}\gamma\mathbf{k}^ip^j + \frac{\gamma^2}{1+\gamma}\mathbf{p}\cdot\varphi_S'^{(1)}\frac{p^i - \mathbf{k}^i\mathbf{k}\cdot\mathbf{p}}{R_{AB}}\\
&\quad - \frac{\gamma}{1+\gamma}[\gamma(1-\mathbf{k}\cdot\mathbf{p}) + 1]\mathbf{p}\cdot\partial_{Bj}\varphi_S'^{(1)}\left[\delta_{ij} + \frac{\gamma^2}{1+\gamma}p^ip^j\right] - \frac{\gamma}{1+\gamma}[\gamma(1-\mathbf{k}\cdot\mathbf{p}) + 1]\mathbf{p}\cdot\partial_{Aj}\varphi_S'^{(1)}\gamma\mathbf{k}^ip^j,
\end{aligned} \tag{82}$$

$$\partial_{t_B}\varphi^{(1)}(\mathbf{x}_A,t_B,\mathbf{x}_B)=-c\gamma p^j\Big\{\gamma(1-\mathbf{k}\cdot\mathbf{p})\partial_{Bj}\varphi_I'^{(1)}+\mathbf{k}\cdot\partial_{Bj}\varphi_S'^{(1)}-\frac{\gamma}{1+\gamma}[\gamma(1-\mathbf{k}\cdot\mathbf{p})+1]\mathbf{p}\cdot\partial_{Bj}\varphi_S'^{(1)}$$
$$+\gamma(1-\mathbf{k}\cdot\mathbf{p})\partial_{Aj}\varphi_I'^{(1)}+\mathbf{k}\cdot\partial_{Aj}\varphi_S'^{(1)}-\frac{\gamma}{1+\gamma}[\gamma(1-\mathbf{k}\cdot\mathbf{p})+1]\mathbf{p}\cdot\partial_{Aj}\varphi_S'^{(1)}\Big\},\tag{83}$$

where $\partial_{A/Bj}\varphi_{I/S}'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})$ is the expression of the derivative of the static phase function,

$$\partial_{Aj}\varphi_{I/S}'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})=\frac{\partial\varphi_{I/S}'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})}{\partial x_{bA}^j},\tag{84}$$

$$\partial_{Bj}\varphi_{I/S}'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})=\frac{\partial\varphi_{I/S}'^{(1)}((\mathbf{x}_{bA},\mathbf{x}_{bB})}{\partial x_{bB}^j}.\tag{85}$$

The expressions of the derivatives of the phase function in the moving case are also obtained by inserting into Eqs. (81)–(83) the static phase function and its derivatives, keeping in mind the conversions (76).

In the case of a spherically symmetric body, the scalar potential and the vector potential can be expressed, respectively, as

$$w(\mathbf{y})=\frac{M}{|\mathbf{y}|},\tag{86}$$

$$\mathbf{w}(\mathbf{y})=-\frac{M}{2|\mathbf{y}|^3}(\mathbf{y}\times\mathbf{S}),\tag{87}$$

where M is the mass of the body and $\mathbf{S}$ is the body's spin moment (angular momentum per unit of mass). Therefore, the $\varphi_I'^{(1)}$ and $\varphi_S'^{(1)}$ are given, respectively, by

$$\varphi_I'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})=-\frac{2Mk_0}{c^2}\ln\frac{r_{bA}+r_{bB}+R_{AB}}{r_{bA}+r_{bB}-R_{AB}}\tag{88}$$

and

$$\varphi_S'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})=\frac{4Mk_0}{c^3}\mathbf{S}\times\frac{(\mathbf{n}_{bB}+\mathbf{n}_{bA})R_{AB}}{(r_{bA}+r_{bB})^2-R_{AB}^2}.\tag{89}$$

By inserting Eqs. (88) and (89) into Eq. (78) and using the conversions (76), we can obtain the one-order phase function in the case of a rotating, uniformly moving, and spherically symmetric body as

$$\varphi^{(1)}(\mathbf{x}_A,t_B,\mathbf{x}_B)=-\frac{2Mk_0\gamma(1-\mathbf{k}\cdot\mathbf{p})}{c^2}\ln\frac{|\mathbf{X}_{bA}+\gamma\mathbf{p}R_{AB}|+|\mathbf{X}_{bB}|+R_{AB}\gamma(1-\mathbf{k}\cdot\mathbf{p})}{|\mathbf{X}_{bA}+\gamma\mathbf{p}R_{AB}|+|\mathbf{X}_{bB}|-R_{AB}\gamma(1-\mathbf{k}\cdot\mathbf{p})}$$
$$+\frac{4Mk_0}{c^3}\mathbf{k}\cdot\left(\mathbf{S}\times\frac{(\mathbf{N}_{bB}+\mathbf{N}_{bA-G})R_{AB}\gamma(1-\mathbf{k}\cdot\mathbf{p})}{(|\mathbf{X}_{bA}+\gamma\mathbf{p}R_{AB}|+|\mathbf{X}_{bB}|)^2-(R_{AB}\gamma(1-\mathbf{k}\cdot\mathbf{p}))^2}\right)$$
$$-\frac{\gamma}{1+\gamma}[\gamma(1-\mathbf{k}\cdot\mathbf{p})+1]\mathbf{p}\cdot\left(\mathbf{S}\times\frac{(\mathbf{N}_{bB}+\mathbf{N}_{bA-G})R_{AB}\gamma(1-\mathbf{k}\cdot\mathbf{p})}{(|\mathbf{X}_{bA}+\gamma\mathbf{p}R_{AB}|+|\mathbf{X}_{bB}|)^2-(R_{AB}\gamma(1-\mathbf{k}\cdot\mathbf{p}))^2}\right).\tag{90}$$

In order to compute the derivatives of the phase function in the case of a rotating and uniformly moving body, we need to give the derivatives of the phase function in the static case. From the expressions (88) and (89), the derivatives of $\varphi_I'^{(1)}$ and $\varphi_S'^{(1)}$ are given by

$$\partial_{x_{bA}^i}\varphi_I'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})=\frac{4Mk_0}{c^2}\frac{\mathbf{k}^i(r_{bA}+r_{bB})+n_{bA}^iR_{AB}}{(r_{bA}+r_{bB})^2-R_{AB}^2},\tag{91}$$

$$\partial_{x_{bB}^i}\varphi_I'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})=-\frac{4Mk_0}{c^2}\frac{\mathbf{k}^i(r_{bA}+r_{bB})-n_{bB}^iR_{AB}}{(r_{bA}+r_{bB})^2-R_{AB}^2},\tag{92}$$

and

$$\partial_{x_{bA}^i}\varphi_S'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})=\frac{4Mk_0}{c^3}\mathbf{S}\times\left[\frac{-(\mathbf{n}_{bB}+\mathbf{n}_{bA})\mathbf{k}^i}{(r_{bA}+r_{bB})^2-R_{AB}^2}-\frac{(\mathbf{n}_{bB}+\mathbf{n}_{bA})R_{AB}}{((r_{bA}+r_{bB})^2-R_{AB}^2)^2}(2(r_{bA}+r_{bB})n_{bA}^i+2R_{AB}\mathbf{k}^i)\right],\tag{93}$$

$$\partial_{x_{bB}^i}\varphi_S'^{(1)}(\mathbf{x}_{bA},\mathbf{x}_{bB})=\frac{4Mk_0}{c^3}\mathbf{S}\times\left[\frac{(\mathbf{n}_{bB}+\mathbf{n}_{bA})\mathbf{k}^i}{(r_{bA}+r_{bB})^2-R_{AB}^2}-\frac{(\mathbf{n}_{bB}+\mathbf{n}_{bA})R_{AB}}{((r_{bA}+r_{bB})^2-R_{AB}^2)^2}(2(r_{bA}+r_{bB})n_{bB}^i-2R_{AB}\mathbf{k}^i)\right].\tag{94}$$

The derivatives (81)–(83) of the phase function in the case of a rotating, uniformly moving, and spherically symmetric body are given by combining Eqs. (88), (89), and (91)–(94) with the conversions (76). We can easily obtain the TTF and derivatives of the TTF from the phase function and the derivatives of the phase function.

As an example, we use the results presented in the previous section to give estimates of the relativistic corrections on the observables for the TianQin mission [25] and BEACON mission [26,31]. TianQin is a space-borne gravitational wave detector, which relies on a constellation of three identical spacecraft, placed on nearly identical geocentric orbits with a semimajor axis of $\sim 10^8$ m, and forming a nearly equilateral triangle. Using the parameters of TianQin, we can estimate the contributions of the mass monopole of Earth on the range. The contributions can be split into two parts: (1) a part related to the case where Earth is static; (2) a part proportional to Earth's velocity. The magnitude of part (1) is 10^{-2} m. The magnitude of part (2) is 10^{-8} m, which is not neglected for the accuracy of TianQin. These estimates suggest that the contributions of the motion of Earth need to be considered carefully. Then, the contributions of the spin of Earth on the range also can be split into two parts: (3) a part related to the case where Earth is static; (4) a part proportional to Earth's velocity. The magnitude of part (3) is 10^{-10}–10^{-9} m, which is a measurable part for measurement. The magnitude of part (4) is 10^{-14}–10^{-13} m. This part is related to the motion of Earth respect to the Sun, which has a period of one year. It is an important part for the space-borne gravitational wave detection missions in the future.

The BEACON concept is a space-borne experiment designed to test the metric nature of gravitation—a fundamental postulate of Einstein's general relativity. Its architecture is based on a constellation of four small spacecraft placed on the circular Earth orbit at a radius of $\sim 8 \times 10^7$ m. Using the parameters of the preliminary BEACON mission concept, we can estimate the contributions of the mass monopole of Earth on the range. The contributions can be split into two parts: (1) a part related to the case where Earth is static; (2) a part proportional to Earth's velocity. The contribution of part (1) is a few centimeters. The magnitude of part (2) is 10^{-8} m, which is not neglected for the accuracy of 0.1 nm on the range. These estimates suggest that the contributions of the motion of Earth need to be considered carefully. Then, the contributions of the spin of Earth on the range also can be split into two parts: (3) a part related to the case where Earth is static; (4) a part proportional to Earth's velocity. The magnitude of part (3) is 10^{-8} m, which is measurable for the accuracy of 0.1 nm on the range. The variation of this part is $\sim 10^{-11}$ m under the spacecraft's modulation, which is close to the accuracy of BEACON. This contribution needs to be considered carefully in the future. The magnitude of part (4) is less than 1 pm, which can be omitted for BEACON. This part is related to the motion of spacecraft respect to Earth, which has a period of a few days. It is clear that our results can be applied to high-precision space missions.

VI. CONCLUSION

In this paper, we study the propagation of light traveling through the gravitational field by the phase function method in the post-Minkowskian approximation of general relativity. By solving the eikonal equation, we give the general post-Minkowskian expansion of the phase function. Any nth-order perturbation $\varphi^{(n)}(x_A, x_B)$ is an integral taken along the zeroth-order curve joining x_A and x_B.

The phase function contains all the information about light propagation in the gravitational field, such as the time transfer function, frequency shift, astrometric observables, and so on. As the applications of the phase function, we have determined the specific phase function $\varphi(x_A, x_B)$ and time transfer function $T(x_A, x_B)$ in the field of a static, spherically symmetric body at the second post-Minkowskian approximation. We also determine the frequency shift up to the order of c^{-3} in this gravitational field. A rough estimate demonstrates that the effects of the second order in G must be taken into account for some space missions, such as Global Astrometric Interferometer for Astrophysics (GAIA) [32] and Space Interferometric Mission (SIM) [33]. As another application, we develop a highly accurate relativistic model that describes observations of the modern space missions. This model is more suitable for high-precision space missions in the Solar System, since it contains the effects due to the motion of rotating bodies. The phase function in the gravitational field of rotating and uniformly moving bodies can be derived from its expression in a stationary gravitational field. We use our model to give some estimates of the relativistic corrections on the observables for the TianQin mission and BEACON. The contribution of Earth's spin on the range reaches 10^{-8} for the BEACON, which need to be considered for high-precision space missions testing general relativity. The contribution of the motion of Earth's spin on the range reaches 10^{-13} m for TianQin, and this contribution must be considered for future space-borne gravitational wave missions.

ACKNOWLEDGMENTS

The authors thank the anonymous referee and Jean-Michel Le Floch for the useful comments that helped to improve the readability of the manuscript. This work is supported by the National Natural Science Foundation of China (Grant No. 91636221).

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